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Conservation laws in the field-theoretic representation of the Dirac theory

By **Cornel Lanczos** in Berlin

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Translated by D. H. Delphenich

It will be shown that the Dirac "current vector" in the new representation does not correspond to a vector, but a tensor: the "stress-energy tensor." The Dirac conservation law corresponds to the conservation law of impulse and energy. The stress-energy tensor is composed of two parts: an "electromagnetic" part that has the same structure as the stress-energy tensor of the Maxwell tensor and a "mechanical" one that corresponds to the hydrodynamical Ansatz. The coupling between these two tensors that appears organically here raises the known contradiction that has adhered to the dynamics of the electron theory.

In two previous papers [*] the author showed a new possibility for the representation of the Dirac theory that was constructed exclusively from the conventional relativistic spacetime structure and worked only with known field-theoretic concepts. In one respect, the new representation exhibited a peculiar lack: It did not succeed in singling out a vector that would correspond to the fundamental divergence-free "current vector" of the Dirac theory. The vector that one could regard as the analogous object to the Dirac current vector [cf., expression (90) in the first paper] was not divergence-free, while the object that was actually divergence-free [cf., expression (13) in the second paper] did not represent a vector. Now – as the author has recognized in the meantime – this difficulty resolves into a state of affairs that admits much larger perspective for the field-theoretic representation and seems to strengthen the intrinsic reality of the entire argument.

We mentioned that the two Dirac equations for H and H' [see equation (8) in the second paper] of our total system of field equations are completely equivalent. One can thus define the "current vector" for these two Dirac equations (which is not complex and therefore actually represents a vector), and in this way obtain two divergence-free objects. We may not say "vectors," since the vector property under the transformation properties of F and G that were that were assumed by us no longer exists. The divergence-free character is, however, merely a consequence of the field equations and is independent of the transformation properties.

We thus have two divergence-free quaternions:

$$H\bar{H}^*, \quad H'\bar{H}'^*, \tag{1}$$

[*] ZS. f. Phys. **57**, 447 and 474, 1929.

and obviously any linear combination of them is also divergence-free.

For the moment, if we set:

$$A = \tfrac{1}{2}(F + G), \qquad B = \tfrac{1}{2}(F - G) \tag{2}$$

then we have:

$$\left.\begin{aligned} H &= A + iBj_z, \\ H' &= A - iBj_z. \end{aligned}\right\} \tag{3}$$

The divergence-free property is therefore also true for the following two quaternions:

$$\left.\begin{aligned} A\bar{A}^* + B\bar{B}^* &= \tfrac{1}{2}(F\bar{F}^* + G\bar{G}^*), \\ Bj_z\bar{A}^* + Aj_z\bar{B}^* &= \tfrac{1}{2}(Fj_z\bar{F}^* - Gj_z\bar{G}^*). \end{aligned}\right\} \tag{4}$$

Now, it is, however, obvious that the unit quaternion j_z is in no way distinguished from the other spatial units. The singling out of j_z came about only through a special notation of the Dirac equation and the arrangement of the ψ quantities that it demanded. We may therefore just as well set j_x or j_y in place of j_z .

In this way, we thus obtain four divergence-free quaternions, which we may also write collectively in the following way:

$$F\, j_\alpha \bar{F}^* + G\, \bar{j}_\alpha \bar{G}^*, \tag{5}$$

where j_α means any of the four unit quaternions.

The fact that the Dirac divergence equation has been quadrupled here leads one to conjecture that one is not dealing with a scalar, but a vector divergence. If this were true then the four quaternions (5) would be collectively equivalent to a tensor. Now, this is actually the case.

Namely, with the help of a vector V, one can define a vector from an anti-symmetric tensor F by means of the following quaternion product:

$$F\, V \bar{F}^*. \tag{6}$$

In fact:

$$F'\, V'\, \bar{F}'^* = p\, F\, \bar{p}\, p V\, \bar{p}^*\, p^* \bar{F}^* \bar{p}^* = p\,(F\, V\, \bar{F}^*)\, \bar{p}^*. \tag{7}$$

However, one can write:

$$F\, V \bar{F}^* = (F\, j_\nu \bar{F}^*) V_\nu \tag{8}$$

for (6), and one can also express the fact that a quaternion Q has the vector property by saying that:

$$Q_\mu\, U_\mu = \text{invariant}, \tag{9}$$

where the components of the quaternion Q are denoted by Q_μ and U is also a vector. We thus have:

$$(F\, j_\nu \bar{F}^*)_\mu\ U_\mu\, V_\nu = \text{invariant}, \tag{10}$$

and from the definition of a tensor this means that the quantities:

$$(F\, j_i \bar{F}^*)_k \tag{11}$$

are tensor components. In other words: If one writes the components of any quaternion $F\, j_\alpha \bar{F}^*$ one after the other in a row then the four rows, when taken together, would yield a matrix of 16 quantities that are tensor components.

One may proceed in a completely analogous way with the vector *G*. One can then likewise define a vector by means of the product $G\bar{V}G$, or also by means of:

$$G\bar{V}\bar{G}^* = (G\, \bar{j}_\nu \bar{G}^*)V_\nu\,. \tag{12}$$

One then has:

$$G'\bar{V}'\bar{G}'^* = p\, G\ \bar{p}^*\ p^*\,\bar{V}\ \bar{p}\ p\,\bar{G}^*\bar{p}^* = p\,(G\bar{V}\,\bar{G}^*)\,\bar{p}^*\,. \tag{13}$$

Thus the quantities:

$$(G\, \bar{j}_i \bar{G}^*)_k \tag{14}$$

also define the components of a tensor. It proves to be convenient to include the factor –1/2, and we then set:

$$T_{ik} = -\tfrac{1}{2}(F\, j_i \bar{F}^*)_k\,, \tag{15}$$

$$U_{ik} = -\tfrac{1}{2}(G\, \bar{j}_i \bar{G}^*)_k\,. \tag{16}$$

The divergence-free tensor that we shall denote by W_{ik} [†] is composed of these two tensors by addition, such that we have:

$$W_{ik} = T_{ik} + U_{ik}\,. \tag{17}$$

The Dirac conservation law for the four quaternions (5) thus takes the form of a divergence equation for this tensor:

$$\operatorname{div}(W_{ik}) = \frac{\partial W_{i\nu}}{\partial x_\nu} = 0, \tag{18}$$

which represents nothing but the conservation law for impulse and energy.

One may rightfully speak of the tensor W_{ik} that emerges here, and whose divergence vanishes, as the "stress-energy tensor," and we then arrive at the following remarkable result:

[†] The symbol *W* should not suggest "probability" (German = Wahrscheinlichkeit). When one can interpret the Dirac vector as the "probability flux," there is no longer any sense in giving an analogous meaning to a tensor of second order that enters here in place of the Dirac vector. I therefore believe that a compromise between the "reactionary" standpoint that is taken here, which aspires to a complete field-theoretic description on the basis of the conventional spacetime structure, and the probability-theoretic (statistical) standpoint no longer seems possible at this stage.

In place of the Dirac current vector, one has the stress-energy tensor, and in place of the Dirac conservation law, one has the impulse-energy law.

The Dirac current vector was an extension of the scalar $\psi\psi^*$ that Schrödinger has interpreted as the "density of electricity." Here, the vector will now be extended by a step, namely, to a second-order tensor [†]. However, the larger manifold of quantities can be accepted willingly when one reflects on the fundamental meaning of the stress-energy tensor for dynamics, and the fundamental meaning of the Riemann curvature tensor, by which one can conjecture a metric background for the entire theory that is given here, and a latent relationship to the most important and all-encompassing of all the disciplines that we are presented with: the general theory of relativity.

It is noteworthy that the stress-energy tensor that was introduced in (17) is not symmetric.

We first consider the tensor (15). It is written vector-analytically in the following form:

$$T_{ik} = S\,F_{ik} - M\tilde{F}_{ik} - (S^2 + M^2)\,g_{ik} + (F_i^{\ \nu} F_{k\nu} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} g_{ik}) \,. \tag{15′}$$

The first two terms are anti-symmetric, while the other one is symmetric.

The other tensor (16) takes the following form:

$$U_{ik} = (M_i\,S_k - M_k\,S_i) + M_i\,M_k + S_i\,S_k - \tfrac{1}{2}(M_\nu M^\nu + S_\nu S^\nu)\,g_{ik} \,. \tag{16′}$$

An anti-symmetric term is also present here that comes about by the interaction of the two vectors S_i and M_i .

It is now very remarkable that the stress-energy tensor immediately becomes symmetric when all of the quantities drop out that are foreign to the Maxwell theory, and thus, when the scalars S and M, as well as the magnetic vector M_i , are set equal to zero. In the first paper, we proved – in the absence of external fields – that this restriction is, in fact, possible, while in the second paper it was shown that the same thing can no longer be done with the introduction of vector potentials. From this page on, we shall also once more prove that the introduction of the vector potential into the equations is not happening in the correct way. The fundamental meaning of the stress-energy tensor would be lost if one would sacrifice its symmetry; as far as that is concerned, this is without a doubt possible.

If we preserve only the electromagnetic field strengths F_{ik} and the electric current vector S_i as the remaining fundamental quantities then the stress-energy tensor, which shall be symmetric from now on, seems to be composed of two parts.

The first – "electromagnetic" – part T_{ik} is completely identical with the Maxwell stress-energy tensor of electromagnetic fields:

$$T_{ik} = F_i^{\ \nu} F_{k\nu} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} g_{ik} \,. \tag{19}$$

One can also ascribe a certain meaning to the second part U_{ik} in light of a similar object in mechanics. It is:

[†] The process recalls the development of the theory of gravitation, where Newton's scalar potential was built up by Einstein into a second-order tensor.

$$U_{ik} = S_i\, S_k - \tfrac{1}{2}\, S_\nu S^\nu g_{ik}\,. \tag{20}$$

We may speak of this tensor as the "mechanical" stress-energy tensor. In fact, in the field-theoretic representation of the mechanical impulse-energy current, the symmetric tensor:

$$\mu_0\, u_i\, u_k\,, \tag{21}$$

is introduced as the "kinetic impulse-energy tensor" (Minkowski) [*]; μ_0 is therefore the scalar mass density, and u_i the velocity vector. Obviously, the object $S_i\, S_k$ is completely analogous, except that the current vector appears in place of the velocity. Moreover, this analogy is even more far-reaching, when we observe that for a static, spherically-symmetric solution the mean value of the spatial components of S_i necessarily vanish and only a temporal component can remain – thus, the mean value of the vector S_i is, in fact, known in direction of the velocity (which likewise has only a temporal component in the rest-system).

The second term of (20) is well-known, and from hydrodynamics, in fact. The additional Ansatz pg_{ik} for the matter tensor arises when p means the hydrostatic pressure (which is a scalar). The coupling (20) *now says that a hydrostatic pressure of magnitude* $\mu_0 / 2$ *is coupled with the mechanical mass density* μ_0. This pressure is enormous when one realizes that in the [CGS] system one must multiply by c^2. For water, one comes to the unimaginable magnitude of 4.5×10^{14} atmospheres [**]! Thus, this pressure is not to be interpreted as macroscopic for neutral matter. Moreover, we must regard it as a "cohesive pressure" that is required for the construction of electrons in order to maintain equilibrium with the large repulsive forces [***].

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[*] Cf., e.g., W. Pauli, *Relativitätstheorie*, Leipzig and Berlin, Teubner, 1921, pp. 675. M. v. Laue, *Die Relativitätstheorie* I, 4th ed., Braunschweig, Friedr. Vieweg and Sohn, 1921, pp. 207.

[**] This remarkable result recalls the known consequence of the theory of relativity, by which any mass m is linked with the immense energy $m\,c^2$. Just as the kinetic energy in mechanics represents only a small differential increment when compared to the rest energy, the ordinary hydrostatic pressure of gravitational origin seems to be a second-order quantity compared to the enormous "eigen-pressure" of matter here. This expression actually proves to be positive here, so it is directed inward, despite the apparent opposite sign in the last terms of equation (20). The square of the length of the velocity vector $u^k = i\, dx^k / ds$ is not $+ 1$, but $- 1$.

[***] Whether the electron is "smeared" in a self-explanatory way might be of great interest as a rough calculation in orders of magnitude in order to make a comparison with electron theory possible. We imagine a spherical shell of radius a that is uniformly endowed with charge and mass. The hydrostatic pressure that is coupled with the mass density $M / 4\pi a^2$ then give rise to an inward-directed force of magnitude $Mc^2 / 4\pi a^3$ per unit area. The electric pull from the outside has the magnitude $\frac{1}{2}\left(\frac{e}{4\pi a^2}\right)^2$. The equilibrium of the two forces demands that:

$$\frac{Mc^2}{4\pi a^3} = \frac{1}{2}\left(\frac{e}{4\pi a^2}\right)^2,$$

which gives the mechanical mass:

$$M = \frac{e^2}{8\pi c^2 a}\,.$$

The electrostatic energy of the field divided by c^2 – which is what electron theory refers to as the "electromagnetic mass" – is just as large.

If we imagine that the divergence of the Maxwell tensor gives the Lorentz force and the divergence of the mechanical tensor gives the inertial force then we see that on this basis, the dynamics of the electrons emerges from this as a closed, harmonious whole, which is never the case in classical field theory. Indeed, one has the conjecture that one still has to extend the electromagnetic quantities by mechanical ones in order to obtain the "cohesive force;" i.e., in order to avoid the well-known "blowing-up" of the electron and to make possible a differential formulation of dynamics. However, one has no reference point for an organic fusion of mechanics and electrodynamics.

Such an intrinsic coupling is present in the field equations that emerge here, as a result of the double coupling of field strengths and the current vector. The current vector thus ceases to be a "material" quantity that is forced upon one from the outside, and which does not actually belong to the field and only serves to introduce a singularity. Here, it further defines an actual field quantity that is determined by the field equations. Likewise, the vanishing divergence of the matter tensor is no heuristic principle, in order to obtain dynamics along with the field equations, but these basic dynamical equations seem *necessary consequences of the field equations.* The intrinsic coherence then has a structure that one finds in the theory of general relativity, where the divergence equation represents a mathematical identity of the curvature tensor, and the principal of the geodetic line is already established by the field equations [*].

Thus, the coupling (17) of the two essentially different tensors (15) and (16) is not superficial, but is uniquely determined by the structure of the theory. Then neither the one ("electromagnetic") nor the other ("mechanical") part has a vanishing divergence, but only the given sum, which leaves no factor or sign free.

The essential difficulties and intrinsic contradictions of the electron theory are thus eliminated and the hitherto-unsuspected connection is so compelling that one can scarcely doubt that this path leads to deeper results. The fact that the problem of the electron is already soluble is excluded, since it obviously lacks some essential traction. On the one hand, this is not by any means conceivable when one is dealing with a linear system of equations, and on the other hand (just like the situation with the classical field equations), these equations possess no regular "eigen-solutions" of a type that they would give rise to stationary energy nodes, as one might expect in a truly satisfactory "theory of matter."

However, the most serious objection that one can raise against the closely-related conjecture that would follow naturally from the connection between the Dirac theory and the Maxwell equations that was found here – viz., that quantum theory consists of the tail

[*] Whether it seems very plausible to place the dynamics of the electron on this basis, cannot be established on the basis of our Ansätze. The divergence equation says nothing as a consequence of the field equations that did not emerge from this. However, the field equations are linear and thus allow the superposition principle, which excludes any dynamic influence, *a priori*. It is quite clear that this dichotomy is very closely connected with the complication that we have often mentioned before that the incorporation of the vector potential into the equations, which originally happens on the grounds of the quantum-mechanical prescription, obviously leads to unsatisfactory consequences. Such an incorporation is not at all requisite, since the field quantities clearly exist in sufficient variety, and, in particular, the current vector already plays the role of a vector potential in the "back-reaction," such that this should not be specially introduced as a foreign element. The extension of the equations by the vector potential seems, in that regard, to be only a stopgap for a currently-unknown nonlinearity in the system. The principle of motion can, in fact, lie in the divergence equation, without contradicting the superposition principle (which is then no longer true).

end of a correction to classical field theory – consists in the fact that the theory of the electrons itself does not emerge as a "first approximation." In comparison to the theory of electrons, would like to once more write down a reduced system of equation (98) from the first paper, when we drop all quantities that are foreign to the theory of electrons. As the single difference, we would only like to introduce another field quantity:

$$\varphi_i = \frac{S_i}{\alpha}, \tag{22}$$

in place of the current vectors S_i – viz., the vector potential – as we might say as a result of the back-reaction equation.

We then have the equations:

$$\left.\begin{aligned} \frac{\partial F_{i\nu}}{\partial x_\nu} &= \alpha^2 \varphi_i, \\ \frac{\partial \varphi_i}{\partial x_k} - \frac{\partial \varphi_k}{\partial x_i} &= F_{ik}. \end{aligned}\right\} \tag{23}$$

In the vacuum equations for the theory of the electrons, the term on the right-hand side of the first equation is missing. We then obtain the classical field equations when we make the constant α go to 0. However, since the constant $\alpha = 2\pi mc / h$ includes Planck's constant in the denominator, this passage to the limit does not mean $h \to 0$, but $h \to \infty$. The macroscopic behavior of the electron will thus be characterized, not by the expected transition $h \to 0$, but by the unnatural transition $h \to \infty$. In fact, one could only consider the theory of the electron as the first approximation when the constant α of the theory is very small. In reality, however, this constant is very large: $\alpha = 2.59 \times 10^{10}\ \text{cm}^{-1}$. Thus, even if – as is indeed necessary in all cases – the equation were completed by hitherto-unknown quadratic terms (cf., the remark * on pp. 6), this would still not alter the fact that the macroscopic behavior of the electron would certainly not be given correctly. Then, at a large distance, where the quadratic terms have certainly already died off and the linear approximation seems correct, one would not obtain a potential that falls off like $1/r$, and furthermore, the potential would not behave like $e^{-\alpha r} / r$, with the strong damping constant α, which certainly contradicts experience, because it excludes any interaction of the electrons at a great distance whatsoever.

If we plausibly regard the quantum-mechanical reactions of the individual electrons (spin interactions, etc.) as an interaction at a very small distance then we can also say: The free electron behaves at small distances as if the constant α were very large, and at large distances, it behaves as if the same constant were very small. If we would not like to believe in the very unrealistic dualism that, in addition to the ordinary field-theoretic processes, there are also special "quantum-mechanical" processes then we would inevitably come to the demand that the constant α^2 of our theory should not be regarded as a real constant, but as a field function that depends upon the fundamental field quantities in a hitherto-unknown way *.

* In light of Einstein's theory of gravitation, the conjecture of a connection with the scalar Riemannian curvature (which likewise has the dimension cm^{-2}) is hardly known to be close at hand.

Then, in fact, everything becomes orderly. The term in α^2 is then no longer a linear term, but one of higher order. For the linear approximation, one then obtains the vacuum equations of the classical theory of the electron. In practice, the function α^2 would drop down to zero in the peripheral domain, while in the central region in the immediate neighborhood of the electron center, a practically flat function-range of the given order of magnitude would be expected. One can then understand why the de Broglie-Schrödinger wave equation with constant α cannot be characteristic of a single electron, but a large "swarm" of electrons. Then, as a result of the multiple spatial relationships between the different electrons, one can be required to take the statistical mean of the over a larger region with a sufficiently constant range to the function α, while for the single electron α goes to zero very rapidly.

In a comprehensive field theory, it would also be scarcely conceivable that one might introduce a quantity as a "universal constant" that includes the mass of the electron. It would then be completely hopeless for one to understand the mass difference between the electron and proton.

Naturally, the matter tensor would also come into contact with this new assumption. In fact, the constant α enters into the mechanical part of the matter tensor, as long as one introduces the vector potential into (22), in place of the current vector. However, we then have:

$$U_{ik} = \alpha^2 \left(\varphi_i \, \varphi_k - \tfrac{1}{2} \varphi_\nu \, \varphi^\nu \, g_{ik}\right). \tag{24}$$

In the peripheral region, where α dies off to zero, in practice, the mechanical part drops out and all that remains is the ordinary electromagnetic stress-energy tensor. In the central region, however (precisely where the constitution of the electron would demand!), the strictly mechanical part would come into action, and particularly the large force of cohesion, as well. Naturally, the Ansatz (24) (the sum defined by (17), resp.) for the matter tensor also comes into question now only in the first approximation, for slowly varying α. The divergence-free character of this tensor would then indeed be proved under the assumption of a constant α.

Should the possibilities that were anticipated here prove to be actually viable then quantum mechanics would prevail as a self-standing discipline. It would then fuse with a deeper "theory of matter" that is constructed from regular solutions of differential equations – in the final analysis, it would then go into the "world-equations" of the universe. The "matter-field" duality would then be just as obsolete as the "corpuscle-wave" duality.

Berlin-Nikolassee, August 1929.