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The tensor-analytic aspects of the Dirac equation

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The Dirac equation for the electron is treated in a new formalism with the assistance of quaternions, which seems adapted to the problem to a far-reaching degree. The transformation of the equations, as well as the invariant and covariant objects of the Dirac theory will be developed in a unified and systematic manner. It also uniquely shows a path to a covariant formulation of equations by means of ordinary tensor calculus when one carries out a doubling of the Dirac equation. The resulting system decomposes into two parts. The one part corresponds to a coupling of an anti-symmetric tensor with a vector of the same structure, like the ones that enter into the Maxwell equations today, as a relationship between electromagnetic field strengths and current vectors. The other part yields a “back-coupling” in the form of a back-reaction of the current vector on the field strengths of the same structure as is known for the electromagnetic field in the form of a connection between the vector potential and the field strengths.

1. Introduction. The Dirac equation is based upon two premises: On the one hand, it shall be a linear differential equation of first order, and on the other hand, the iterated application of the equation (in the field-free case) shall deliver the Schrödinger wave equation. The second viewpoint already practically includes the invariance under Lorentz transformations, without which, one would say that the transformations that thus arise must necessarily come about in the usual vector-analytic sense. The matrix calculus (the operator method, resp.) has the advantage that the solution of the problem can be delivered without having to worry about the demands of conventional tensor analysis. This advantage is thus contrasted with the awkwardness that arises when one must have recourse to quantities in a physical theory that do not allow an interpretation in terms of the usual tensor-analytic constructions that have long since proved to be far-reaching in the description of nature. Many different topics of research are undertaken on the basis of getting around the operator method and putting the Dirac equation into a form that one can interpret in terms of normal, vector-analytic constructions. C. G. Darwin [*] has established an analogy with Maxwell’s equations in the special case in which the mass term of the Dirac equation vanishes. On the other hand, Madelung [**] has developed a system of equations that is to be regarded as a generalization of the Maxwell equations and allows one to obtain a natural interpretation of the Maxwell equations. The invariance of this system under Lorentz transformations remains problematic [***].

[*] Proc. Roy. Soc. **120**, 621, 1928. Nature **123**, 203, 1929.

[**] ZS. f. Phys. **54**, 303, 1929.

[***] Remark by the editor: From friendly correspondence with Prof. Madelung, I learned that the invariance still exists. However, the transformation thus obtained – even if it is linear – has no useful

In the present paper, the author addresses the question from a somewhat different viewpoint. Without heuristically looking for analogies with the classical field equations, he examines the transformation properties of the Dirac equation in full generality with the use of a formalism that seems adapted to the problem to a high degree and is constructed from the "quaternions" that Hamilton introduced. The quaternion calculus has never gained any real currency in physics. The union of the vector and scalar multiplication into a single operation does not prove to be sufficiently elastic for the purposes of vector calculus when compared to the magnificently unified and consistent viewpoints of tensor analysis that operates with quantities that were based in special properties of three- or four-dimensional spaces, and must ultimately yield to the benefits of a purely component-wise representation. Nonetheless, the fact remains that quaternions can afford considerable practical simplifications in terms of organizing the general transformations of the Lorentz group; on the one hand, with their help, an arbitrary Lorentz transformation can be represented very simply, and on the other, those constructions that are to be interpreted as invariant (covariant, resp.) quantities in the sense of tensor analysis are also distinguished by special and easily-described properties in the quaternion calculus.

The author came to the present examination by the circumstance that ten years ago, in his doctoral work [*], he dealt with precisely the problem of establishing the connection between quaternions and Lorentz group and exhibiting how, with their help, the laws of special relativity theory come to be represented in a formally very simple way, insofar as they relate to the electromagnetic field. Later, he did not pursue the topic further along this path, since a random observation then led him to an investigation that was likewise undertaken in regard to the Maxwell equations with the same tools that are now applied to the Dirac equation, and, in fact, this path proved itself to be natural and practical. The observation was that the system of equations that he was treating, which was regarded as a generalization of the Maxwell equations, is remarkably equivalent to the Dirac equation for the case in which the mass term vanishes. Now, the addition of this term does not introduce any complications and only allows one, in a very simple way, to overlook the transformation properties of functions, while also giving a new outlook on the tensor-analytic sense of the Dirac equation, and can possibly lead an entirely new point of view. In fact, the following developments are of such almost trivial simplicity, and the direction of advance is described so uniquely that one can scarcely resist forming the impression that one possesses a *via regia* here that allows one to make inroads into the intrinsic essence of the Dirac equation in a more adequate way than is possible in the general operator method when it is not restricted to the particular problem.

Before we come to speak of our actual problem, permit us to briefly recapitulate the foundations of the less general quaternion calculus, and further sketch out the methods and results of the aforementioned doctoral work in their essential details (Sections 2, 3, 4).

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vector-analytic character, and therefore the system does not fall within the purview of the method followed here.

[*] "Die funktionstheoretischen Beziehungen der Maxwellschen Äthergleichungen," Publishing firm of Josef Nemeth, Budapest 1919. (Due to the difficulties associated with the post-war conditions, this work appeared in only fifty lithographed copies.) Inaugural dissertation at the University of Szeged, Hungary.

2. The quaternions. By a *quaternion*, one understands this to mean the combination of four quantities – viz., *components* – in the form:

$$Q = Xj_x + Yj_y + Zj_z + Tj_l\,. \tag{1}$$

The quantities (j_x, j_y, j_z, j_l) are four *unit vectors.* The notation l is chosen for the fourth component, instead of t, because we would like to reserve t for the real time, while one sets *:

$$l = i\,c\,t. \tag{2}$$

Outside of the self-explanatory addition, there is a fundamental operation of multiplication. It is determined once the multiplication of unit vectors is determined. We establish:

$$\left.\begin{array}{c} j_x j_y = j_z,\ j_y j_x = -j_x j_y,\ j_x j_l = j_l j_x = j_x, \\ j_x^2 = j_y^2 = j_z^2 = -j_l^2 = -j_l. \end{array}\right\} \tag{3}$$

The unwritten equations are provided here by cyclic permutation of the x, y, z. The fourth unit vector j_l , which is in the direction of the imaginary "time axis," behaves like the ordinary unit. One can therefore also set $j_l = 1$ (omit j_l as a factor, resp.).

The multiplication is associative, but not commutative. Moreover, in place of the simple commutative law, here, we have the law:

$$GF = \overline{\bar{F}\bar{G}} \tag{4}$$

or:

$$\overline{GF} = \bar{G}\bar{F}\,, \tag{4a}$$

where the overbar means the following: One goes to the *conjugate* of the quaternion; i.e., one takes the spatial components – the *spatial part*, as we would like to say – with the opposite sign:

$$\bar{F} = -Xj_x - Yj_y - Zj_z + T. \tag{5}$$

For an arbitrary choice of the number of factors, one has the rule that one obtains the conjugate of the product when one writes the sequence of factors in the opposite order and takes the conjugate everywhere.

One easily sees that the quantity $F\bar{F}$ is merely a number (the spatial components = 0).

3. Four-dimensional rotations and quaternions. One can regard the multiplication of quaternions F with the quaternions p in the sense of:

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* Naturally, in the pre-relativistic era Hamilton did not relate the fourth unit quaternion to physical time, and it is added as a scalar unit to the three spatial ones. We have likewise introduced the fourth dimension as the time dimension with regard to the applications that are of interest to us here.

$$F' = p\,F, \tag{6}$$

as a transformation of the line segment *F* when one represents the quaternion as a vector in four-dimensional space. The matrix of this transformation reads:

$$\begin{vmatrix} p_4 & -p_3 & p_2 & p_1 \\ p_3 & p_4 & -p_1 & p_2 \\ -p_2 & p_1 & p_4 & p_3 \\ -p_1 & -p_2 & -p_3 & p_4 \end{vmatrix}. \tag{7}$$

(Here, the components of the quaternion *p* are indicated by indices, instead of separate symbols.) If we establish that the square of the length of *p* shall be:

$$p\,\bar{p} = p_1^2 + p_2^2 + p_3^2 + p_4^2 = 1 \tag{8}$$

then we see that the transformation is orthogonal. We would like to briefly refer to an orthogonal transformation of this character as a *p-transformation.* As a result, the associative law of multiplication defines the structure of a *group* on the *p*-transformations within the general orthogonal transformations.

We obtain a second group when we multiply the quaternions, not on the left, but on the right:

$$F' = F\,q. \tag{9}$$

This *q-transformation* * has the matrix:

$$\begin{vmatrix} q_4 & q_3 & -q_2 & q_1 \\ -q_3 & q_4 & q_1 & q_2 \\ q_2 & -q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{vmatrix}. \tag{10}$$

The two groups are oppositely extended when one forms their composition in the most general group of orthogonal transformations. An arbitrary orthogonal transformation can be represented in the form:

$$F' = p\,F\,q, \tag{11}$$

with the auxiliary condition that:

$$p\,\bar{p} = 1, \qquad q\,\bar{q} = 1. \tag{12}$$

An arbitrary four-dimensional rotation is characterized by its six parameters. In fact, there are six parameters at our disposal between the two quaternions, since their lengths have been normalized to 1.

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* An alteration of this notation from the usual for the generalized coordinates and impulse should not be a source of concern, in view of the entirely different situation.

The purely spatial rotations, for which the time axis remains unchanged, define a subgroup of the general transformations. Thus, it is obvious that $\bar{F}'$ must be connected with $\bar{F}$, just as F' is connected with F. Since it follows from (11), by reversing the factors that:

$$\bar{F}' = \bar{q}\,\bar{F}\,\bar{p}\,, \tag{11a}$$

one must then have:

$$p = \bar{q}\,, \qquad q = \bar{p}\,. \tag{13}$$

The purely spatial rotations are thus characterized by the fact that the former quaternion is equal to the conjugate of the latter one.

In the four-dimensional space of reality, one of the dimensions is imaginary. Correspondingly, an orthogonal transformation cannot have all real coefficients, and we must regard the two characteristic quaternions as complex quantities. On the other hand, they may not be completely arbitrary complex quantities. Indeed, one is concerned with only real Lorentz transformations, for which the real (x, y, z, t) again goes to a real (x', y', z', t'). This restriction may be characterized in the following way: We consider the "position vector" $R = (x, y, z, l)$. We go to the complex conjugate, which we would always like to denote by a "star" (*). Due to the imaginary nature of l, only the temporal part of R changes its sign. We thus have:

$$R^* = -\bar{R}\,. \tag{14}$$

This peculiarity must also remain preserved in the primed system. Therefore, if:

$$R' = pR\,q \tag{15}$$

then one must have:

$$p^*\,R^*\,q^* = -\,\bar{q}\,\bar{R}\,\bar{p}\,, \tag{16}$$

and therefore:

$$p^* = \bar{q}\,, \qquad q^* = \bar{p}\,. \tag{17}$$

The real Lorentz transformations are then characterized by the fact that the former quaternion is the complex conjugate and the "overbarred" value of the latter one. It thus suffices to give p, since q is already determined. A real Lorentz transformation is thus represented in the following way:

$$F' = p\,F\,\bar{p}^*\,. \tag{18}$$

Since the latter quaternion must be equal to $\bar{p}$ in the case of spatial rotations we then see that the spatial rotations always belong to the real p.

4. The Hamiltonian operator. We now introduce the following differential operator (viz., the *gradient*):

$$\nabla = \left(j_x \frac{\partial}{\partial x} + j_y \frac{\partial}{\partial y} + j_z \frac{\partial}{\partial z} + \frac{\partial}{\partial l} \right), \tag{19}$$

which we would like to refer to as the *Hamiltonian operator* [†]. Under coordinate transformations, it behaves like a vector.

We apply this operator to a quaternion F in the following way:

$$\bar{\bar{\nabla}} F = \left(-\frac{\partial}{\partial x} j_x - \frac{\partial}{\partial y} j_y - \frac{\partial}{\partial z} j_z + \frac{\partial}{\partial x} j_l\right)(X\,j_x + Y\,j_y + Z\,j_z + T). \tag{20}$$

If we now set:

$$\bar{\bar{\nabla}} F = 0 \tag{21}$$

then we perform virtually the same processes in four dimensions as we do in the two dimensions of complex function theory when we write down the Cauchy-Riemann differential equations by setting:

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iy) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right) = 0.$$

This formal analogy was, in its own right, what led me to investigate equation (21).

When written component-wise, we obtain the following sequence of equations:

$$\left.\begin{aligned}
\frac{\partial X}{\partial l} - \frac{\partial T}{\partial x} + \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} &= 0,\\
\frac{\partial Y}{\partial l} - \frac{\partial T}{\partial y} + \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z} &= 0,\\
\frac{\partial Z}{\partial l} - \frac{\partial T}{\partial z} + \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} &= 0,\\
\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} + \frac{\partial T}{\partial l} &= 0.
\end{aligned}\right\} \tag{21a}$$

This operator is closely connected with the Maxwell equations of empty space. There, we have:

$$\left.\begin{aligned}
\frac{1}{c}\frac{\partial \mathfrak{E}}{\partial t} - \operatorname{rot}\mathfrak{H} &= 0,\\
\frac{1}{c}\frac{\partial \mathfrak{H}}{\partial t} + \operatorname{rot}\mathfrak{E} &= 0,\\
\operatorname{div}\mathfrak{E} &= 0,\\
\operatorname{div}\mathfrak{H} &= 0.
\end{aligned}\right\} \tag{22}$$

[†] For Hamilton himself, only the spatial part of this operator entered into consideration. Strictly speaking, we must then refer to an *extended Hamiltonian operator*. For the sake of brevity, we would thus like to distance ourselves from that; all the more so, since this extension is entirely self-explanatory for the relativistic applications of quaternions.

If one subtracts the second equation, multiplied by *i*, then one obtains the complex equation:

$$\frac{1}{c}\frac{\partial(\mathfrak{H}+i\mathfrak{E})}{i\partial t}-\mathrm{rot}(\mathfrak{H}+i\mathfrak{E}) = 0, \tag{23}$$

in which only the combination:

$$\mathfrak{F} = \mathfrak{H} + i\mathfrak{E} \tag{24}$$

appears. Likewise, one can combine the last two equations into the form:

$$\mathrm{div}\,(\mathfrak{H} + i\mathfrak{E}) = 0. \tag{25}$$

If one now denotes the components of $\mathfrak{F}$ by *X*, *Y*, *Z* (which can therefore be regarded as complex numbers) and writes down the component-wise equations then one sees, with no further assumptions, that the system thus obtained represents nothing but equation (21a) when one sets $T = 0$ in it.

Equation (21), which is Lorentz-invariant and delivers the wave equation for all components, just like the Maxwell equations, can therefore be regarded as its natural formal extension. A second application of the ∇-operation yields the wave equation in the form:

$$\nabla(\bar{\nabla}F) = (\nabla\bar{\nabla})F = 0. \tag{26}$$

In fact, the operator $\nabla\bar{\nabla}$ is obviously a scalar and nothing but the Laplacian Δ in four dimensions:

$$\nabla\bar{\nabla} = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial l^2}\,. \tag{27}$$

5. The Dirac equation for the case of vanishing mass. If one adds the second equations in equation (21a), multiplied by *i*, to the first one, and similarly, the fourth equation to the third one, then this yields two equations, into which only the combinations $X + iY$ and $Z + iT$ enter:

$$\left.\begin{aligned}\frac{\partial(X+iY)}{\partial l}+i\frac{\partial(Z+iT)}{\partial x}-\frac{\partial(Z+iT)}{\partial y}-i\frac{\partial(X+iY)}{\partial z}=0,\\ \frac{\partial(Z+iT)}{\partial l}+i\frac{\partial(X+iY)}{\partial x}+\frac{\partial(X+iY)}{\partial y}+i\frac{\partial(Z+iT)}{\partial z}=0.\end{aligned}\right\} \tag{28}$$

Naturally, these two equations cannot replace the original four. The halving of the number of equations from the eight of the Maxwellian schema to four comes about due to the fact that we went from real to complex variables. Here, however, the *X*, *Y*, *Z*, *T* themselves are already complex. However, we can still add two further equations when we consider the system of equations for the complex conjugate quantities, and apply the same operations. Obviously, we then have to provide all components with a star and to observe, in addition, that the term with $\partial / \partial l$ changes sign. We then obtain the same equations for the "starred" quantities, except with the opposite signs in the first terms.

If we now make the following association:

$$\left.\begin{aligned} X + iY = \psi_4, \;\; X^* + iY^* = \psi_2, \\ Z + iT = \psi_3, \;\; Z^* + iT^* = \psi_1 \end{aligned}\right\} \tag{29}$$

then we see, with no further assumptions, that we have the Dirac equations before us in the form that Weyl † wrote down explicitly in his textbook, when we drop the mass term. We then see that the operator ∇ can completely replace the Dirac operator, which is linked with the advantage that the Hamiltonian operator has a much closer relationship with vector-analytic quantities.

Let it be remarked that the association of the Dirac equations to the Maxwellian ones was treated in a similar manner by C. G. Darwin ††, except that one sets $T = 0$ in the Maxwellian case. The complication then arises that ψ_1 and ψ_2 are not independent of each other, and furthermore, one of them is already given in terms of the other one. By contrast, equations (21) admit just as many degrees of freedom as would correspond to the Dirac equations, and are, in fact, equivalent to them.

Before we go on to the consideration of the completed Dirac equations, we would like to establish an important peculiarity of equation (21) that we shall address later on. The transformation of the quantity F under a Lorentz transformation will obviously not be prescribed uniquely by the equations. One already has, with no rotation of the system of axes, that the associative law of multiplication implies a transformation of the form:

$$F' = Fk \tag{30}$$

(where k is an arbitrary quaternion) is possible, without which, something would change in the system. In general, one can only say so much about Lorentz transformation:

$$\nabla' = p \, \nabla \, \bar{p}^* \tag{31}$$

for the transformation of F, namely, that one must have:

$$F' = p \, F \, k, \tag{32}$$

where k may still be an arbitrary quaternion.

This multi-valuedness does not appear in the special case of the Maxwell equations, because $T = 0$ from the outset, and one arrives at the fact that $T' = 0$ must also remain true in the new system. The rotation that F experiences under the transformation (32) must necessarily be purely spatial, and we obtain:

† Hermann Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, Verlag S. Hirzel, 1928. pp. 271.
†† loc. cit.; see the remark * on pp. 1.

$$k = \bar{p}\,. \tag{33}$$

We know that the electromagnetic field strength transforms like an anti-symmetric tensor (six-vector) in the four-dimensional schema. We thus infer: If we know that a complex quaternion transforms like:

$$p\,F\,\bar{p} \tag{34}$$

then the temporal part is an invariant of the transformation, while the real (imaginary, resp.) part of the spatial components can be combined into a six-vector. Naturally, one can also immediately give the proof of this type of transformations for an anti-symmetric tensor, so, for the sake of brevity, we shall not do so.

6. The Dirac equation for an imaginary mass. It seems natural to attempt to introduce the mass term in such a way that the right-hand side of equation (21) is not set to 0, but a term that is proportional to F is introduced. One must thus observe that it is not the two-fold repetition of the $\bar{\nabla}$-operation, but the $\nabla\bar{\nabla}$ operation, that delivers the wave equation. On the other hand, we can make use of the specific peculiarity of the four-dimensional spacetime continuum that one has:

$$\bar{\nabla} = -\,\nabla^*. \tag{35}$$

We now make the Ansatz:

$$\bar{\nabla}F = \alpha\, i\, F^*, \tag{36}$$

where α is any real number. From this equation, it now follows, upon consideration of the rule (35):

$$\bar{\nabla}F^* = \alpha\, i\, F, \tag{37}$$

and we obtain, when we apply the operator ∇ to the first equation:

$$\nabla\bar{\nabla}F = -\alpha^2 F. \tag{38}$$

We thus obtain, in effect, the desired Schrödinger equation, but with the opposite signs on the mass terms, which is possible only for imaginary masses. The change of sign is – as we will see later on – deeply rooted, and may not be removed by any introduction of a numerical factor into the equation. Despite the fact that an imaginary mass obviously has no real meaning, it is still of heuristic value to briefly discuss eq. (36). If we combine any two equations again in precisely the schema of equations (28), and we again introduce the ψ-quantities in the sense of the association (29), then we obtain precisely the Dirac equation when we set the mass m equal to the imaginary value:

$$m = -\,\frac{h}{2\pi c}\,\alpha\, i. \tag{39}$$

The transformation law for the quaternion F in the case of a Lorentz transformation may be found quite easily. If we set:

$$F' = x\,F\,y, \tag{40}$$

then this yields:

$$\bar{\nabla}' F' = p^* \bar{\nabla}\, \bar{p}\, x F y\,. \tag{41}$$

One must then have:

$$x^* = p^*, \qquad x = p, \quad y = y^*, \tag{42}$$

so equation (36) also remains valid in the transformed system. The second condition is equivalent to the first one. The third condition says that y must be a real quaternion.

The rule for the transformation of F then reads:

$$F' = p\,F\,r, \tag{43}$$

with arbitrary real r, whose length we would, however, like to think of as normalized to 1. One can obviously assign no invariant sense to a quantity that transforms in this way. p is real under purely spatial rotations, so one can set $r = \bar{p}$ and F would be a vector. For general Lorentz transformations, however, such an arrangement is impossible.

This is also nothing too surprising. In presenting equation (36), we are essentially exploiting the property of the spacetime continuum that one of its dimensions is imaginary. In a purely real four-dimensional manifold, an equation with the character of (36) would lose its meaning. The conceptual structures of tensor analysis are, however, of such a type that they nowhere exploit the reality of the fundamental quadratic form, and the peculiarity + 3, –1 of the index of inertia of reality can be regarded as a random accident. Here, however, it is precisely this peculiarity that will be exploited.

If F itself also has no invariant sense then one can construct quantities with its help that take on such a sense. We first have an invariant:

$$F'\bar{F}' = p\,F\,r\;\bar{r}\;\bar{F}\;\bar{p} = p\,F\;\bar{F}\;\bar{p} = F\bar{F}\,. \tag{44}$$

It appears as a complex quantity, and is therefore equivalent to two real invariants, which are precisely the two invariants of Dirac theory. Furthermore, the quantities:

$$F\bar{F} \tag{45}$$

define a vector. Indeed, the transformation law reads:

$$F'\bar{F}'^* = p\,F\,r\;\bar{r}^*\bar{F}^*\bar{p}^* = p(F\,\bar{F}^*)\,\bar{p}^*, \tag{46}$$

since r is real. This vector, whose spatial part is purely imaginary and whose temporal part is purely real, is nothing but the *probability current vector* in the Dirac theory.

There are no other covariant constructions, as long as one places no restriction on r, which cannot be done without some degree of arbitrariness.

7. The Dirac equation for real masses. One can also arrive at the Schrödinger equation in the following general way: We do not set $\overline{\nabla}F$ equal to a quantity that is proportional to *F*, but further introduce a new quantity *G:*

$$\overline{\nabla}F = \alpha\, G. \tag{47}$$

If we now require that:

$$\nabla G = \beta\, F \tag{48}$$

then this obviously yields:

$$\nabla\overline{\nabla}F = \alpha\,\beta\, F. \tag{49}$$

In this, the product $\alpha\beta$ can be positive, as well as negative. By absorbing a factor into *G*, one can arrive at the fact that the constants on the right-hand side are either equal to each other or equal and opposite, and thus bring the system into two normal forms. Namely, one first has:

$$\left.\begin{aligned}\overline{\nabla}F &= \alpha G,\\ \nabla G &= \alpha F,\end{aligned}\right\} \tag{50}$$

or secondly:

$$\left.\begin{aligned}\overline{\nabla}F &= \alpha G,\\ \nabla G &= -\alpha F.\end{aligned}\right\} \tag{51}$$

We first consider the second case. If we go over to complex conjugate quantities in the second equation then we can also replace (51) with:

$$\left.\begin{aligned}\overline{\nabla}F &= \alpha G,\\ \overline{\nabla}G^{*} &= \alpha F^{*}.\end{aligned}\right\} \tag{51a}$$

One can now add these two equations and write:

$$\overline{\nabla}\,(F + iG^{*}) = \alpha(F + G^{*})^{*}, \tag{52}$$

or also:

$$\overline{\nabla}\,(F + iG^{*}) = i\alpha(F + G^{*})^{*}, \tag{53}$$

and thus we have again found our previous equation (36) for the combination $H = F + iG^{*}$.

Therefore, in the first case, which corresponds to a real mass precisely, such a combination, which should lead to an equation for only one single quantity, does not seem possible with no further assumptions [†]. This case shall be treated by us in the sequel.

We assume, as is convenient, that our basic equation takes the following form:

[†] We will see later that, in general, corresponding combinations also exist here when one introduces quaternions as factors.

$$\left.\begin{aligned}\bar{\bar{\nabla}} F &= \alpha G^*, \\ \bar{\bar{\nabla}} G &= -\alpha F^*.\end{aligned}\right\} \tag{54}$$

We can now once more arrive at the Dirac equation when we write down the combination (I + *i*II, III + *i*IV) in the first system, as well as the second one. We thus have the following arrangement before us:

$$\left.\begin{aligned}X_1 + iY_1 = \psi_4, \quad X_1^* + iY_2^* = \psi_2, \\ Z_1 + iT_1 = \psi_3, \quad Z_2^* + iT_2^* = \psi_1,\end{aligned}\right\} \tag{55}$$

and for the mass, we have:

$$m = -\frac{h}{2\pi c}\,\alpha. \tag{56}$$

The indices 1 and 2 refer to the quaternions *F* and *G*, resp. However, the system (54), which indeed consists of eight equations, is therefore obviously still not exhausted. We can, moreover, now switch the roles of *F* and *G* and address the following arrangement:

$$\left.\begin{aligned}X_2 + iY_2 = \sigma_4, \quad X_1^* + iY_2^* = \sigma_2, \\ Z_2 + iT_2 = \sigma_3, \quad Z_2^* + iT_2^* = \sigma_1.\end{aligned}\right\} \tag{57}$$

There exists a Dirac system of equations for the σ, as well, which are independent of the ψ, and are indeed completely identical to the former ones, except that the sign of the mass term has been switched, as a result of the change of sign in the second equation (54). These two systems together are now equivalent to the system (54) †.

The transformation of *F* and *G* under a Lorentz transformation (31) is easily given in the following form: One must have:

$$\left.\begin{aligned}F' &= pFk, \\ G' &= p^* G k^*,\end{aligned}\right\} \tag{58}$$

where *k* might mean an arbitrary quaternion.

Here, as well, a peculiar indeterminacy of the transformation arises that does not correspond to anything in the Dirac theory, since the transformation of the ψ quantities is unique there. This has the following basis: In general, the ψ and σ quantities will be mixed under a transformation of the (X_1, …, T_1), (X_2, …, T_2), when it is converted into the ψ and σ. However, if we assume only one Dirac equation – e.g., the one that involves ψ – then only those transformations in which the σ quantities do not enter come into question, which thus transform the ψ among themselves. That is, we then distinguish a subgroup of the system of equations (54) and arrive at the fact that this subgroup shall be

† A doubling of the Dirac equation is also present in the Madelung schema (loc. cit.), where eight complex equations were written down.

transformed into itself. The transformation of the ψ quantities is uniquely determined by this (up to a common factor), so the distinguished subgroup takes on no meaning, while one can conjecture about the complete system. In fact, this is found to be true, and we then succeed in formulating the extended system in a generally covariant way. However, before we go on to this, in regard to the importance of the Dirac theory, it is of interest for us to focus our attention on the specific transformation properties of the Dirac subgroup more closely.

8. Unified derivation of the covariants of the Dirac theory. We thus now assume that we have the system of equations in terms of just the ψ, and consider only those transformations that take the ψ to other ψ. We can easily show that with this condition the undetermined quaternion k in equation (58) can be set equal to 1 in practice. Namely, we need only to show that now the combinations $X + iY$ and $Z + iT$ can transform only in these combinations, since otherwise this would also be required of the quantities $X - iY$ ($Z - iT$, resp.) that arise from the σ quantities. For the matrix of the transformation this means that one can easily show the following: If we subdivide the matrix into four small blocks by means of a horizontal and vertical cut through the middle then in each of these blocks, we must have that the two terms in the diagonal are equal to each other, while the other two are equal and opposite. If we test the matrix (7) of a p-transformation for this property then we see that this condition is, in fact, fulfilled. By contrast, for the matrix (10), this is only the case in two of its blocks under a q-transformation, but not in the other two. We must then set k_1 and k_2 equal to 0, and all that remains is multiplication of all the ψ by one and the same complex number.

These transformations are trivial, due to the linearity of the equation. It is natural that we eliminate them by setting $k = 1$. Such a similarity transformation, which is also only possible without any rotation of the coordinate system, is then excluded, and the transformation of F and G is made single-valued. However, in quantum mechanics the normalization does not extend as far. There, only the absolute value of any complex is normalized, but not, however, its phase. This is based in the fact that for quantum mechanics only "Hermitian operations" play a special role, and phase has no influence on them. Therefore, when we set $k = 1$, we consider a more restricted group of transformations than the one that is appropriate to quantum mechanics. In this way, we certainly obtain all quantum-mechanically meaningful covariants, but possibly others. Now, one can proceed in such a way one examines all contravariant structures under the narrower transformation group $k = 1$, and then, in turn, single out the ones for which the so-called "phase transformations" do not apply.

We would thus like to consider the transformation:

$$\left.\begin{aligned} F' &= pF, \\ G' &= pG \end{aligned}\right\} \tag{59}$$

uniquely from now on. We therefore do not, by any means, have the quantities F and G at our disposal, but only those combinations of their components that appear in equations

(55). In their place, we now introduce a single quaternion H, which we associate with the ψ quantities in the following way:

$$\left.\begin{aligned} X+iY=\psi_4,\;\; X^*+iY^*=\psi_2, \\ Z+iT=\psi_3,\;\; Z^*+iT^*=\psi_1. \end{aligned}\right\} \tag{60}$$

One can easily that deduce from this that H is composed from F and G in the following manner:

$$2H = F + G + i(F - G)\, j_z\,. \tag{61}$$

If we set:

$$G + H = I, \qquad G - F = K \tag{62}$$

then we have:

$$2H = I - iK\, j_z\,. \tag{61a}$$

One may also easily derive a differential equation for H. Namely, if we apply the operation $\bar{\bar{\nabla}}$ to (61a), while considering that equations (54) are just as valid for I and K as they are for F and G, then this yields:

$$\begin{aligned} 2\,\bar{\bar{\nabla}}\, H &= \alpha(K^* + i\, I^* j_z) = \alpha(-K^* j_z + i\, I^*)\, j_z \\ &= \alpha\, i(I^* + i\, K^* j_z)\, j_z = \alpha\, i\, H^* j_z\,, \end{aligned}$$

so:

$$\bar{\bar{\nabla}}\, H = \alpha\, i\, H^* j_z\,. \tag{63}$$

This equation is – as one can easily prove directly – equivalent to the Dirac equation in the ψ[†]. If we would not like to leave an invariant formulation of the Dirac equation up in the air then we could start with this equation from the outset and base our investigation upon it.

For H, one also has the transformation:

$$H' = p\, H, \tag{64}$$

and we can immediately see that:

$$H'\,\bar{H}' = p\, H\,\bar{H}\,\bar{p} = H\,\bar{H} \tag{65}$$

is an invariant. If we separate the real and imaginary parts by setting:

$$H\,\bar{H} = A + Bi \tag{66}$$

then we obtain two invariants – viz., the two fundamental invariants of the Dirac theory, precisely.

We can likewise see that the quantity:

[†] Instead of the j_z on the right-hand side, j_x or j_y can also appear naturally, so one has only to carry out a cyclic permutation of X, Y, Z in the arrangement of the components for the ψ in equations (60).

$$H' \bar{H}'^{*} = p\,(H\bar{H})\,\bar{p}^{*} \tag{67}$$

defines a vector. This vector represents the "probability current" in the Dirac theory, and its divergence-free character was proved by Dirac †.

Remarkably, this vector can be extended to three other ones, which we obtain as follows: Let V be an arbitrary vector. We define the product:

$$\bar{H}\,V\,H^{*} \tag{68}$$

and show that it is an invariant. In fact, we have:

$$\bar{H}'V'\,H'^{*} = \bar{H}\;\bar{p}\;pV\;\bar{p}^{*}\;p^{*}H^{*} = \bar{H}\,V\,H^{*}\,. \tag{69}$$

We now write this product in the following form:

$$\bar{H}\,V\,H^{*} = \sum_{\alpha=1}^{4}(\bar{H}\;j_{\alpha}H^{*})V_{\alpha}\,, \tag{70}$$

if we denote the components of V by V_α. However, if $\sum B_\alpha V_\alpha$ is an invariant then the B_α must necessarily define the components of a vector. We thus obtain a vector with the components:

$$B_\alpha = \bar{H}\;j_\alpha\,H^{*}\,. \tag{71}$$

In reality, we obtain not just one, but four, vectors in this way if the invariant (68) is indeed itself a quaternion, and is thus equivalent to four vectors, since any coefficient in the j_i remains unchanged. If we write the components of the four quaternions in a sequence then we obtain a square matrix in which each column of components defines a vector.

However, one can now exhibit the schema (71) very simply. Namely, we can regard it as an orthogonal transformation of the j_α, and indeed we are obviously dealing with a real Lorentz transformation (since the latter quaternion is the complex conjugate and overbar of the former). We thus define a p-matrix out of $\bar{H}$ and a q-matrix out of H^*, and multiply these two matrices. The resulting matrix, when applied to the j_α, gives a matrix in which the columns have simply been switched with the rows. However, the components of the individual vectors define the columns. The fourth one is therefore identical with previously-obtained current vector.

The transformation matrix is, however, an orthogonal one. We thus obtain four mutually perpendicular vectors at each point; one of them is a current vector. The lengths of all these vectors are equal to each other ††.

Namely, for all four of them, the square of the length is:

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† Proc. Roy. Soc. **118**, 251, 1928.

†† This geometric result is very interesting methodologically, especially in view of Einstein's "teleparallelism," which is indeed constructed from such "local *n*-beins."

$$= (H\,\bar{H})(H^*\bar{H}^*) = A^2 + B^2.$$

The divergence of these vectors may also be computed very simply, since it must indeed be an invariant. We then have to form:

$$\sum \frac{\partial B_\alpha}{\partial x_\alpha} = \sum \left(\frac{\partial}{\partial x_\alpha} \bar{H}\, j_\alpha H^* \right) = (\bar{H}\nabla)H^* + \bar{H}(\nabla H^*)$$

$$= \overline{(\bar{\nabla} H)}\, H^* - \bar{H}(\bar{\nabla} H)^* = U - \bar{U}^*, \tag{72}$$

where we have set:

$$U = \overline{(\bar{\nabla} H)}\, H^*. \tag{73}$$

Since the differential equation (63) exists for H, it follows that:

$$U = -\,\alpha\, i\, j_z \bar{H}^* H^* = -\,\alpha\, i(A - i\,B)\, j_z\,, \tag{74}$$

and we then obtain:

$$\sum \frac{\partial B_\alpha}{\partial x_\alpha} = -\,2\,\alpha B\, j_z\,. \tag{75}$$

Therefore, only the vector that belongs to the third row of the matrix has a non-zero divergence.

The remaining three orthogonal vectors – and among them, the current vector – are divergence-free.

We can also write the four vectors in the following form:

$$B^{(i)} = H\, j_i \bar{H}^*, \tag{76}$$

where j_i is any one of the four unit quaternions. Only the third and fourth vectors are impervious to phase transformations. The fourth one is the aforementioned Dirac current vector. The third one:

$$B^{(3)} = H\, j_z \bar{H}^*, \tag{77}$$

is the one whose divergence is non-zero.

In a similar way, we can discover the tensors of the Dirac theory. We define:

$$H' j_z\, \bar{H}' = p\, H\, j_\alpha\, \bar{H}\, \bar{p}\,, \tag{78}$$

where j_α shall be one of the three spatial quaternion units. (The fourth unit delivers only the invariant that we already know.) We already know that the spatial part a quantity that transforms in this way can be regarded as an anti-symmetric tensor when one separates its real and imaginary parts (precisely as we say for the electromagnetic field strengths). The temporal component gives an invariant. This temporal component is then equal to 0 for all three structures, and thus yields nothing.

The phase transformation affects only the quantity:

$$H\, j_z\, \bar{H} \tag{79}$$

that is defined by j_z, which is thus the only quantum-mechanically allowable one. This anti-symmetric tensor was presented by C. G. Darwin [†].

There is yet another possibility for arriving at a tensor. With the help of the two vectors U and V, we define the invariants:

$$\bar{H}'U'\bar{V}'H' = \bar{H}\,\bar{p}\,pU\,\bar{p}^*\,p^*\bar{V}\,\bar{p}\,p\,H = \bar{H}\,U\,\bar{V}\,H\,. \tag{80}$$

If we write this in the form:

$$\sum_{\mu,\nu}(\bar{H}\, j_\mu\, \bar{j}_\nu\, H)U_\mu V_\nu \tag{81}$$

then we obtain a bilinear form whose coefficients must be tensor components:

$$T_{ik} = \bar{H}\, j_i\, \bar{j}_k\, H\,. \tag{82}$$

Here, as well, we likewise obtain four tensors, since the components emerge as quaternions, and each unit vector can be solved for one tensor component.

We can also just as well start with the invariants:

$$\bar{H}^*\bar{U}\,V\,H^*, \tag{83}$$

and then correspondingly obtain the tensor:

$$T'_{ik} = \bar{H}^*\,\bar{j}_i\, j_k\, H^*. \tag{84}$$

We thus have enumerated all covariant structures of order zero, one, and two that can be constructed quadratically from the fundamental quantities for the restricted group of k = 1.

The covariants of the Dirac theory have already been treated in the literature [††]. Therefore, the unified treatment that was given here, by its clarity and simplicity, can serve as the methodological driving force for other representations.

9. The breakdown of the current vector under strict covariance. If we once more look back upon our train of thought then we can say the following: We started from a large system of equations and found that the transformation of the functions was not uniquely defined by the equations. However, we succeeded in showing that a certain

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[†] Proc. Roy. Soc. **120**, 621, 1928.

[††] Cf., in particular, J. v. Neumann, ZS. f. Phys. **48**, 868, 1928.

subgroup of the system should transform into itself, and by means of this demand, it was possible to lift the indeterminacy of the transformation.

When we are only dealing with the examination of the Dirac equation, we can therefore just as well base it on equation (63) from the outset, which we have derived from the larger system for a certain combination of *F* and *G*, so the arrangement (60) can be done, and therefore show that equation (63) is, in fact, equivalent to the Dirac system. If we now embark upon this path and would like to immediately calculate the transformation of the function *H* from this system then this would show a peculiar discrepancy with our last results that is perhaps not without meaning. If one represents the transformation of *H* in the form:

$$H' = p\,H\,k \tag{85}$$

then one obtains the following condition equation for the quaternion *k*:

$$j_z\,k = k^*\,j_z\,. \tag{86}$$

It is now in no way the case that this condition would imply a far too comprehensive restriction on *k* such that, say, only a trivial similarity transformation remains. Furthermore, the condition (86) only says that the *x* and *y* parts of *k* are purely imaginary, while the *z* and *i* parts must be pure real, and when we also carry out the natural length normalization, which explicitly excludes the trivial similarity transformation, then what remains is a very essential 3-parameter group of transformations, which has nothing to do with the "phase indeterminacy" of ψ. Moreover, one observes that under this transformation, which is possible without any rotation of the axes, the new ψ are expressed in terms of not only the previous ψ, but also in terms of their conjugates ψ^*. Since the ψ play only the role of auxiliary variables, one can give no objective grounds for the exclusion of this transformation, and especially since it is not at all a normal transformation for the (*X*, *Y*, *Z*, *T*) (without introducing the conjugates), and nothing would suggest this, so one should not introduce these quantities in place of the ψ as fundamental quantities, since obviously the Hamiltonian operator is in closer contact with tensor-analytic quantities than the Dirac operator. However, if one allows this transformation then a large part of the covariants described are sacrificed. Only the invariant $H\bar{H}$ and a vector remain. However, it is not the divergence-free current vector, but any vector $B^{(3)}$, whose divergence has been shown to be non-zero. We find, in fact, that:

$$H' j_z \bar{H}'^* = p\,H\,k\,j_z\,\bar{k}^*\bar{H}^*\bar{p}^* = p\,H\,j_z\,\bar{H}^*\bar{p}^*, \tag{87}$$

because from (86), one has:

$$j_z = k^*\,j_z\bar{k} = k\,j_z\bar{k}^*. \tag{88}$$

The invariance of the current vector under this *k*-transformation has not been proved. The vector $B^{(3)}$ is then the only one that comes into question as actually being strictly covariant, while for the other structure (in particular, for any current vector, as well) the covariance is generated by a restriction that is arbitrary, after all †.

This situation seems to show very strongly that the Dirac equation should not be regarded as a closed system in itself, but as a component of a larger system.

10. Covariant formulation of the doubled Dirac equation. We would now like to consider the system (54) as a whole and investigate its transformation properties. The system is, as we say, equivalent to simultaneous Dirac equations that involve two independent groups of quantities: ψ and σ. From the standpoint of the total system, such a decomposition would be entirely artificial and unnatural, and would also offer no advantage as a mathematical tool, because it amounts to the singling out of a subgroup that is not inherently preferred. We would like to leave this behind and take the viewpoint that the quantities F and G that enter into the equations should have some invariant sense. This principle will thus lead us to lift the indeterminacy in the transformation that resides in the freedom in the quaternion k, while we have achieved this up to now exactly by the singling out of an in itself non-covariant subgroup of equations.

We once more write down the transformation equations of our functions F and G:

$$\left.\begin{aligned} F' &= pFk, \\ G' &= pGk^*, \end{aligned}\right\} \tag{58}$$

where k can be arbitrary and the only one that is possible because naturally the normalization $k\bar{k} = 1$ can be assumed. Before we go into this, after establishing the principle of covariance for k, we would like to remark that certain invariants and covariants are also possible when k is left arbitrary.

Namely, one first has the two invariants:

$$F\bar{F}, \quad G\bar{G}, \tag{89}$$

which are now equivalent to four real invariants.

One then has the two vectors:

$$F\bar{G}^*, \qquad G\bar{F}^*, \tag{90}$$

so one has, e.g.:

$$F'\bar{G}'^* = p\,F\,k\,\bar{k}\,\bar{G}^*\bar{p}^* = p(F\bar{G}^*)\bar{p}^*. \tag{91}$$

† Remark by the editor: The author has not suggested that the transformation behavior can change under the introduction of a vector potential. This is, however, in fact the case, such that under the extension of the system by the external field the special group of transformations that was found breaks down here. The objection that the usual transformation theory of ψ functions would be invalid thus becomes untenable. The details will be found in the paper "Zur kovarianten Formulierung der Diracschen Gleichung" that appears next.

The two vectors (90) are thus not independent of each other, and furthermore, the second one is equal to the complex conjugate and overbarred value of the first one. It then suffices to consider only $F\bar{G}^*$.

One can conjecture that this vector – its real or imaginary part, resp. – must correspond to the current vector, because it represents the analogous object to $H\bar{H}^*$. However, if we calculate its divergence from an entirely similar process to the one that we applied in (72) then we do not find that this is true. Moreover, this yields:

$$\mathrm{Div}(F\bar{G}^*) = \alpha[(F\bar{F}) + (G\bar{G})^*]. \tag{92}$$

We might regard this finding as a negative factor when compared to the understanding of the field that we desire here. In fact, however, this objection would not be justified. This vector, which is distinguished by the fact that it nevertheless always remains covariant under any unrestricted transformation of the equations, indeed corresponds, not to the current vector of the Dirac theory, at all, but to a vector (77) whose divergence does not vanish there either. As compared to the Dirac current vector, one cannot make it correspond to a covariant object in a unique way, because it is not distinguished by invariant properties, but its covariance, as we found in the last section, comes about through an unnecessary restriction of the possible transformations.

We can interpret the formal introduction of two quantities tensor-analytically. When $k = \bar{p}^*$, we are dealing with a vector. When $k = \bar{p}$, we can regard the object as a skew-symmetric tensor, which is invariant in its temporal part. Both objects appear singularly, which is a choice that we also might encounter, except that F and G switch roles:

We would then like to choose:

$$k = \bar{p}. \tag{93}$$

By this choice, we have made F into an anti-symmetric tensor and G into a vector. Now, only covariant quantities appear in our equation. Furthermore, both systems of equations are already covariant in themselves, not just the total system.

We start with the first system. This system shows a far-reaching kinship with the Maxwell equations of the electromagnetic field. We shall therefore not imagine the vacuum equations, but the complete equations, in which the current vector is on the right-hand side. In fact, when a current vector is present, one can bring the Maxwell equations into the following form:

$$\bar{\nabla}F = S^*, \tag{94}$$

by means of the complex combination (24), where S means the current vector, which is regarded as a quaternion. This is, however, precisely our first equation, which therefore takes on a very simple interpretation by way of a classical analogy, except that for the Maxwell schema one would set $F_4 = 0$, but the appearance of this term means merely that on the left-hand side one further adds the gradient of a scalar. In addition, the complex nature of our vector G must be interpreted as saying that it can be considered to be not only an electric, but also a magnetic current.

The appearance of the second equation is essentially new and foreign to classical theory. It means that there is a "back-coupling" of the current and field strengths, a reaction of the field to the current. The appearance of this equation is to be regarded as the actual effect that leads to the existence of the Schrödinger equation.

The second equation also has an invariant sense. An equation of this form is well-known to us from the theory of electromagnetic fields. Namely, if we represent the electromagnetic field strength with the help of a vector potentials Φ_i then we have the equation:

$$\bar{\bar{\nabla}}\Phi = -F^*, \tag{95}$$

in which the vector Φ_i is considered to be a quaternion, but that is precisely the second of our equations, in which one must only consider that for us Φ must be regarded as a complex quaternion. This implies that in addition to the usual "rotational structure," a further "dual" structure also comes into view that corresponds to the appearance of a "magnetic vector potential."

We can thus translate our system of equations into the usual language of physics at our leisure by using the known vector symbols, as well as with the relations of tensor analysis, so we need only to bring into play the corresponding equations of the electromagnetic field and replace the corresponding quantities. The only difference is that the continuity condition for the current cannot be read off from the general structure of the equations, due to the appearance of a surplus of invariants that are absent in the Maxwell theory. This invariant ensures that the vanishing of the divergence for the current vector is not a necessary consequence of the equations, since the system of equations involves no intrinsic dependencies (i.e., a lack of identities.).

For the sake of better understanding, we would like to write down the resulting equations in tensor-analytic form explicitly. The following quantities appear:

1. An anti-symmetric tensor: $F_{ik} = -F_{ik}$ (viz., the "electromagnetic field strength"). Let the "dual" tensor be denoted by $\tilde{F}_{ik}$ [*].

2. Two vectors: S_i and M_i (viz., "electric and magnetic current").

3. Two invariants: S and M.

We obtain, in total, 16 equations that decompose into two groups (A) and (B). The group (A) includes two vector equations. The group (B) includes an anti-symmetric tensor equation (six equations) and two equations.

With the usual symbolism of tensor analysis, the equations read as follows:

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[*] The dual association results from the following schema:

$$\tilde{F}_{12} = iF_{34}, \ldots, \; \tilde{F}_{14} = iF_{23}, \ldots$$

(The ellipses mean cyclic permutations of 1, 2, 3.)

$$\left.\begin{array}{l} (A)\begin{cases} \dfrac{\partial S}{\partial x_i} + \dfrac{\partial F_{i\mu}}{\partial x_\mu} = \alpha S_i, \\[2ex] \dfrac{\partial M}{\partial x_i} - \dfrac{\partial \tilde{F}_{i\mu}}{\partial x_\mu} = \alpha M_i; \end{cases} \\[4ex] (B)\begin{cases} \dfrac{\partial S}{\partial x_i} - \dfrac{\partial S_k}{\partial x_i} + \left(\dfrac{\partial M_i}{\partial x_k} - \dfrac{\partial M_k}{\partial x_i}\right) = \alpha F_{ik}, \\[2ex] \dfrac{\partial S_\mu}{\partial x_\mu} = \alpha S, \\[2ex] \dfrac{\partial M_\mu}{\partial x_\mu} = \alpha M. \end{cases} \end{array}\right\} \tag{96}$$

The constant α is thus defined by:

$$\alpha = \frac{2\pi mc}{h}. \tag{97}$$

The total system exhibits a far-reaching symmetry, like the Dirac equations. It is therefore questionable whether all quantities are actually meaningful. One can conjecture that the two scalars S and M do not occur in reality. The first group of equations would then be completely equivalent to the Maxwellian schema. The continuity condition for the current would be a consequence of the equations. Despite this specialization, the aforementioned complication that the Darwin association yielded (see pp. 8) would now no longer occur. The two groups of ψ quantities (viz., ψ and σ) are just the field strengths, but the field strengths and current vector combined together, and both types of quantities occur in each group [as equations (55) and (57) show]. Therefore, despite the specialization, there is no algebraic relationship between the ψ now.

Furthermore, the Maxwell equations also allow one to arrive at another heuristic viewpoint, in fact, for the "magnetic current vector" M_i, which vanishes there. Here, one must generally observe that one can in no way make the same demands on the quantities as freely as one can in ordinary field theory. As a result of the double coupling, there are then no dependent and independent quantities here, and the number of equations is just as large as the number of unknowns. It is precisely characteristic of the equations of quantum mechanics that one works with homogeneous equations (viz., "eigensolutions"), and the foreign functions (e.g., energy, vector potential) do not enter in as "right-hand sides" of equations, but as factors. If we set certain quantities equal to zero in our system then an indeterminacy would arise: The number of equations would be larger than the number of unknowns. In order to not arrive at a contradiction, one must then have such a nullification simultaneously compensated by the dropping of some equations – one then has that a number of identities come about, under which the excess equations seem to be mere consequences of the remaining ones.

From a mathematically aesthetic viewpoint, it would then be generally coupled with an essential benefit. This is not to deny that the far-reaching symmetry of the Maxwell equations relative to the duality between electric and magnetic field strengths exists only as long as one writes down the equations in the symbolism of three-dimensional vector analysis. When viewed from the four-dimensional tensor-analytic standpoint, there is an intrinsic difference between the two systems of Maxwell equations, which also manifests itself in reality in the absence of magnetic currents. In the one system, a normal divergence comes about, while in the other one the divergence includes the "dual" field strengths. The tensor-analytic formulation is completely free of objections, so it would be a coincidence that relates to the use of four-dimensional space that the "dual" tensor to a tensor of second degree is also again of precisely second degree. The very occurrence of this situation is logically less satisfying.

If one now sets the magnetic current $M_i = 0$ in our equations (96) then one must likewise drop exactly those equations that involve the divergence of the dual tensors in group (A), which corresponds to the Maxwell system. These equations are then consequences of group (B), and no longer belong to the defining data of the system. The "dual" object in the system would then not occur at all. We have ten equations for ten quantities, namely, the four Maxwell equations:

$$\frac{\partial F_{i\mu}}{\partial x_\mu} = \alpha S_i , \tag{98A}$$

on the one hand, and the six reaction equations:

$$\frac{\partial S_i}{\partial x_k} - \frac{\partial S_k}{\partial x_i} = \alpha F_{ik} . \tag{98B}$$

The divergence equation for charge is now a consequence of these equations, just like the missing four Maxwell equations.

In general, this system is so asymmetrical that the existence of a direct relationship to the Dirac equations seems to be called into question, whether or not this system is also of first order, and the Schrödinger wave equation also applies to each component here. In fact, we also again arrive at the Dirac equation when we append to our system, along with the Maxwell equations of the second kind, also the divergence equation for the current, since identities indeed actually exist when one also adds nothing new to the system. The eight Dirac equations for the ψ quantities again define a subgroup of our total system, and say something less than our ten equations. Now, however, a restriction already enters into the ψ quantities. Then, from our association (55), we recognize that – since the vector G is now no longer complex, but has a real spatial part and an imaginary temporal part – the quantity $\psi_1 = S_z - iS_i$ becomes a pure real quantity. A similar specialization does not come about for the remaining ψ.

It is self-explanatory that the extension of the equations for the presence of an external field can first decide whether the intuitions that we have given up on here do or do not take on any real meaning. Our goal here was more specific. We have addressed the question of whether one can formulate the Dirac theory in such a way that one thus operates exclusively with quantities that are field-theoretically reasonable. We found a way that was almost uniquely prescribed and, in fact, led to the desired results. It produced a covariant system of great intrinsic coherence that admitted new insights into many relationships. As was pointed out, a certain arbitrariness that also adheres to the Dirac transformation theory, and is based in deeper considerations, was thus neutralized, since here any indeterminacy in the transformations has been removed.

Should the path that we embarked upon here prove to be fruitless then that would prove with a high degree of probability that it would seem hopeless to conjecture a field-theoretic foundation for the Dirac theory. Our developments then leave scarcely any doubt that this can only lie in the direction that we found here, if it is present, to begin with.

In terms of formal relationships, it should be accounted as a profit that a method was developed that allowed one to present the transformation properties of the Dirac quantities in a very unified and transparent manner.

Berlin-Nikolassee, July 1929.