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Quantum electrodynamics in configuration space

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The electromagnetic field and its interaction with matter is described by a Schrödinger equations in the configuration space of light quanta. The results are identical with those of Heisenberg and Pauli.

Introduction. Heisenberg and Pauli [*] have presented a quantum theory of the electromagnetic field and its interaction with matter. In it, they employed the methods of quantized waves. It seems to us preferable to introduce the configuration space of light quanta, analogous to the usual quantum mechanics. By doing this, one shows that the form of the equations can be derived from a few physically plausible assumptions. The considerations of Dirac et. al. [**] lead one to conjecture that the equations are equivalent to each other, which we will also confirm in the fourth section by direct calculation.

In particular, the complication that the interaction of a particle with itself becomes infinitely large is not eliminated here. The equations are thus certainly still not the physically correct ones, and we also do not believe that this drawback can be removed by any purely formal alteration.

1. Wave equation for a light quantum. The opinion has occasionally been expressed that it is not possible to present a wave equation for light quanta, because there is no conservation of number for them. Pauli and Heisenberg [***] first expressed the idea that this difficulty could be easily overcome: One must consider a system of functions in 0, 3, 6, …-dimensional space, of which, the N^{th} one delivers the probability that N particles are present and that the configuration of these particles is given by the point in question in $3N$-dimensional space. These functions are then coupled by a system of simultaneous equations.

As a preliminary, we consider the case in which no matter is present. These functions will then be independent of each other and one may restrict oneself to the consideration of "single quanta." Of the three-dimensional equations, one must demand that their solutions correspond to all possible states of motion and polarization; i.e., all solutions of the Maxwell equations:

$$\dot{\mathfrak{E}} = c \operatorname{rot} \mathfrak{H}, \tag{1a}$$

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[*] W. Heisenberg and W. Pauli, Zeit. Phys. **56** (1929), 1; **59** (1930), 168 (cited as *loc. cit.* I and II).

[**] P. A. M. Dirac, Proc. Roy. Soc. (A) **114** (1927), 243; P. Jordan, Zeit Phys. **45** (1927), 766; O. Klein and P. Jordan, *ibid*., **45** (1927), 751); P. Jordan and E. Wigner, *ibid.*, **47** (1928), 631.

[***] W. Heisenberg and W. Pauli, *loc. cit.*II, pp. 190.

$$\dot{\mathfrak{H}} = -\,c\ \mathrm{rot}\ \mathfrak{E}, \tag{1b}$$

$$\mathrm{div}\ \mathfrak{E} = \mathrm{div}\ \mathfrak{H} = 0. \tag{2}$$

However, of these solutions, our functions must be distinguished by the fact that they have a time dependency of $e^{i\omega t}$ for a light quantum of frequency ω, in order that conservation of energy should be valid (in their interaction with matter). We therefore may not require that $\mathfrak{E}$ and $\mathfrak{H}$ must be real. In place of that, we must, however, pose another auxiliary condition, which expresses the idea that only quanta of positive energy are present. We examine this condition for the case of a linearly-polarized plane wave:

$$\mathfrak{E} = \mathfrak{E}_0\, e^{i[\omega t + (\mathfrak{k}\,\mathfrak{r})]}, \qquad \mathfrak{H} = \mathfrak{H}_0\, e^{i[\omega t + (\mathfrak{k}\,\mathfrak{r})]}.$$

In the case where ω is negative, we can replace it with $(-\,c\,|\,\mathfrak{k}\,|)$. (In the usual notation, a negative frequency belongs to a positive energy.) From the Maxwell equations, it follows in this case that:

$$-\,ik\mathfrak{E} = \mathrm{rot}\ \mathfrak{H}, \qquad k = |\,\mathfrak{k}\,|. \tag{3}$$

For any other solution, we must demand that (3) is true for each Fourier component of $\mathfrak{E}$ and $\mathfrak{H}$. To abbreviate, we introduce an operator $\sqrt{\Delta}$, which is defined by saying that it takes a function:

$$\Phi(\mathfrak{r}) = \int \varphi(\mathfrak{k})\, e^{i(\mathfrak{k}\,\mathfrak{r})}\, d\Omega, \qquad d\Omega = dk_x\, dk_y\, dk_z$$

to:

$$\sqrt{\Delta}\,\Phi = \int ik \cdot \varphi(\mathfrak{k})\, e^{i(\mathfrak{k}\,\mathfrak{r})}\, d\Omega. \tag{4}$$

The notation $\sqrt{\Delta}$ is justified by the fact that the repetition of this operator obviously leads to the Laplace operator. We must then impose the auxiliary condition that:

$$-\,\mathrm{rot}\ \mathfrak{H} = \sqrt{\Delta}\,\mathfrak{E}. \tag{5}$$

This condition is trivially compatible with the Maxwell equations. Now, however, that $\mathfrak{H}$ is uniquely determined by $\mathfrak{E}$, such that one only needs to determine one of the two functions. Much less remains to be said regarding the relationship with the field quantities, so we no longer denote the wave function by $\mathfrak{E}$, but by $\mathfrak{F}$. From (1a), (2), and (5), it then follows that:

$$\frac{1}{c}\dot{\mathfrak{F}} = -\,\sqrt{\Delta}\,\mathfrak{F}, \tag{6a}$$

$$\mathrm{div}\ \mathfrak{F} = 0. \tag{6b}$$

However, we must add a normalization condition, as well, which expresses the idea that one is dealing with a quantum. It seems reasonable to do this in the following way: If

one has, in turn, a monochromatic wave then in the classical Maxwell theory the total energy is given by:

$$\frac{1}{2}\int (\mathfrak{E}^2 + \mathfrak{H}^2)\, dV\,,$$

so the number of light quanta is given by:

$$\frac{1}{2h\,|\,\omega\,|}\int (\mathfrak{E}^2 + \mathfrak{H}^2)\, dV$$

($h = \frac{1}{2\pi}\cdot$ Planck's constant). For a monochromatic wave, we will then denote the light quantum number by †:

$$\frac{1}{2h\,|\,\omega\,|}\int \mathfrak{F}\mathfrak{F}^*\, dV\,. \tag{7}$$

Due to our auxiliary condition, we can again replace $|\,\omega\,|$ with ck, and then for an arbitrary field function, which we assume to have the form:

$$\mathfrak{F}(\mathfrak{r}) = \int \mathfrak{f}(\mathfrak{k})\, e^{i(\mathfrak{k}\,\mathfrak{r})}\, d\Omega$$

at a give time, calculate the number of quanta:

$$N = \frac{1}{2h}\int dV \int \frac{\mathfrak{f}(\mathfrak{k})\mathfrak{f}^*(\mathfrak{k}')}{ck}\, e^{i(\mathfrak{k}-\mathfrak{k}')\mathfrak{r}}\, d\Omega d\Omega'\,. \tag{8}$$

Corresponding to (4), we define the operator:

$$\frac{1}{\sqrt{\Delta}}\ \Phi(\mathfrak{r}) = \frac{1}{2\pi^2 i}\int \frac{\Phi(\mathfrak{r})}{|\,\mathfrak{r}-\mathfrak{r}'\,|^2}\, dV'\,. \tag{9a}$$

With this definition, (8) becomes:

$$N = \frac{i}{2hc}\int \mathfrak{F}^*(\mathfrak{r})\frac{1}{\sqrt{\Delta}}\mathfrak{F}(\mathfrak{r})\, dV \ = \frac{1}{4\pi^2 hc}\int \frac{\mathfrak{F}^*(\mathfrak{r})\mathfrak{F}(\mathfrak{r}')}{|\,\mathfrak{r}-\mathfrak{r}'\,|^2}\, dV dV'\,, \tag{10}$$

and the normalization condition says that the integral (10) should have the value 1.

One cannot therefore define, perhaps, $\mathfrak{F}^*\frac{1}{\sqrt{\Delta}}\mathfrak{F}$ to be the probability density, since this quantity is not defined to be positive definite.

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† (7) is legitimate, since, from (5) and (2), one has $\int \mathfrak{E}\,\mathfrak{E}^*\, dV = \int \mathfrak{H}\,\mathfrak{H}^*\, dV$.

However, we have still not succeeded in finding the correct expression for the probability density.

2. Many particles without interaction. We now go on to the case of many light quanta and electrons. Now, in that case it suffices to write down the equations for the case of one electron, since for many electrons, they can be obtained by a simple generalization. Let N be the number of quanta, so one must present an equation in $(3N + 3)$-dimensional space. The solutions must be products of solutions of equations (6) with a solution for the force-free Dirac equation. Thus, one must have a quantity with $4 \cdot 3^N$ components. We denote them by:

$$F^N_{m_1 m_2 \cdots m_N} \Big| (\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_N, \mathfrak{Q}, t) \,. \tag{11}$$

ρ shall refer to the index of the Dirac function, m_ν shall denote the three spatial directions in the space of the ν^{th} quantum, $\mathfrak{Q} = Q_1, Q_2, Q_3$ should be the coordinates of the electron, and $\mathfrak{q}^\nu = q_1^\nu, q_2^\nu, q_3^\nu$ should be those of the ν^{th} quantum. For the sake of clarity, we will thus often omit the indices and simply write F^N. It is then self-explanatory that a differential operator that acts on the coordinates of the ν^{th} particle always also refers to the index m_ν; e.g., $\text{div}_\nu = \sum_{m_\nu} \frac{\partial}{\partial q^\nu_{m_\nu}}$. Differential operators with no index refer to the coordinates of the electron. Likewise, the Dirac matrices $a^1, a^2, a^3, a^4 = a$, the first three of which we can also write as the vector $\mathfrak{a}$, shall be employed with no indices, and then they always act on the function that has the index ρ.

By their definition, and due to (6), the function F then satisfies the equation:

$$\left(\frac{1}{c} \frac{\partial}{\partial t} + (\mathfrak{a}, \text{grad}) + \frac{imc}{h} \alpha \right) F^N + \sum_\nu \sqrt{\Delta}\, F^N = 0. \tag{12}$$

In a well-known way, we naturally do not choose the simple product functions, but linear combinations of them that remain unchanged when one permutes the points q_ν and the associated indices m_ν in the same way (symmetry principle).

For the sake of clarity, we write out these equations explicitly for the case $N = 0$ and $N = 1$:

$$\frac{1}{c} \frac{\partial}{\partial t} F^0_\rho + (\mathfrak{a}_{\rho\sigma} \text{grad}) F^0_\sigma + \frac{imc}{h} \alpha_{\rho\sigma} F^0_\sigma = 0, \tag{13}$$

$$\frac{1}{c} \frac{\partial}{\partial t} \mathfrak{F}^1_\rho + (\mathfrak{a}_{\rho\sigma} \text{grad}) \mathfrak{F}^1_\sigma + \frac{imc}{h} \alpha_{\rho\sigma} \mathfrak{F}^1_\sigma + \sqrt{\Delta_1}\, \mathfrak{F}^1_\rho = 0. \tag{14}$$

In addition, as always, the following auxiliary condition is valid:

$$\text{div}, F^N = 0; \tag{12a}$$

i.e., e.g.:

$$\operatorname{div}_1 F^1(\mathfrak{q}_1, \mathfrak{Q}) = 0. \tag{14a}$$

As a normalization, we must demand that:

$$J_N = \left(\frac{i}{2hc}\right)^N \int F^{*N} \frac{1}{\sqrt{\Delta_1}} \cdots \frac{1}{\sqrt{\Delta_N}} \cdot F^N dW dV_1 \cdots dV_N = 1, \tag{15}$$

$$dW = dQ_1\, dQ_2\, dQ_3\,,$$
$$dV_\nu = dq_1^\nu dq_2^\nu dq_3^\nu\,.$$

From the derivation of (12), it is evident that (12a), as well as (15), are compatible with the equations; i.e., they are, in themselves, true for all times when they are fulfilled for time zero.

3. Interaction. If we would like to bring the interaction of particles into the calculation then we would have to alter the equations in several ways. First, there is certainly no longer any conservation law for the number of light quanta. We can therefore no longer require of the integral J^N in (15) that it remain constant. It should, moreover, give the probability that precisely N quanta are present. However, we must still require that a condition of the form:

$$\sum_{N=0}^{\infty} J_N = 1 \tag{16}$$

exists. This condition must naturally be true for all time and we will see that the possibilities will be reduce considerably by the definition of the equations.

Furthermore, we have to observe that the classical equation (2) is true only for the matter-free case and otherwise has to be replaced by:

$$\operatorname{div} \mathfrak{E} = \rho. \tag{17}$$

We would first like to leave open how this equation is to be formulated in general in our theory; i.e., what we generally must set in place of (12a). However, for the special case $N = 1$ – i.e., for (14a) – this question is easily answered. Indeed, this expression means the charge density $\mathfrak{q}$, under the assumption that the electron is found at the point $\mathfrak{Q}$. It is indeed scarcely possible to find another plausible expression for this besides:

$$\operatorname{div}_1 \mathfrak{F}_\rho^1(\mathfrak{q}\mathfrak{Q}) = e \cdot \delta(\mathfrak{q} - \mathfrak{Q}) \cdot F_\rho^0(\mathfrak{Q})\,. \tag{18}$$

$\delta(\mathfrak{r}) = \delta(x) \cdot \delta(y) \cdot \delta(z)$ is the three-dimensional Dirac singular function [*].

[*] P. A. M. Dirac, Proc. Roy. Soc. (A) **113** (1927), 621.

However, we must now observe that the relation (5) relates to only the divergence-free part of $\mathfrak{E}$. If one splits $\mathfrak{E}$ into a divergence-free and a rotation-free (i.e., a transversal and a longitudinal) part then one easily realizes that the first one will be given by:

$$-\frac{\text{rot rot}}{\Delta}\,\mathfrak{E}.$$

We must then replace the operator $\sqrt{\Delta}$ in (12) with:

$$-\frac{\text{rot}_\nu\ \text{rot}_\nu}{\sqrt{\Delta_\nu}}. \tag{19}$$

Likewise, we can only demand that the normalization integral for the transversal part of F preserves the form (10). There is an arbitrary factor for the contribution of the longitudinal part. However, the equations take their simplest form when one keeps the integral (10) for the total F. Intuitively, this notation means that a longitudinal quantum corresponds to an electrical field strength of equal amplitude, just as a transversal quantum corresponds to one of equal wavelength.

In addition, we have to find an expression for the action of the field on the electron. A single electron must perform a force-free motion as long as no quanta whatsoever are present – i.e., as long as only F^0 is different from zero – but, by contrast, it will generally be deflected whenever a quantum is present. We must then add terms that depend upon $\mathfrak{F}^1$ to equation (13). These terms must naturally be analogous to the potential terms in the Dirac equation, but with a characteristic deviation. Namely, if one considers an electron under the influence of an external electrical field that is assumed to be constant then its classical energy is $e \cdot V$. However, if we consider the total field as being generated by electrons then we must replace the energy of each electron with $\frac{1}{2}eV$ in order to obtain the correct interaction, because otherwise we would count the same energy twice. This well-known situation that occurs often in the classical theory brings with it the idea that we have to substitute only one-half the potential in the Dirac equation. This determination is only unique when we pose an auxiliary condition for the potential, for which purpose, the condition:

$$\text{div}\ \mathfrak{A} = 0 \tag{20}$$

proves to be most convenient. With this condition and the equations:

$$\mathfrak{E} = -\,\text{grad}\ \varphi - \frac{1}{c}\dot{\mathfrak{A}}\,, \quad \mathfrak{H} = \text{rot}\ \mathfrak{A},$$

the potential may be represented as [*]:

[*] The meaning of the operator $1/\Delta$ is, from the foregoing, immediately understandable.

$$\mathfrak{A} = -\frac{\mathrm{rot}}{\Delta}\mathfrak{H}, \qquad \varphi = -\frac{\mathrm{div}}{\Delta}\mathfrak{E}. \tag{21}$$

The quantities that we substitute in place of $\varphi \cdot \psi_\rho$, $\mathfrak{A} \cdot \psi_\rho$ in the Dirac equation will thus have, in hindsight of (5), (19), and (21), the form:

$$-\frac{1}{2}\frac{\mathrm{div}_1}{\Delta_1}F^1_\rho(\mathfrak{q},\mathfrak{Q}), \qquad -\frac{1}{2}\frac{\mathrm{rot}_1\mathrm{rot}_1}{\Delta_1^{3/2}}F^1_\rho(\mathfrak{q},\mathfrak{Q}), \tag{22}$$

such that equation (13) goes to:

$$\frac{1}{c}\frac{\partial}{\partial t}F^0 + (\mathfrak{a},\mathrm{grad})F^0 + \frac{imc}{h}\alpha F^0 + \frac{ie}{2hc}\left[\frac{\mathrm{div}_1}{\Delta_1}F^1(\mathfrak{Q},\mathfrak{Q}) + \mathfrak{a}\frac{\mathrm{rot}\ \mathrm{rot}_1}{\Delta_1^{3/2}}F^1(\mathfrak{Q},\mathfrak{Q})\right] = 0. \tag{23}$$

This extra term in the expression for $\partial F^0 / \partial t$ also delivers a contribution to $\partial J^0 / \partial t$. Due to the preservation of the normalization, this change must be compensated by a corresponding change in J_1. We must then add extra terms to (14) that depend upon F^0. (Physically, this means that the existence of an effect of the field on the electron necessarily implies the existence of a corresponding action of the electron on the field.) In order to give the form of these terms, we calculate $\partial J^0 / \partial t$. However, one can thus easily convince oneself that the longitudinal terms of (22), with hindsight of (18), deliver no contribution, such that we drop the first term of (22) and can replace the second one with:

$$\frac{ie}{2hc}\mathfrak{a}\frac{1}{\sqrt{\Delta}}F^1(\mathfrak{Q},\mathfrak{Q}).$$

Finally, one obtains:

$$\frac{1}{c}\frac{\partial J_0}{\partial t} = +\int F^{*0}\frac{ie}{2hc}\mathfrak{a}\frac{1}{\sqrt{\Delta_1}}F^1(\mathfrak{Q},\mathfrak{Q})\,dW + \int F^0(\mathfrak{Q})\frac{ie}{2hc}\mathfrak{a}^*\frac{1}{\sqrt{\Delta_1}}F^{*1}(\mathfrak{Q},\mathfrak{Q})\,dW.$$

This may be converted into:

$$\frac{i}{2hc}\int\frac{1}{\sqrt{\Delta_1}}F^{*1}(\mathfrak{q},\mathfrak{Q})\cdot\delta(\mathfrak{q}-\mathfrak{Q})\,e\,\mathfrak{a}\,F^0(\mathfrak{Q})\,dV_1dW + \ldots$$

One thus sees that it is necessary for the constancy of $\sum J_N$ that we add a term to (14) that looks like:

$$e\ \mathfrak{a}\ \delta(\mathfrak{q}-\mathfrak{Q})\cdot F^0(\mathfrak{Q}). \tag{24}$$

The appearance of this term is now precisely what one would expect at this point if $e\mathfrak{a}\delta(\mathfrak{q}-\mathfrak{Q})$ were indeed the expression for the current density at the location $\mathfrak{q}$ in the Dirac theory and the term corresponded precisely to the current density in the Maxwellian

equations. With this, we also achieve a further confirmation of the factor 1/2 in the potential terms in the zeroth equation.

One can also exclude the form of this extra term from the compatibility of the divergence condition with the equations. Namely, if one takes the divergence of equation (14) with the addition of (24), and with consideration of (18), then one obtains (13), multiplied by a factor $\delta(\mathfrak{q} - \mathfrak{Q})$. The compatibility of the equations is thus proved, up to the additional terms that distinguish (23) from (13). We must then make two more changes: First, we must add completely analogous extra terms in (14) and, in addition, we must assume that there is an auxiliary condition between F^2 and F^1 that is analogous to (18). However, it then follows once more that the normalization is constant, so we must add terms that are analogous to (24) in the equation for F^3. Finally, one must then generally set down the following equations:

$$\left.\begin{aligned}
&\left(\frac{1}{c}\frac{\partial}{\partial t} + (\mathfrak{a}\,\mathrm{grad}) + \frac{imc}{h}\alpha\right)F^N - \sum_\nu \frac{\mathrm{rot}_\nu\,\mathrm{rot}_\nu}{\sqrt{\Delta_\nu}}F^N \\
&+\frac{ie}{2hc}\left[\frac{\mathrm{div}_{N+1}}{\Delta_{N+1}}F^{N+1}(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{Q},\mathfrak{Q}) + \frac{\mathfrak{a}\,\mathrm{rot}_{N+1}\,\mathrm{rot}_{N+1}}{\Delta_{N+1}^{3/2}}\cdot F^{N+1}(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{Q},\mathfrak{Q})\right] \\
&+e\,\mathfrak{a}\sum_\nu F^{N-1}(\mathfrak{q}_1,\ldots,\mathfrak{q}_{\nu-1},\mathfrak{q}_{\nu+1},\ldots,\mathfrak{q}_N,\mathfrak{Q})\cdot\delta(\mathfrak{q}_\nu-\mathfrak{Q}) = 0,
\end{aligned}\right\} \tag{25}$$

and thus demand the auxiliary condition:

$$\mathrm{div}_\nu F^N = e\,\delta(\mathfrak{q}_\nu - \mathfrak{Q})\,F^{N-1}(q_1,\ldots,\mathfrak{q}_{\nu-1},\mathfrak{q}_{\nu+1},\ldots,\mathfrak{q}_N,\mathfrak{Q}). \tag{25a}$$

We now also immediately indicate how the equations are to be generalized in the case of many electrons: One does not need to assume the interaction between the electrons, but one can add the interaction terms between the individual electrons and the light quanta. The number of equations will not be increased either, since we assume the conservation of particle number for matter. In this case, one obtains [*]:

$$\left.\begin{aligned}
&\left(\frac{1}{c}\frac{\partial}{\partial t} + \sum_p(\mathfrak{a}_p\,\mathrm{grad}_p) + \frac{imc}{h}\alpha_p\right)F^N(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{Q}_1,\ldots,\mathfrak{Q}_n) - \sum_\nu \mathrm{rot}_\nu\mathrm{rot}_\nu\frac{1}{\sqrt{\Delta_\nu}}F^N \\
&+\frac{ie}{2hc}\sum_p\left[\frac{\mathrm{div}_{N+1}}{\Delta_{N+1}}F^{N+1}(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{Q}_p,\mathfrak{Q}_1,\ldots,\mathfrak{Q}_n)\right. \\
&\left.+\mathfrak{a}_p\frac{\mathrm{rot}_{N+1}\mathrm{rot}_{N+1}}{\Delta_{N+1}^{3/2}}F^{N+1}(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{Q}_p,\mathfrak{Q}_1,\ldots,\mathfrak{Q}_n)\right] \\
&+e\sum_{p,\nu}\mathfrak{a}_p\delta(\mathfrak{q}_1,\ldots,\mathfrak{q}_{\nu-1},\mathfrak{q}_{\nu+1},\ldots,\mathfrak{q}_N,\mathfrak{Q}_1,\ldots,\mathfrak{Q}_n) = 0,
\end{aligned}\right\} \tag{26}$$

[*] The matrices $\mathfrak{a}_p$, a_p act on the function with the index ρ_p of the p^{th} electron.

and

$$\operatorname{div}_\nu F^N(\mathfrak{q}_1, \ldots, \mathfrak{q}_N, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_n)$$

$$= e \sum_p \delta(\mathfrak{q}_\nu - \mathfrak{Q}_p)\ F^N(\mathfrak{q}_1, \ldots, \mathfrak{q}_{\nu-1}, \mathfrak{q}_{\nu+1}, \ldots, \mathfrak{q}_N, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_n). \tag{26a}$$

4. Relationship to the Heisenberg-Pauli theory. In this section, we would like to show that the equations that were presented here are completely equivalent in their content to those of Heisenberg and Pauli. With that, we likewise verify the relativistic invariance of our equations, which does not strictly follow, from the type of derivation in three-dimensional notation, which is particular to a coordinate system. Naturally, one can also verify the invariance directly when one computes with the operator Λ (cf., *loc. cit.* II) that belongs to a Lorentz transformation for our equations.

In order to obtain the comparison between the two theories directly, it is recommended that one subject the Heisenberg-Pauli equations to a conversion. Namely, for our purpose, it is preferable to operate, not with the standing proper oscillations of a cavity, but with travelling waves that one subjects to a suitable periodic boundary condition in order to obtain discrete proper oscillations.

One then observes that we have employed Heaviside units for the field strengths and replaced $h/2\pi$ with h. One then obtains, up to the matter part, whose treatment will not be altered [cf., *loc. cit.* II, equation (42)], the Hamilton function:

$$H = \int dV \left[\frac{1}{4}\left(\frac{\partial \Phi_i}{\partial x_k} - \frac{\partial \Phi_k}{\partial x_i} \right)^2 + \frac{c^2}{2}\Pi_k^2 + e\Phi_k \alpha^k_{\rho\sigma} \psi^*_\rho \psi_\sigma \right]. \tag{27}$$

We develop Φ and Π into a Fourier series:

$$\Phi_i = \frac{1}{\sqrt{4\pi}} \cdot L^{-3/2} \cdot \sum_{\mathfrak{k}} e^{\frac{2\pi i \mathfrak{k}\mathfrak{r}}{L}} \sqrt{\frac{cL}{k}} [\mathfrak{e}_i^{\mathfrak{k}_1}(A_{\mathfrak{k}1} + A_{\mathfrak{k}-1}) + \mathfrak{e}_i^{\mathfrak{k}_2}(A_{\mathfrak{k}2} + A_{\mathfrak{k}-2}) + \mathfrak{e}_i^{\mathfrak{k}_3} q_{23}\sqrt{2}], \tag{28}$$

$$\Pi_i = L^{-3/2} \sum_{\mathfrak{k}} e^{\frac{2\pi i \mathfrak{k}\mathfrak{r}}{L}} \sqrt{\frac{cL}{k}} [-\mathfrak{e}_i^{\mathfrak{k}_1}(A_{\mathfrak{k}1} - A_{\mathfrak{k}-1}) - \mathfrak{e}_i^{\mathfrak{k}_2}(A_{\mathfrak{k}2} - A_{\mathfrak{k}-2}) + \mathfrak{e}_i^{\mathfrak{k}_3} q_{23} \cdot \sqrt{2}]. \tag{29}$$

Here, $\mathfrak{e}^{\mathfrak{k}_1}$, $\mathfrak{e}^{\mathfrak{k}_2}$ are vectors of magnitude 1 that are perpendicular to $\mathfrak{k}$ and $\mathfrak{e}^{\mathfrak{k}_3} \parallel \mathfrak{k}$. Moreover, one shall have $\mathfrak{e}^{\mathfrak{k}_1} = \mathfrak{e}^{-\mathfrak{k}_1}$, $\mathfrak{e}^{\mathfrak{k}_2} = \mathfrak{e}^{-\mathfrak{k}_2}$, but $\mathfrak{e}^{\mathfrak{k}_3} = -\mathfrak{e}^{-\mathfrak{k}_3}$. The following commutation relations must then be valid:

$$[A_{\mathfrak{k},\lambda}, A_{\mathfrak{k},-\lambda}] = 0; \qquad [A_{\mathfrak{k},\lambda}, A_{-\mathfrak{k},\lambda}] = 0; \qquad [A_{\mathfrak{k},\lambda}, A_{-\mathfrak{k},-\lambda}] = 0; \qquad (\lambda = 1, 2), \tag{30}$$

$$[p_{\mathfrak{k},3}, q_{-\mathfrak{k},3}] = -ih. \tag{30a}$$

These relations, as well as the remark that Φ, Π are real quantities, so their operators are Hermitian, lead one to the Ansatz:

$$A_{\mathfrak{k},\lambda} = \sqrt{h} e^{-\frac{i\Theta_{\mathfrak{k}\lambda}}{h}} \cdot N_{\mathfrak{k}\lambda}^{1/2}; \qquad A_{\mathfrak{k},\lambda} = \sqrt{h}\, N_{\mathfrak{k}\lambda}^{1/2} e^{\frac{i\Theta_{\mathfrak{k}\lambda}}{h}}. \tag{31}$$

Another process is preferable for the longitudinal part: One indeed still has the auxiliary condition, which is written in our variables:

$$\left(p_{\mathfrak{k}3} + \frac{i}{\sqrt{8\pi^3 c k^3}} e^{-\frac{2\pi i \mathfrak{k}\mathfrak{Q}}{L}} \right) \varphi(N_{\mathfrak{k}\lambda}, P_{\mathfrak{k}3}, \mathfrak{Q}) = 0. \tag{32}$$

The functionals that satisfy this auxiliary condition define a subsystem of functionals that do not combine with each other, and for all *q*-numbers only those matrix elements are physically meaningful that belong to two states that are compatible with the auxiliary condition.

In place of (32), we would now like to introduce the condition:

$$\left(B_{\mathfrak{k}3} + \frac{i}{\sqrt{8\pi^3 c k^3}} e^{-\frac{2\pi i \mathfrak{k}\mathfrak{Q}}{L}} \right) \varphi = 0; \qquad B_{\mathfrak{k}3} = p_{\mathfrak{k}3} + i\, q_{-\mathfrak{k},3}, \tag{32a}$$

and correspondingly replace $p_{\mathfrak{k}3}$ with $B_{\mathfrak{k}3}$ in the Hamilton function. We indeed obtain functions in this way, but the relations between all gauge-invariant quantities remain unchanged, since, as one easily sees, all matrix elements of $q_{\mathfrak{k}3}$ vanish in the subsystem that is singled out by (32a) when one writes down (32a) for $-\mathfrak{k}$ and goes over to the complex conjugates on both sides. This possibility is based upon the fact that the reality of the longitudinal part of the field strength is not a new condition, but already follows from div $\mathfrak{E} = \rho$. Commutation relations are now valid for the quantities $B_{\mathfrak{k}3}$ that are analogous to (30), such that we can set:

$$B_{\mathfrak{k}3} = -\sqrt{h}\, e^{-\frac{i}{h}\Theta_{k3}} N_{\mathfrak{k}3}^{1/2}. \tag{33}$$

In the term with $B_{\mathfrak{k}3}\ B_{-\mathfrak{k},3}$ in the Hamilton function, one expresses a factor from (32a) beforehand, as appropriate. One finally comes to:

$$\left.\begin{aligned}
&[(-E + \sum N_{\mathfrak{k}\lambda} h\omega_{\mathfrak{k}\lambda})\delta_{\rho\sigma} - ikc(\mathfrak{a}_{\rho\sigma}\ \mathrm{grad}) + mc^2\alpha_{\rho\sigma}]\varphi_\sigma\{N_{\mathfrak{k}\lambda}, \mathfrak{Q}) \\
&-\frac{1}{4\pi}\sum_{\mathfrak{k}} \sqrt{\frac{h}{\pi c k^3}}\, \omega_{\mathfrak{k}} (N_{\mathfrak{k}3}+1)^{1/2} e^{\frac{2\pi i \mathfrak{k}\mathfrak{Q}}{L}} \varphi_\rho(\cdots, N_{\mathfrak{k}3}+1, \cdots) \\
&+\frac{1}{\sqrt{4\pi}} \cdot \sum_{k,\lambda=1,2} e\mathfrak{a}_{\rho\sigma} \sqrt{\frac{hc}{k}} \cdot \frac{1}{L} \cdot \mathfrak{e}_{\mathfrak{k}\lambda} \Big[N_{\mathfrak{k}\lambda}^{1/2} e^{-\frac{2\pi i \mathfrak{k}\mathfrak{Q}}{L}} \varphi_\sigma(\cdots, N_{\mathfrak{k}\lambda}-1, \cdots, \mathfrak{Q}) \\
&+(N_{\mathfrak{k}\lambda}+1)^{1/2} e^{\frac{2\pi i \mathfrak{k}\mathfrak{Q}}{L}} \varphi_\sigma(\cdots, N_{\mathfrak{k}\lambda}+1, \cdots, \mathfrak{Q}) \Big] \\
&+\frac{1}{\sqrt{4\pi}} \sum_{\mathfrak{k}} e\mathfrak{a}_{\rho\sigma} \sqrt{\frac{hc}{k}} \frac{1}{L} \mathfrak{e}_{\mathfrak{k}3} e^{-\frac{2\pi i \mathfrak{k}\mathfrak{Q}}{L}} N_{\mathfrak{k}3}^{1/2} \varphi_\sigma(\cdots, N_{\mathfrak{k}3}-1, \cdots, \mathfrak{Q}) = 0
\end{aligned}\right\} \tag{34}$$

in place of the final equation (*loc. cit.*, 69), with the auxiliary condition:

$$\sqrt{h}(N_{\mathfrak{k}3}+1)^{1/2}\cdot\varphi(N_{\mathfrak{k}3}+1,\ldots,\mathfrak{Q}) = e\cdot\frac{e^{-\frac{2\pi i\mathfrak{k}\mathfrak{Q}}{L}}}{\sqrt{8\pi^3 ck^3}}\varphi(N_{\mathfrak{k}3},\ldots,\mathfrak{Q})\,. \tag{34a}$$

We now define the functions:

$$F_\rho^0 = \varphi_\rho(0,\,0,\,\ldots,\,0,\,\mathrm{Q}),$$

$$F_\rho^1(\mathfrak{q},\mathfrak{Q}) = \sqrt{4\pi}\cdot icL^{-3/2}\sum_{\mathfrak{k},\lambda}\mathfrak{e}_{\mathfrak{k}\lambda}\sqrt{\frac{kh}{Lc}}\,e^{-\frac{2\pi i\mathfrak{k}\mathfrak{q}}{L}}\varphi_\rho(0,\ldots,1_{\mathfrak{k}\lambda},\ldots,\mathfrak{Q})\,,$$

$$\begin{aligned}F_\rho^2(\mathfrak{q},\mathfrak{q}',\mathfrak{Q}) = -\,4\pi c^2 L^{-3}\\ \Bigg\{\sum_{\mathfrak{k},\mathfrak{k}',\lambda,\lambda'}\frac{1}{\sqrt{2}}\sqrt{kk'}\,\frac{h}{Lc}\mathfrak{e}_{\mathfrak{k}\lambda}\mathfrak{e}_{\mathfrak{k}'\lambda'}e^{-\frac{2\pi i(\mathfrak{k}\mathfrak{q}+\mathfrak{k}'\mathfrak{q}')}{L}}\varphi_\rho(0,\ldots,1_{\mathfrak{k}\lambda},\ldots,1_{\mathfrak{k}'\lambda'},\ldots)\\ +\sum_{\mathfrak{k},\lambda}k\cdot\frac{h}{Lc}\mathfrak{e}_{\mathfrak{k}\lambda}\mathfrak{e}_{\mathfrak{k}\lambda}e^{-\frac{2\pi i}{L}\mathfrak{k}(\mathfrak{q}+\mathfrak{q}')}\varphi_\rho(0,\ldots,2_{\mathfrak{k}\lambda},\ldots)\Bigg\},\end{aligned}$$

etc.,

in the well-known way [†]. We now examine (34) for the case in which all $N = 0$; we express the $\varphi_\rho(0, \ldots, 1, 0, \ldots, \mathfrak{Q})$ that appear in this equation in terms of F^1. In this way, we obtain (25) for $N = 0$. We likewise come to (25.1) when we set all $N = 0$ in (33), up to a certain $\mathfrak{k}\lambda$, for which $N_{\mathfrak{k}\lambda} = 1$, and then multiply by:

$$i\sqrt{4\pi}cL^{-3/2}\sqrt{\frac{kh}{Lc}}\mathfrak{e}_{\mathfrak{k}\lambda}e^{\frac{2\pi i\mathfrak{k}\mathfrak{q}}{L}}$$

and sum over all $\mathfrak{k}$ and λ.

Moreover, one can easily convince oneself in the same way that (33a) is identical with (25a) and that the normalization condition:

$$\sum_{N_1}\sum_{N_2}\cdots\sum_{N_{\mathfrak{k}\lambda}}\int\varphi_\rho\{N_{\mathfrak{k}\lambda},\mathfrak{Q}\}\varphi_\rho^*\{N_{\mathfrak{k}\lambda},\ldots,\mathfrak{Q}\}\,dW = 1$$

emerges from (16).

With that, the identity of the two systems of equations is proved.

5. Impulse and field operators. We would expect that the operator of the total impulse would be given here by:

$$-\,i\,h\sum_n \mathrm{grad}_n \tag{35}$$

[†] Cf., e.g., P. A. M. Dirac, Proc. Roy. Soc. London (A) **114** (1927), 243.

here, where the summation is taken over all particles (matter and light quanta). One now sees immediately that (35) is actually an integral of the equations of motion, since only coordinate differences enter into (25) explicitly. One can also appreciate that (35) is the one quantity that comes into question for this purpose.

It is further interesting to exhibit the operator of electric and magnetic field strength at a point $\mathfrak{r}$. We thus remark that Maxwell's equations must be fulfilled for these operators as *q*-number relations. In particular, that must be true for the equation div $\mathfrak{E} = \rho$, and one sees from this that the longitudinal part of the electric field must be represented by the operator:

$$\mathfrak{E}_{\text{long}}(\mathfrak{r})\, F^N(\mathfrak{q}_1, \ldots, \mathfrak{q}_N, \mathfrak{Q}) = F^{N+1}_{\text{long}}(\mathfrak{q}_1, \ldots, \mathfrak{q}_N, \mathfrak{Q}).$$

This suggests that in general one might replace the field operator as follows:

$$F^N \to F^{N+1}(\mathfrak{q}_1, \ldots, \mathfrak{q}_N, \mathfrak{Q}). \tag{36}$$

However, this is not permissible, since the longitudinal part of (36) is indeed Hermitian, due to the auxiliary condition (25a), although the transversal part is not. We must then make the transversal part of (36) Hermitian, as well, and will then be led to the operator [*]:

$$\left.\begin{aligned}
&\mathfrak{E}_l(\mathfrak{r}) F^N_{m\rho}(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{Q}) \\
&= \frac{1}{2}\left\{ F^{N+1}_{m_1\cdots m_N l\rho}(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{r},\mathfrak{Q}) + \frac{\operatorname{grad}\operatorname{div}_{N+1}}{\Delta_{N+1}} F^{N+1}_{m_1\cdots m_N l\rho}(\mathfrak{q}_1,\ldots,\mathfrak{q}_N,\mathfrak{r}_1,\mathfrak{Q}) \right. \\
&\left. + ih\cdot\sum_\nu \left[\sqrt{\Delta_\nu}\,\delta_{m_\nu l} - \frac{1}{\sqrt{\Delta_\nu}} \frac{\partial^2}{\partial x_l \partial q^\nu_{m_\nu}} \right] \delta(\mathfrak{q}_\nu - \mathfrak{r}) F^{N-1}_{m_1\cdots m_{\nu-1} m_{\nu+1} m_N \rho}(\mathfrak{q}_1,\ldots,\mathfrak{q}_{\nu-1},\mathfrak{q}_{\nu+1},\ldots\mathfrak{q}_N,\mathfrak{Q}). \right\}
\end{aligned}\right. \tag{37}$$

One can check that (37) actually fulfills the Maxwell equations and that one will be led to (37) in any event by calculating with the operators of Heisenberg and Pauli.

The operator of the magnetic field strength can be defined in an analogous way when one observes that, due to (5), our function $\mathfrak{F}$ is simultaneously the representative of the electric and magnetic field strengths, and is again Hermitian. Naturally, the same commutation relations are true for these operators as are true for those of Heisenberg and Pauli.

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Zürich, Physikal. Institut der Eidgenössischen Techn. Hochschule, January 1930.

[*] Due to the circumstances that were discussed in the fourth section, the longitudinal part of the field operator is not determined uniquely. One can also define an operator that is identically Hermitian, so it satisfies the Maxwell equations only as a result of the auxiliary condition.