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# The angular deformations of continuous media

By J. LE ROUX

Translated by D. H. Delphenich

**1. –** The expressions for the angular modifications in the deformation of a continuous medium take a remarkably simple form when one compares the sine of the deformed angle to the sine of the initial angle.

As I did in my previous notes on deformation that were published in the *Comptes rendus de l'Académie des Sciences*, I will call an arbitrary material line a *fiber* and a fiber of infinitely-small length, an *elementary fiber*. A *sheet* will be a portion of matter that is extended over a surface, but with negligible thickness.

In the neighborhood of a point, the direction of a fiber is determined by its tangent; similarly, one can associate the direction of an element of a sheet with either the tangent plane or the normal to the surface that carries that element.

Having posed these definitions, we shall consider at a point:

1. The angle between two elementary fibers.
2. The angle between a fiber and a sheet.
3. The angle between two sheets.

**2. – Angle between two elementary fibers.** – Let $ds$, $ds'$ be the lengths of the two fibers in the deformed state, and let $\theta$ be their angle; let $ds_0$, $ds'_0$, $\theta_0$ be the analogous quantities in the initial state.

The expression:

$$ds \cdot ds' \sin \theta$$

represents the area of an infinitesimal triangle that is defined by two fibers; the ratio:

$$\frac{ds \cdot ds' \cdot \sin \theta}{ds_0 \cdot ds'_0 \cdot \sin \theta_0}$$

will then be equal to $1 + E$, where $E$ denotes the surface dilatation of the plane of that triangle at the point considered. If one calls the linear dilatations of the two fibers $e$ and $e'$ then one will have:

$$\frac{ds \cdot ds'}{ds_0 \cdot ds'_0} = (1 + e)\,(1 + e'),$$

and consequently, one will get:

$$(1 + e)\,(1 + e')\,\frac{\sin\theta}{\sin\theta_0} = 1 + E.$$

We write that relation in the form:

$$\frac{\sin\theta}{\sin\theta_0} = \frac{1+E}{(1+e)(1+e')}. \tag{1}$$

**3. – Angle between a fiber and a sheet.** – Let $ds$ be an elementary fiber at a point $M$ of the deformed medium, let $d\sigma$ be an element of the sheet, and let $\varphi$ be the angle between them. The same letters, when affected with the index 0, will continue to denote the analogous quantities for the initial medium; here, we will have to consider the linear dilatation $e$ of the fiber, the surface dilatation $E$ of the sheet, and the cubic dilatation $\Theta$ of the medium at $M$.

The product:

$$ds \cdot d\sigma \cdot \sin\varphi$$

represents the volume of an infinitely-small cylinder that has $ds$ for its base and $d\sigma$ for the generator of the fiber. One will then have:

$$\frac{ds \cdot d\sigma \cdot \sin\varphi}{ds_0 \cdot d\sigma_0 \cdot \sin\varphi_0} = (1 + \Theta);$$

that is:

$$\frac{\sin\varphi}{\sin\varphi_0} = \frac{1+\Theta}{(1+e)(1+E)}. \tag{2}$$

In the calculations that relate to flexure, one has to consider the angle $\varphi'$ of a fiber with the normal to a sheet. Since that angle $\varphi'$ is the complement of the angle $\varphi$, the relation (2) will take the form:

$$\frac{\cos\varphi'}{\cos\varphi'_0} = \frac{1+\Theta}{(1+e)(1+E_1)}. \tag{2$'$}$$

Equation (2) presents a remarkable analogy with equation (1) and with the analogous formula that relates to the angle between two sheets that we shall establish later on. However, it differs by a peculiarity that merits special attention.

The ratio $\dfrac{\sin\varphi}{\sin\varphi_0}$ is determined entirely at a point $M$ when one knows the linear dilatation of the fiber and the surface dilatation of the sheet. On the contrary, the calculation of the ratio $\dfrac{\sin\theta}{\sin\theta_0}$ that is given by formula (1) demands that one must know, not only the linear dilatations of the two elementary fibers that define the angle $\theta$, but also the *plane* of the two fibers, or at least, the surface dilatation of that plane.

**4. – Angle between two sheets.** – Consider two elementary sheets $d\sigma$ and $d\sigma'$ at $M$ whose planes form the dihedral angle $\psi$ between them. Call the surface dilatations of the two sheets $E$ and $E'$, and let $e$ denote the linear dilatation of the fiber that is directed along their intersection. In order to apply the line of reasoning to this problem that served for us in the first two cases, we shall appeal to the following elementary proposition, whose proof is immediate:

*The volume of an arbitrary parallelepiped is equal to the product of the areas of two contiguous faces, multiplied by the sine of their dihedral angle, and divided by the length of their edge of intersection.*

Having said that, start from the point $M$ in the planes of the two sheets that are considered and take two infinitely-small parallelograms $du$, $du'$ that have a common $ds$ that is obviously directed along the line of intersection. Upon continuing to distinguish the initial values by the index zero, we will have:

$$\frac{du\cdot du'\cdot\sin\psi}{ds}:\frac{du_0\cdot du_0'\cdot\sin\psi_0}{ds_0} = 1 + \Theta,$$

or rather:

$$\frac{(1+E)(1+E')}{(1+e)}\frac{\sin\psi}{\sin\psi_0} = 1 + \Theta,$$

and finally:

$$\frac{\sin\psi}{\sin\psi_0} = \frac{(1+\Theta)(1+e)}{(1+E)(1+E')}. \tag{3}$$

Equations (1), (2), and (3) show that there exist true ratios of dilatations for the sines of the angles of the various types that are analogous to the ones that exist for the lengths, surface areas, and volumes.

**5. – Review of the formulas for the dilatation.** – The proof of our formulas by a direct calculation is very simple.

Let us first recall the fundamental formulas that relate to the linear and surface dilatations.

We suppose that the coordinates $x$, $y$, $z$ at each point of the deformed medium are expressed as functions of the coordinates $x_0$, $y_0$, $z_0$ of the point considered in the initial medium. Moreover, in order to avoid redundancy, we agree to let the letters without indices denote the quantities that relate to the deformed state and the same letters, when affected with the index zero, will refer to the corresponding quantities in the initial state.

Let $ds$ be an arc element that has $dx$, $dy$, $dz$ for its projections onto the axes and $\alpha$, $\beta$, $\gamma$ for its direction cosines.

One has:

$$(4)\qquad \begin{cases} dx = \dfrac{\partial x}{\partial x_0}dx_0 + \dfrac{\partial x}{\partial y_0}dy_0 + \dfrac{\partial x}{\partial z_0}dz_0, \\ dy = \dfrac{\partial y}{\partial x_0}dx_0 + \dfrac{\partial y}{\partial y_0}dy_0 + \dfrac{\partial y}{\partial z_0}dz_0, \\ dz = \dfrac{\partial z}{\partial x_0}dx_0 + \dfrac{\partial z}{\partial y_0}dy_0 + \dfrac{\partial z}{\partial z_0}dz_0. \end{cases}$$

Hence, upon dividing by $ds_0$ and setting $ds / ds_0 = 1 + e$, one will infer:

$$(5)\qquad \begin{cases} (1+e)\alpha = \dfrac{\partial x}{\partial x_0}\alpha_0 + \dfrac{\partial x}{\partial y_0}\beta_0 + \dfrac{\partial x}{\partial z_0}\gamma_0, \\ (1+e)\beta = \dfrac{\partial y}{\partial x_0}\alpha_0 + \dfrac{\partial y}{\partial y_0}\beta_0 + \dfrac{\partial y}{\partial z_0}\gamma_0, \\ (1+e)\gamma = \dfrac{\partial z}{\partial x_0}\alpha_0 + \dfrac{\partial z}{\partial y_0}\beta_0 + \dfrac{\partial z}{\partial z_0}\gamma_0. \end{cases}$$

Upon adding these latter equations together after squaring them, one will obtain the expression for the square of $1 + e$ as a homogeneous function of degree two in $\alpha_0, \beta_0, \gamma_0$. We set:

$$(6)\qquad (1 + e)^2 = e_{11}\alpha_0^2 + e_{22}\beta_0^2 + e_{33}\gamma_0^2 + 2\, e_{23}\,\beta_0\,\gamma_0 + 2\, e_{31}\,\gamma_0\,\alpha_0 + 2\, e_{12}\,\alpha_0\,\beta_0\,,$$

and we denote the right-hand side of equation (6) by $f(\alpha_0, \beta_0, \gamma_0)$.

Equations (5) define the transformation of the direction cosines of the lines. There is good reason to introduce the equations of the transformation of the direction cosines of the normals to the sheets, along with these formulas. Let $\xi, \eta, \zeta$ denote the direction cosines of the normal to an elementary fiber whose surface dilatation will be denoted by $E$; the cosines $\xi_0, \eta_0, \zeta_0$ define the direction of the normal to the corresponding initial sheet. Upon representing the functional determinant of the functions $u, v$ with respect to the variables $x, y$ by the general notation $\dfrac{d(u,v)}{d(x,y)}$, we will then have the transformation formulas for the normals in the form:

$$(7)\qquad \begin{cases} (1+E)\,\xi = \dfrac{d(y,z)}{d(y_0,z_0)}\xi_0 + \dfrac{d(y,z)}{d(z_0,x_0)}\eta_0 + \dfrac{d(y,z)}{d(x_0,y_0)}\zeta_0, \\ (1+E)\,\eta = \dfrac{d(z,x)}{d(y_0,z_0)}\xi_0 + \dfrac{d(z,x)}{d(z_0,x_0)}\eta_0 + \dfrac{d(z,x)}{d(x_0,y_0)}\zeta_0, \\ (1+E)\,\zeta = \dfrac{d(x,y)}{d(y_0,z_0)}\xi_0 + \dfrac{d(x,y)}{d(z_0,x_0)}\eta_0 + \dfrac{d(x,y)}{d(x_0,y_0)}\zeta_0. \end{cases}$$

The square of $(1 + E)$ is also expressed by a quadratic form in the cosines $\xi_0$, $\eta_0$, $\zeta_0$ :

(8) $$(1 + E)^2 = E_{11}\xi_0^2 + E_{22}\eta_0^2 + E_{33}\zeta_0^2 + 2\,E_{23}\,\eta_0\,\zeta_0 + 2\,E_{31}\,\zeta_0\,\xi_0 + 2\,E_{22}\,\xi_0\,\eta_0\,.$$

The quadratic form on the right-hand side of equation (8) is the adjoint form to $f(\alpha_0, \beta_0, \gamma_0)$; we denote it by $F(\xi_0, \eta_0, \zeta_0)$.

One knows that the cubic dilatation $\Theta$ is defined by the equality:

$$1 + \Theta = \frac{d(x, y, z)}{d(x_0, y_0, z_0)}\,.$$

The discriminant of the form $f(\alpha_0, \beta_0, \gamma_0)$ is equal to $(1 + \Theta)^2$, and that of the form $F(\xi_0, \eta_0, \zeta_0)$ is equal to $(1 + \Theta)^4$.

**6. – Calculation of the angles.**

*Angle between two fibers.* – Let $\alpha, \beta, \gamma$ and $\alpha', \beta', \gamma'$ be the direction cosines of the two fibers, and let $e$ and $e'$ be their dilatations, respectively.

The use of the formulas (5) permits one to calculate the cosine of their angle $\theta$, which is expressed by a bilinear form in the cosines that relate to the initial state. One then finds:

$$(1 + e)\,(1 + e')\cos\theta = \frac{1}{2}\left\{\alpha_0'\frac{\partial f}{\partial\alpha_0} + \alpha_0'\frac{\partial f}{\partial\beta_0} + \gamma_0'\frac{\partial f}{\partial\gamma_0}\right\};$$

one then infers that:

(9) $$(1 + e)^2\,(1 + e')^2\sin^2\theta = f(\alpha_0, \beta_0, \gamma_0)\;f(\alpha_0', \beta_0', \gamma_0') - \frac{1}{4}\left[\alpha_0' f_{\alpha_0}' + \beta_0' f_{\beta_0}' + \gamma_0' f_{\gamma_0}'\right]^2.$$

The right-hand side of equation (9) is expressed with the aid of the adjoint form and the binary determinants that are deduced from two rows of elements:

$$\begin{matrix} \alpha_0 & \beta_0 & \gamma_0 \\ \alpha_0' & \beta_0' & \gamma_0' \end{matrix}$$

by the identity:

(10) $$\begin{aligned} & f(\alpha_0, \beta_0, \gamma_0)\;f(\alpha_0', \beta_0', \gamma_0') - \frac{1}{4}\left[\alpha_0' f_{\alpha_0}' + \beta_0' f_{\beta_0}' + \gamma_0' f_{\gamma_0}'\right]^2 \\ &= F\left[(\beta_0\gamma_0' - \gamma_0\beta_0'), (\gamma_0\alpha_0' - \alpha_0\gamma_0'), (\alpha_0\beta_0' - \beta_0\alpha_0')\right]. \end{aligned}$$

On the other hand, if we let $\xi, \eta, \zeta$ denote the direction cosines of the normal to the plane of the two fibers then we will have:

$$\beta_0\gamma_0' - \gamma_0\beta_0' = \xi_0 \sin\theta_0\,,$$

$$\gamma_0\alpha_0' - \alpha_0\gamma_0' = \eta_0 \sin\theta_0 ,$$
$$\alpha_0\beta_0' - \beta_0\alpha_0' = \zeta_0 \sin\theta_0 .$$

If we substitute these values into the expression for the quadratic form $F$ then we will find that:

$$F\left[(\beta_0\gamma_0' - \gamma_0\beta_0'), (\gamma_0\alpha_0' - \alpha_0\gamma_0'), (\alpha_0\beta_0' - \beta_0\alpha_0')\right] = \sin^2\theta_0\, F(\xi_0, \eta_0, \zeta_0),$$

and since the form $F(\xi_0, \eta_0, \zeta_0)$ represents the square of the expression $1 + E$ relative to the surface dilatation of the plane of the fibers, equation (9) will finally be converted into the form:

$$(1 + e)^2 (1 + e')^2 \sin^2\theta = (1 + E)^2 \sin^2\theta_0 ,$$

which is equivalent to our formula (9).

*Angle between a fiber and a sheet.* – The cosine of the angle $\left(\frac{\pi}{2} - \varphi\right)$ that is formed between a fiber and the normal to a sheet can be calculated with the aid of formulas (5) and (7). The binary functional determinants that figure in it as coefficients in the right-hand sides of formulas (7) are the coefficients of the partial derivatives:

$$\frac{\partial x}{\partial x_0}, \frac{\partial x}{\partial y_0}, \ldots$$

in the development of the ternary functional determinant:

$$\frac{d(x, y, z)}{d(x_0, y_0, z_0)}$$

in the elements of its rows or columns.

Upon taking that remark into account, one will find immediately that:

$$(1 + e)(1 + E)(\alpha\xi + \beta\eta + \gamma\zeta) = (1 + \Theta)(\alpha_0\,\xi_0 + \beta_0\,\eta_0 + \gamma_0\,\zeta_0),$$

and one will have, consequently:

$$\frac{\sin\varphi}{\sin\varphi_0} = \frac{1+\Theta}{(1+e)(1+E)} .$$

*Angle between two sheets.* – The calculation of the angle between two sheets is absolutely similar to that of the calculation of the angle between two fibers. All that one needs to do is replace equations (5) with equations (7) and replace the identity (10) with the following one:

$$F(\xi_0, \eta_0, \zeta_0)\; F(\xi_0', \eta_0', \zeta_0') - \frac{1}{4}\left[\xi_0' F_{\xi_0}' + \eta_0' F_{\eta_0}' + \zeta_0' F_{\zeta_0}'\right]$$

$$= (1 + \Theta)^2\; f\left[(\eta_0\zeta_0' - \zeta_0\eta_0'), (\zeta_0\xi_0' - \xi_0\zeta_0'), (\xi_0\eta_0' - \eta_0\xi_0')\right]. \tag{11}$$

One will then find the equation:

$$(1 + E)^2 (1 + E')^2 \sin^2 \psi = (1 + \Theta)^2 (1 + e)^2 \sin^2 \psi_0 , \tag{12}$$

which is equivalent to our formula (3).

*Another form for the calculations.* – The calculation of the sine of the angles between two fibers or two sheets can be further carried out by a somewhat different procedure that exhibits the relation that exists between the systems (5) and (7).

Recall equations (5) and the analogous equations that relate to the direction $\alpha', \beta', \gamma'$, and form the binary determinant:

$$\begin{aligned} \beta\gamma' - \gamma\beta' &= \xi \sin\theta, \\ \gamma\alpha' - \alpha\gamma' &= \eta \sin\theta, \\ \alpha\beta' - \beta\gamma' &= \zeta \sin\theta. \end{aligned}$$

We have:

$$(1 + e)(1 + e')(\beta\gamma' - \gamma\beta') = \begin{vmatrix} \frac{\partial y}{\partial x_0}\alpha_0 + \frac{\partial y}{\partial y_0}\beta_0 + \frac{\partial y}{\partial z_0}\gamma_0 & \frac{\partial z}{\partial x_0}\alpha_0 + \frac{\partial z}{\partial y_0}\beta_0 + \frac{\partial z}{\partial z_0}\gamma_0 \\ \frac{\partial y}{\partial x_0}\alpha_0' + \frac{\partial y}{\partial y_0}\beta_0' + \frac{\partial y}{\partial z_0}\gamma_0' & \frac{\partial z}{\partial x_0}\alpha_0' + \frac{\partial z}{\partial y_0}\beta_0' + \frac{\partial z}{\partial z_0}\gamma_0' \end{vmatrix}. \tag{13}$$

The determinant in the right-hand side can be put into the form of a sum of products of second-order determinants; equation (13) then takes the form:

$$(1 + e)(1 + e')(\beta\gamma' - \gamma\beta') =$$
$$\frac{d(y,z)}{d(y_0,z_0)}(\beta_0\gamma_0' - \gamma_0\beta_0') + \frac{d(y,z)}{d(z_0,x_0)}(\gamma_0\alpha_0' - \alpha_0\gamma_0') + \frac{d(y,z)}{d(x_0,y_0)}(\alpha_0\beta_0' - \beta_0\alpha_0'),$$

and that result will become:

$$(1 + e)(1 + e') \sin\theta \cdot \xi = \sin\theta_0 \left[ \frac{d(y,z)}{d(y_0,z_0)}\xi_0 + \frac{d(y,z)}{d(z_0,x_0)}\eta_0 + \frac{d(y,z)}{d(x_0,y_0)}\zeta_0 \right]$$

by an immediate transformation, or, upon dividing by $\sin\theta_0$ :

$$\frac{(1+e)(1+e')\sin\theta}{\sin\theta_0} \cdot \xi = \frac{d(y,z)}{d(y_0,z_0)}\xi_0 + \frac{d(y,z)}{d(z_0,x_0)}\eta_0 + \frac{d(y,z)}{d(x_0,y_0)}\zeta_0 .$$

We thus find the first of equations (7) in an equivalent form, and the interpretation of the result that is obtained will give formula (1) immediately.

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