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Research into the geometry of finite deformations

By J. LE ROUX.

Translated by D. H. Delphenich

Introduction

This paper has the objective of extending to finite deformations the geometric theory of torsion and flexure of continuous media that I had studied in a previous work for the case of infinitesimal deformations ([1]). The definitive results have exactly the same form, and the calculations present only insignificant differences at several points. Moreover, after establishing the fundamental formulas that relate to the flexure of fibers and sheets, I have deemed it pointless to recall the study of geometric properties that one can deduce; I refer that question to my previous paper.

Although I principally have the ulterior applications to mechanics in mind, it is obvious that this theory will present an exclusively geometric character. In some regards, one can consider it to be a branch of geometry that is strongly analogous to the theory of the curvature of lines and surfaces.

CHAPTER I

DILATATION

1-2-3. Definitions and generalities. Tangent homogeneous deformation. – 4. Relations between the coefficients of the homogeneous deformation. – 5. Linear dilatation. – 6. Surface deformation. – 7. Relations between the linear dilatation and the surface dilatation. – 8. Variation of the thickness of a layer. – 9. Angular dilations. – 10. Connection to deformation.

1. We consider a continuous medium in two different states, which we call the *initial state* and the *final – or deformed – state.* Points will be assumed to be defined by their coordinates referred to rectangular axes. However, for the questions that we will be occupied with, it is not at all necessary that the two states of the medium must be referred to the same axes. In order to simplify the presentation and avoid repetition, we agree, once and for all, to denote the corresponding elements of two media by the same symbols, but to affect the quantities that relate to the initial state with the index zero. For

([1]) Ann. de l'É.N.S (3) **28** (1911), 523-579.

example, if $M(x, y, z)$ is a point of the deformed medium then M_0 (x_0, y_0, z_0) will be the homologous point of the initial medium.

We suppose that the coordinates of each system are functions of the coordinates of the other system, and that those functions are continuous and differentiable, at least up to order two.

2. We first regard the coordinates x, y, z of a point M of the deformed medium as functions of the coordinates x_0, y_0, z_0 of the corresponding point of the initial medium.

We will then have the relations:

$$\text{(1)} \qquad \left\{ \begin{aligned} dx &= \frac{\partial x}{\partial x_0} dx_0 + \frac{\partial x}{\partial y_0} dy_0 + \frac{\partial x}{\partial z_0} dz_0, \\ dy &= \frac{\partial y}{\partial x_0} dx_0 + \frac{\partial y}{\partial y_0} dy_0 + \frac{\partial y}{\partial z_0} dz_0, \\ dz &= \frac{\partial z}{\partial x_0} dx_0 + \frac{\partial z}{\partial y_0} dy_0 + \frac{\partial z}{\partial z_0} dz_0 \end{aligned} \right.$$

between the differentials. We let Δ denote the functional determinant:

$$\text{(2)} \qquad \frac{d(x, y, z)}{d(x_0, y_0, z_0)} = \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial y}{\partial x_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial z_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix} = \Delta.$$

That determinant must be expressly supposed to be non-zero in the domain considered, and similarly, if the two media are referred to the same coordinate trihedron or to superposable trihedra then it will be necessary that Δ must remain positive in order for our transformation to have any real mechanical significance. Otherwise, it would be impossible to pass from the first state to the second one by a continuous deformation without annulling the volumes. For a similar reason, the determinant Δ must remain negative when the two coordinate trihedra are not superposable.

Upon considering the initial coordinates to be functions of the final coordinates, we will have:

(3)
$$\begin{cases} dx_0 = \dfrac{\partial x_0}{\partial x} dx + \dfrac{\partial x_0}{\partial y} dy + \dfrac{\partial x_0}{\partial z} dz, \\ dy_0 = \dfrac{\partial y_0}{\partial x} dx + \dfrac{\partial y_0}{\partial y} dy + \dfrac{\partial y_0}{\partial z} dz, \\ dz_0 = \dfrac{\partial z_0}{\partial x} dx + \dfrac{\partial z_0}{\partial y} dy + \dfrac{\partial z_0}{\partial z} dz. \end{cases}$$

Now, the values of the differentials dx_0, dy_0, dz_0 that are expressed by equations (3) are obviously identical to the ones that one will obtain by solving equations (1). One thus has the following identities, as well as the other analogous identities that one will deduce by permuting the coordinates:

(4)
$$\begin{cases} \dfrac{\partial x_0}{\partial x} = \dfrac{1}{\Delta} \dfrac{d(y,z)}{d(y_0,z_0)}, \\ \dfrac{\partial x_0}{\partial y} = \dfrac{1}{\Delta} \dfrac{d(z,x)}{d(y_0,z_0)}, \\ \dfrac{\partial x_0}{\partial z} = \dfrac{1}{\Delta} \dfrac{d(x,y)}{d(y_0,z_0)}. \end{cases}$$

The notation $\dfrac{d(y,z)}{d(y_0,z_0)}$ denotes the binary functional determinant:

$$\begin{vmatrix} \dfrac{\partial y}{\partial y_0} & \dfrac{\partial z}{\partial y_0} \\ \dfrac{\partial y}{\partial z_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix} = \frac{\partial \Delta}{\partial \left(\dfrac{\partial x}{\partial x_0} \right)}.$$

3. One attaches the notion of the tangent homogeneous deformation to a point M of the medium to the consideration of equations (1) between the two systems of differentials. One knows that what one thus calls the homogeneous deformation (T) is defined by the following equations, in which the symbols X, Y, Z, X_0, Y_0, Z_0 denote the current coordinates:

(T)
$$\begin{cases} X - x = \dfrac{\partial x}{\partial x_0}(X_0 - x_0) + \dfrac{\partial x}{\partial y_0}(Y_0 - y_0) + \dfrac{\partial x}{\partial z_0}(Z_0 - z_0), \\ Y - y = \dfrac{\partial y}{\partial x_0}(X_0 - x_0) + \dfrac{\partial y}{\partial y_0}(Y_0 - y_0) + \dfrac{\partial y}{\partial z_0}(Z_0 - z_0), \\ Z - z = \dfrac{\partial z}{\partial x_0}(X_0 - x_0) + \dfrac{\partial z}{\partial y_0}(Y_0 - y_0) + \dfrac{\partial z}{\partial z_0}(Z_0 - z_0). \end{cases}$$

That expression for the *tangent homogeneous at M* is convenient and sufficiently explicit. Meanwhile, one agrees to observe that one does not have a simple point M in view, but a pair of corresponding points (M_0, M), or, more exactly, the material element that that is transferred from M_0 to M.

4. Relations between the coefficients of the homogeneous deformations. – The coefficients of the homogeneous deformation (T) and those of the inverse deformation verify nine fundamental identities. We write that the values of dx_0, dy_0, dz_0 that are inferred from equations (T) will satisfy equations (1) identically:

$$\begin{aligned}
&\frac{\partial x}{\partial x_0}\frac{\partial x_0}{\partial x}+\frac{\partial x}{\partial y_0}\frac{\partial y_0}{\partial x}+\frac{\partial x}{\partial z_0}\frac{\partial z_0}{\partial x}=1,\\
&\frac{\partial x}{\partial x_0}\frac{\partial x_0}{\partial y}+\frac{\partial x}{\partial y_0}\frac{\partial y_0}{\partial y}+\frac{\partial x}{\partial z_0}\frac{\partial z_0}{\partial y}=0,\\
&\frac{\partial x}{\partial x_0}\frac{\partial x_0}{\partial z}+\frac{\partial x}{\partial y_0}\frac{\partial y_0}{\partial z}+\frac{\partial x}{\partial z_0}\frac{\partial z_0}{\partial z}=0,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

One can summarize these relations in a unique form by denoting the coordinates x_1, x_2, x_3, instead of x, y, z:

$$\frac{\partial x_i}{\partial x_0}\frac{\partial x_0}{\partial x_k}+\frac{\partial x_i}{\partial y_0}\frac{\partial y_0}{\partial x_k}+\frac{\partial x_i}{\partial z_0}\frac{\partial z_0}{\partial x_k}=\begin{cases}1 & \text{for } i=k,\\ 0 & \text{for } i\neq k.\end{cases}\tag{5}$$

Upon proceeding in the same fashion for the inverse deformation, we will obtain a second system of nine analogous identities, which are, moreover, consequences of the first ones; we further summarize them in the following formula, where x_{10}, x_{20}, x_{30} denote the initial coordinates:

$$\frac{\partial x_{i0}}{\partial x}\frac{\partial x}{\partial x_{k0}}+\frac{\partial x_{i0}}{\partial y}\frac{\partial y}{\partial x_{k0}}+\frac{\partial x_{i0}}{\partial z}\frac{\partial z}{\partial x_{k0}}=\begin{cases}1 & \text{for } i=k,\\ 0 & \text{for } i\neq k.\end{cases}\tag{6}$$

Formulas (5) and (6) obviously include the relations that exist between the nine cosines of an orthogonal transformation as a special case.

5. Linear dilatation. – A fiber, or a material line that passes through the point M_0 in the initial medium, is transformed into a fiber that passes through the point M of the deformed medium. Let ds be the length of an infinitely-small fiber that issues from the point M, let α, β, γ be its direction cosines, and let dx_1, dx_2, dx_3 be its projections onto the axes. One has:

$$\begin{aligned}
dx &= \alpha\, ds,\\
dy &= \beta\, ds,
\end{aligned}$$

$$dz = \gamma\, ds.$$

We denote the linear dilatation of the fiber by e, so:

$$\frac{ds}{ds_0} = 1 + e.$$

The relations between the position of the initial fiber and that of the deformed fiber can be deduced from equations (1) or (3). Upon dividing by ds_0, the first system will give:

$$\text{(7)} \qquad \begin{cases} (1+e)\,\alpha = \dfrac{\partial x}{\partial x_0}\alpha_0 + \dfrac{\partial x}{\partial y_0}\beta_0 + \dfrac{\partial x}{\partial z_0}\gamma_0, \\[2mm] (1+e)\,\beta = \dfrac{\partial y}{\partial x_0}\alpha_0 + \dfrac{\partial y}{\partial y_0}\beta_0 + \dfrac{\partial y}{\partial z_0}\gamma_0, \\[2mm] (1+e)\,\gamma = \dfrac{\partial z}{\partial x_0}\alpha_0 + \dfrac{\partial z}{\partial y_0}\beta_0 + \dfrac{\partial z}{\partial z_0}\gamma_0. \end{cases}$$

In the same manner, upon dividing by ds, one will deduce from the second one that:

$$\text{(8)} \qquad \begin{cases} \dfrac{\alpha}{1+e} = \dfrac{\partial x_0}{\partial x}\alpha + \dfrac{\partial x_0}{\partial y}\beta + \dfrac{\partial x_0}{\partial z}\gamma, \\[2mm] \dfrac{\beta}{1+e} = \dfrac{\partial y_0}{\partial x}\alpha + \dfrac{\partial y_0}{\partial y}\beta + \dfrac{\partial y_0}{\partial z}\gamma, \\[2mm] \dfrac{\gamma}{1+e} = \dfrac{\partial z_0}{\partial x}\alpha + \dfrac{\partial z_0}{\partial y}\beta + \dfrac{\partial z_0}{\partial z}\gamma. \end{cases}$$

We deduce the following equations, which define the dilatation, from these two systems:

$$\text{(9)} \qquad (1 + e)^2 = e_{11}^0\,\alpha_0^2 + e_{22}^0\,\beta_0^2 + e_{33}^0\,\gamma_0^2 + 2e_{23}^0\,\beta_0\gamma_0 + 2e_{31}^0\,\gamma_0\alpha_0 + 2e_{12}^0\,\alpha_0\beta_0\,,$$

$$\text{(10)} \qquad \frac{1}{(1+e)^2} = e_{11}\,\alpha^2 + e_{22}\,\beta^2 + e_{33}\,\gamma^2 + 2\,e_{23}\,\beta\gamma + 2\,e_{31}\,\gamma\alpha + 2\,e_{12}\,\alpha\beta\,.$$

In the first formula, we have set:

$$e_{11}^0 = \left(\frac{\partial x}{\partial x_0}\right)^2 + \left(\frac{\partial y}{\partial x_0}\right)^2 + \left(\frac{\partial z}{\partial x_0}\right)^2,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$e_{23}^0 = \frac{\partial x}{\partial y_0}\frac{\partial x}{\partial z_0} + \frac{\partial y}{\partial y_0}\frac{\partial y}{\partial z_0} + \frac{\partial z}{\partial y_0}\frac{\partial z}{\partial z_0}\,,$$

and in the second one:

$$e_{11} = \left(\frac{\partial x_0}{\partial x}\right)^2 + \left(\frac{\partial y_0}{\partial x}\right)^2 + \left(\frac{\partial z_0}{\partial x}\right)^2,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$e_{23} = \frac{\partial x_0}{\partial y}\frac{\partial x_0}{\partial z} + \frac{\partial y_0}{\partial y}\frac{\partial y_0}{\partial z} + \frac{\partial z_0}{\partial y}\frac{\partial z_0}{\partial z},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

The quadratic cosine forms (9) and (10) can obviously be replaced with differential forms that express the linear elements of each of the media as functions of the relative coordinates of the other one:

(9′) $$ds^2 = \sum c_{ik}^0 \, dx_{i0} \, dx_{k0},$$

(10′) $$ds_0^2 = \sum e_{ik} \, dx_i \, dx_k .$$

One knows how the consideration of formulas (9) or (10) leads one to represent the dilation by a second-order indicatrix – viz., the ellipsoid of dilatations – that one can consider in either of the two media ([1]).

6. Surface dilatation. – I call a portion of matter that extends over a surface, but with negligible thickness, a *sheet*. An elementary sheet can be associated with an infinitely-small piece of the material surface.

Consider two elementary fibers that issue from the same origin M, and denote their component in the deformed state by:

$$d_1x, \quad d_1y, \quad d_1z$$

and

$$d_2x, \quad d_2y, \quad d_2z,$$

respectively. Set:

$$d(y, z) = \begin{vmatrix} d_1y & d_1z \\ d_2y & d_2z \end{vmatrix},$$

$$\dots\dots\dots\dots\dots\dots$$

The determinant:

$$d(y, z) = \begin{vmatrix} d_1y & d_1z \\ d_2y & d_2z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y}{\partial x_0} d_1x_0 + \frac{\partial y}{\partial y_0} d_1y_0 + \frac{\partial y}{\partial z_0} d_1z_0 & \frac{\partial z}{\partial x_0} d_1x_0 + \frac{\partial z}{\partial y_0} d_1y_0 + \frac{\partial z}{\partial z_0} d_1z_0 \\ \frac{\partial y}{\partial x_0} d_2x_0 + \frac{\partial y}{\partial y_0} d_2y_0 + \frac{\partial y}{\partial z_0} d_2z_0 & \frac{\partial z}{\partial x_0} d_2x_0 + \frac{\partial z}{\partial y_0} d_2y_0 + \frac{\partial z}{\partial z_0} d_2z_0 \end{vmatrix},$$

([1]) E. and F. COSSERAT, Annales de la Faculté des Sciences de Toulouse, t. X.

and the other analogous determinants $d\,(z, x)$, $d\,(z, y)$ can be developed into a sum of products of binary determinants:

$$
(11)\qquad \begin{cases}
d(y,z) = \dfrac{d(y,z)}{d(y_0,z_0)}d(y_0,z_0) + \dfrac{d(y,z)}{d(z_0,x_0)}d(z_0,x_0) + \dfrac{d(y,z)}{d(x_0,y_0)}d(x_0,y_0), \\
d(z,x) = \dfrac{d(z,x)}{d(y_0,z_0)}d(y_0,z_0) + \dfrac{d(z,x)}{d(z_0,x_0)}d(z_0,x_0) + \dfrac{d(z,x)}{d(x_0,y_0)}d(x_0,y_0), \\
d(x,y) = \dfrac{d(x,y)}{d(y_0,z_0)}d(y_0,z_0) + \dfrac{d(x,y)}{d(z_0,x_0)}d(z_0,x_0) + \dfrac{d(x,y)}{d(x_0,y_0)}d(x_0,y_0).
\end{cases}
$$

The determinant $d(y, z)$ represents, in magnitude and sign, the area of the projection onto the plane YOZ of the infinitely-small parallelogram $d\sigma$ that is defined by the two elementary fibers considered. Thus, let ξ, η, ζ be the direction cosines of the normal to the element ds, where that normal is assumed to be drawn in the sense of the positive axis of rotation of the first fiber to the second one. One will have:

$$
\xi\, d\sigma = d\,(y, z),\qquad
\eta\, d\sigma = d\,(z, x),\qquad
\zeta\, d\sigma = d\,(x, y).
$$

We let E denote the dilatation of the corresponding elementary sheet:

$$
\frac{d\sigma}{d\sigma_0} = 1 + E,
$$

and we let ξ_0 , η_0, ζ_0 denote the direction cosines of the normal to the initial sheet $d\sigma_0$. There is thus reason to observe that the two systems of cosines (ξ, η, ζ) and (ξ_0, η_0, ζ_0) do not refer to two states of the same fiber. The direction cosines of the fibers and the direction cosines of the normals to the sheets form two contragredient systems, to use the expression that Sylvester employed in the theory of algebraic forms.

Upon dividing by $d\sigma_0$, we will obtain equations (11), and upon introducing the notations that were defined above, we will get the relations:

$$
(12)\qquad \begin{cases}
(1+E)\,\xi = \dfrac{d(y,z)}{d(y_0,z_0)}\xi_0 + \dfrac{d(y,z)}{d(z_0,x_0)}\eta_0 + \dfrac{d(y,z)}{d(x_0,y_0)}\zeta_0, \\
(1+E)\,\eta = \dfrac{d(z,x)}{d(y_0,z_0)}\xi_0 + \dfrac{d(z,x)}{d(z_0,x_0)}\eta_0 + \dfrac{d(z,x)}{d(x_0,y_0)}\zeta_0, \\
(1+E)\,\zeta = \dfrac{d(x,y)}{d(y_0,z_0)}\xi_0 + \dfrac{d(x,y)}{d(z_0,x_0)}\eta_0 + \dfrac{d(x,y)}{d(x_0,y_0)}\zeta_0.
\end{cases}
$$

The inverse transformation will lead to similar formulas, and we write down simply the first of them:

(13) $$\frac{\xi_0}{1+E} = \frac{d(y_0,z_0)}{d(y,z)}\xi + \frac{d(y_0,z_0)}{d(z,x)}\eta + \frac{d(y_0,z_0)}{d(x,y)}\zeta\,,$$

………………………………………………..

If one takes equations (4) into account then the preceding results can be put into the form:

(12′) $$\begin{cases} \dfrac{1+E}{\Delta}\xi = \dfrac{\partial x_0}{\partial x}\xi_0 + \dfrac{\partial y_0}{\partial x}\eta_0 + \dfrac{\partial z_0}{\partial x}\zeta_0, \\ \dfrac{1+E}{\Delta}\eta = \dfrac{\partial x_0}{\partial y}\xi_0 + \dfrac{\partial y_0}{\partial y}\eta_0 + \dfrac{\partial z_0}{\partial y}\zeta_0, \\ \dfrac{1+E}{\Delta}\zeta = \dfrac{\partial x_0}{\partial z}\xi_0 + \dfrac{\partial y_0}{\partial z}\eta_0 + \dfrac{\partial z_0}{\partial z}\zeta_0, \end{cases}$$

(13′) $$\frac{\Delta\xi_0}{1+E} = \frac{\partial x}{\partial x_0}\xi + \frac{\partial y}{\partial x_0}\eta + \frac{\partial z}{\partial x_0}\zeta\,,$$

………………………………

There is a remarkable analogy between equations (7) and (8), which relate to the transformation of fibers, and systems (12) and (13), which relate to the transformation of sheets. The surface dilatation will be likewise defined by formulas that are similar to the ones that we found for the linear dilatation.

Set:

$$E_{11}^0 = \left[\frac{d(y,z)}{d(y_0,z_0)}\right]^2 + \left[\frac{d(z,x)}{d(y_0,z_0)}\right]^2 + \left[\frac{d(x,y)}{d(y_0,z_0)}\right]^2,$$

…………………………………………………,

$$E_{23}^0 = \frac{d(y,z)}{d(z_0,x_0)}\frac{d(y,z)}{d(y_0,z_0)} + \frac{d(z,x)}{d(z_0,x_0)}\frac{d(z,x)}{d(x_0,y_0)} + \frac{d(x,y)}{d(z_0,x_0)}\frac{d(x,y)}{d(x_0,y_0)},$$

……………………………………………………………………...,

and similarly:

$$E_{11} = \left[\frac{d(y_0,z_0)}{d(y,z)}\right]^2 + \left[\frac{d(z_0,x_0)}{d(y,z)}\right]^2 + \left[\frac{d(x_0,y_0)}{d(y,z)}\right]^2,$$

…………………………………………………,

$$E_{23} = \frac{d(y_0,z_0)}{d(z,x)}\frac{d(y_0,z_0)}{d(y,z)} + \frac{d(z_0,x_0)}{d(z,x)}\frac{d(z_0,x_0)}{d(x,y)} + \frac{d(x_0,y_0)}{d(z,x)}\frac{d(x_0,y_0)}{d(x,y)},$$

……………………………………………………………………...

We will then have the following formulas for the surface dilatations:

(14) $$(1+E)^2 = E_{11}^0\,\xi_0^2 + E_{22}^0\,\eta_0^2 + E_{33}^0\,\zeta_0^2 + 2E_{23}^0\,\eta_0\zeta_0 + 2E_{31}^0\,\zeta_0\xi_0 + 2E_{12}^0\zeta_0\eta_0\,,$$

(15) $$\frac{1}{(1+E)^2} = E_{11}\,\xi^2 + E_{22}\,\eta^2 + E_{33}\,\zeta^2 + 2E_{23}\,\eta\zeta + 2E_{31}\,\zeta\xi + 2E_{12}\,\xi\eta\,.$$

7. Relation between the linear dilatation and the surface dilatation. – One easily verifies that each of the quadratic forms that relate to the surface dilatation is the adjoint to the corresponding form that relates to the linear dilatation. That property will result immediately from calculating with the coefficients. For example, one will have:

$$E_{11}^0 = e_{22}^0 e_{33}^0 - (e_{23}^0)^2,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$E_{23}^0 = e_{12}\,e_{31} - e_{11}\,e_{22}\,,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Since the discriminant of the form (7) is equal to Δ^2, that of the form (14) will be equal to Δ^4. A similar relationship exists between the forms (10) and (15), whose discriminants are equal to $1 / \Delta^2$ and $1 / \Delta^4$, respectively.

Moreover, a very simple geometric reasoning will show that this must be true. Consider the ellipsoid of dilatations that relates to the point M of the deformed media. If one denotes the quadratic form that figures in the right-hand side of equation (10) by φ (α, β, γ) then the ellipsoid considered will be represented by:

$$\varphi(X, Y, Z) = 1.$$

Each ray of the ellipsoid is obtained by starting at the origin and associating a measured length with the value of the ratio $1 + e$ that corresponds to the direction of that ray in the deformed medium.

Let (D) be a diametral plane of the ellipsoid, let E be the surface dilatation of an elementary sheet that passes through M and parallel to the plane (D). In that diametral place, the area of the parallelogram that is constructed from two conjugate rays of the ellipsoid will be constant and equal to $1 + E$. Draw a plane (P) that is tangent to the ellipsoid and parallel to (D), and let δ be the distance to the origin of the plane (P). The product of the area $(1 + E)$ with the distance δ is equal to the volume of the parallelpepiped that is constructed from three conjugate rays of the ellipsoid. One then has:

(16) $$(1 + E)\,\delta = \Delta.$$

If one lets $\Phi(u, v, w)$ denote the adjoint quadratic form to φ (α, β, γ) then the tangential equation to the ellipsoid will be put into the form:

$$\Phi(u, v, w) = \frac{h^2}{\Delta^2},$$

in which u, v, w, h are the homogeneous coordinates of a tangent plane.

Now, replace the coordinates u, v, w with the direction cosines ξ, η, ζ of the normal, and replace h with $-\delta$, we have the relation:

$$\Phi(\xi, \eta, \zeta) = \frac{\delta^2}{\Delta^2},$$

which then becomes:

$$\Phi(\xi, \eta, \zeta) = \frac{1}{(1+E)^2},$$

by virtue of equation (16). One thus find formula (15).

8. Variation of the thickness of the layer. – The quantity $\delta = \Delta / (1 + E)$ has a simple significance in the deformation. Consider an infinitely-small cylinder that contains the point M. Let dV be its volume, let $d\sigma$ be its area, and let dh be its height. One has:

(17) $$dV = d\sigma\, dh.$$

In the initial medium, one will likewise have:

(17′) $$dV_0 = d\sigma_0\, dh_0$$

for the corresponding cylinder.

The ratio dV / dV_0 is equal to Δ, while the ratio of the areas $d\sigma / d\sigma_0$ is equal to $1 + E$, in which E denotes the surface dilatation of the base. Upon dividing both sides of equations (17) by both sides of (17′), one will then find:

$$\Delta = (1 + E)\frac{dh}{dh_0},$$

and when that result is compared to equation (16) that will give:

(18) $$\frac{dh}{dh_0} = d = \frac{\Delta}{1+E}.$$

From that, if one cuts an infinitely-thin material layer in the medium that passes through the point M and has a height of dh at that point then the ratio of the height of the deformed layer to that of the initial layer – viz., dh / dh_0 – will be equal to δ. The variation of $1 / \delta$ as a function of the direction cosines of the normal is proportional to that of the surface ratio $1 + E$. One will deduce equations (12′) and (13′) directly.

9. Angular dilatations. – The calculation of angles can be performed analytically with the aid of the quadratic forms that enter into the expression for the linear or surface dilatation and the corresponding polar forms ([1]). However, the results will be obtained more rapidly by geometric considerations.

1. *Angle between fibers.* – Consider two elementary fibers that issue from the same point M. Let ds, ds' be their lengths, let θ be the angle between them, let e and e' be the corresponding linear dilatations, and let E be the surface dilatation of the sheet that they determine.

One will have:

$$\frac{ds\,ds'\sin\theta}{ds_0\,ds'_0\sin\theta_0} = 1 + E,$$

so

$$\frac{\sin\theta}{\sin\theta_0} = \frac{1+E}{(1+e)(1+e')}. \tag{19}$$

1. *Angle between a fiber and a sheet.* – Start from the point M, and take an infinitely-small length ds on the fiber, and a surface element $d\sigma$ on the sheet.

Let φ be the angle between the fiber and the sheet. The volume of the infinitesimal cylinder that has $d\sigma$ for its base and ds for its edge will be equal to:

$$ds\,d\sigma\sin\varphi.$$

Upon letting Θ denote the cubic dilatation at the point considered, letting e denote the linear dilatation of the fiber, and if one lets E denote the surface dilatation of the sheet then one will have, in turn:

$$\frac{ds\,d\sigma\sin\varphi}{ds_0\,d\sigma_0\sin\varphi_0} = 1 + \Theta,$$

$$(1 + e)\,(1 + E)\,\frac{\sin\varphi}{\sin\varphi_0} = 1 + \Theta,$$

$$\frac{\sin\varphi}{\sin\varphi_0} = \frac{1+\Theta}{(1+e)(1+E)}. \tag{19'}$$

If one considers the angle φ between the fiber and the normal to the sheet, instead of the angle φ' between the fiber and the sheet, then the preceding relation will become:

$$\frac{\cos\varphi'}{\cos\varphi'_0} = \frac{1+\Theta}{(1+e)(1+E)}.$$

([1]) One will find these calculations performed in a Note: "Sur les déformations angulaires" (Travaux Scientifique de l'Université de Rennes, 1911).

3. *Angle between two sheets.* – The volume of an arbitrary parallelepiped is equal to the product of the areas of the two contiguous faces, multiplied by the sines of their dihedral angles, and divided by the length of their common edge.

From that, consider two elementary sheets that intersect at *M* and form the dihedral angle ψ between them. Let *E* and E' be their surface dilatations, let *e* be the linear dilatation of the fiber that is directed along their intersection, let Θ be the cubic dilatation at the point *M*. One will immediately finds, by an argument that is analogous to the one that we employed for the two preceding cases:

$$\frac{(1+E)(1+E')}{1+e}\frac{\sin\psi}{\sin\psi_0} = 1 + \Theta,$$

so

$$\frac{\sin\psi}{\sin\psi_0} = \frac{(1+\Theta)(1+e)}{(1+E)(1+E')}. \tag{19'}$$

The three formulas (19), (19′), (19″) are remarkable in their simplicity and their similarity. They show that *deformation ratios* for the angles of the sines that are analogous to those of the lines, surfaces, and volumes.

10. Deformation ratios. – The name of *deformation ratio* seems convenient and significant to us as a way of representing the ratio of a quantity of the deformed medium to the corresponding quantity of the initial medium: The ratio of the linear deformation is represented by $1 + e$, the *ratio of surface deformation*, by $1 + E$, and the *ratio of cubic dilatation*, by $1 + \Theta = \Delta$. In the calculations that relate to finite deformations, the dilatation will generally enter into only the corresponding deformation ratio; for example, we will not have to consider the linear dilatation *e* except in the linear deformation ratio $1 + e$.

CHAPTER II

SECOND-ORDER DIFFERENTIAL ELEMENTS

11. The differential deformation (dT). – Let (T) and (T') be the tangent homogeneous deformations at two infinitely-close points M and M'. One can consider the second (T') as the resultant of the first (T) and an infinitesimal deformation (dT) that we call the *differential deformation* at the point M.

The deformation (T) is defined by the equations (T) of no. **3**, and the deformation (T'), by the analogous equations:

$$\begin{cases} X'-x' = \left(\dfrac{\partial x}{\partial x_0}+d\dfrac{\partial x}{\partial x_0}\right)(X_0-x_0)+\left(\dfrac{\partial x}{\partial y_0}+d\dfrac{\partial x}{\partial y_0}\right)(Y_0-y_0)+\left(\dfrac{\partial x}{\partial z_0}+d\dfrac{\partial x}{\partial z_0}\right)(Z_0-z_0), \\ Y'-y' = \left(\dfrac{\partial y}{\partial x_0}+d\dfrac{\partial y}{\partial x_0}\right)(X_0-x_0)+\left(\dfrac{\partial y}{\partial y_0}+d\dfrac{\partial y}{\partial y_0}\right)(Y_0-y_0)+\left(\dfrac{\partial y}{\partial z_0}+d\dfrac{\partial y}{\partial z_0}\right)(Z_0-z_0), \\ Z'-z' = \left(\dfrac{\partial z}{\partial x_0}+d\dfrac{\partial z}{\partial x_0}\right)(X_0-x_0)+\left(\dfrac{\partial z}{\partial y_0}+d\dfrac{\partial z}{\partial y_0}\right)(Y_0-y_0)+\left(\dfrac{\partial z}{\partial z_0}+d\dfrac{\partial z}{\partial z_0}\right)(Z_0-z_0). \end{cases}$$

The differential deformation (dT) will be represented by the equations that one obtains by replacing the values of the initial current coordinates X_0, Y_0, Z_0 in the system (T) with their values that are inferred from the system (T).

Now, upon solving the system (T) for the differences $X_0 - x_0$, $Y_0 - y_0$, $Z_0 - z_0$, one will find:

$$X_0 - x_0 = \frac{\partial x_0}{\partial x}(X-x)+\frac{\partial x_0}{\partial y}(Y-y)+\frac{\partial x_0}{\partial z}(Z-z),$$
$$Y_0 - y_0 = \frac{\partial y_0}{\partial x}(X-x)+\frac{\partial y_0}{\partial y}(Y-y)+\frac{\partial y_0}{\partial z}(Z-z),$$
$$Z_0 - z_0 = \frac{\partial z_0}{\partial x}(X-x)+\frac{\partial z_0}{\partial y}(Y-y)+\frac{\partial z_0}{\partial z}(Z-z).$$

When these values are substituted in equations (T), while taking the identities (5) into account, that will give the defining equations for the differential transformation (dT) in the following form:

$$(dT)\left\{\begin{aligned} X' - x' &= (1+da_{11})(X-x) + da_{12}\ (Y-y) + da_{13}\ (Z-z),\\ Y' - y' &= da_{21}\ (X-x) + (1+da_{22})(Y-y) + da_{23}\ (Z-z),\\ Z' - z' &= da_{31}\ (X-x) + da_{22}\ (Y-y) + (1+da_{33})(Z-z). \end{aligned}\right.$$

We have set:

$$\begin{aligned} da_{11} &= \frac{\partial x_0}{\partial x} d\left(\frac{\partial x}{\partial x_0}\right) + \frac{\partial y_0}{\partial x} d\left(\frac{\partial x}{\partial y_0}\right) + \frac{\partial z_0}{\partial x} d\left(\frac{\partial x}{\partial z_0}\right)\\ &= -\left[\frac{\partial x}{\partial x_0} d\left(\frac{\partial x_0}{\partial x}\right) + \frac{\partial x}{\partial y_0} d\left(\frac{\partial y_0}{\partial x}\right) + \frac{\partial x}{\partial z_0} d\left(\frac{\partial z_0}{\partial x}\right)\right], \end{aligned}$$

$$\begin{aligned} da_{12} &= \frac{\partial x_0}{\partial y} d\left(\frac{\partial x}{\partial x_0}\right) + \frac{\partial y_0}{\partial y} d\left(\frac{\partial x}{\partial y_0}\right) + \frac{\partial z_0}{\partial y} d\left(\frac{\partial x}{\partial z_0}\right)\\ &= -\left[\frac{\partial x}{\partial x_0} d\left(\frac{\partial x_0}{\partial y}\right) + \frac{\partial x}{\partial y_0} d\left(\frac{\partial y_0}{\partial y}\right) + \frac{\partial x}{\partial z_0} d\left(\frac{\partial z_0}{\partial y}\right)\right], \end{aligned}$$

$$\begin{aligned} da_{13} &= \frac{\partial x_0}{\partial z} d\left(\frac{\partial x}{\partial x_0}\right) + \frac{\partial y_0}{\partial z} d\left(\frac{\partial x}{\partial y_0}\right) + \frac{\partial z_0}{\partial z} d\left(\frac{\partial x}{\partial z_0}\right)\\ &= -\left[\frac{\partial x}{\partial x_0} d\left(\frac{\partial x_0}{\partial z}\right) + \frac{\partial x}{\partial y_0} d\left(\frac{\partial y_0}{\partial z}\right) + \frac{\partial x}{\partial z_0} d\left(\frac{\partial z_0}{\partial z}\right)\right], \end{aligned}$$

$$\begin{aligned} da_{21} &= \frac{\partial x_0}{\partial x} d\left(\frac{\partial y}{\partial x_0}\right) + \frac{\partial y_0}{\partial x} d\left(\frac{\partial y}{\partial y_0}\right) + \frac{\partial z_0}{\partial x} d\left(\frac{\partial y}{\partial z_0}\right)\\ &= -\left[\frac{\partial y}{\partial x_0} d\left(\frac{\partial x_0}{\partial x}\right) + \frac{\partial y}{\partial y_0} d\left(\frac{\partial y_0}{\partial x}\right) + \frac{\partial y}{\partial z_0} d\left(\frac{\partial z_0}{\partial x}\right)\right], \end{aligned}$$

$$\begin{aligned} da_{22} &= \frac{\partial x_0}{\partial y} d\left(\frac{\partial y}{\partial x_0}\right) + \frac{\partial y_0}{\partial y} d\left(\frac{\partial y}{\partial y_0}\right) + \frac{\partial z_0}{\partial y} d\left(\frac{\partial y}{\partial z_0}\right)\\ &= -\left[\frac{\partial y}{\partial x_0} d\left(\frac{\partial x_0}{\partial y}\right) + \frac{\partial y}{\partial y_0} d\left(\frac{\partial y_0}{\partial y}\right) + \frac{\partial y}{\partial z_0} d\left(\frac{\partial z_0}{\partial y}\right)\right], \end{aligned}$$

$$\begin{aligned} da_{23} &= \frac{\partial x_0}{\partial z} d\left(\frac{\partial y}{\partial x_0}\right) + \frac{\partial y_0}{\partial z} d\left(\frac{\partial y}{\partial y_0}\right) + \frac{\partial z_0}{\partial z} d\left(\frac{\partial y}{\partial z_0}\right)\\ &= -\left[\frac{\partial y}{\partial x_0} d\left(\frac{\partial x_0}{\partial z}\right) + \frac{\partial y}{\partial y_0} d\left(\frac{\partial y_0}{\partial z}\right) + \frac{\partial y}{\partial z_0} d\left(\frac{\partial z_0}{\partial z}\right)\right], \end{aligned}$$

$$da_{31} = \frac{\partial x_0}{\partial x} d\left(\frac{\partial z}{\partial x_0}\right) + \frac{\partial y_0}{\partial x} d\left(\frac{\partial z}{\partial y_0}\right) + \frac{\partial z_0}{\partial x} d\left(\frac{\partial z}{\partial z_0}\right)$$

$$= -\left[\frac{\partial z}{\partial x_0} d\left(\frac{\partial x_0}{\partial x}\right) + \frac{\partial z}{\partial y_0} d\left(\frac{\partial y_0}{\partial x}\right) + \frac{\partial z}{\partial z_0} d\left(\frac{\partial z_0}{\partial x}\right)\right],$$

$$da_{32} = \frac{\partial x_0}{\partial y} d\left(\frac{\partial z}{\partial x_0}\right) + \frac{\partial y_0}{\partial y} d\left(\frac{\partial z}{\partial y_0}\right) + \frac{\partial z_0}{\partial y} d\left(\frac{\partial z}{\partial z_0}\right)$$

$$= -\left[\frac{\partial z}{\partial x_0} d\left(\frac{\partial x_0}{\partial y}\right) + \frac{\partial z}{\partial y_0} d\left(\frac{\partial y_0}{\partial y}\right) + \frac{\partial z}{\partial z_0} d\left(\frac{\partial z_0}{\partial y}\right)\right],$$

$$da_{33} = \frac{\partial x_0}{\partial z} d\left(\frac{\partial z}{\partial x_0}\right) + \frac{\partial y_0}{\partial z} d\left(\frac{\partial z}{\partial y_0}\right) + \frac{\partial z_0}{\partial z} d\left(\frac{\partial z}{\partial z_0}\right)$$

$$= -\left[\frac{\partial z}{\partial x_0} d\left(\frac{\partial x_0}{\partial z}\right) + \frac{\partial z}{\partial y_0} d\left(\frac{\partial y_0}{\partial z}\right) + \frac{\partial z}{\partial z_0} d\left(\frac{\partial z_0}{\partial z}\right)\right].$$

These formulas are summarized in the following ones:

(20)
$$da_{ij} = \frac{\partial x_0}{\partial x_j} d\left(\frac{\partial x_i}{\partial x_0}\right) + \frac{\partial y_0}{\partial x_j} d\left(\frac{\partial x_i}{\partial y_0}\right) + \frac{\partial z_0}{\partial x_j} d\left(\frac{\partial x_i}{\partial z_0}\right)$$

$$= -\left[\frac{\partial x_i}{\partial x_0} d\left(\frac{\partial x_0}{\partial x_j}\right) + \frac{\partial x_i}{\partial y_0} d\left(\frac{\partial y_0}{\partial x_j}\right) + \frac{\partial x_i}{\partial z_0} d\left(\frac{\partial z_0}{\partial x_j}\right)\right].$$

12. The coefficients a_{ijk} . – One notes the analogy between the differentials da_{ij} and the infinitely-small rotations that relate to the displacement of a tri-rectangular trihedron. Our calculation is, moreover, applicable to the case of two arbitrary, infinitely-close translations that are independent of the parameters that serve to define them.

In what follows, we shall suppose that one takes the independent variable to be the coordinates x, y, z of a point M of the deformed medium, and we shall set:

(21)
$$da_{ij} = a_{ij1}\, dx + a_{ij2}\, dy + a_{ij3}\, dz.$$

It results immediately from equation (22) that one can invert the order of the last two indices in the three-index coefficients:

$$a_{ijk} = a_{ikj} .$$

For the infinitely-small deformations that one usually considered in elasticity, and which one assumes to be defined by equations of the form:

$$x' = x + u, \qquad y' = y + v, \qquad z' = z + w,$$

the coefficients a_{ijk} reduce to second derivatives of the displacements u, v, w.

The number of independent coefficients a_{ijk} is equal to the number of second derivatives of the coordinates of the one system with respect to those of the other – viz., 18. Later on, we shall study the relations that exist between these coefficients and the various elements of the deformation. However, we shall first occupy ourselves with the dilatations in the differential dilatation (dT).

13. Linear dilatation in the deformation (dT). – If we let e denote the linear dilatation of a fiber under the homogeneous deformation (T), and let e' denote the dilatation of the *same fiber* or *a parallel fiber* under the deformation (T') then the corresponding deformation ratio for the differential deformation (dT) will be equal to $\frac{1+e'}{1+e}$. The transformation of the direction cosines will be obtained by formulas that are analogous to equations (7). Let α', β', γ' be the cosines that the transformation (dT) makes correspond to α, β, γ. We will get three equations of the form:

$$\frac{1+e'}{1+e}\alpha' = (1 + da_{11})\,\alpha + da_{12}\,\beta + da_{13}\,\gamma.$$

However, since the new direction is infinitely close to the first one, and the difference between the dilatations is infinitely small, it is natural to set:

$$e' = e + de, \qquad \alpha' = \alpha + d\alpha, \quad \beta' = \beta + d\beta, \quad \gamma' = \gamma + d\gamma.$$

If we substitute these values into the equation above, while neglecting second-order infinitesimals, then we will find:

$$\left(1+\frac{de}{1+e}\right)\alpha + d\alpha = (1 + da_{11})\,\alpha + da_{12}\,\beta + da_{13}\,\gamma.$$

That will finally give us the system:

$$\text{(23)} \qquad \left\{\begin{aligned} d\alpha + \alpha\frac{de}{1+e} &= \alpha\,da_{11} + \beta\,da_{12} + \gamma\,da_{13}, \\ d\beta + \beta\frac{de}{1+e} &= \alpha\,da_{21} + \beta\,da_{22} + \gamma\,da_{23}, \\ d\gamma + \gamma\frac{de}{1+e} &= \alpha\,da_{31} + \beta\,da_{32} + \gamma\,da_{33}. \end{aligned}\right.$$

Moreover, one will recover the same results by differentiating formulas (7), where one regards α_0, β_0, γ_0 as constants, and upon then replacing these direction cosines of the initial fiber with their values as functions of α, β, γ that are given by equations (8).

Add both sides of equations (23), after having multiplied them by α, β, γ, respectively; that will give:

$$\frac{de}{1+e} = \alpha^2\, da_{11} + \beta^2\, da_{22} + \gamma^2\, da_{33} + \beta\gamma(da_{32} + da_{23}) \tag{24}$$

$$+ \gamma\alpha\,(da_{13} + da_{31}) + \alpha\beta\,(da_{21} + da_{12})\ .$$

The logarithmic differential $\frac{de}{1+e}$, when taken while considering α_0, β_0, γ_0 to be constants, is thus calculated with the aid of the coefficients of the differential deformation (dT) as the linear dilatation under ordinary infinitesimal deformations.

14. Mean rotation. – The mean rotation of the differential deformation has the components:

$$\begin{cases} dp_1 = \frac{1}{2}(da_{32} - da_{23}), \\ dp_2 = \frac{1}{2}(da_{13} - da_{31}), \\ dp_3 = \frac{1}{2}(da_{21} - da_{12}). \end{cases} \tag{25}$$

In the case of infinitely-small deformations, the quantities dp_i are the differentials of the components of the mean rotation, but for finite deformations, those quantities are not exact differentials, at least, in general. Nevertheless, we can, with no inconvenience, let $\frac{\partial p_i}{\partial x}$, $\frac{\partial p_i}{\partial y}$, $\frac{\partial p_i}{\partial z}$ denote the coefficients of dx, dy, dz, respectively, in the expression for dp_i.

One will then have:

$$\frac{\partial p_i}{\partial x} = \tfrac{1}{2}(a_{321} - a_{231}), \quad \frac{\partial p_i}{\partial y} = \tfrac{1}{2}(a_{322} - a_{232}), \quad \frac{\partial p_i}{\partial z} = \tfrac{1}{2}(a_{323} - a_{233}). \tag{26}$$

The other analogous expressions are deduced from these by permuting indices.

The identities $a_{ijk} = a_{ikj}$ give the relation:

$$\frac{\partial p_1}{\partial x} + \frac{\partial p_2}{\partial y} + \frac{\partial p_3}{\partial z} = 0.$$

We believe that is it useful to remark here that if the homogeneous deformations (T) and (T') are pure deformations then the rotation of the differential deformation (dT) will nevertheless be non-zero, unless the principal dilatation axes of (T) and (T') are not parallel. That result will correspond to the fact that the finite, pure deformations do not, in general, constitute a group of transformations.

15. Cubic dilatation. – Let dV_0 be an infinitely-small volume element that surrounds the point M_0 of the initial medium; dV and dV' are the transforms of that same volume under the homogeneous deformations (T) and (T'), respectively.

The cubic dilatation of the differential deformation (dT) is equal to $dV' / dV - 1$. Now, we have:

$$dV = (1 + \Theta)\, dV_0$$

and

$$dV' = (1 + \Theta + d\Theta)\, dV_0 \,;$$

consequently:

$$\frac{dV'}{dV} - 1 = \frac{d\Theta}{1+\Theta}.$$

On the other hand, one knows that the cubic dilatation of the deformation (dT) is represented by the sum:

$$da_{11} + da_{22} + da_{33}\,.$$

Hence, one has the identity:

$$da_{11} + da_{22} + da_{33} = \frac{d\Theta}{1+\Theta}. \tag{27}$$

It is easy to verify that result directly by calculation. In fact, upon differentiating the equation:

$$1 + \Theta = \begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial y_0} & \dfrac{\partial x}{\partial z_0} \\ \dfrac{\partial y}{\partial x_0} & \dfrac{\partial y}{\partial y_0} & \dfrac{\partial y}{\partial z_0} \\ \dfrac{\partial z}{\partial x_0} & \dfrac{\partial z}{\partial y_0} & \dfrac{\partial z}{\partial z_0} \end{vmatrix},$$

one will find:

$$d\Theta = \frac{d(y,z)}{d(y_0,z_0)}\, d\left(\frac{\partial x}{\partial x_0}\right) + \frac{d(y,z)}{d(z_0,x_0)}\, d\left(\frac{\partial x}{\partial y_0}\right) + \frac{d(y,z)}{d(x_0,y_0)}\, d\left(\frac{\partial x}{\partial z_0}\right) + \ldots$$

If we replace the binary functional determinants in the right-hand sides with their values that one infers from equations (4) then we will have:

$$d\Theta = (1 + \Theta)\left[\frac{\partial x_0}{\partial x}\, d\left(\frac{\partial x}{\partial x_0}\right) + \frac{\partial y_0}{\partial x}\, d\left(\frac{\partial x}{\partial y_0}\right) + \frac{\partial z_0}{\partial x}\, d\left(\frac{\partial x}{\partial z_0}\right) + \cdots\right];$$

finally, by virtue of formulas (20):

$$d\Theta = (1 + \Theta)\,(da_{11} + da_{22} + da_{33}).$$

16. Surface dilatation. – Differentiate equations (23), while regarding ξ_0, η_0, ζ_0 as constants in them.

The first one gives:

$$\left(\frac{1+E}{1+\Theta}\right)d\xi+\xi d\left(\frac{1+E}{1+\Theta}\right) = \xi_0 d\left(\frac{\partial x_0}{\partial x}\right)+\eta_0 d\left(\frac{\partial y_0}{\partial x}\right)+\zeta_0 d\left(\frac{\partial z_0}{\partial x}\right).$$

Upon replacing the cosines ξ_0, η_0, ζ_0 in the right-hand side of that equation with their values that are inferred from equations (13′) and proceeding in the same manner in regard to the other two equations of the system (12′), one will obtain the system:

$$(28)\qquad \begin{cases} d\xi+\xi\, d\log\left(\dfrac{1+E}{1+\Theta}\right)=-(\xi da_{11}+\eta da_{21}+\zeta da_{31}), \\ d\eta+\eta\, d\log\left(\dfrac{1+E}{1+\Theta}\right)=-(\xi da_{12}+\eta da_{22}+\zeta da_{32}), \\ d\zeta+\zeta\, d\log\left(\dfrac{1+E}{1+\Theta}\right)=-(\xi da_{13}+\eta da_{23}+\zeta da_{33}). \end{cases}$$

One deduces from this that:

$$(29)\qquad d\log\left(\frac{1+\Theta}{1+E}\right) = \xi^2\, da_{11} + \eta^2\, da_{22} + \zeta^2\, da_{33} + \eta\zeta\,(da_{32} + da_{23}) + \zeta\xi\,(da_{13} + da_{31}) + \xi\eta\,(da_{21} + da_{12})\,.$$

The quadratic form that figures in the right-hand side of equation (29) is exactly the same as the one that represents the linear dilatation in formula (24). That result can be explained when one refers to the significance of the ratio $\delta = \dfrac{1+\Theta}{1+E}$ that we already occupied ourselves with in no. **8**. The dilation of the thickness of a layer will not, in general, correspond to a linear dilatation under a finite dilatation, because the normal fiber to the initial layer does not correspond to the normal fiber to the deformed layer. On the contrary, the normal fibers will correspond to each other under an infinitely-small deformation, or more precisely, each of them will correspond to a fiber that is infinitely close to the other one. It will then result that the dilatation of the thickness of a layer must then be represented by the same quadratic form as the linear dilatation of the fiber that is normal to that layer.

17. Expressing the second derivatives of the initial coordinates as functions of the coefficients a_{ijk}. – Consider the three equations:

$$- da_{1j} = \frac{\partial x}{\partial x_0} d\left(\frac{\partial x_0}{\partial x_j}\right) + \frac{\partial x}{\partial y_0} d\left(\frac{\partial y_0}{\partial x_j}\right) + \frac{\partial x}{\partial z_0} d\left(\frac{\partial z_0}{\partial x_j}\right),$$

$$- da_{2j} = \frac{\partial y}{\partial x_0} d\left(\frac{\partial x_0}{\partial x_j}\right) + \frac{\partial y}{\partial y_0} d\left(\frac{\partial y_0}{\partial x_j}\right) + \frac{\partial y}{\partial z_0} d\left(\frac{\partial z_0}{\partial x_j}\right),$$

$$- da_{3j} = \frac{\partial z}{\partial x_0} d\left(\frac{\partial x_0}{\partial x_j}\right) + \frac{\partial z}{\partial y_0} d\left(\frac{\partial y_0}{\partial x_j}\right) + \frac{\partial z}{\partial z_0} d\left(\frac{\partial z_0}{\partial x_j}\right).$$

That system can be solved for the three differentials that figure in the right-hand sides and yield the following values for these quantities:

$$- d\left(\frac{\partial x_0}{\partial x_j}\right) = \frac{\partial x_0}{\partial x} da_{1j} + \frac{\partial x_0}{\partial y} da_{2j} + \frac{\partial x_0}{\partial z} da_{3j},$$

$$- d\left(\frac{\partial y_0}{\partial x_j}\right) = \frac{\partial y_0}{\partial x} da_{1j} + \frac{\partial y_0}{\partial y} da_{2j} + \frac{\partial y_0}{\partial z} da_{3j},$$

$$- d\left(\frac{\partial z_0}{\partial x_j}\right) = \frac{\partial z_0}{\partial x} da_{1j} + \frac{\partial z_0}{\partial y} da_{2j} + \frac{\partial z_0}{\partial z} da_{3j}.$$

If one lets u denote any of the initial coordinates x_0, y_0, z_0, when considered as a function of the final coordinates x, y, z, then one will have, as a consequence:

$$\text{(30)} \qquad - d\left(\frac{\partial u}{\partial x_j}\right) = \frac{\partial u}{\partial x} da_{1j} + \frac{\partial u}{\partial y} da_{2j} + \frac{\partial u}{\partial z} da_{3j}.$$

Upon equating this with the coefficients of dx, dy, dz, one will obtain an expression for each of the second derivatives $\frac{\partial^2 u}{\partial x_j\,\partial x_k}$ as a linear and homogeneous function of the coefficients a_{ijk} :

$$\text{(31)} \qquad \begin{cases} -\dfrac{\partial^2 u}{\partial x_j\,\partial x_k} = \dfrac{\partial u}{\partial x} a_{1jk} + \dfrac{\partial u}{\partial y} a_{2jk} + \dfrac{\partial u}{\partial z} a_{3jk} \\ [x_1 = x,\ x_2 = y,\ x_3 = z,\ u = (x_0, y_0, z_0)]. \end{cases}$$

If one knows the numerical values of the first derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ at a point and those of the 18 coefficients a_{ijk} then one can calculate the 18 numerical values of the second derivatives $\frac{\partial^2 u}{\partial x_j\,\partial x_k}$. The coefficients a_{ijk} are thus subject to no restriction insofar as their *numerical values* at a fixed point are concerned. However, the same thing is not true for their derivatives. Indeed, the 54 first derivatives are expressed with the aid of the 30 third derivatives of the functions x_0, y_0, z_0, and consequently, must verify at least 24 condition equations that we shall determine.

18. Partial differential equations that the coefficients a_{ijk} must satisfy. – That question comes down to the search for compatibility conditions for a system of partial differential equations. If one supposes that the 18 functions a_{ijk} are known then the three initial coordinates x_0, y_0, z_0, when considered to be functions of the variables x, y, z, will verify a system of six second-order partial differential equations:

$$\frac{\partial^2 u}{\partial x_j\,\partial x_k}+\frac{\partial u}{\partial x}a_{1jk}+\frac{\partial u}{\partial y}a_{2jk}+\frac{\partial u}{\partial z}a_{3jk}=0 \qquad (j, k = 1, 2, 3). \tag{32}$$

We thus have to express the idea that the system (32) admits three distinct solutions whose functional determinant with respect to the variables x, y, z is non-zero.

The condition equations that relate to the second derivatives:

$$\frac{\partial^2 u}{\partial x_j\,\partial x_k}=\frac{\partial^2 u}{\partial x_k\,\partial x_j}$$

are consequences of the identities:

$$a_{ijk} = a_{ijk}\,.$$

It will then remain for us to simply write that by virtue of the system (32) the third derivatives will satisfy the conditions:

$$\frac{\partial}{\partial x_i}\frac{\partial^2 u}{\partial x_j\,\partial x_k}=\frac{\partial}{\partial x_k}\frac{\partial^2 u}{\partial x_j\,\partial x_i}\,. \tag{33}$$

Write equations (32) in the form:

$$\frac{\partial^2 u}{\partial x_j\,\partial x_k}+\sum_r a_{rjk}\frac{\partial u}{\partial x_r}=0.$$

Differentiate this with respect to x_i and replace the second derivatives in the result with their values as inferred from the same system (32). Finally, permute the indices k and l and write that the condition (32) is verified. One will obtain the equation:

$$\sum_i\left[\frac{\partial a_{ijk}}{\partial x_l}-\frac{\partial a_{ijl}}{\partial x_k}+\sum_r(a_{irk}a_{rjl}-a_{irl}a_{rjk})\right]\frac{\partial u}{\partial x_i}=0.$$

The coefficients of the three partial derivatives $\partial u / \partial x_i$ must be separately zero. Indeed, if that were not true then the functions u would verify the same linear equation that would be homogeneous of order one and have the form:

$$A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}+C\frac{\partial u}{\partial z} = 0.$$

It would thus be impossible to find three solutions whose functional determinant is non-zero.

The compatibility conditions for the system (32) thus finally come down to the form:

$$(34) \qquad \frac{\partial a_{ijk}}{\partial x_l}-\frac{\partial a_{ijl}}{\partial x_k}+a_{i1k}a_{1jl}+a_{i2k}a_{2jl}+a_{i3k}a_{3jl}-a_{i1l}a_{1jk}-a_{i1l}a_{1jk}-a_{i1l}a_{1jk} = 0.$$

From the theory of partial differential equations, one knows, moreover, that equations (34) are sufficient. If they were not verified then the conditions of compatibility that relate to the derivatives of higher order of the known function u would themselves be satisfied by virtue of equations (34) and the ones that result by differentiation.

The number of equations (34) is equal to 27; however, if one takes the three identities:

$$\frac{\partial a_{i12}}{\partial z}-\frac{\partial a_{i12}}{\partial y}+\frac{\partial a_{i23}}{\partial x}-\frac{\partial a_{i21}}{\partial z}+\frac{\partial a_{i31}}{\partial y}-\frac{\partial a_{i32}}{\partial x}=0 \qquad (i = 1, 2, 3)$$

into account then one will recognize that they reduce to twenty-four distinct conditions.

19. Deformations that correspond to a given system of coefficients a_{ijk}. – Suppose that the compatibility conditions are satisfied. In order to solve the system of six equations (32), one can give the values of the function u and its first derivatives at a point arbitrarily. The derivatives of higher order are then determined and expressed as a linear and homogeneous functions of the initial derivatives. The system admits an obvious first solution of $U_0 = 1$. If one knows three other ones U_1, U_2, U_3 whose functional determinant is non-zero then the most general solution will have the form:

$$u = a_0 + a_1U_1 + a_2U_2 + a_3U_3,$$

where a_0, a_1, a_2, a_3 denote arbitrary constants.

The initial coordinates x_0, y_0, z_0 will thus be finally defined as functions of the x, y, z by three expressions of the form:

$$\begin{aligned} x_0 &= a_0 + a_1U_1 + a_2U_2 + a_3U_3,\\ y_0 &= b_0 + b_1U_1 + b_2U_2 + b_3U_3,\\ z_0 &= c_0 + b_1U_1 + c_2U_2 + c_3U_3 .\end{aligned}$$

Since the constants a_i, b_i, c_i are arbitrary, we have this proposition:

If one knows an arbitrary deformation that corresponds to a given system of functions a_{ijk} then one will obtain the most general deformation of the system by combining the deformation considered with an arbitrary homogeneous deformation that is performed on the initial medium.

20. Calculation of the differentials of the coefficients of the linear dilatation. – In order to study the various questions that relate to deformations, it is useful to express the differentials of the coefficients of the dilatation as functions of the differentials da_{ij} . We shall not enter into the details of the calculation, which is extremely simple. One will find that:

$$
(35)\qquad \begin{cases}
-de_{11} = 2(e_{11}da_{11} + e_{12}da_{21} + e_{13}da_{31}),\\
-de_{22} = 2(e_{21}da_{12} + e_{22}da_{22} + e_{23}da_{32}),\\
-de_{33} = 2(e_{31}da_{12} + e_{32}da_{23} + e_{33}da_{33}),\\
-de_{23} = (e_{21}da_{13} + e_{22}da_{23} + e_{23}da_{33}) + (e_{31}da_{12} + e_{32}da_{22} + e_{23}da_{32}),\\
-de_{31} = (e_{31}da_{11} + e_{32}da_{21} + e_{33}da_{31}) + (e_{11}da_{13} + e_{12}da_{23} + e_{13}da_{33}),\\
-de_{12} = (e_{11}da_{12} + e_{12}da_{22} + e_{13}da_{32}) + (e_{21}da_{11} + e_{22}da_{21} + e_{23}da_{31}).
\end{cases}
$$

One finds the following results for the coefficients of the surface dilatation:

$$
(36)\qquad \begin{cases}
dE_{11} = 2[E_{11}da_{11} + E_{12}da_{12} + E_{13}da_{13} - E_{11}(da_{11} + da_{22} + da_{33})],\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,\\
dE_{23} = E_{31}da_{21} + E_{32}da_{22} + E_{33}da_{23}\\
\qquad\quad + E_{21}da_{31} + E_{22}da_{32} + E_{23}da_{33} - 2E_{23}(da_{11} + da_{22} + da_{23}),\\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{cases}
$$

A remarkable property that is common to all of these expressions is that the coefficients of the differential da_{ij} are the same coefficients of the quadratic forms that intervene in the formulas for the dilatation.

21. Inverse problem. Expressing the coefficients a_{ijk} as functions of the coefficients of the dilatation. – We treat the inverse problem only for the linear dilatation. Introduce three auxiliary differentials $d\omega_1$, $d\omega_2$, $d\omega_3$ that are defined by the formulas:

$$
(37)\qquad \begin{cases}
e_{31}da_{12} + e_{32}da_{22} + e_{33}da_{32} - (e_{21}da_{13} + e_{22}da_{23} + e_{23}da_{33}) = d\omega_1,\\
e_{11}da_{13} + e_{12}da_{23} + e_{13}da_{33} - (e_{31}da_{11} + e_{32}da_{21} + e_{33}da_{31}) = d\omega_2,\\
e_{21}da_{11} + e_{22}da_{21} + e_{23}da_{31} - (e_{11}da_{12} + e_{12}da_{22} + e_{13}da_{32}) = d\omega_3,
\end{cases}
$$

With the aid of the systems (35) and (37), we form the combinations of the following form, which contain only three unknowns:

(38)
$$\begin{cases} e_{11}da_{11}+e_{12}da_{21}+e_{13}da_{31} = & -\tfrac{1}{2}de_{11}, \\ e_{21}da_{11}+e_{22}da_{21}+e_{23}da_{31} = & \tfrac{1}{2}d\omega_3-\tfrac{1}{2}de_{12}, \\ e_{31}da_{11}+e_{32}da_{21}+e_{33}da_{31} = & -\tfrac{1}{2}d\omega_2-\tfrac{1}{2}de_{31}. \end{cases}$$

The determinant of equations (38) is equal to $\frac{1}{\Delta^2} = \frac{1}{(1+\Theta)^2}$; the minors are the coefficients E_{ik} of the adjoint form. One will then have:

(39)
$$\begin{cases} -\dfrac{2da_{21}}{(1+\Theta)^2} = E_{21}de_{11}+E_{22}de_{12}+E_{23}de_{13}+E_{23}d\omega_2-E_{22}d\omega_3, \\ -\dfrac{2da_{11}}{(1+\Theta)^2} = E_{11}de_{11}+E_{12}de_{12}+E_{13}de_{13}+E_{13}d\omega_2-E_{12}d\omega_3, \\ -\dfrac{2da_{31}}{(1+\Theta)^2} = E_{31}de_{11}+E_{32}de_{12}+E_{33}de_{13}+E_{33}d\omega_2-E_{32}d\omega_3. \end{cases}$$

The other quantities da_{ij} will obviously be calculated by a similar process. In order to then have expressions for the coefficients a_{ijk}, it will suffice to develop the two sides of each of equations (39) as linear functions of the dx, dy, dz, and to equate the corresponding coefficients of the same differentials. It is then necessary that we must first develop each of the quantities $d\omega_i$. Now, upon referring to equations (35) and (37), one will easily verify the following identities:

(40)
$$\begin{cases} d\omega_1 = \left(\dfrac{\partial e_{21}}{\partial z}-\dfrac{\partial e_{31}}{\partial y}\right)dx+\left(\dfrac{\partial e_{22}}{\partial z}-\dfrac{\partial e_{32}}{\partial y}\right)dy+\left(\dfrac{\partial e_{23}}{\partial z}-\dfrac{\partial e_{33}}{\partial y}\right)dz, \\ d\omega_2 = \left(\dfrac{\partial e_{31}}{\partial x}-\dfrac{\partial e_{11}}{\partial z}\right)dx+\left(\dfrac{\partial e_{32}}{\partial x}-\dfrac{\partial e_{12}}{\partial z}\right)dy+\left(\dfrac{\partial e_{33}}{\partial x}-\dfrac{\partial e_{13}}{\partial z}\right)dz, \\ d\omega_3 = \left(\dfrac{\partial e_{11}}{\partial y}-\dfrac{\partial e_{21}}{\partial x}\right)dx+\left(\dfrac{\partial e_{12}}{\partial y}-\dfrac{\partial e_{22}}{\partial x}\right)dy+\left(\dfrac{\partial e_{13}}{\partial y}-\dfrac{\partial e_{23}}{\partial x}\right)dz. \end{cases}$$

Substituting these values into equations (39) and analogous equations, we finally find that:

(41)
$$-\frac{2a_{ijk}}{(1+\Theta)^2} = E_{i1}\left(\frac{\partial e_{1j}}{\partial x_k}+\frac{\partial e_{1k}}{\partial x_j}-\frac{\partial e_{jk}}{\partial x_1}\right) + E_{i2}\left(\frac{\partial e_{2j}}{\partial x_k}+\frac{\partial e_{2k}}{\partial x_j}-\frac{\partial e_{jk}}{\partial x_2}\right)+E_{i3}\left(\frac{\partial e_{3j}}{\partial x_k}+\frac{\partial e_{3k}}{\partial x_j}-\frac{\partial e_{jk}}{\partial x_3}\right).$$

22. Relationship between the coefficients a_{ijk} and the Christoffel brackets. – The form of the result offers a special interest in the manner by which it is attached to the theory of quadratic forms of differentials ([1]).

Consider an arbitrary quadratic form of the differentials dx_i ($i = 1, 2, \ldots, n$):

$$f = \sum e_{ik}\, dx_i\, dx_k ,$$

where we let E denote the discriminant and let E_{ik} denote the coefficients of the adjoint form.

Upon introducing a notation that is due to Christoffel, we set:

$$\begin{bmatrix} k & l \\ & i \end{bmatrix} = \frac{1}{2}\left(\frac{\partial e_{ik}}{\partial x_l} + \frac{\partial e_{il}}{\partial x_k} - \frac{\partial e_{kl}}{\partial x_i}\right),$$

and in turn:

$$\begin{Bmatrix} j & k \\ & i \end{Bmatrix} = \sum \frac{E_{ir}}{E} \begin{bmatrix} j & k \\ & r \end{bmatrix}.$$

Apply these formulas to the quadratic form (10′) that represents the linear element of the initial medium as a function of the coordinates of the deformed medium:

$$\sum e_{ik}\, dx_i\, dx_k = ds_0^2 ,$$

and recall that the discriminant E of the form is equal to $\dfrac{1}{(1+\Theta)^2}$.

We find immediately that:

$$a_{ijk} = -\begin{Bmatrix} j & k \\ & i \end{Bmatrix}.$$

Our coefficients with three indices thus coincide, up to sign, with the Christoffel brackets that relate to the quadratic form considered. That agreement is particularly interesting, since the starting points are entirely different.

The determination of the deformation when the dilatation is given is a well-known problem. The preceding results give the immediate solution. One first calculates the coefficients a_{ijk} by formulas (41) or with the aid of the Christoffel symbols, and one then forms the linear equations (32); the discussion of no. **19** applies. Nonetheless, the constants $a_0, a_1, \ldots$ are no longer entirely arbitrary, but are subject to the condition that they must give a well-defined value to the linear element. It then results that if one

([1]) CHRISTOFFEL, "Transformation der homogenen Differentialausdrücke zweiten Grades," Journal der Crelle, Bd. 70, pp. 46. – LIPSCHITZ, "Untersuchungen in Betreff der ganzen homogenen Funktionen von n Differentialen," *ibid.*, pp. 71. – DARBOUX, *Leçons sur les systèmes orthogonaux et les coordinées curvilignes,* Livre II, Chap. II.

knows a solution to the problem then all of the other ones will be deduced from it by a Euclidian displacement.

CHAPTER III

FUNDAMENTAL SECOND-DEGREE COVARIANTS

23. Generalities. – 24. Second dilatation. – 25. Definition of torsion. – 26. Analytical expression for the torsion. – 27. Torsion indicatrix. – 28. Asymmetric character of torsion. – 29. Expressing the coefficients of torsion as functions of the dilatation. – 30. Application. – 31. Derived rotation. Rotation of the rotation. – 32. Components of the vector Φ. – 33. Expressing the coefficients a_{ijk} as functions of the covariants.

23. The consideration of the fundamental covariants that we shall occupy ourselves with will permit us to express the eighteen coefficients a_{ijk} as functions of the coefficients of three algebraic forms that each have a geometric and mechanical significance that is independent of the coordinates. The first of them is a ternary cubic form that represents what we call the *second dilatation*; it involves ten components. The second one is a quadratic form that defines the distribution of mechanical torsions; the six coefficients of that form are coupled by one linear relation, which reduces the number of parameters upon which they depend to five. Finally, the third one is a linear form whose consideration can be replaced with that of a vector; it depends upon three independent parameters. The total number of arbitrary quantities that figure in the expression for the three forms considered is thus equal to eighteen, like that of the coefficients a_{ijk} .

The set of these three forms presents an obvious analogy with the geometric quantities that were introduced by Woldemar Voigt in his study of the linear relations between a vector and a tensor.

24. Second dilatation. – If one replaces the quantities da_{ik} in the expression for the logarithmic differential $\frac{de}{1+e}$ that is defined by equation (24) with their values in (21) then one will obtain an expression that is linear and homogeneous with respect to the differentials dx, dy, dz. Now, suppose that the direction of the infinitely-small displacement that is defined by these differentials coincides with the direction α, β, γ. Upon letting ds denote the elementary arc that corresponds to that displacement:

$$\frac{1}{1+e}\frac{de}{ds} = \sum a_{ijk}\,\alpha_i\,\alpha_j\,\alpha_k \qquad (i, j, k = 1, 2, 3), \tag{42}$$

$$(\alpha_1 = \alpha,\ \alpha_2 = \beta,\ \alpha_3 = \gamma).$$

We give the name of *second dilatation* to that logarithmic derivative $\frac{1}{1+e}\frac{de}{ds}$, which is taken as we have already indicated by regarding α_0, β_0, γ_0 as constants – i.e., upon displacing in the deformed medium along the lines that correspond to the lines of the initial medium. We let D_2 (α, β, γ) denote the cubic form of cosines that represents the second dilatation in formula (42), and set:

(43) $$D_2(\alpha, \beta, \gamma) = \sum c_{111}\,\alpha^3 + 3\sum c_{112}\,\alpha^2\beta + 6\,c_{112}\,\alpha\beta\gamma;$$

the coefficients c_{ijk} do not change when one permutes their three indices in an arbitrary manner.

Their values are expressed as functions of the a_{ijk} by the formula:

(44) $$3\,c_{ijk} = a_{ijk} + a_{jkl} + a_{kij}\,.$$

Following a procedure that is currently employed in geometry and mechanics, one can represent the variation of the second dilatation at a point M as a function of the direction α, β, γ by a surface of third order that we call the *indicatrix* of the second dilatations. It suffices to start with the point M (or any other origin) and draw a vector MI in the direction considered whose length is defined by the equality:

$$MI = \frac{1}{\sqrt[3]{D_2(\alpha, \beta, \gamma)}}\,.$$

When the direction varies, the point I will describe the indicatrix. The asymptotic cone to the indicatrix of second dilatations is defined by the tangents to the fibers for which the second dilatation is zero.

That indicatrix is independent of the reference axes with the same names as those of the ellipsoid of dilatations.

The form $D_2(\alpha, \beta, \gamma)$ is a differential covariant of the deformation with respect to the group of Euclidian displacements – i.e., in simpler, but equivalent terms: with respect to the coordinate transformations.

The proper invariants of the form D_2, or the simultaneous invariants of that form and any other invariant form, are thus differential invariants of the deformation with respect to the Euclidian group.

25. Definition of torsion. – In the mechanics of slender bodies, the *torsion* of a rectilinear fiber is the deformation that is produced when one of the extremities of the fiber remain fixed, while the right section to the other extremity is turned through a certain angle around the axis of the fiber.

For a fiber that is directed along the Oz axis, that would be the deformation that is defined by the following equations:

(45) $$\begin{cases} x = x_0 \cos\dfrac{z_0}{a} + y_0 \sin\dfrac{z_0}{a}, \\ y = x_0 \sin\dfrac{z_0}{a} + y_0 \cos\dfrac{z_0}{a}, \\ z = z_0. \end{cases}$$

The right section that is drawn through the origin remains fixed, while the other one turns through an angle that is proportional to the edge z_0 .

The consideration of the differential deformation (*dT*) permits one to easily extend that notation of torsion to the three-dimensional continuous media.

We first remark that in the neighborhood of a point *M* of the medium, the orientation of the fibers or sheets that issue from that point is determined by the homogeneous deformation (*T*) that is tangent to *M*. Having said that, consider an infinitely-small fiber *MM′* that issues from *M*. The deformation (*T*), when extended to all of the medium, will bring the point *M* to its defining position and orient all of the infinitesimal elements that issue from that point. If one then leaves the point *M* fixed, as well as the directions that issue from that point, then one will apply the differential deformation (*dT*) to the elements that issue from *M′*, as the final orientation of these latter elements will be found to be obtained likewise. The mean rotation of the deformation (*dT*) does not, in general, have its axis directed along *MM′*, but one can decompose it into two rotations whose axes are parallel and perpendicular to *MM′*, respectively. It is the former component that produces the torsion of the fiber.

From that, we shall call the projection of the mean rotation of the corresponding infinitesimal deformation (*dT*) onto the direction of the fiber the *total torsion* of the elementary fiber *MM′* and the ratio of the total torsion to the length of the fiber the *mean torsion*.

For the elementary fibers, we will hardly have to consider the mean torsion, which we will then call, more simply, the *torsion of the fiber* ([1]).

26. Analytical expression for the torsion. – The analytical expression for torsion results immediately from these considerations. Preserving the notations of Chapter II, we let *ds* denote the length of the elementary fiber *MM′*, let α, β, γ denote its direction cosines, and let $\tau(\alpha, \beta, \gamma)$ denote the corresponding mean torsion. The total torsion will then be:

$$\tau\, ds = \alpha\, dp_1 + \beta\, dp_2 + \gamma\, dp_3 ,$$

and the mean torsion will be:

$$\tau = \alpha \frac{dp_1}{ds} + \beta \frac{dp_2}{ds} + \gamma \frac{dp_3}{ds} .$$

Now, one has:

$$\frac{dp_i}{ds} = \alpha \frac{\partial p_i}{\partial x} + \beta \frac{\partial p_i}{\partial y} + \gamma \frac{\partial p_i}{\partial z} ,$$

and consequently, one will have:

$$\tau(\alpha, \beta, \gamma) = \alpha^2 \frac{\partial p_1}{\partial x} + \beta^2 \frac{\partial p_2}{\partial y} + \gamma^2 \frac{\partial p_3}{\partial z} \tag{46}$$

([1]) See *Comptes rendus de l'Académie des Sciences*, 30 May 1910 and 10 April 1911.

$$+ \beta\gamma\left(\frac{\partial p_3}{\partial y}+\frac{\partial p_2}{\partial z}\right)+\gamma\alpha\left(\frac{\partial p_1}{\partial z}+\frac{\partial p_3}{\partial x}\right)+\alpha\beta\left(\frac{\partial p_2}{\partial x}+\frac{\partial p_1}{\partial y}\right).$$

The expression for the torsion is then a function of degree two in the direction cosines of the fiber; it is formed from the differential coefficients of the rotations dp_i, and like the linear dilatation for an infinitesimal deformation, from the partial derivatives of the displacements. We set:

(47) $$\tau(\alpha, \beta, \gamma) = \tau_{11}\,\alpha^2 + \tau_{22}\,\beta^2 + \tau_{33}\,\gamma^2 + 2\tau_{22}\,\beta\gamma + 2\tau_{31}\,\gamma\alpha + 2\tau_{12}\,\alpha\beta.$$

The coefficients τ_{ij} are expressed immediately with the aid of either the differential coefficients dp_i / dx_k or with the aid of the coefficients α_{ijk} :

(48) $$\begin{cases} \tau_{11} = \dfrac{\partial p_1}{\partial x} = a_{321} - a_{231}, \\ \tau_{22} = \dfrac{\partial p_2}{\partial y} = a_{132} - a_{312}, \\ \tau_{33} = \dfrac{\partial p_3}{\partial z} = a_{213} - a_{123}, \\ 2\tau_{23} = \dfrac{\partial p_3}{\partial y} + \dfrac{\partial p_2}{\partial z} = a_{212} - a_{122} + a_{133} - a_{313}, \\ 2\tau_{31} = \dfrac{\partial p_1}{\partial z} + \dfrac{\partial p_3}{\partial x} = a_{323} - a_{233} + a_{221} - a_{121}, \\ 2\tau_{12} = \dfrac{\partial p_2}{\partial x} + \dfrac{\partial p_1}{\partial y} = a_{131} - a_{211} + a_{321} - a_{231}. \end{cases}$$

As in the case of infinitesimal deformations, the relation:

(49) $$\tau_{11} + \tau_{22} + \tau_{33} = 0$$

is satisfied identically.

27. Torsion indicatrix. – The variation of the torsion at a point as a function of the direction of the fiber is represented by a quadratic expression whose equation will have the form:

$$\tau(x, y, z) = \pm 1$$

when one takes the point M to be the origin.

The asymptotic cone of the indicatrix is always real and admits an inscribed tri-rectangular trihedron. It is formed from the tangents to the fibers for which the mechanical torsion at M is zero. That is why we have given it the name of the *cone of intorsion.*

The consideration of the torsion indicatrix immediately exhibits the elements that enjoy some important properties relative to the torsion. We call the principal axes and principal planes of the indicatrix the *principal axes* and *principal planes* of torsion, and the torsions of the fibers that are directed along those axes will be the *principal torsions*.

28. Asymmetric character of torsion. – By virtue of the relation (49), the algebraic sum of the principal torsions will be zero.

The sign of the torsion of a fiber depends essentially upon the sense that is chosen to define the positive rotations. Consequently, it will vary with the orientation of the trihedron of the coordinate axes. A symmetry transformation that has the effect of changing the senses of the rotations will also change the signs of the torsions. If we, with Voigt, give the name of *tensors* to the geometric quantities that are defined by the quadratic forms in the cosines then we will see that the torsion is represented by a tensor, but it is an axial tensor.

The asymmetric character of torsion is, moreover, exhibited by formulas (48) when one considers the expressions for coefficients τ_{ij} as functions of the coefficients a_{ijk} .

29. Expression of the coefficients of torsion as functions of those of the dilatation. – By replacing the coefficients a_{ijk} in formulas (48) with their values that are inferred from equations (41), one will obtain the values of the coefficients τ_{ij} as functions of the coefficients of the linear dilatation. According to the notation of Christoffel, we will have:

$$\tau_{11} = \begin{Bmatrix} 3\ 1 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 2\ 1 \\ 3 \end{Bmatrix},$$

$$\tau_{23} = \begin{Bmatrix} 1\ 3 \\ 2 \end{Bmatrix} - \begin{Bmatrix} 3\ 3 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 2\ 2 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 1\ 2 \\ 3 \end{Bmatrix}.$$

These expressions simplify in the case of infinitesimal deformations, and one will have simply:

$$\tau_{11} = \begin{bmatrix} 3\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2\ 1 \\ 3 \end{bmatrix},$$

$$\tau_{23} = \begin{bmatrix} 1\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1\ 2 \\ 3 \end{bmatrix},$$

in which the square brackets have replaced the curly ones.

If one would like to apply these formulas to the coefficients of dilatation that one usually considers in the theory of elasticity then one must observe that all of our calculations have been performed on the quadratic form of the formula (10), which gives the ratio $\frac{1}{(1+e)^2}$. The value that is approached by that ratio for infinitesimal deformations is $1 - 2e$, while the value that is approached by the inverse ratio $(1 + e)^2$ that one considers in the usual calculations is 1 + 2*e*. A change of sign will then result that will have to be taken into account in the calculations.

30. Application to an example. – The application of our theory of torsion to an example will show that our definition is not arbitrary, but that it still corresponds to the ordinary sense of the word *torsion*. First, consider the deformation that is defined by equations (45). The inverse deformation will be given by the equations:

$$\begin{aligned} x_0 &= x\cos\frac{z}{a} + y\sin\frac{z}{a}, \\ y_0 &= -x\sin\frac{z}{a} + y\cos\frac{z}{a}, \\ z_0 &= z. \end{aligned}$$

The calculation of the linear element gives:

$$dx_0^2 + dy_0^2 + dz_0^2 = dx^2 + dy^2 + \frac{x^2+y^2+z^2}{a^2}dz^2 + \frac{2y}{a}dx\,dz - \frac{2x}{a}dy\,dz.$$

One finds the following expressions for the coefficients da_{ij} of the differential deformation (dT):

$$da_{11} = da_{22} = da_{33} = 0,$$

$$da_{12} = -\frac{dz}{a}, \qquad da_{13} = -\frac{dy}{a} + x\frac{dz}{a^2},$$

$$da_{21} = \frac{dz}{a}, \qquad da_{22} = \frac{dx}{a} + y\frac{dz}{a^2},$$

$$da_{31} = da_{32} = 0.$$

Consequently, the components of the corresponding infinitesimal rotation are:

$$\begin{aligned} dp_1 &= -\frac{1}{2}\left(\frac{dx}{a} + y\frac{dz}{a^2}\right), \\ dp_2 &= \frac{1}{2}\left(-\frac{dy}{a} + x\frac{dz}{a^2}\right), \\ dp_3 &= \frac{dz}{a}, \end{aligned}$$

from which, one will deduce the expression for the torsion:

$$\tau(\alpha, \beta, \gamma) = \frac{\gamma^2}{a} - \frac{1}{2}\frac{\alpha^2+\beta^2}{a} + \frac{(\beta x - \alpha y)\gamma}{a^2}. \tag{50}$$

Formula (50) shows that the torsion of the fibers that are parallel to *Oz* is constant and equal to $1 / a$. That quantity indeed represents the ratio that is obtained by dividing the angle of rotation of each right section by the distance from the section considered to the invariable right section $z = 0$. The fibers that are perpendicular to *Oz* also have a constant torsion; it has a sign that is opposite to the former and is equal to $-1 / 2a$.

It will be easy to succeed in applying the general formulas that we have established to the deformation (45) and to verify their exactness in that simple example. I have considered some other examples that relate to the case of infinitesimal deformations in my previous paper.

31. Derived rotation. – We call the rotation about a given direction α, β, γ that has the ratios $\frac{dp_1}{ds}, \frac{dp_2}{ds}, \frac{dp_3}{ds}$ for its components, where *ds* denotes an elementary arc that is carried in the direction considered, the *derived rotation*. Consequently, the components of the derived rotation will have the following values:

$$\frac{dp_1}{ds} = \alpha\frac{\partial p_1}{\partial x} + \beta\frac{\partial p_1}{\partial y} + \gamma\frac{\partial p_1}{\partial z},$$
$$\frac{dp_2}{ds} = \alpha\frac{\partial p_2}{\partial x} + \beta\frac{\partial p_2}{\partial y} + \gamma\frac{\partial p_2}{\partial z},$$
$$\frac{dp_3}{ds} = \alpha\frac{\partial p_3}{\partial x} + \beta\frac{\partial p_3}{\partial y} + \gamma\frac{\partial p_3}{\partial z}.$$

The consideration of torsion permits us to apply the Helmholtz decomposition into symmetric and asymmetric parts to the derived rotation.

Set:

$$\varphi_1 = \frac{1}{2}\left(\frac{\partial p_3}{\partial y} - \frac{\partial p_2}{\partial z}\right),$$
$$\varphi_2 = \frac{1}{2}\left(\frac{\partial p_1}{\partial z} - \frac{\partial p_3}{\partial x}\right),$$
$$\varphi_3 = \frac{1}{2}\left(\frac{\partial p_2}{\partial x} - \frac{\partial p_1}{\partial y}\right).$$

The expressions for the differential ratios dp_i / ds can then be written:

$$\frac{dp_1}{ds} = \frac{1}{2}\frac{\partial \tau}{\partial \alpha} + \varphi_2\,\gamma - \varphi_3\,\beta,$$
$$\frac{dp_2}{ds} = \frac{1}{2}\frac{\partial \tau}{\partial \beta} + \varphi_3\,\alpha - \varphi_1\,\gamma,$$

,

$$\frac{dp_3}{ds} = \frac{1}{2}\frac{\partial \tau}{\partial \gamma} + \varphi_1\,\beta - \varphi_2\,\alpha\,.$$

.

The vector Φ, which has the components φ_1, φ_2, φ_3, presents itself as the rotation of the rotation; it intervenes in the study of flexure, as we shall find. Along with the second dilatation and the torsion, it constitutes the system of our three fundamental second-order covariants.

The coefficients of each of these three covariants change values when one effects a coordinate transformation, but the new coefficients of each transformed form are expressed uniquely with the aid of coefficients of the analogous form that relate to the first system of axes.

From the algebraic viewpoint, the consideration of the vector Φ can obviously be replaced with that of the linear form in the cosines:

$$\varphi_1\,\alpha + \varphi_2\,\beta + \varphi_3\,\gamma.$$

32. Calculation of the components of the vector Φ. – The expressions for the components φ_1, φ_2, φ_3 of the vector Φ are obtained immediately by replacing the differential coefficients of the rotation with their values:

$$(51)\qquad \begin{cases} \varphi_2 = \dfrac{1}{2}\left(\dfrac{\partial p_1}{\partial z} - \dfrac{\partial p_3}{\partial x}\right) = \dfrac{1}{4}(a_{323} - a_{233} - a_{211} + a_{121}), \\ \varphi_1 = \dfrac{1}{2}\left(\dfrac{\partial p_3}{\partial y} - \dfrac{\partial p_2}{\partial z}\right) = \dfrac{1}{4}(a_{212} - a_{122} - a_{132} + a_{313}), \\ \varphi_3 = \dfrac{1}{2}\left(\dfrac{\partial p_2}{\partial x} - \dfrac{\partial p_1}{\partial y}\right) = \dfrac{1}{4}(a_{131} - a_{311} - a_{322} + a_{232}). \end{cases}$$

If one adds and subtracts the same quantity a_{111} in the expression for φ_1 then one will find, upon taking into account the permutability of the last two indices in the coefficients a_{ijk} , that:

$$4\varphi_1 = a_{111} + a_{221} + a_{331} - (a_{111} + a_{122} + a_{133}).$$

Now, one infers from equation (27) that:

$$\frac{\partial \log(1+\Theta)}{\partial u} = a_{111} + a_{221} + a_{331}\,.$$

It remains for us to transform the sum $a_{111} + a_{122} + a_{133}$.

Let $\Delta(u)$ generally denote the second-order Lamé differential parameter that relates to the function u:

$$\Delta(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\,.$$

Upon replacing the coefficients a_{ijk} with their values, we will find that:

$$a_{111} + a_{122} + a_{133} = \frac{\partial x}{\partial x_0}\Delta(x_0) + \frac{\partial x}{\partial y_0}\Delta(y_0) + \frac{\partial x}{\partial z_0}\Delta(z_0),$$

and equations (51) become:

$$(52)\qquad \left\{\begin{aligned} 4\varphi_1 &= \frac{1}{1+\Theta}\frac{\partial\Theta}{\partial x} + \frac{\partial x}{\partial x_0}\Delta(x_0) + \frac{\partial x}{\partial y_0}\Delta(y_0) + \frac{\partial x}{\partial z_0}\Delta(z_0), \\ 4\varphi_2 &= \frac{1}{1+\Theta}\frac{\partial\Theta}{\partial y} + \frac{\partial y}{\partial x_0}\Delta(x_0) + \frac{\partial y}{\partial y_0}\Delta(y_0) + \frac{\partial y}{\partial z_0}\Delta(z_0), \\ 4\varphi_3 &= \frac{1}{1+\Theta}\frac{\partial\Theta}{\partial z} + \frac{\partial z}{\partial x_0}\Delta(x_0) + \frac{\partial z}{\partial y_0}\Delta(y_0) + \frac{\partial z}{\partial z_0}\Delta(z_0). \end{aligned}\right.$$

In the case of infinitesimal deformations, these formulas will reduce to the following form, which I gave in my first paper:

$$4\varphi_1 = \frac{\partial\Theta}{\partial x} - \Delta u,$$
$$4\varphi_2 = \frac{\partial\Theta}{\partial y} - \Delta v,$$
$$4\varphi_3 = \frac{\partial\Theta}{\partial z} - \Delta w.$$

In equations (52), the derivatives $\partial x / \partial x_0$, …, which are taken with respect to the initial variables, can be replaced with their values that are inferred from the inverse system to formulas (4):

$$\frac{\partial x}{\partial x_0} = (1 + \Theta)\frac{d(y_0, z_0)}{d(y, z)}, \ \ldots,$$

in such a manner that the transformed expressions no longer contain derivatives that are taken with respect to the same system of variables x, y, z. One will thus find:

$$4\varphi_1 = \frac{1}{1+\Theta}\frac{\partial\Theta}{\partial x} + (1+\Theta)\begin{vmatrix} \Delta(x_0) & \Delta(y_0) & \Delta(z_0) \\ \dfrac{\partial x_0}{\partial y} & \dfrac{\partial y_0}{\partial y} & \dfrac{z_0}{\partial y} \\ \dfrac{\partial x_0}{\partial z} & \dfrac{\partial y_0}{\partial z} & \dfrac{\partial z_0}{\partial z} \end{vmatrix}.$$

33. Expressing the coefficients a_{ijk} as functions of the coefficients of the covariants. – Equations (44), (48), and (51) express the components of the three covariants as functions of the coefficients a_{ijk} ; conversely, one can express these coefficients with the aid of the covariants. The calculation is quite simple and gives the following result:

$$(53)\quad \left\{\begin{aligned}
&a_{111} = c_{111},\\
&a_{122} = c_{122} - \frac{4}{3}\tau_{23} - \frac{4}{3}\varphi_1,\\
&a_{133} = c_{133} + \frac{4}{3}\tau_{23} - \frac{4}{3}\varphi_1,\\
&a_{123} = a_{132} = c_{123} - \frac{2}{3}\tau_{33} + \frac{2}{3}\tau_{22},\\
&a_{113} = a_{131} = c_{113} + \frac{2}{3}\tau_{12} + \frac{2}{3}\varphi_3,\\
&a_{112} = a_{121} = c_{112} + 2\tau_{31} + \frac{2}{3}\varphi_2,\\
&a_{211} = c_{112} + \frac{4}{3}\tau_{31} - \frac{4}{3}\varphi_2,\\
&a_{222} = c_{222},\\
&a_{223} = a_{232} = c_{233} - \frac{2}{3}\tau_{12} + \frac{2}{3}\varphi_3,\\
&a_{231} = a_{213} = c_{123} - \frac{2}{3}\tau_{11} + \frac{2}{3}\tau_{33},\\
&a_{212} = a_{321} = c_{122} + \frac{2}{3}\tau_{23} + \frac{2}{3}\varphi_1,\\
&a_{311} = c_{113} - \frac{4}{3}\tau_{12} - \frac{4}{3}\varphi_3,\\
&a_{322} = c_{223} + \frac{4}{3}\tau_{12} - \frac{4}{3}\varphi_3,\\
&a_{333} = c_{333},\\
&a_{323} = a_{332} = c_{233} + \frac{2}{3}\tau_{31} + \frac{2}{3}\varphi_2,\\
&a_{331} = a_{313} = c_{331} + \frac{2}{3}\tau_{23} + \frac{2}{3}\varphi_1,\\
&a_{312} = a_{321} = c_{123} - \frac{2}{3}\tau_{22} + \frac{2}{3}\tau_{11}.
\end{aligned}\right.$$

This set of eighteen formulas is summarized in three identities, and we write just the first of them:

$$(54)\qquad a_{111}\,\alpha^2 + a_{122}\,\beta^2 + a_{133}\,\gamma^2 + 2\,a_{123}\,\beta\gamma + 2\,a_{121}\,\gamma\alpha + 2\,a_{112}\,\alpha\beta$$

$$= \frac{1}{3}\frac{\partial D_2}{\partial \alpha} + \frac{2}{3}(\gamma\tau'_\beta - \beta\tau'_\gamma) - \frac{4}{3}\varphi_1 + \frac{4}{3}\alpha(\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3)\,.$$

The other two identities are deduced from this by permuting the indices and the cosines α, β, γ.

CHAPTER IV

FLEXURE OF FIBERS AND SHEETS

34. Compositions of the incurvations. – 35. Another representation of the curvature. – 36. Curvature of a deformed fiber. – 37. Calculation of the logarithmic differential $\frac{de}{1+e}$. – 38. Decomposition of the curvature. Definition of the flexure. – 39. Decomposition of the total flexure into its three components. – 40. Another form of the formulas. – 41. Geometric elements of flexure. – 42. Incurvation and flexure of sheets. – 43. Remark on the transform of the initial curvature. - 44. Geodesic flexure.

34. Composition of incurvations. – One knows that the study of the motion of Serret trihedra that are coupled to a skew curve leads one to represent curvature by a rotation ([1]) whose axis is perpendicular to the osculating plane and whose angular velocity is measured by the inverse of the radius of curvature. That mode of representation lends itself to the composition by geometric addition.

Let R be the figurative rotation of the curvature ω of an infinitely-small arc ds. If R is the resultant of the other two rotations R', R'' that have their axes in the normal plane to the element ds then the corresponding curvature ω can itself be considered to be the resultant of the curvature ω' and ω'', which figure in the rotations R' and R'', respectively.

Let x, y, z be the coordinates of a point M of the curve; let:

$$\begin{matrix} \alpha, & \beta, & \gamma, \\ \alpha', & \beta', & \gamma', \\ \alpha'', & \beta'', & \gamma'' \end{matrix}$$

be the system of direction cosines of the tangent, the principal normal, and the binormal, and let ρ be the radius of curvature. The components of the figurative rotation of the curvature around the coordinate axes are:

$$R_1 = \frac{\alpha''}{\rho}, \qquad R_2 = \frac{\beta''}{\rho}, \qquad R_3 = \frac{\gamma''}{\rho}.$$

One can represent all of the elements that relate to curvature with the aid of these quantities.

The axis of curvature is the locus of points that remain immobile under the resultant motion of the rotation R and a translation whose velocity, which is equal to unity, is directed along the tangent. If one calls the current coordinates X, Y, Z then the points of the axis will consequently verify the following relations:

$$\begin{aligned} \alpha + R_2 (Z - z) - R_3 (Y - y) &= 0, \\ \beta + R_3 (X - x) - R_1 (Z - z) &= 0, \\ \gamma + R_1 (Y - y) - R_2 (X - x) &= 0, \end{aligned}$$

([1]) DARBOUX, *Leçons sur la Théorie générale des surfaces*, Livre I, Chap. I.

which are compatible and reduce to two distinct conditions by virtue of the equality:

$$R_1\,\alpha + R_2\,\beta + R_3\,\gamma = 1.$$

These equations can be replaced with the system:

(55) $$\alpha\,(X - x) + \beta\,(Y - y) + \gamma(Z - z) = 0,$$

(56) $$\begin{vmatrix} X - x & Y - y & Z - z \\ \alpha & \beta & \gamma \\ R_1 & R_2 & R_3 \end{vmatrix} + 1 = 0,$$

whose geometric significance is obvious.

One can associate each curvature component R', R'', … with an element that is analogous to the resultant curvature, namely, a curvature plane, which is perpendicular to the axis of figurative rotation, a center, a radius, a circle, a curvature axis, and even a principal normal of curvature.

In my previous paper on infinitesimal deformations, I indicated a very simple construction for the axis of curvature that is the resultant of two given curvatures when one knows the axes of the component curvature, and I showed that this construction is the transform of the geometric addition of values by polar reciprocals.

35. Another representation of the curvature. – From the viewpoint of the composition of curvatures, there exists a second representation that offers the same advantages as the figurative rotation. It consists of endowing the principal normal of curvature with a length that measures the value of the curvature. That length can, moreover, be taken to have either the same or the opposite sense as the radius, provided that one adopts the same convention for all of the components curvatures. Upon denoting the components of the vector thus-obtained by H_1, H_2, H_3, one will then have:

$$H_1 = -\frac{\alpha'}{\rho}, \qquad H_2 = -\frac{\beta'}{\rho}, \qquad H_3 = -\frac{\gamma'}{\rho}.$$

The relations between the new figurative vector H (H_1, H_2, H_3) and the rotation R (R_1, R_2, R_3) are given by the formulas:

$$\begin{aligned} R_1 &= H_2\,\gamma - H_3\,\beta, & H_1 &= \beta R_3 - \gamma R_2\,, \\ R_2 &= H_3\alpha - H_1\,\gamma, & H_2 &= \gamma R_1 - \alpha R_3\,, \\ R_3 &= H_1\,\beta - H_2\,\alpha, & H_3 &= \alpha R_2 - \beta R_1\,. \end{aligned}$$

With the use of the new notations, equation (56) becomes:

(56′) $$H_1\,(X - x) + H_2\,(Y - y) + H_3\,(Z - z) + 1 = 0.$$

It represents the polar plane to the extremity of the vector H with respect to the imaginary sphere:

$$(X - x)^2 + (Y - y)^2 + (Z - z)^2 + 1 = 0.$$

It results from this that the curvature axis is the polar to the extremity of the vector H with respect to the imaginary circle that is obtained by cutting the preceding sphere with the plane normal to the curve. If one represents the composition of the curvature, on the one hand, by the geometric addition of the corresponding vectors H', H'', …, and on the other hand, by the construction of the axes that were pointed out in my previous paper ([1]), then the correspondence by polar reciprocals between the two figures will become obvious.

36. Curvature of a deformed fiber. – The formulas that relate to the transformation of the curvature of the fibers are obtained easily by making use of the Frenet formulas. We preserve the notations that were indicated in no. **35** for the curvature and the direction cosines of the tangent, the principal normal, and the binormal in the deformed medium. When the same letters are affected with the index zero, they will denote the analogous quantities for the initial medium. In no. **5**, we established equations of the form:

$$(1 + e)\,\alpha = \frac{\partial x}{\partial x_0}\alpha_0 + \frac{\partial x}{\partial y_0}\beta_0 + \frac{\partial x}{\partial z_0}\gamma_0 .$$

Differentiate this, while regarding α_0, β_0, γ_0 as variables and taking the Frenet formulas into account; we find:

$$(57)\qquad (1 + e)\frac{\alpha' ds}{\rho} + \alpha\, de = \left(\frac{\partial x}{\partial x_0}\alpha_0' + \frac{\partial x}{\partial y_0}\beta_0' + \frac{\partial x}{\partial z_0}\gamma_0'\right)\frac{ds_0}{\rho_0}$$
$$+ \alpha_0\, d\left(\frac{\partial x}{\partial x_0}\right) + \beta_0\, d\left(\frac{\partial x}{\partial y_0}\right) + \gamma_0\, d\left(\frac{\partial x}{\partial z_0}\right).$$

If one replaces α_0, β_0, γ_0 with their values that one infers from formulas (8) then one will get:

$$\alpha_0\, d\left(\frac{\partial x}{\partial x_0}\right) + \beta_0\, d\left(\frac{\partial x}{\partial y_0}\right) + \gamma_0\, d\left(\frac{\partial x}{\partial z_0}\right) = (1 + e)\,(\alpha\, da_{11} + \beta\, da_{12} + \gamma da_{13}).$$

It remains for us to transform the parentheses:

$$\frac{\partial x}{\partial x_0}\alpha_0' + \frac{\partial x}{\partial y_0}\beta_0' + \frac{\partial x}{\partial z_0}\gamma_0' .$$

([1]) Annales de l'École Normale, 1911, pp. 541.

The homogeneous deformation (T) that is tangent to M makes a direction MN correspond to the initial principal normal that generally differs from the principal normal of the deformed curvature. We call the direction cosines of that transform α_1', β_1', γ_1' and the linear dilatation of the corresponding elementary fiber e_1; by virtue of equations (7) in no. **5**, we will then have:

$$(1 + e_1)\,\alpha_1' = \frac{\partial x}{\partial x_0}\alpha_0' + \frac{\partial x}{\partial y_0}\beta_0' + \frac{\partial x}{\partial z_0}\gamma_0'.$$

When one divides both sides of equation (57) and the other two analogous equations by $(1 + e)\,ds = (1 + e)^2\,ds_0$, they will become:

$$(58)\qquad \left\{\begin{aligned} \frac{\alpha'}{\rho} + \frac{\alpha}{1+e}\frac{de}{ds} &= \frac{1+e_1}{(1+e)^2}\frac{\alpha_1'}{\rho_0} + \alpha\frac{da_{11}}{ds} + \beta\frac{da_{12}}{ds} + \gamma\frac{da_{13}}{ds}, \\ \frac{\beta'}{\rho} + \frac{\beta}{1+e}\frac{de}{ds} &= \frac{1+e_1}{(1+e)^2}\frac{\beta_1'}{\rho_0} + \alpha\frac{da_{21}}{ds} + \beta\frac{da_{22}}{ds} + \gamma\frac{da_{23}}{ds}, \\ \frac{\gamma'}{\rho} + \frac{\gamma}{1+e}\frac{de}{ds} &= \frac{1+e_1}{(1+e)^2}\frac{\gamma_1'}{\rho_0} + \alpha\frac{da_{31}}{ds} + \beta\frac{da_{32}}{ds} + \gamma\frac{da_{33}}{ds}. \end{aligned}\right.$$

37. Calculation of the logarithmic differential $\frac{de}{1+e}$. – The logarithmic derivative $\frac{1}{1+e}\frac{de}{ds}$, which figures in the left-hand side of equations (58), is taken by considering α_0, β_0, γ_0 as variables. Consequently, it differs from the second dilatation, which is a derivative of the same quantity, but it is taken by regarding α_0, β_0, γ_0 as constants. The difference between these two quantities is calculated easily with the aid of equations (58), when one adds them together after having multiplied them by α, β, γ, respectively.

One finds:

$$\frac{1}{1+e}\frac{de}{ds} = \frac{(1+e_1)(\alpha\alpha_1' + \beta\beta_1' + \gamma\gamma_1')}{(1+e)^2}\frac{1}{\rho_0} + D_2(\alpha, \beta, \gamma),$$

or rather:

$$(59)\qquad \frac{1}{1+e}\frac{de}{ds} - D_2(\alpha, \beta, \gamma) = \frac{(1+e_1)}{(1+e)^2}\frac{\cos\theta_1}{\rho_0},$$

in which θ_1 denotes the angle that is formed in the deformed medium by the direction of the tangent MT and the direction MN, which is the transform of the initial principal normal.

38. Decomposition of the curvature. Definition of the flexure. – We perform a first transformation in the formulas that are obtained for the curvature by eliminating the logarithmic derivative $\frac{de}{(1+e)\,ds}$ from equations (58). In order to write the result of the elimination in a simple form, we first remark that one has:

$$\beta\gamma' - \gamma\beta' = \alpha'', \ldots$$

The expressions $\beta\gamma_1' - \gamma\beta_1'$, … transform in the same manner by introducing the direction cosines of the common perpendicular to the tangent MT and to the direction MN_1 that is the transform of the initial principal normal. The plane (P) that is determined by these two directions is the transform of the initial osculating plane by the homogeneous deformation (T) that is tangent to M. We let α_1'', β_1'', γ_1'' denote the direction cosines of the normal to the plane (P), and we will have, in turn:

$$\begin{aligned}\beta\gamma_1' - \gamma\beta_1' &= \alpha_1'' \sin\theta_1,\\ \gamma\alpha_1' - \alpha\gamma_1' &= \beta_1'' \sin\theta_1,\\ \alpha\beta_1' - \beta\alpha_1' &= \gamma_1'' \sin\theta_1,\end{aligned}$$

where θ_1 denotes the angle that was already defined above. Upon introducing these notations, the elimination of the derivative $\frac{de}{(1+e)\,ds}$ between the last two equations (58) will give us:

$$(60)\qquad \begin{aligned}\frac{\alpha''}{\rho} = \frac{1+e_1}{(1+e)^2}\sin\theta_1\frac{\alpha_1''}{\rho_0} &+ \beta\left(\alpha\frac{da_{31}}{ds} + \beta\frac{da_{32}}{ds} + \gamma\frac{da_{33}}{ds}\right)\\ &- \gamma\left(\alpha\frac{da_{21}}{ds} + \beta\frac{da_{22}}{ds} + \gamma\frac{da_{23}}{ds}\right).\end{aligned}$$

We now remark that the two directions MT and MN are the transforms of the two rectangular directions in the initial medium. Consequently, if we let E denote the surface dilatation of the elementary sheet that is applied to M_0 on the initial osculating plane then we will have:

$$(1 + e)\,(1 + e_1)\sin\theta_1 = (1 + E).$$

Equation (60) then becomes:

$$(61)\qquad \begin{aligned}\frac{\alpha''}{\rho} = \frac{1+E}{(1+e)^3}\frac{\alpha_1''}{\rho_0} &+ \beta\left(\alpha\frac{da_{31}}{ds} + \beta\frac{da_{32}}{ds} + \gamma\frac{da_{33}}{ds}\right)\\ &- \gamma\left(\alpha\frac{da_{21}}{ds} + \beta\frac{da_{22}}{ds} + \gamma\frac{da_{23}}{ds}\right).\end{aligned}$$

One will find, in the same way, that:

$$
(61')\quad \begin{cases} \dfrac{\beta''}{\rho} = \dfrac{1+E}{(1+e)^3}\dfrac{\beta_1''}{\rho_0} + \gamma\left(\alpha\dfrac{da_{11}}{ds} + \beta\dfrac{da_{12}}{ds} + \gamma\dfrac{da_{13}}{ds}\right) - \alpha\left(\alpha\dfrac{da_{31}}{ds} + \beta\dfrac{da_{32}}{ds} + \gamma\dfrac{da_{33}}{ds}\right), \\[2ex] \dfrac{\gamma''}{\rho} = \dfrac{1+E}{(1+e)^3}\dfrac{\gamma_1''}{\rho_0} + \alpha\left(\alpha\dfrac{da_{21}}{ds} + \beta\dfrac{da_{22}}{ds} + \gamma\dfrac{da_{23}}{ds}\right) - \beta\left(\alpha\dfrac{da_{11}}{ds} + \beta\dfrac{da_{12}}{ds} + \gamma\dfrac{da_{13}}{ds}\right). \end{cases}
$$

Formulas (61) and (61′) represent the projections of the figurative rotation of the curvature of the fiber in the deformed medium. An examination of their right-hand sides will show that this rotation decomposes into two other ones. The one that equals $\dfrac{1+E}{(1+e)^3}\dfrac{1}{\rho_0}$ represents the transform of the curvature of the initial fiber by the homogeneous deformation (T) that is tangent to M; it is zero for originally rectilinear fibers. The second rotation is independent of the initial curvature; it is the same for all straight or curved fibers that admit the same tangent. It is to that second component of the curvature that we given the name of *flexure of the fiber*.

One must remark the analogy that our decomposition of curvature presents with Meusnier's theorem. Here, the flexure plays a role that is comparable to the one that is played by the curvature of normal sections in the theory of surfaces.

39. Decomposition of the total flexure. – By making use of the identity (54) and two other analogous identities, one can transform the expression for the components of the figurative rotation of the flexure. We denote these components by F_1, F_2, F_3 and we will have:

$$
(62)\quad \begin{cases} F_1 = \beta\sum\alpha\, da_{31} - \gamma\sum\alpha\, da_{21} \\[1ex] \quad = \dfrac{1}{3}\left(\beta\dfrac{\partial D_2}{\partial\gamma} - \gamma\dfrac{\partial D_2}{\partial\beta}\right) + \dfrac{4}{3}\left(\dfrac{1}{3}\tau'_\alpha - \alpha\tau\right) + \dfrac{4}{3}(\varphi_2\gamma - \varphi_3\beta), \\[2ex] F_2 = \dfrac{1}{3}\left(\gamma\dfrac{\partial D_2}{\partial\alpha} - \alpha\dfrac{\partial D_2}{\partial\gamma}\right) + \dfrac{4}{3}\left(\dfrac{1}{3}\tau'_\beta - \beta\tau\right) + \dfrac{4}{3}(\varphi_3\alpha - \varphi_1\gamma), \\[2ex] F_3 = \dfrac{1}{3}\left(\alpha\dfrac{\partial D_2}{\partial\beta} - \beta\dfrac{\partial D_2}{\partial\alpha}\right) + \dfrac{4}{3}\left(\dfrac{1}{3}\tau'_\gamma - \gamma\tau\right) + \dfrac{4}{3}(\varphi_1\beta - \varphi_\alpha\alpha). \end{cases}
$$

Equations (62) show that the flexure can, in turn, be decomposed into three partial flexures that are attached to the three fundamental covariants, and which we distinguish by the names of *flexure of the second dilatation*, *flexure of torsion*, and the *cyclic* or polar *flexure*, respectively. The projection of the corresponding figurative rotations onto Ox are:

For the flexure of the second dilatation:

$$
F_1' = \frac{1}{3}\left(\beta\frac{\partial D_2}{\partial\gamma} - \gamma\frac{\partial D_2}{\partial\beta}\right),
$$

for the flexure of torsion:

$$F_1'' = \frac{4}{3}\left(\frac{1}{2}\frac{\partial \tau}{\partial \alpha} - \alpha\tau\right),$$

and for the cyclic flexure:

$$F_1''' = \frac{4}{3}(\varphi_2\,\gamma - \varphi_3\,\beta).$$

40. Another form of the formulas for incurvation. – Let α_2', β_2', γ_2' denote the direction cosines of the principal normal relative to the first component of curvature – i.e., the curvature of the transform of the initial fiber by the homogeneous deformation that is tangent to *M*. One will have:

$$\alpha_2' = \beta_1''\gamma - \gamma_1''\beta, \qquad \beta_2' = \gamma_1''\alpha - \alpha_1''\gamma, \qquad \gamma_2' = \alpha_1''\beta - \beta_1''\alpha,$$

and similarly, upon considering the principal normal, properly speaking, of the deformed fiber:

$$\alpha' = \beta''\gamma - \gamma''\beta, \qquad \beta' = \gamma''\alpha - \alpha''\gamma, \qquad \gamma' = \alpha''\beta - \beta''\alpha.$$

If one takes these relations into account, along with the identity:

$$D_2(\alpha, \beta, \gamma) = \sum \alpha\left(\alpha\frac{da_{11}}{ds} + \beta\frac{da_{12}}{ds} + \gamma\frac{da_{13}}{ds}\right),$$

then one will deduce the new system from equations (61) and (61′):

$$(63)\qquad \left\{\begin{aligned}
\frac{\alpha'}{\rho} &= \frac{1+E}{(1+e)^3}\frac{\alpha_2'}{\rho_0} - \alpha D_2(\alpha,\beta,\gamma) + \alpha\frac{da_{11}}{ds} + \beta\frac{da_{12}}{ds} + \gamma\frac{da_{13}}{ds},\\
\frac{\beta'}{\rho} &= \frac{1+E}{(1+e)^3}\frac{\beta_2'}{\rho_0} - \beta D_2(\alpha,\beta,\gamma) + \alpha\frac{da_{21}}{ds} + \beta\frac{da_{22}}{ds} + \gamma\frac{da_{23}}{ds},\\
\frac{\gamma'}{\rho} &= \frac{1+E}{(1+e)^3}\frac{\gamma_2'}{\rho_0} - \gamma D_2(\alpha,\beta,\gamma) + \alpha\frac{da_{31}}{ds} + \beta\frac{da_{32}}{ds} + \gamma\frac{da_{33}}{ds}.
\end{aligned}\right.$$

The introduction of fundamental covariants will give the following equivalent form to the right-hand sides of equations (63):

$$(63')\qquad \left\{\begin{aligned}
\frac{\alpha'}{\rho} &= \frac{1+E}{(1+e)^3}\frac{\alpha_2'}{\rho_0} + \frac{1}{3}\frac{\partial D_2}{\partial \alpha} - \alpha D_2 + \frac{2}{3}\left(\gamma\frac{\partial \tau}{\partial \beta} - \beta\frac{\partial \tau}{\partial \gamma}\right)\\
&\quad - \frac{4}{3}\varphi_1 + \frac{4}{3}\alpha(\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3),\\
&\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right.$$

Systems (63) or (63′) can also be deduced easily by a direct calculation from formulas (58) by taking equation (59) into account and the obvious relations:

$$\alpha_1' = \alpha \cos \theta_1 + \alpha_2 \sin \theta_1 , \qquad \dots$$

From the viewpoint of the decomposition of curvature and flexure, it is obvious that equations (63′) have exactly the same significance and scope as equations (61), (61′), and (62).

41. Geometric elements that relate to flexure. – Since flexure is one of the components of the curvature, there is reason to make it correspond to the same geometric elements that the curvature corresponds to. For each direction of the fibers at a given point, we will then have an axis of flexure, a plane of flexure that is analogous to the osculating plane, a principal normal of flexure, a radius, a circle, and a center of flexure.

If one represents the flexure according to the method that was indicated in no. 35 as a vector H that is carried by the principal normal of flexure in the opposite direction to the radius of flexure then one will have the following expressions for the components of the that vector:

$$(64)\qquad \begin{cases} H_1 = \alpha D_2 - \dfrac{1}{3}\dfrac{\partial D_2}{\partial \alpha} + \dfrac{2}{3}\left(\beta \dfrac{\partial \tau}{\partial \gamma} - \gamma \dfrac{\partial \tau}{\partial \beta}\right) + \dfrac{4}{3}[\varphi_1 - \alpha(\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3)], \\ H_2 = \beta D_2 - \dfrac{1}{3}\dfrac{\partial D_2}{\partial \beta} + \dfrac{2}{3}\left(\gamma \dfrac{\partial \tau}{\partial \alpha} - \alpha \dfrac{\partial \tau}{\partial \gamma}\right) + \dfrac{4}{3}[\varphi_2 - \beta(\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3)], \\ H_3 = \gamma D_2 - \dfrac{1}{3}\dfrac{\partial D_2}{\partial \gamma} + \dfrac{2}{3}\left(\alpha \dfrac{\partial \tau}{\partial \beta} - \beta \dfrac{\partial \tau}{\partial \alpha}\right) + \dfrac{4}{3}[\varphi_3 - \gamma(\alpha\varphi_1 + \beta\varphi_2 + \gamma\varphi_3)]. \end{cases}$$

The axis of flexure will be defined by the two equations:

$$(65)\qquad \begin{cases} \alpha(X-x) + \beta(Y-y) + \gamma(Z-z) = 0; \\ -\dfrac{1}{3}\left[(X-x)\dfrac{\partial D_2}{\partial \alpha} + (Y-y)\dfrac{\partial D_2}{\partial \beta} + (Z-z)\dfrac{\partial D_2}{\partial \gamma}\right] \\ \qquad + \dfrac{2}{3}\begin{vmatrix} X-x & Y-y & Z-z \\ \alpha & \beta & \gamma \\ \dfrac{\partial \tau}{\partial \alpha} & \dfrac{\partial \tau}{\partial \beta} & \dfrac{\partial \tau}{\partial \gamma} \end{vmatrix} \\ + \dfrac{4}{3}[\varphi_1(X-x) + \varphi_2(Y-y) + \varphi_3(Z-z)] + 1 = 0. \end{cases}$$

The first one represents the normal plane; in the second one, we have separated the terms that provide the different components of flexure.

The plane of flexure, which is perpendicular to the axis of the figurative rotation, is represented by the equation:

$$F_1\,(X - x) + F_2\,(Y - y) + F_3\,(Z - z) = 0.$$

It is pointless to insist upon the calculation of the other elements, which are deduced easily from our formulas.

42. Incurvation and flexure of the sheets. Normal flexure. – When a fiber belongs to a given sheet S, the curvature of the fiber can be decomposed, conforming to the theory of surfaces, into a normal curvature and a tangential or geodesic curvature. The principal normal of the first one is normal to the sheet, while that of the second one is tangent. We shall first occupy ourselves with the normal curvature.

Let a, b, c be the direction cosines of the normal to the sheet S in the deformed medium, let E be the surface dilatation of the sheet of the point considered M, let ω be the angle that the osculating plane of the fiber forms with the normal to the sheet, let ω_1 be the angle of the plane that is the transform of the initial osculating plane with that same normal, and let ω_0 be the angle between the initial osculating plane and the normal to the initial sheet.

The angles ω_0 and ω_1 are the complements of the angles that are formed in the two media between the elementary sheet considered and the elementary sheet that is situated in the original osculating plane, respectively. Consequently, if we let E' denote the surface dilatation of the original osculating plane at M then, by virtue of equation (19′) of no. **9**, we will have:

(66)
$$\frac{\cos\omega_1}{\cos\omega_0} = \frac{(1+\Theta)(1+e)}{(1+E)(1+E')}.$$

Having said that, start with equations (63), which we add together, after multiplying them by a, b, c, respectively, and replacing E with E' above. We will find:

$$\frac{\cos\omega}{\rho} = \frac{(1+E')}{(1+e)^3}\frac{\cos\omega_1}{\rho_0} + \sum a\left(\alpha\frac{da_{11}}{ds} + \beta\frac{da_{12}}{ds} + \gamma\frac{da_{13}}{ds}\right),$$

or rather, by virtue of equation (66):

(67)
$$\frac{\cos\omega}{\rho} = \frac{1+\Theta}{(1+E)(1+e)^2}\frac{\cos\omega_0}{\rho_0} + \sum a\left(\alpha\frac{da_{11}}{ds} + \beta\frac{da_{12}}{ds} + \gamma\frac{da_{13}}{ds}\right).$$

If we let R denote the radius of the normal curvature of the sheet along the fiber considered and let R_0 denote the analogous radius for the initial medium then we will have:

(68)
$$\frac{1}{R} = \frac{1+\Theta}{(1+E)(1+e)^2}\frac{1}{R_0} + \sum a\left(\alpha\frac{da_{11}}{ds} + \beta\frac{da_{12}}{ds} + \gamma\frac{da_{13}}{ds}\right).$$

Finally, by introducing the fundamental covariants, we infer that:

(69)
$$\frac{1}{R} = \frac{1+\Theta}{(1+E)(1+e)^2}\frac{1}{R_0} + \frac{1}{3}\left(\alpha\frac{\partial D_2}{\partial \alpha} + \beta\frac{\partial D_2}{\partial \beta} + \gamma\frac{\partial D_2}{\partial \gamma}\right) + \frac{2}{3}\begin{vmatrix} a & b & c \\ \alpha & \beta & \gamma \\ \frac{\partial \tau}{\partial \alpha} & \frac{\partial \tau}{\partial \beta} & \frac{\partial \tau}{\partial \gamma} \end{vmatrix} - \frac{4}{3}(a\,\varphi_1 + b\,\varphi_2 + c\,\varphi_3).$$

Equations (68) and (69) further exhibit the decomposition of the normal curvature into a sum of two curvatures: The first one is the transform of the initial normal curvature by the homogeneous deformation that is tangent to M. The second one, which is independent of the initial curvature, is the normal flexure to the sheet along the fiber considered. Equation (69) also gives the decomposition of the normal flexure into its three components relative to the fundamental covariants.

43. Remark on the transform of the initial curvature. – The form of the first component of the curvature is remarkable in its simplicity. Set:

$$\frac{1}{R_1} = \frac{1+\Theta}{(1+E)(1+e)^2}\frac{1}{R_0}.$$

If we let ε denote the dilatation of the thickness of a layer that is applied to the surface considered at M then we will have:

$$\frac{1+\Theta}{1+E} = 1 + \varepsilon,$$

and the preceding equation will become:

$$\frac{1}{R_1} = \frac{1+\varepsilon}{(1+e)^2}\frac{1}{R_0}.$$

If one varies the direction of the fiber on the sheet around the point M_0 in the initial medium then the curvature $1 / R_0$ will be expressed by a homogeneous function of degree two in the direction cosines $\alpha_0, \beta_0, \gamma_0$. Upon replacing these cosines with their values as functions of α, β, γ, one will obtain an expression for the ratio $\frac{1}{(1+e)^2}\frac{1}{R_0}$ that is also homogeneous and of degree two in α, β, γ, and whose consideration will yield the Dupin indicatrix that relates to the transformed curvature $1 / R_1$. The axes of that new indicatrix will be the transformed directions of the conjugate diameters that are common to the initial indicatrix and to the ellipse along which the tangent plane to the sheet will cut the ellipsoid of linear dilatations at the point M_0 in the initial medium.

44. Geodesic flexure. – For the calculation of the geodesic curvature, it is simpler to start with the figurative rotation whose components are given by equations (61) and (61′). The axis of the figurative rotation of the geodesic curvature is normal to the surface. One will then obtain the expression for that rotation by adding equations (61) and (61′) together, after having multiplied them by a, b, c, respectively. The result further decomposes into two parts, one of which depends upon the initial curvature of the fiber and the position of the osculating plane, while the other one depends uniquely on the flexure, and for that reason, we shall call it the *geodesic flexure*. From the viewpoint of the first component, there is, nevertheless, a difference in the result obtained for the normal curvature, in the sense that the geodesic curvature of the transformed fiber under the homogeneous deformation (T) will no longer be expressed uniquely with the aid of the initial geodesic curvature and the linear dilatation of the fiber.

The figurative rotation of the geodesic flexure has the expression:

$$F_n = a\,F_1 + b\,F_2 + c\,F_3\,,$$

where F_1, F_2, F_3 have the values that were defined by formulas (62).

One sees that by the use of fundamental covariants the formulas that relate to the flexure of fibers and sheets take on a form that is exactly similar to the one that we obtained previously for the infinitesimal deformations. The consequences that we have deduced from the viewpoint of geometric properties thus persist entirely without the slightest modification, and it is pointless to reproduce them here.

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