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Special relativity and the geometry of wave systems

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INTRODUCTION

The present article is a contribution to the critical examination of the theory of relativity. Following Brillouin ([1]), Lecornu ([2]), Painlevé ([3]), and many others, I will examine some of the grave objections that have been raised against Einstein's construction ([4]).

As a result of the negative result of the celebrated Michelson-Morley experiment, it seems legitimate to assume temporarily, by way of *postulate*, the impossibility of detecting the motion of the Earth with respect to the ether, and it would seem useful to examine all of the consequences that this hypothesis implies. That was the viewpoint that was adopted by Poincaré in *Dynamique de l'Électron* ([5]). Poincaré did not conceal the conventional, and somewhat arbitrary, character of the theory of relativity. On the subject of the Lorentz hypothesis, he remarked that in that theory, two equal lengths are, by definition, two lengths such that light takes the same time to traverse them, and he added that it might suffice to renounce that definition in order for Lorentz's theory to be upset as completely as Ptolemy's system was by the intervention of Copernicus.

Poincaré's reservations seem to be justified.

There are some inconveniences to abusing principles. When one qualifies the principle of a scientific proposition then one will neither add to its degree of certainty nor to the precision of the observations that it summarizes.

The principle of relativity is quite contestable. It differs from the other fundamental propositions of science by its negative character. The only thing that one can assert in regard to the Michelson experiment is that the phenomena occur in a different manner from what one had supposed. Therefore, one must seek the reason for the result obtained in a more exact and complete analysis of those phenomena. To declare in advance that one must find nothing and that one will never find anything along those lines is an arbitrary statement that is, in a sense, intended to avoid the difficulty.

([1]) *Propos sceptiques du sujet du principe de relativité*, Scientia, 1913.
([2]) *La Mécanique. Les Idées et les Faits*, 1918, pp. 45-54. – Comptes rend. Acad. Sci. **174**, pp. 337.
([3]) Comptes rend. Acad. Sci. 2nd semester, 1921.
([4]) See Comptes rend. Acad. Sci. **172**, pp. 1227 and 1467; *ibid.*, **173**, pp. 1074 and 1343; *ibid.*, **174**, pp. 924.
([5]) Rend. di Palermo, **21** (1906), pp. 129.

The principle of special relativity, as it was stated by Einstein ([1]), can be formulated, to some extent, in the following manner:

1. *The laws of Nature, when referred to a system of reference to which the laws of Newtonian mechanics apply, keep the same form of expression in any other system that is deduced from the first one by a motion of uniform translation.*

2. *The velocity of light is a universal constant that is independent of the velocity of uniform translation of the focus.*

When one examines how that principle is applied, one confirms one of its contradictions from the outset and some of the ambiguities that frequently reappear in the published work on relativity. After having said "a uniform translation" in the statement of the principle, one replaces that translation with a Lorentz transformation in its application. They are not the same thing.

One perceives, moreover, that the only natural laws that Einstein considered in special relativity are the ones that refer to the partial differential equation of wave propagation in an isotropic medium when the velocity of propagation is equal to that of light.

In effect, the phenomena to which the theory of special relativity applies are the ones whose study comes down to that equation. It is wrong for one to apply that to all phenomena without exception.

The extraordinary facts, which are paradoxical in appearance, that one deduces from Einstein's theory are based upon that abuse of extension, and principally upon a confusion between Lorentz's local time and ordinary time.

Since they are two different things, one does not have the right to extend something that is true for one of them to the other one. The same observations apply to velocity. One pretends to show that no velocity can exceed that of light. That proposition is true for the "pseudo-velocity," in the Einstein sense, but it does not apply to velocity in the ordinary sense of the term; once more, there are two different things that are referred to by the same name. One can multiply the sources of confusion of that kind. In most cases, the words employed have a certain meaning in the arguments and calculations, and one then gives them a different meaning in their physical application.

The principle of special relativity, in the Einstein sense, seems completely useless. One can deduce many absurdities. The definite facts to which it applies can be obtained much more simply by a regular analytical study of the equation of wave propagation in an isotropic medium. One can even argue that the assortment of rulers and ideal chronometers that is introduced into Einstein's arguments constitutes a bizarre procedure for representing certain integrals of the equation in question.

Analysis has no need for that somewhat crude artifice. The determination of the integrals with moving pole provides a very neat and simple representation of the Lorentz

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([1]) Cf. LORENTZ, EINSTEIN, MINKOWSKI, *Das Relativitäts Prinzip.* Eine sammlung von Abhandlungen, 1920.

WEYL, *Raum, Zeit, Materie*, 1921.

EDDINGTON, *Espace, Temps, Gravitation*, Paris, Hermann, 1921.

BECQUEREL, *Le principe de Relativité et la Théorie de la Gravitation*, Paris, 1922.

transformation and the apparent Fitzgerald-Lorentz contraction. Those integrals lead to the consideration of ellipsoidal interference waves that are flattened like Lorentz's electrons.

For an observer in motion in an isotropic vibrating medium, the phenomena are similar to the ones that are observed in a medium at rest, except for the replacement of the spherical progressive wave with the flattened ellipsoidal interference wave.

It is the interference wave that replaces the progressive wave in reflection phenomena. In the propagation of a train of plane waves that is referred to a moving reference system, the ray is not perpendicular to the wave front, but it is parallel to the conjugate diameter to the front in the interference ellipsoid. Once more, it is the interference ellipsoid that intervenes as the director element in the determination of the interference sheets of two wave trains of the same period. Finally, if one measures the distances in terms of the radii of the ellipsoid then the analytical formulas will preserve exactly the same form for moving systems that they have for systems at rest.

Since those results are simple consequences of the equation of wave propagation, they will be applicable to the extent that the equation considered represents observed phenomena precisely. It applies to sound, as well as light. It will cease to be applicable when the motion of the source produces a dynamical effect on the medium that destroys its isotropy.

The calculations to which this study leads are exactly the same as those of the theory of special relativity, but one has no need for any new principle or hypothesis. If one does not lose sight of the starting point then one will not feel tempted to extend the results that are obtained to phenomena for which they have not been proved. Analysis shows very neatly that one has no right to deduce the equality of relative velocities of propagation in every sense of the term from the Michelson experiment, which thus exhibits the error in logic that has been at the origin of Einstein's theory.

I shall not concern myself with the theory of general relativity in this first article, and it will be the subject of another article.

CHAPTER I

TIME

1. – In order to discuss any scientific matter properly, it is necessary to agree on the meaning of words. Without that elementary precaution, any discussion would be in vain.

The symbols of language have no significance by themselves, and one is not generally sure that the same formulas represent the same ideas for the schools of two different people. Verbal definition can obviously determine the meanings of certain words, but that always means reducing them to other terms whose significance has been supposed previously. Upon reassembling them step-by-step, one will necessarily arrive at a set of fundamental symbols whose significance cannot be defined by other words. We will nonetheless recognize the agreement that exists regarding the meaning of those symbols upon confirming that one can apply them to the same facts of experience in the same manner.

It is, moreover, obvious that we have learned from the outset in our use of language to constantly attach the same word to the consideration of similar objects or facts. The constant association of symbol and the thing that hearing the symbol spoken recalls will remind one of the images of the experimental facts that were perceived before.

The initial definitions then reduce essentially to the confirmation of the agreement that exists between the fundamental words and the common facts of experience. Hence, it will result that the surest method for recognizing whether one is in accord in regard to the meaning of the essential terms of a theory consists of associating them with the common basis of experimental observations.

Certain logicians, who take the viewpoint of formal logic, consider words to be simple symbols that are initially devoid of any concrete sense. The argument then consists of combining those symbols according to well-defined rules, like algebraic symbols. Questions of form have their utility, like the rules of algebraic calculations, but they constitute only one facet of the problem. In scientific matters, one must always start from experiments and return to them.

An algebraic calculation can be exact and nevertheless lead to false conclusions. For example, that will be the case when one denotes two different quantities by the same symbol; that is a very common error amongst the novices.

That is precisely the fundamental error in Einstein's theory. Only its confrontation with experiments can reveal that fact.

2. Time in classical mechanics. – In his treatise on the principles of mechanics, H. Hertz ([1]) began with a set of considerations that he assumed were foreign to all experiments and based solely upon the laws of intrinsic intuition, in the Kantian sense. There is obviously a great deal of illusion in that, since the words will acquire a communicable meaning only when they correspond to experimental facts. Science is not preoccupied with the ideas of Kant on time and space, to which some philosophers accord, perhaps, far too much importance.

([1]) *Die Prinzipien der Mechanik*, 1894, pp. 53.

Metaphysical considerations do not enter into either calculations or the interpretation of results. What is important to know is the practical procedure that is employed effectively in order to attach a precise numerical value that all observers will agree upon to the consideration of time.

Chronometers are only auxiliary instruments of comparison whose indications are valid only for a limited duration. One calibrates chronometers by means of astronomical observations. By definition, the practical determination of the number *t* amounts to a measurement of the hour angle. The observation of diurnal motion then provides us with a sort of common chronometer that is valid for all terrestrial observers. It would be premature for us to occupy ourselves with any other one.

3. Isochronicity and simultaneity. – The correspondence between time and any physical phenomenon is based upon the notion of simultaneity. The analysis of that motion presents no difficulty when one is dealing with phenomena that are produced in the proximity of the observer. For the phenomena that are produced at a distance, the question will necessitate a certain degree of examination, since it is in that subject that we will observe the first point of divergence between relativistic language and common language. The whole world knows that when one fires a gun, a distant observer will note a very appreciable time interval between seeing the flash and hearing the sound. Nevertheless, those two phenomena are simultaneous for a close observer. There is then good reason to distinguish between the simultaneity of the production and the simultaneity of perception for two phenomena when one is dealing with distant observers. It remains for us to examine how we can establish the correspondence of the simultaneity of production, in the ordinary sense of the word, between phenomena that happen at two different locations.

A phenomenon happens in New York (e.g., City Hall) on 7 October 1921 at 8h 9m 23s mean solar time for the meridian through the location; what is the time at the Paris Observatory *at the same moment?* The dihedral angle subtended by the meridian planes of the two locations is the difference in longitude. It is $76^{\circ}\ 20'\ 38''$, which corresponds to a time difference of 5h 5m 23s. The time of the Paris meridian that corresponds to the event considered in New York is then 13h 14m 46s.

The determination of the correspondence necessitates simply the measurement of the difference in longitude; i.e., a dihedral angle. Once one has determined the necessary elements, one can then see the correspondence of simultaneity between events that happen at the two different locations, and with a precision that is comparable to the most delicate physical measurements.

It is, nonetheless, obvious that the determination of simultaneity or the order of succession in the production of phenomena will necessitate the knowledge of a set of elements that are sometimes invalid. The order of perception can differ from one observer to the other and can consequently the order of production can differ, as well. All of this is well-known. However, the uncertainty that governs the epoch likewise governs the position, and has the same order of magnitude. That fact that one ignores a date is not a sufficient motive for declaring that the notion of date does not exist. All of the predictions of astronomical phenomena are resolved by questions of simultaneity.

4. Correspondence of motions. – If one would like to attach the practical determination of time to a more general notion then one can remark that the correspondence of simultaneity between the positions of moving body, on the one hand, and what is indicated by a chronometer, on the other, comes down to a correspondence between two motions.

One of them is a *typical motion* to which one compares all of the others. Suppose that the position of the moving body that defines the typical motion is determined with the aid of a parameter t. One can obviously determine the correspondence of simultaneity by employing the same parameter for the study of the other motions.

5. Time in the expression of physical laws. – When one has that simple correspondence of simultaneity in mind, it is obvious that one is allowed a great deal of latitude in one's choice of parameter. For example, one can employ the true or mean solar angle with indifference. That is no longer true, however, when time must intervene in the statement of a physical law. A linear function of the mean solar angle is not linear with respect to the true solar angle. The notion of uniform motion that enters into the statement of the principle of inertia then supposes a well-defined calibration of time. The study of the consequences of the principle must indicate whether one can take the mean solar angle to be the calibration of time with sufficient precision or if there is good reason to apply certain corrections to it.

On that subject, one can remark: If one takes the true solar angle in order to calibrate time then one must replace the ordinary statement of the principle of inertia with another one that states that the velocity of a moving point, when it is free from all external actions, will be a variable function of time. The laws of mechanics are independent of time only for a certain choice of parameter.

6. Isochronicity in Einstein's theory. – In Einstein's theory, the definition of time rests upon other considerations. Every observer is assumed to be endowed with an *ideal* chronometer to which he refers the phenomena in his neighborhood. The definition of the correspondence of simultaneity presents difficulties only for the observations that are performed at different locations. Einstein imagined the following process: Two observers A and B are carried along in the translational motion of a system of axes with respect to which one can apply the principles of Newton's mechanics. At one instant t_A that is recorded by A' chronometer, one emits a light signal that arrives at B at the epoch t_B, as recorded by the B's chronometer. It reflects from a mirror and returns to A at the epoch t'_A. *By definition*, the two chronometers will be *isochronous* if one has:

$$t_B - t_A = t'_A - t_B, \qquad \text{so} \qquad t_B = \frac{t_A + t'_A}{2}.$$

One assumes, in addition, that the ratio $\dfrac{2\overline{AB}}{t'_A - t_A}$ is a universal constant V, which represents the velocity of light *in vacuo*.

The time of a phenomenon in the reference system considered is the time that is recorded by an ideal chronometer that is situated at the location at which the phenomenon is produced. Two phenomena are called *simultaneous* in the system S when they correspond to identical readings of chronometers that are placed at the locations where they are produced. After having thus defined simultaneity, Einstein then observed that two chronometers that appear to be simultaneous in one system of reference will no longer be so in another system S' that is in uniform translational motion with respect to the first one, and consequently the conditions of simultaneity will differ from one system to another.

Indeed, suppose that a bar AB displaces uniformly with respect to S with a velocity v in its own direction. At the epoch t_A , one emits a light signal that arrives at B at the epoch t_B, and after reflection, returns to A at the epoch t'_A. Since the times t_A , t_B , t'_A are recorded by the chronometers in the system S, one will have:

$$t_A - t_B = \frac{AB}{V - v}, \qquad t'_A - t_B = \frac{AB}{V + v}.$$

By virtue of the displacement of the bar, they will not satisfy the condition:

$$t_A - t_B = t'_A - t_B .$$

However, if one considers the system S' that participates in the translational motion of the bar with respect to the system S then the conditions of the experiment must be exactly the same as if the system S' were at rest. Therefore, let θ_A, θ_B, θ'_A be the times that are analogous to t_A , t_B , t'_A, but as recorded by the chronometers of the observers that are linked to the system S'. The condition of isochronicity in the system S' must be expressed by the equality:

$$\theta_B - \theta_A = \theta'_A - \theta_B .$$

Moreover, the measurement of the length AB in the system S' must likewise satisfy the condition:

$$\frac{2AB}{\theta'_A - \theta_A} = V,$$

in which V has the same *numerical* value for S and S'.

The numbers θ are different from the number t then, and equal values of θ will not correspond to equal values of t.

The definition of isochronicity that Einstein gave was inspired by the idea that the velocity of light must be a universal constant, as well as in all directions and also for systems S and S' that are deduced from each other by a uniform, translational motion.

From the conditions that he imposed, Einstein deduced the relations that must exist between the coordinates and time, as envisioned in the two systems, in order for those conditions to be verified. He then obtained the formulas for the Lorentz transformation.

Suppose, to simplify the calculations, that the axes *Ox* and $O'x'$ of the two systems coincide, and that the origin O' of the system S' describes the origin *Ox* as having a velocity *v*. Let *x*, *y*, *z*, *t* be the coordinates and time of the first system, while x', y', z', t' are the corresponding quantities of the second one; in addition, let *c* be the speed of light. Under that hypothesis, the formulas for the Lorentz transformation are:

$$(1) \qquad \begin{cases} x' = \dfrac{1}{\beta}(x - vt), \\ y' = y, \\ z' = z, \\ t' = \dfrac{1}{\beta}\left(t - \dfrac{vx}{c^2}\right). \end{cases}$$

One has set:

$$\beta = \sqrt{1 - \frac{v^2}{c^2}}\ .$$

I shall refrain from reproducing Einstein's proof of this. We shall recover the same results later on by a different method and with a precise interpretation of the parameter t'.

One knows that the Lorentz formulas are invertible. If one solves equations (1) for *x*, *y*, *z*, *t* then one will have:

$$x = \frac{1}{\beta}(x' + vt'), \qquad t = \frac{1}{\beta}\left(t' + \frac{vx'}{c^2}\right),$$

$$y = y', \qquad z = z'.$$

7. Comparison of Einstein's definition with what has been determined experimentally. – The definition of isochronicity that Einstein adopted and the condition that he imposed on the conservation of the *numerical value* of the velocity of light then implies this necessary consequence: Time is not an element that is independent of the reference system; it has no invariant character that one can generally attribute to it in relation to the change of coordinates.

As a result, we find ourselves confronting two different notions: Common time, which is common to all observers no matter what the relative displacement of one with respect to the other, and that of Einstein, which varies with the system of reference. One is led to demand to know which of those notions is the more correct one. If one goes back to the experimental conditions, whether real or assumed, then one will immediately recognize that one is dealing with two different things, and not two different conceptions of the same thing. Indeed, we have seen that common time, sidereal time, or mean solar time is determined by the observation of a sort of universal chronometer that is the same for all observers. On the contrary, under Einstein's hypothesis, each observer has an *ideal* chronometer, and the correspondence of isochronicity between two different chronometers is determined uniquely by the method of light signals. The conditions are not the same, and there is no reason to assume that the numbers that are determined by the two methods are identical.

It is interesting to compare that discussion with an argument of Pascal in regard to the various definitions of time.

Upon repeating it almost word-for-word, we can say that:

"As a result of that definition, there will be two things that one can refer to by the name of *time:* One of them is the one that the whole world naturally means by that word, and that everyone that speaks our language refers to by that term. The other one will be Einstein's parameter *t*, which the relativists also refer to by that name, according to a new convention. One must then avoid ambiguity and not confuse the consequences, because it does not follow from it that the thing that one naturally means by the word *time* is, in fact, Einstein's parameter. He was free to call those two things the same, but he did not make their characters agree, along with their names."

A large part of the theory of special relativity rests upon a consistent ambiguity that consists of applying results that are applicable only to Einstein's pseudo-time to mean solar time.

8. Critique of the definition of isochronicity. – Since the notions of isochronicity and simultaneity are the origin of any relativistic ambiguity, it is indispensible to insist upon the artificial and arbitrary character of Einstein's definitions. There are some notions that one will alter upon applying rules of logical definitions to them. They are the ones whose general idea results from the concordant observation of a large multiplicity of natural facts and which end up being sufficiently familiar to us that we no longer remember having acquired them. Such is the case for the simultaneity that is determined in practice by a set of observations of various forms. The method that Einstein supposed is never applied under the conditions that he indicated. Telegraphic signals can obviously be used for the problem of longitudes, but only to the extent that the errors that can result are less than the measurement errors that are provided by the application of other methods. The conclusions that Einstein inferred are based precisely upon the consideration of quantities whose order of smallness is that of the errors. The problem of longitudes can be posed for a sphere whose dimensions are much larger than those of Earth, and for which, consequently, the results that are provided by the method of light signals must submit to a very appreciable correction. Einstein's ideal chronometers have no existence in reality. Since they are the products of pure imagination, the author could endow them with all the qualities that he pleased, because he had arranged them like all of his work.

When one remains at the same location, one can always replace mean solar time with a proportional number – such as sidereal time, for instance – and Einstein's definition will present no inconvenience. The same thing is not true for displacements. The Lorentz transformation formula supposes a law of time shift for the records that are provided by moving chronometers. It is obvious that one needs considerable credulity in order to assume that the advance of chronometers is governed by that law. We shall see, moreover, that the time shift in question presents itself in the study of wave systems with a completely different significance from the one that was assumed by Einstein.

There is another criticism that one can formulate in regard to the confusion that comes about between Einstein's pseudo-time and ordinary time. It is not concerned with merely the difference between the numerical values, which is a secondary question: It refers to the character itself of the notion of time. Astronomers take the minutest precautions in order to render the chronometric observation of various observatories rigorously comparable. The numbers that are measured directly submit to various corrections that are destined to eliminate the purely local circumstances of the observations. In that way, the calculated numbers will present a truly invariant character in regard to the changes of coordinates or observer. Now, it is to precisely that invariant character that one must attribute the special role of time in mechanics. Duration is a sort of integral invariant. In the expression for the theorem of *vis viva*, one considers the ratio of two invariants.

Einstein's pseudo-time does not present the same character, so it is an abuse of terminology for it to keep the same role in the expression of the laws of mechanics. There is yet another serious consequence: From Einstein's argument, the constancy of the velocity of light will not be a natural law – i.e., a relation between observed facts. It results simply from the arbitrary *convention* by which the observers enforce the correspondence between their chronometers.

9. Ordinary time and the relativistic interval. – In the theory of relativity, the element that enjoys the same invariance properties as ordinary time is the quantity that is referred to by the name of Minkowski's *interval* or *proper time*.

The elementary interval $d\sigma$ is defined by the equality:

$$d\sigma^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2.$$

It is unaltered by a Lorentz transformation. Under the motion of a material point, the integral $\int d\sigma$, when evaluated between two positions or two states of the moving body, is likewise independent of the reference system. That property corresponds to the invariant character of duration in classical mechanics.

If one establishes a correspondence between two motions that is expressed by a relation between the intervals then that correspondence will persist when one performs a change of reference system that is represented by a Lorentz transformation.

Suppose that one takes the typical motion to be a motion that is observable in the different systems, and that the correspondence of simultaneity is expressed by a relation between the intervals of the various moving points and the interval that corresponds to the typical motion: One will find oneself in a case that is analogous to that of classical mechanics. In the general theory of relativity, it is, indeed, the interval that plays the role of common time, but without bearing that name.

Einstein's conclusions in regard to time do not result from the principle of relativity; they come from the special convention that serves as the basis for his definition of isochronicity, and we have seen that his convention contradicts the determination of simultaneity in the ordinary sense of the word. He had to suppose that the observers displaced in a space that was marked out with chronometers and that each of them observed the reading of the chronometer that he passed, instead of consulting his own chronometer. Since those readings are independent of the velocity of the observer with respect to the space marked out, it is obvious that Einstein's conclusions will have to be different. That hypothesis, with its paradoxical appearance, is perhaps closer to reality than Einstein's, since astronomical phenomena provide precisely a sort of chronometric calibration of space. One can attach that concept to the method that is employed at sea in order to determine the longitude by the observation of lunar distances.

Nevertheless, we shall see that the study of wave phenomena leads to the introduction of a parameter that is analogous to Lorentz's local time and enjoys the same properties from the standpoint of transformations. However, that parameter is not mean solar time.

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CHAPTER II

WAVES WITH MOVING POLES

10. – In rectangular coordinates, the equation of wave propagation in a homogeneous, isotropic medium takes the following form:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}=0. \tag{3}$$

The constant c is the velocity of propagation, which depends upon the constitution of the medium, while the variable t is deemed to denote ordinary time.

That equation admits integrals with fixed poles that one obtains easily by Poisson's method. I shall briefly recall the calculations.

For example, look for the integrals that admit a fixed pole at the origin of coordinates. We set:

$$r^2 = x^2 + y^2 + z^2,$$

and we will be led to study the integrals that depend upon only r and t. They satisfy the celebrated equation of Euler and Poisson that Darboux made a profound study of:

$$\frac{\partial^2 V}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}+\frac{2}{r}\frac{\partial V}{\partial r}=0. \tag{4}$$

An easy calculation will bring it into the form:

$$\frac{\partial^2}{\partial r^2}(rV)-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rV)=0,$$

so

$$r\,V = f(r + ct) + \varphi(r - ct),$$

in which f and φ denote arbitrary functions of the characteristic variables $r + ct$ and $r - ct$.

One will then have:

$$V = \frac{f(r+ct)+\varphi(r-ct)}{r}.$$

That integral admits characteristic singularities that are defined by the form of the functions f and φ, along with the fixed pole $r = 0$. The latter singularity can disappear for certain combinations of arbitrary functions. In particular, if one has:

$$f(r + ct) = \frac{1}{r+ct}, \qquad \varphi(r - ct) = \frac{1}{r-ct},$$

then the integral V will come down to the fundamental integral ([1]):

$$\frac{1}{r^2 - c^2 t^2}.$$

11. – Now consider the case of a pole that displaces along the Ox axis with uniform velocity v. In order to convert the problem of the wave with moving pole into that of the wave with fixed pole, it is natural to perform a change of coordinates by setting $x = x_1 + vt$.

The partial differential equation (3) will then become:

$$\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\left(\frac{\partial^2 V}{\partial t^2} - 2v\frac{\partial^2 V}{\partial x_1\,\partial t} + v^2\frac{\partial^2 V}{\partial x_1^2}\right) = 0. \tag{5}$$

Equation (5) contains a term in $\frac{\partial^2 V}{\partial x_1\,\partial t}$. One makes them disappear by a new transformation that is performed on the variable t. Supposing that $v^2 < c^2$, we will have:

$$\theta = \beta\, t - \frac{v\, x_1}{c^2 \beta}, \tag{6}$$

in which β denotes the Lorentz factor $\sqrt{1 - \frac{v^2}{c^2}}$.

If we take v to be a new variable, in place of t then we will convert equation (5) into the symmetric form:

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 V}{\partial x_1^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial \theta^2} = 0. \tag{7}$$

By a further transformation:

$$x' = \frac{x_1}{\beta} = \frac{x - vt}{\beta}, \tag{8}$$

we will finally recover the form of the original equation:

$$\frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{\partial^2 V}{\partial \theta^2} = 0. \tag{9}$$

([1]) HADAMARD, "Sur les solutions fondamentales et l'intégration des équations linéaires aux dérivées partielles," Ann. Ec. Norm. Sup. (1904) and (1905).

It is clear that formulas (6) and (8) define the Lorentz transformation. That is obvious for formula (8). In formula (6), when one replaces x_1 with its value $x_1 = x - vt$, one will find that:

$$\theta = \beta t - \frac{v(x - vt)}{c^2 \beta} = \frac{1}{\beta}\left(t - \frac{vx}{c^2}\right). \tag{10}$$

We integrate equation (9) by Poisson's method, while setting:

$$r' = \sqrt{x'^2 + y'^2 + z'^2}\,,$$

and get the general form for the desired integral with two arbitrary functions:

$$V = \frac{f(r' - c\theta) + \varphi(r' + c\theta)}{r'}. \tag{11}$$

12. Modulus and parameter of radiation. – The denominator r' takes the form:

$$r' = \sqrt{\frac{(x - vt)^2}{\beta^2} + y^2 + z^2}$$

when it is expressed as a function of the original variables.

We give that quantity the name of the *modulus* of the integral V.

The points that correspond to the same value of the modulus in the moving system are situated on the surface of an ellipsoid of revolution that has Ox for its axis and the moving pole for its center.

That surface is flattened in proportion to the Lorentz contraction factor.

The characteristic arguments $v' \pm c\ \theta$, upon which the arbitrary functions depend, correspond to series of waves; the positive waves, which dilate concentrically around the pole, and the negative waves, which, by contrast, contract when one approaches that point. For one of them, the pole constitutes a focus for emission, and for the other one, a focus for absorption.

In the motion of the wave, the variation of modulus is always equal to that of the argument $c\theta$ in absolute value. In what follows, we shall set $c\theta = u'$ and it give the name of *radiation parameter* to that quantity.

The expression for the radiation parameter depends upon only the constant c and the velocity of dragging of the moving system, when both the magnitude and direction of that velocity are considered.

We likewise let $u = ct$ denote the radiation parameter of a system of fixed axes.

15. Ellipsoidal interference waves. – Consider the case of a *doublet* that is comprised of the juxtaposition of a focus for emission and a focus for absorption that correspond to equal periodic waves.

For example, let:

$$f = A \sin \frac{2\pi(r' - u' + \alpha)}{\lambda}, \qquad \varphi = A \sin \frac{2\pi(r' + u' + \beta)}{\lambda},$$

in which A, α, β, λ denote constants.

The corresponding value of V will take the following form:

$$V = \frac{2A}{r'} \sin 2\pi \frac{r' + \frac{\alpha+\beta}{2}}{\lambda} \cos 2\pi \frac{u' + \frac{\beta-\alpha}{2}}{\lambda}. \tag{12}$$

That form exhibits a series of stationary interference surfaces in the moving system.

The function V will be annulled for any radiation parameter at the points where the modulus verifies the condition:

$$r = n\frac{\lambda}{2} \qquad (n \text{ integer}).$$

There is interference at those points.

For that reason, we give the name of *interference waves* to the ellipsoidal waves that correspond to the constant values of the modulus.

If one considers a system of moving axes with the same directions as the fixed axes that have their origins at the pole of the wave then the interference wave surface will be represented by the equation:

$$\frac{x_1^2}{\beta^2} + y_1^2 + z_1^2 = u^2 = \text{const.}$$

14. Physical significance of the Lorentz transformation – We call the particular surface that corresponds to $u = 1$ the *interference ellipsoid of the moving medium.*

The wavelengths of the interference waves will vary with direction by virtue of the contraction of the interference wave in the direction of displacement of the pole. However, if we evaluate the distances in radii of the ellipsoid (or, more precisely, if we refer them in each direction to a unit that varies in proportion to the radius of the interference ellipsoid that corresponds to that direction) then the axes of the ellipsoid will be measured in terms of equal numbers. That change of units is equivalent to the transformation:

$$\frac{x_1}{\beta} = x', \qquad y_1 = y', \qquad z_1 = z'.$$

We now have a very simple and precise physical interpretation for the Lorentz transformation.

Let u and u' denote the radiation parameters for the waves that are attached to two reference systems, respectively, and set $v / c = \alpha$. The Lorentz formulas will then take the following form:

$$x' = \frac{1}{\beta}(x - \alpha u), \qquad u' = \frac{1}{\beta}(u - \alpha x),$$
$$y' = y, \qquad z' = z, \qquad \beta = \sqrt{1-\alpha^2}\,.$$

In each system, the rectilinear distances are measured in radii of the interference ellipsoid that relates to the system considered. We give the name of *Lorentz variables* to those values of the coordinates.

15. Phase ([1]) of the radiation parameter. – The values of the radiation parameter are simply proportional to time in the fixed system. In the moving system, those values will propagate or shift by way of plane waves. Upon expressing u' as a function of t and x, one will have:

$$u' = \frac{c}{\beta}\left(t - \frac{vx}{c^2}\right).$$

The phase velocity is then equal to c^2 / v.

In the system of variables x_1, y_1, z, t that relates to the axes that are comoving with the translational motion of the moving pole, but whose lengths will preserve their usual significance, one will have:

$$u' = \frac{c}{\beta}\left(\beta^2 t - \frac{vx_1}{c^2}\right).$$

The phase velocity is equal to $c^2\,\beta / v$ when it is measured in wave lengths of the interference wave and in time. It is the geometric mean of two others.

16. Periods. – Consider a periodic function of the radiation parameter; let U' be its period. The corresponding period referred to time will not be the same for the two systems of axes. Let T be the period, when referred to the fixed system, and consequently take x = const., by hypothesis. One will have:

$$T = \frac{U'\beta}{c}.$$

The period T_1, which refers to the moving system, and when one takes x_1 = const., by hypothesis, will be given by the formula:

$$T_1 = \frac{U'}{c\beta}.$$

The ratio of the periods T / T_1 is equal to:

([1]) An expression that was employed by VARCOLLIER, "Les déplacements dans les champs de vecteurs et la Théorie de la Relativité," Revue générale des Sciences (1918), pp. 101-114 and 135-146.

$$\beta^2 = 1 - \frac{v^2}{c^2}.$$

The product of the periods TT_1 is equal to U'^2 / c^2. The ratio U' / c will be the period, when referred to the variable θ that is defined by equation (10), and it will correspond to Einstein's pseudo-time in the case of light vibrations.

17. Progressive waves. – Instead of taking the viewpoint of interferences, consider the locus of points that are attained at an epoch t by an instantaneous perturbation that issues from moving pole at a previous instant t_0. The problem amounts to the search for the locus of points for which the characteristic argument $r' - u'$ of the positive wave has a constant value that is given at the epoch considered t. The initial value is equal to:

$$u'_0 = \frac{\beta}{c}\left(t_0 - \frac{vx_0}{c^2}\right),$$

in which one can suppose that $x_0 = ct_0$, moreover.

At the epoch t, one must then have:

$$r' - u' = -u'_0,$$

or

$$r'^2 = (u' - u'_0)^2.$$

By a simple calculation, one infers from this that:

$$(x - x_0)^2 + y^2 + z^2 = c^2 (t - t_0)^2.$$

As one would have expected, the locus is a sphere that its center at the point of origin of the perturbation and a radius that grows in proportion to duration. That result is the same as if the pole were fixed. In order to distinguish these ordinary waves from interference waves, we give them the name of *progressive waves*. In the case of a fixed pole, the interference waves will coincide with progressive waves.

18. Analytical character of the functions that represent the modulus and radiation parameter of the moving wave. – When one introduces the radiation parameter in place of time, the partial differential equation for the wave propagation will take on the form:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{\partial^2 V}{\partial u^2} = 0.$$

The characteristic multiplicities are defined by the first-order equation:

$$\left(\frac{\partial\varphi}{\partial x^2}\right)^2+\left(\frac{\partial\varphi}{\partial y^2}\right)^2+\left(\frac{\partial\varphi}{\partial z^2}\right)^2-\left(\frac{\partial\varphi}{\partial u^2}\right)^2=0.$$

We denote the left-hand side of that equation by $D\,(\varphi)$. The modulus r' and that radiation parameter u' satisfy the following relations:

$$D\,(r') = 1, \qquad D\,(u') = -1,$$

$$\frac{\partial r'}{\partial x}\frac{\partial u'}{\partial x}+\frac{\partial r'}{\partial y}\frac{\partial u'}{\partial y}+\frac{\partial r'}{\partial z}\frac{\partial u'}{\partial z}-\frac{\partial r'}{\partial u}\frac{\partial u'}{\partial u}=0.$$

Using a notation that is frequently employed, the last of these relations can be represented by:

$$D\,(r', u') = 0.$$

It is easy to deduce from these results that the functions $r' - u'$ and $r' + u'$ satisfy the partial differential equation for characteristic multiplicities.

Upon then attaching the Lorentz transformation to the theory of characteristics and the motion of the propagation of waves with moving pole, one sees how the problem can be generalized to arbitrary motions. The difficulties that one encounters are of a purely analytical order and have the same character for all of the problems that relate to the integration of partial differential equations.

19. Extension to the case of arbitrary velocities. – We have consistently supposed that the velocity of translation v is less than the speed of propagation of c. That restrictive condition is not obligatory. For speeds that are greater than c, the interference waves will be hyperbolic, and the modulus will be annulled on the surface of a real cone. The Lorentz coefficient of contraction must be replaced with $\sqrt{\alpha^2-1}$; however, aside from that detail, the calculation will be identical. The case of $v = c$ is a singular case that will necessitate a special study.

The fact that the Lorentz transformation can be presented in the study of interferences of any wave system shows sufficiently that the conclusions that one believes can be inferred in the special case of light are not well-founded.

Moreover, mathematical theories have no mysterious power to govern phenomena. Einstein's pretense of enacting *a restrictive condition to which the laws of nature* ***must be subject*** is difficult to assume.

The analytical operations have neither the significance nor the scope that one thus attributes to them. They are not facts that one bends to our formulas; they are formulas that must be adapted to observation, and we cannot pretend to attribute a precision to the result of our calculations that is greater than that of the experimental data upon which they are founded.

20. Anisotropic, homogeneous systems. – The fact that the Maxwell equations preserve their form under Lorentz transformations seems to indicate that interference waves and the radiation parameter must be very important in the study of optical phenomena and electromagnetism when one has to consider moving foci. From the purely analytically viewpoint, their consideration permits us to generalize our results quite simply, in addition. Indeed, we can envision the much more general case of the propagation of waves in an anisotropic, homogeneous medium. If one always takes the interference ellipsoid to be the quadric directrix for the measurement of distances then, as we have pointed out, the equations will preserve exactly the same form as for isotropic systems. The calculations and the expression for the integral itself will remain identical. We will always have an elliptic modulus r and a radiation parameter u that is a linear function of time and the coordinates.

The anisotropy of the waves can be revealed only by measurements that are made with instruments that do not participate in the apparent deformation of the waves. If the lengths are evaluated in terms of radii of the interference ellipsoid then any distinction between isotropic and anisotropic systems will disappear.

We then arrive at a result that is quite interesting, and which explains the role of the theory of special relativity in the intrinsic study of wave systems. One can have a notion of the displacement of the focus in the medium only by comparison with other phenomena.

21. Remark on the Michelson experiment. – In the Michelson experiment, the interference phenomena are observed with the aid of instruments that are alleged to be rigid. One does not find them then in the case of homogeneous phenomena that we just spoke of. The study of experiments can be facilitated to a certain degree by the results that we have established already.

The usual discussion is obviously defective and can allow some doubts about the validity of the conclusion to persist. Meanwhile, we assume that they are exact. The consequences are interesting, but have no relationship with Einstein's theory.

The experiment must exhibit the flattening of the interference waves. On the contrary, the results obtained tend to show that under the conditions and limitations of observation, the interference wave is rigorously spherical, in the physical sense of the term, since the measurements are made with terrestrial instruments.

The modulus r then denotes a true distance. The equation of the characteristic multiplicity in four variables:

$$v^2 = (u - u_0)^2,$$

gives the interference wave that we just envisioned when it is interpreted by supposing that u = const.. The same equation gives another quadric that represents the progressive wave when it is interpreted by supposing that t = const.. Since the radiation parameter u is a linear function of the coordinates, that quadric will generally be an ellipsoid of revolution that has the moving pole for its focus.

We then arrive at an interesting result that was pointed out by Poincaré ([1]) in a less general form, rediscovered by Ch.-E. Guillame ([2]), and likewise corresponds to the research of Sagnac ([3]).

It is the progressive wave that gives the law of variation of the relative velocities of propagation. *From that result, one does not therefore have the right to say that those velocities are equal in all directions. The isotropy of the interference wave does not imply the isotropy of the progressive wave.*

At the same time, we obtain another very important consequence.

The ellipsoidal form of the progressive wave no longer permits us to assume that the medium of propagation is isotropic in the restricted domain in which one performs the experiment.

In that domain, *the Earth will then influence the medium, and consequently, the propagation of light.*

As for the form and nature of that influence, the scope of possible hypotheses is extremely vast. We shall not formulate any. The only legitimate consequence that we can infer from the Michelson experiment is the isotropy of the interference wave. We have no information about the expression for the parameter u. The result is quite interesting, but it presents nothing paradoxical when one examines it with no preconceived ideas.

In our calculations, we have supposed that the medium can be considered to be homogeneous in the domain of the experiment and that the measurements are performed with the aid of terrestrial instruments.

In order to explain the result of Michelson's experiment, Einstein supposed that, in addition to the terrestrial observers, there exist other observers that are carried along by the motion of the Earth. Those observers will be endowed with an appropriate set of compressible rulers and retarded chronometers whose readings will correspond to those of the terrestrial instruments by way of the Lorentz formulas.

The intervention of these hypothetical personages seems hardly useful in the context of the question, because we can speak only in terrestrial language of terrestrial measurements and terrestrial observers. Moreover, in the Michelson experiment, one does not have to consult the chronometer or establish the comparison with the other chronometers that do not exist and are carried by observers that do not exist either. Those elements are entirely foreign to the interpretation of phenomena.

([1]) *Science et Méthode*, pp. 239.
([2]) *Comptes rendus du Congrès Int. des Math. de Strasbourg*, pp. 602.
([3]) Compt. rend. Acad. Sci. **174** (1922), pp. 29.

CHAPTER III

GEOMETRY OF WAVE SYSTEMS

22. – The geometry of wave systems in an isotropic medium comes down to the study of Lorentz transformations when it is considered from the most general viewpoint. In that study, one encounters geometric properties that are quite interesting.

We have given the precise significance of the Lorentz formulas independently of any considerations of metaphysical chronometers and enchanted rulers.

If one is given two systems of axes in uniform, translational motion with respect to each other then they will each correspond to an interference ellipsoid and a radiation parameter. Since lengths are assumed to be measured in terms of radii of the interference ellipsoid, we have the Lorentz formulas in the normal form:

$$x' = \frac{1}{\beta}(x - \alpha u), \qquad u' = \frac{1}{\beta}(u - \alpha x), \qquad y' = y, \qquad u' = z.$$

Since the factor $1 / \beta$ is greater than unity, we set:

$$\frac{1}{\beta} = \cosh \varphi, \quad \frac{\alpha}{\beta} = \sinh \varphi, \quad \alpha = \tanh \varphi,$$

upon introducing the hyperbolic functions.

The first two Lorentz formulas become:

$$(13) \qquad \begin{cases} x' = \quad x \cosh \varphi - u \sinh \varphi, \\ u' = -x \sinh \varphi + u \cosh \varphi. \end{cases}$$

By their form, they recall formulas of rotation, with the substitution of hyperbolic functions for circular functions.

One has identically:

$$u'^2 - x'^2 - y'^2 - z'^2 \equiv u^2 - x^2 - y^2 - z^2.$$

Let $d\sigma^2$ denote the common value of the two sides of that identity. The quantity $d\sigma$ that the relativists call the elementary interval is a relative invariant under Lorentz transformations. Conversely, the group of transformations that preserves the invariant $d\sigma$ constitutes the most general Lorentz group.

23. Determination of the transformations of the Lorentz group. – From the geometric viewpoint, it is extremely easy to obtain the most general transformation that preserves the form $d\sigma$.

One knows that the linear transformations with constant coefficients are the only ones that replace a quadratic differential form with constant coefficients with another form of the same nature. We will then have to occupy ourselves with only linear transformations.

We shall even neglect the additive constants in order to confine ourselves to the case of homogeneous forms.

The equation:

(14) $$u^2 - x^2 - y^2 - z^2 = 0$$

can be considered to represent a sphere of radius equal to unity in homogeneous coordinates. Any substitution that reduces the left-hand side to the form:

$$u'^2 - x'^2 - y'^2 - z'^2$$

will be obtained by taking the reference tetrahedron to be a tetrahedron that is conjugate with respect to the sphere. Conversely, any conjugate tetrahedron will correspond to a linear substitution that enjoys the required property.

Of the four summits of a conjugate tetrahedron, there is only one of them that is interior to the sphere; that summit will be the point O', which has the homogeneous coordinates $x' = 0$, $y' = 0$, $z' = 0$ in the new system. The plane $u' = 0$ will be the polar plane to O'.

The trihedron $O'x'\,y'\,z'$ must be conjugate with respect to the imaginary cone whose summit O' circumscribes the sphere. If one would wish that it should be, at the same time, tri-rectangular then it is necessary that one of the new coordinate axes should coincide with the axis of revolution OO' of the cone; the other two will be subject to only the condition that they must be mutually-perpendicular and perpendicular to OO'. Finally, in order for the transformation to be reciprocal – i.e., the first system should be deduced from the second one in the same way that the second is deduced from the first with conservation of orthogonality – it is likewise necessary that one of the axes of the first system must coincide with OO'.

Therefore, take that line to be the common axis of x and x'. Let $u = 1$, $x = \alpha$, $y = 0$, $z = 0$ be the homogeneous coordinates of the point O' in the first system of axes. The polar plane P' to that point has the equation $u - \alpha x = 0$ in the first system and $u' = 0$ in the second. The plane perpendicular to OO' that is drawn through the point O' is represented by the equations:

$$x - \alpha u = 0, \qquad x' = 0, \text{ respectively,}$$

in the two systems of axes.

Upon denoting the constant coefficients by λ and μ, one will then have:

$$u' = \lambda\,(x - \alpha u), \qquad x' = \mu\,(x - \alpha u).$$

The variables u' and x' are expressed as homogeneous functions of u and x. The variables y and z are likewise expressed with the aid of only the variables y', z', since the corresponding coordinate planes pass through the same line OO'.

The proposed identity:

$$u'^2 - x'^2 - y'^2 - z'^2 \equiv u^2 - x^2 - y^2 - z^2$$

then splits into two other ones:

$$u'^2 - x'^2 = u^2 - x^2, \qquad y'^2 + z'^2 \equiv y^2 + z^2.$$

One infers from the first that:

$$\lambda^2 - \mu^2 = \frac{1}{1-\alpha^2}.$$

The second one expresses the idea that the directions of the axes $O'x'$ and $O'z'$ are deduced from the corresponding axes of the first system by a simple rotation.

The conditions of orthogonality and reciprocity obviously restrict the number of parameters upon which the most general transformation will depend. The result obtained translates into a sort of hyperbolic rotation for the variables u and x and an ordinary circular rotation for the variables y and z. Upon neglecting the latter rotation, one will get the Lorentz transformation in its customary form.

24. The modulus. – The cone that circumscribes the sphere (14) and has the point O' for its summit will be represented by the equation:

$$(u - \alpha x)^2 - (1 - \alpha^2)\,(u^2 - x^2 - y^2 - z^2) = 0.$$

The left-hand side of that equation reduces to the following expression:

$$(1 - \alpha^2)\left[\frac{(x-\alpha u)^2}{1-\alpha^2} + y^2 + z^2\right] = (1 - \alpha^2)\,(x'^2 + y'^2 + z'^2),$$

in which one will recognize the square of the modulus of the wave with moving pole, up to a factor.

All of the elements of the transformation, including the modulus, are then defined entirely by a point O' that is assumed to be interior to the fundamental sphere considered.

The abscissa of the polar plane to the point O' is equal to $1 / \alpha$.

It represents the phase velocity of the new radiation parameter u' with respect to u.

25. Composition of Lorentz transformations. – What one calls the composition of velocities in the theory of relativity is, in reality, equivalent to the composition of Lorentz transformations.

In the domain of the four variables u, x, y, z, a displacement will be represented by a variation in those quantities. Consider an infinitely-small displacement du, dx, dy, dz, and let $d\sigma$ be the corresponding interval. Set:

$$ds^2 = dx^2 + dy^2 + dz^2.$$

The equality that defines the interval can be written:

$$d\sigma^2 = du^2 - ds^2,$$

which permits one to set:

$$du = d\sigma \cosh \varphi,$$
$$ds = d\sigma \sinh \varphi,$$

upon letting φ denote the argument of the hyperbolic functions.

The components dx, dy, dz of the displacement along the axes have the form:

$$dx = l\, d\sigma \sinh \varphi, \qquad dy = m\, d\sigma \sinh \varphi, \qquad dz = n\, d\sigma \sinh \varphi,$$

while the parameters l, m, n are coupled by the relation:

$$l^2 + m^2 + n^2 = 1.$$

The ratio ds / du is assigned to a velocity in the theory of relativity; we call it a *pseudo-velocity*. In effect, time, properly speaking, does not enter into the question; as we have seen, it is not necessarily proportional to the radiation parameter.

One has:

$$\frac{ds}{du} = \tanh \varphi,$$

so the components of the pseudo-velocity along the axes will be:

$$\frac{dx}{du} = l \tanh \varphi, \qquad \frac{dy}{du} = m \tanh \varphi, \qquad \frac{dz}{du} = n \tanh \varphi.$$

Any *pseudo-velocity* of that form corresponds to a point O' that is interior to the fundamental sphere that was defined previously, and the point O' likewise corresponds to a Lorentz transformation.

Now, refer the same displacement to a second reference system u', x', y', z'. Upon letting φ' denote the hyperbolic argument that relates to the displacement in that system, one will get formulas that are analogous to the first ones:

$$du' = d\sigma \cosh \varphi', \qquad dx' = l'\, d\sigma \sinh \varphi', \qquad dy' = m'\, d\sigma \sinh \varphi', \qquad du' = n'\, d\sigma \sinh \varphi'.$$

Finally, let θ denote the hyperbolic argument of the Lorentz transformation that establishes the correspondence between the two systems:

$$x = \quad x' \cosh \theta + u' \sinh \theta,$$
$$u = + x' \sinh \theta + u' \cosh \theta.$$

We will immediately get:

(15)
$$\begin{cases} \cosh\theta = \quad \cosh\varphi'\cosh\theta + l'\sinh\varphi'\sinh\theta, \\ l\sinh\theta = l'\sinh\varphi'\cosh\theta + \cosh\varphi'\sinh\theta, \\ m\sinh\theta = m'\sinh\varphi', \\ n\sinh\theta = n'\sinh\varphi'. \end{cases}$$

Since the coefficient l' denotes a cosine, the first two equations of that group recall the fundamental formulas of non-Euclidian trigonometry. We shall see the reason for that later on.

As before, set:

$$\tanh\,\theta = \alpha, \qquad \frac{1}{\cosh\theta} = \beta.$$

Equations (15) give:

(16)
$$\begin{cases} \dfrac{dx}{du} = l\tanh\varphi = \dfrac{dx'/du' + \alpha}{1+\alpha\, dx'/du'}, \\ \dfrac{dy}{du} = \dfrac{\beta\, dy'/du'}{1+\alpha\, dx'/du'}, \\ \dfrac{dz}{du} = \dfrac{\beta\, dz'/du'}{1+\alpha\, dx'/du'}. \end{cases}$$

If the displacement considered is directed along the axis Ox then one will have:

$$l' = 1, \quad m' = n' = 0,$$

which implies that:

$$\varphi = \varphi' + \theta,$$

so

$$\tanh\,\varphi = \frac{\tanh\varphi' + \tanh\theta}{1+\tanh\varphi'\tanh\theta}.$$

26. Case of pseudo-velocities that are greater than that of light. – Up to now, by our use of hyperbolic functions, we have supposed that the components of the pseudo-velocity are all less than unity. Meanwhile, a similar calculation can be performed in the case of arbitrary pseudo-velocities. Consider the case of two pseudo-velocities α and α' that have the same direction. The resultant pseudo-velocity α_1 will be given by the formula:

$$\alpha_1 = \frac{\alpha+\alpha'}{1+\alpha\alpha'}.$$

The difference $1 - \alpha_1$ is no longer necessarily positive. One has:

$$1 - \alpha_1 = \frac{(1-\alpha)(1-\alpha')}{1+\alpha\alpha'}.$$

That quantity will change sign when one of the velocity components crosses unity or also when the denominator is annulled.

Therefore: If one of the components of the pseudo-velocity is equal to unity then the resultant pseudo-velocity will also be equal to unity. If one of the components is equal and opposite to the phase pseudo-velocity of the radiation parameter then the resultant will become infinite.

The sum $1 + \alpha_1$ can be put into a similar form:

$$1 + \alpha_1 = \frac{(1+\alpha)(1+\alpha')}{1+\alpha\alpha'}.$$

27. The Lorentz transformation and Cayley geometry ([1]). – Up to now, we have confined ourselves to the consideration of Lorentz transformations that preserve the Euclidian orthogonality of the axes. However, one can examine the more general case by a simple geometric method that leads to an interesting application of Cayley's notion of angle and distance.

We have shown that the Lorentz transformation can be represented by a homogeneous coordinate transformation that preserves a fixed sphere. In a tetrahedron that is conjugate with respect to the sphere, the interior summit will play a special role when one considers only translational pseudo-velocities that are less than unity.

The center O of the sphere corresponds to a certain fundamental reference system (S). Any other point P that is interior to the sphere will define a translational pseudo-velocity with respect to the system (S) that will be measured by the vector OP. One can take the point P to be the pole; it will correspond to another system (S'). The pseudo-velocities that are referred to the system (S') are not measured by the Euclidian lengths of vectors.

For example, if one considers a point P_1 that is different from P then the pseudo-velocity that corresponds to that point with respect to the system (S') will indeed be measured by the Euclidian length OP_1, but the pseudo-velocity relative to the system (S') will not be measured by the Euclidian length PP_1. It will then seem that there is a difference between the two systems. Now, it is possible to make it disappear by a process of measurement that is applied to all systems indifferently, no matter what their corresponding pole.

Let v denote the vector $\overline{OP}$. The hyperbolic argument θ of the Lorentz transformation that permits one to pass from the system (S) to the system (S') will be defined by the equality:

$$\tanh \theta = \frac{e^{2\theta}-1}{e^{2\theta}+1} = v,$$

and one will consequently have:

([1]) CAYLEY, "A sixth memoir on quantics," Trans. Roy. Phil. Soc. London, 1859.

$$\theta = \frac{1}{2} \ln \frac{1+v}{1-v} .$$

It is easy to give a projective form to that expression that will be preserved under a change of pole. The line *OP* cuts the sphere at two point *M* and *M′*. I have supposed that one has taken the positive sense on the line to the sense of $\overline{M'M}$.

The anharmonic ratio of the four points *M*, *M′*, *O′P* is equal to:

$$\frac{OM}{OM'} : \frac{PM}{PM'} = \frac{1}{-1} : \frac{1-v}{1+v} = \frac{1+v}{1-v} .$$

Upon denoting that anharmonic ratio by (*MM′OP*), one will then have:

$$\theta = \tfrac{1}{2} \ln (MM'OP).$$

The particular role of the position that attributed to the point *O* initially will disappear in this new expression, and one can apply the same formula in order to pass from one point P_1 to another one P_2 .

The line P_1 P_2 cuts the sphere at two points *N*, *N′*.

The hyperbolic argument θ of the corresponding transformation will be given by the formula:

$$\theta = \tfrac{1}{2} \ln (NN'P_1\, P_2).$$

That is precisely the expression for the Cayleyian distance that corresponds to the sphere considered, when it is taken to be the fundamental quadric.

Let $[P_1\, P_2]$ denote that distance. It is obvious that one has a straight line:

$$\overline{[P_1 P_3]} = [P_1\, P_2] + [P_2\ \, P_3\,]$$

for a system of three points P_1, P_2, P_3 .

The Cayleyian distance from any point *P* to a point on the surface of the sphere is infinite.

The pseudo-velocity that corresponds to the argument θ is always equal to tanh θ, no matter what the pole of the reference system. Cayley geometry thus provides a very simple image of the composition of pseudo-velocities.

28. Analytical expression for the Cayleyian distance. – I believe that is useful to briefly recall the calculation of the Cayleyian distance as a function of the coordinates.

Let two points P_1 (x_1, y_1, z_1), P_2 (x_2, y_2, z_2) be interior to the fundamental sphere. The coordinates of any point of the line P_1 P_2 can be put into the form:

$$x = \frac{x_1 + \lambda x_2}{1+\lambda}, \quad y = \frac{y_1 + \lambda y_2}{1+\lambda}, \quad z = \frac{z_1 + \lambda z_2}{1+\lambda} .$$

Upon expressing the idea that the point M is found on the sphere, one will get a equation that has degree two in λ:

$$(1-x_1^2-y_1^2-z_1^2)+2\lambda(1-x_1\,x_2-y_1\,y_2-z_1\,z_2)+\lambda^2(1-x_2^2-y_2^2-z_2^2)\;=0.$$

One sets:

$$F_{ij}=1-x_i\,x_j-y_i\,y_j-z_i\,z_j\,;$$

the equation in λ will take the abbreviated form:

$$F_{11}+2\lambda\,F_{12}+\lambda^2\,F_{22}=0.$$

Let λ_1 and λ_2 be the roots of that equation that correspond to the intersection points M_1, M_2 .

The anharmonic ratio ($M_1\,M_2\,P_1\,P_2$) is equal to λ_1 / λ_2 .

One then has:

$$e^\theta=\sqrt{\frac{\lambda_1}{\lambda_2}}\,,$$

so:

$$\cosh\,\theta=\frac{1}{2}\left(\sqrt{\frac{\lambda_1}{\lambda_2}}+\sqrt{\frac{\lambda_2}{\lambda_1}}\right)\;=\;\frac{\lambda_1+\lambda_2}{2\sqrt{\lambda_1\lambda_2}}\;=-\;\frac{F_{12}}{\sqrt{F_{11}\,F_{22}}}\,.$$

We infer from this that:

$$\sinh^2\,\theta=\frac{F_{12}^2-F_{11}\,F_{22}}{F_{11}\,F_{22}}\,,$$

and as a result:

$$\tanh^2\,\theta=\frac{F_{12}^2-F_{11}\,F_{22}}{F_{12}^2}\,,$$

Upon developing the expression for $\sinh^2\,\theta$, one will find that:

$\sinh^2\,\theta=$

$$\frac{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2-(y_1\,z_2-y_2\,z_1)^2-(z_2\,x_1-z_1\,x_2)^2-(x_1\,y_2-x_2\,y_1)^2}{(1-x_1^2-y_1^2-z_1^2)(1-x_2^2-y_2^2-z_2^2)}\,.$$

The expression for the Cayleyian line element is deduced from this immediately:

$$d\theta^2=\frac{dx^2+dy^2+dz^2-(y\,dz-z\,dy)^2-(z\,dx-x\,dz)^2-(x\,dy-y\,dx)^2}{(1-x^2-y^2-z^2)}\,.$$

These formulas solve the problem of the composition of pseudo-velocities in the most general case.

When the expression for $d\theta^2$ is assigned to a line element, it will lead to the expression for the element that replaces the angle in Cayleyian terminology.

The Cayleyian angle comes down between a logarithm of the anharmonic ratio, like the distance.

Let Δ and Δ' be two lines that intersect at a point P that is interior to the sphere. One can draw two conjugate imaginary tangents T and T' to the sphere through the point P in the plane $\Delta\Delta'$. The Cayleyian angle of the two lines is the product of $1 / 2i$ with the anharmonic ratio of the four lines T, T', Δ, Δ'. That definition, along with that of distance, is obviously inspired by the usual expression for the angle that was given by Laguerre in 1853 ([1]).

29. Cayleyian geometry and Lobachevskian geometry. – F. Klein ([2]) observed in 1871 that Cayleyian geometry provides a simple representation of the propositions of Lobachevski when one takes the elements to be the points and line segments that are interior to the fundamental sphere, and one evaluates the distance using Cayley's definition. It is interesting to point out how that representation is attached to the representation that was utilized by Poincaré in the theory of Fuchsian functions. The correspondence is extremely simple in the case of plane geometry. The fundamental sphere is then replaced by a circle (C) in the fixed plane considered Π.

Imagine a hemisphere (Σ) that is situated above the plane Π and passes through the fundamental circle (C). Any point P in the plane Π that is interior to the circle (C) is the projection of a point P' on the hemisphere. A line segment that is interior to the circle (C) is the projection of a semicircle in Σ. The imaginary tangents that issue from P to the circle (C) are the projections of the rectilinear generators of the sphere that passes through P'.

The anharmonic ratio of the sheaf that is defined by two arbitrary lines that pass through P and the two imaginary tangents to the circle (C) is equal to that of the sheaf of four homologous lines that pass through P' and are drawn in the plane tangent to the sphere at that point. It results immediately from this that the Cayleyian angle between two lines that pass through the point P is equal to the angle at which it cuts the semicircles of the hemisphere (Σ) on the sphere that correspond to those lines, respectively.

It remains to transform the expression for the Cayleyian distance. Let P_1 and P_2 be two points in the plane Π that are interior to the circle (C), and let M and M' be the two points where the line $P_1 P_2$ cuts the fundamental circle (C). Let ρ denote the radius of the semicircle that projects onto the line $M' P_1 P_2 M$, and let φ_1 and φ_2 respectively denote the angles that $M'M$ forms with the radii of that semicircle that ends at the points P_1', P_2' that are homologues of P_1 and P_2. The anharmonic ratio ($M' P_1 P_2 M$) is equal to:

([1]) LAGUERRE, "Sur la théorie des foyers," Nouvelles Annales de Mathématiques, **72** (1853), pp. 57.
([2]) F. KLEIN, Math. Ann. **4** (1871); *ibid.* **6** (1873); *ibid.* **7** (1874).

$$\frac{\rho(1-\cos\varphi_1)}{-\rho(1+\cos\varphi_1)} : \frac{\rho(1-\cos\varphi_2)}{-\rho(1+\cos\varphi_2)} = \frac{\tan^2\varphi_1/2}{\tan^2\varphi_2/2}.$$

The hyperbolic argument that we have denoted by θ and which represents the Cayleyian distance between two points is then equal to $\ln\dfrac{\tan^2\varphi_1/2}{\tan^2\varphi_2/2}$.

Now, if one joins the point M' to the points M, P_1', P_2' and completes that sheaf by adding the tangent to the semicircle at M' then the anharmonic ratio of the sheaf thus-constructed will be the constant anharmonic ratio of the four points M, M', P_1', P_2' of the semicircle.

The angular coefficients of the four lines considered are 0, ∞, $\tan\dfrac{\varphi_1}{2}$, $\tan\dfrac{\varphi_2}{2}$, respectively, so the anharmonic ratio has the value $\tan\dfrac{\varphi_1}{2} / \tan\dfrac{\varphi_2}{2}$, and one will have:

$$\theta = \ln \text{ anh. rat. } M\, M'' P_1'\, P_2'.$$

Now, perform a stereographic projection of the hemisphere (Σ) onto the plane Π. The semicircles whose planes are perpendicular to Π project along the arcs of circles that are normal to the circle (C). The points M_1, M_1' are preserved; P_1', P_2' project onto P_1'', P_2'', respectively. The anharmonic ratio ($M\, M'' P_1''\, P_2''$) on the arc of the circle of the stereographic projection is equal to the anharmonic ratio ($M\, M'' P_1'\, P_2'$) on the semicircle of the sphere (Σ). It is, consequently, equal to the square root of the anharmonic ratio of the four points along the straight line M_1, M_1', P_1, P_2 .

Since the angles are preserved by stereographic projection, the Euclidian angle between two arcs of a circle that is normal to the fundamental circle is equal to the Cayleyian angle between their chords.

One thus recovers the well-known representation of the fundamental elements of non-Euclidian geometry.

The use of the hemisphere (Σ) likewise provides a very simple image of the Lorentz contraction. Let P be a pole in the plane Π, and let P' be its image on (Σ). A small circle on Σ that has the point P' for its pole will project onto an ellipse in Π. The ratio of the axes of that ellipse is equal to the Lorentz contraction coefficient for the translational velocity that corresponds to the point P.

30. Extension to three-dimensional figures. – The extension of the preceding method to three-dimensional figures is extremely simple. The fundamental sphere:

$$(\Sigma) \qquad 1 - x^2 - y^2 - z^2 = 0$$

is the projection onto the space (x, y, z) of the four-dimensional semi-hypersphere (Σ) that is defined by the following equation and inequality:

$$\begin{cases} 1 - x^2 - y^2 - z^2 - w^2 = 0, \\ w > 0. \end{cases}$$

We can repeat the arguments that we applied to the three-dimensional figure that we considered before with that four-dimensional figure, and we will obtain the transformation of rectilinear segments into arcs of circles that are normal to the fundamental sphere.

If one considers the coordinates x, y, z to be the components of a pseudo-velocity in a system (S) then the fourth coordinate w will represent the derivative $d\sigma / du$ of the interval with respect to the radiation parameter. That fourth variable will then have a very precise physical significance.

Of course, the results that we just presented are not essentially new, but it seemed useful to me to recall them in order to accentuate the mathematical interest of the geometry of wave systems.

31. The Minkowski universe ([1])**.** – To conclude this brief study of the Lorentz transformation and the questions that are attached to it, I would like to add some observations on the Minkowski *universe.* One knows that Minkowski considered the three coordinates of space and time to be the elements of a four-dimensional multiplicity that he called the *universe.* The use of geometric language in order to describe sets in which the spatial coordinates and time vary simultaneous is not new. It presents itself advantageously in certain questions, and Hadamard (among others) recently made use of it in his *Leçons sur la propagation des ondes.* One can criticize the terminology employed; it is somewhat ridiculous to wish to represent the universe with the aid of only four variables. However, there are other more serious reproaches. Under the pretext that the four variables appear symmetrically in his formulas, Minkowski presumed to eliminate the physical distinction between those quantities, which is absurd. If one replaces u with si in the quadratic form:

$$- ds^2 = dx^2 + dx^2 + dz^2 - du^2$$

then it will become:

$$- ds^2 = dx^2 + dx^2 + dz^2 + ds^2.$$

That symmetric form has some advantages for certain calculations. In particular, the Lorentz transformations reduce to orthogonal transformations in four variables, which constitutes an interesting reconciliation.

Similarly, the equation of wave propagation takes the symmetric form of the usual potential equation:

([1]) "Die Grundgleichungen für die elektromagnetische Vorgänge in bewegten Körpern," Nachr. der K. Ges. d. Wiss. zu Göttingen, 1908.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial s^2} = 0.$$

However, one should not conceal the fact that this simplification is illusory in the interpretation of real phenomena. The analytical characters of real integrals are very different according to whether the characteristics are real or imaginary. Similarly, the distinction between real and imaginary is essential for Lorentz transformations.

In order to account for the character of the Minkowski transformation, one can compare it to the transformation of $x^2 - y^2$ into $x^2 + y^2$ that replaces an equilateral hyperbola with a circle.

That computational gimmick can be advantageous in certain cases, but it presents some inconveniences in other. However, it is only a gimmick, and the conclusions that one pretends to deduce from the standpoint of physical realities are absolutely unacceptable.

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CHAPTER IV

INTERFERENCE AND REFLECTION OF PLANE WAVES

32. – If plane waves are considered to issue from an infinitely-distant pole then one can refer them to two systems S and S' that are in uniform, translational motion with respect to each other indifferently. We say that S is fixed and S' is in motion.

An integral V of the form:

$$V = A \sin 2\pi \frac{lx + mv + nz + a - u}{\lambda},$$

when referred to the system S, will become:

$$V = A \sin 2\pi \frac{l'x' + m'v' + n'z' + a' - u'}{\lambda'}$$

under the Lorentz transformation when one sets:

$$(17) \qquad l' = \frac{l - \alpha}{1 - l\alpha}, \quad m' = \frac{m\beta}{1 - l\alpha}, \quad n' = \frac{n\beta}{1 - l\alpha}, \quad a' = \frac{a\beta}{1 - l\alpha}, \quad \lambda' = \frac{\lambda\beta}{1 - l\alpha}.$$

The equality $l^2 + m^2 + n^2 = 1$ implies that $l'^2 + m'^2 + n'^2 = 1$.

The first three of formulas (17) are equivalent to formulas (16).

One can compare these results with the condition for two plane wave trains to admit stationary interference planes in a given reference system.

Let two integrals of the same intensity and the same period in the system S be:

$$V = A \sin 2\pi \frac{lx + my + nz + a - u}{\lambda},$$

$$V_1 = A \sin 2\pi \frac{l_1 x + m_1 y + n_1 z + a_1 - u}{\lambda}.$$

The sum $V + V_1$ is annulled at all points of the planes that are represented by the equation:

$$(18) \qquad (l - l_1)\, x + (m - m_1)\, y + (n - n_1)\, z + (a - a_1) = \left(n + \tfrac{1}{2}\right)\lambda \qquad (n \text{ integer}).$$

Consequently, those planes are sheets of stationary interference in the system S.

In the system S', those two wave trains will no longer have the same period in general and will not have to define the sheets of stationary interference.

33. Two sheets of stationary interference in the moving system. – Let us look for the relations that must exist between the periods of two wave trains in the system S in order for them to interfere at S'. The Lorentz transformation yields the result immediately; meanwhile, it seems useful to me to do the calculation directly upon first supposing that S' is deduced from S by a simple translation in the ordinary sense of the word. We let λ and λ_1 denote the periods.

As a consequence, set:

$$x_1 = x - \alpha u$$

and keep the other variables the same.

The characteristic arguments of the integrals will become:

$$\frac{lx_1 + my + nz + a - u(1 - l_1\alpha)}{\lambda},$$

$$\frac{l_1 x_1 + m_1 y + n_1 z + a_1 - u(1 - l_1\alpha)}{\lambda_1}.$$

In order for the radiation parameter u to disappear in the difference between the arguments, it is necessary and sufficient that one must have:

$$\frac{\lambda}{1 - l\alpha} = \frac{\lambda_1}{1 - l_1\alpha}. \tag{19}$$

When that result is compared with the last of formulas (17), it will express the idea that the transformed periods of λ and λ_1 under the Lorentz transformation are equal to each other.

If that condition is realized then the stationary interference planes of the two wave trains, when observed in the system S', will be parallel to the plane:

$$\frac{lx_1 + my + nz}{1 - l\alpha} - \frac{l_1 x_1 + m_1 y + n_1 z}{1 - l_1\alpha} = 0. \tag{20}$$

Equation (20) simplifies by the introduction of the Lorentz transformation.

Indeed, an immediate calculation will convert this into the form:

$$(l' - l_1')\, x' + (m' - m_1')\, y' + (n' + n_1')\, z' = 0. \tag{21}$$

It remains to interpret that result geometrically. If one supposes that the interference waves are true spheres in the system S then the planes of the interference sheets (18) will be parallel to one of the planes that bisect the planes of the plane waves considered.

However, in that case, the interference waves of the system (S') will be ellipsoids; the equation:

$$x'^2 + y'^2 + z'^2 = 1$$

represents an ellipsoid, and equation (21) must be interpreted as a consequence.

The bisecting planes of a dihedral are the planes that are both harmonically conjugate with respect to the faces of the dihedral and with respect to a sphere that has its center on the edge of dihedral.

If the directrix sphere is replaced with an interference ellipsoid then one will find oneself in the case of the moving system S'. Therefore:

The stationary interference planes in the moving system S' are parallel to one of the two planes that are conjugate in direction with respect to the planes of the wave fronts and with respect to the interference ellipsoid.

The expression for the wave front must be made more precise. The use of the coefficients l', m', n' implies that one is dealing with planes that correspond to the hypothesis u' = const., and not to the hypothesis u = const. It is therefore the radiation parameter of the moving system that must enter into the determination of those planes.

The preceding argument does not specify which of the conjugate planes is the one that corresponds to the direction of the stationary interference plane. The process of eliminating u that we employed shows that the plane considered is the one onto which the propagation velocities of the two waves project in the same direction.

34. Rays of propagation. Aberration. – Hadamard has attached the notion of *rays* of propagation to that of *bicharacteristics*, i.e., one-dimensional lines or multiplicities along which the characteristic multiplicities touch their envelope.

In no. **18**, we gave the partial differential equation for the characteristic multiplicities in the system of variables u, x, y, z.

The differential equations for the bicharacteristics that correspond to a given integral are:

$$\frac{dx}{\left(\frac{\partial\varphi}{\partial x}\right)} = \frac{dy}{\left(\frac{\partial\varphi}{\partial y}\right)} = \frac{dz}{\left(\frac{\partial\varphi}{\partial z}\right)} = \frac{-du}{\left(\frac{\partial\varphi}{\partial u}\right)} .$$

In the particular case of plane waves:

$$lx + my + nz - u = \text{const.},$$

one has:

$$\frac{dx}{m} = \frac{dy}{n} = \frac{dz}{n} = \frac{du}{1} .$$

These equations can be applied to the systems of fixed or moving coordinates indifferently, provided that the variable u denotes the radiation parameter relative to the system considered, and that one takes care to interpret them while taking into account the form of the interference ellipsoid.

They express the idea that the ray is parallel to the diameter that is conjugate to the plane wave in the interference ellipsoid. That is what we call the *pseudo-normal direction* to the plane.

The correspondence between the directions of the rays with respect to the two systems of axes in uniform, translational motion with respect to each other will result immediately from formulas (17) when one employs Lorentz variables in the two systems.

A ray of propagation in the first one:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} = \frac{u - u_0}{1}$$

corresponds to a homologous ray in the second one:

$$\frac{x' - x'_0}{l'} = \frac{y' - y'_0}{m'} = \frac{z' - z'_0}{n'} = \frac{u' - u'_0}{1},$$

in which the denominators of the two groups are coupled by formulas (17).

The first three ratios of each system define the stationary spatial direction of the ray, when expressed in Lorentz variables.

Consequently, those equations solve the problem of aberration in the most general case.

35. Reflection. – The results obtained for stationary interference phenomena apply just the same to the reflection of waves from a plane mirror that displaces in the medium considered with a uniform, translational motion.

We must necessarily suppose that the motion of the mirror alters neither the homogeneity nor the isotropy of the mirror. The plane of the mirror is then a stationary interference plane for the incident and reflected waves.

The law of reflection is deduced immediately from that remark. The direction of the plane of reflection corresponds to a *pseudo-normal*, as defined above.

The two rays are harmonic conjugate with respect to the pseudo-normal and the line of intersection of their plane with the plane of the mirror. Therefore, the interference ellipsoid will again replace the sphere in the study of the phenomenon of reflection. In order for the light ray to reflect into its own direction, it is necessary that it must be directed along the pseudo-normal to the plane of the mirror.

One can make an interesting remark on this subject. In the system S', which participates in the translational motion of the mirror, the incident and reflected rays have the same wave length. On the contrary, the wave lengths in the system S generally differ from each other. Even in the case where the two rays are superimposed in S', the two rays of the system S will have different periods. For example, if one is dealing with light vibrations then the incident and reflected rays of the system S will not have the same color. Furthermore, it is convenient to add that the rays considered are not stationary in the system S. In reality, our calculations of the periods apply to the two systems of plane waves of the system S that correspond to the directions of the incident and reflected waves in the system S'.

36. Reflection and interference of elementary polar waves. – One will arrive at similar conclusions in a moving system when one studies the reflection of an elementary that issues from a simple pole or the interference of two identical elementary waves that issue from the different poles that are linked to the same system. In those questions, it is the elliptic modulus of the wave (no. **12**) that replaces the distance and the pseudo-normal that replaces the normal.

The formulas that relate to stationary interference phenomena or reflection are generally obtained by the elimination of time. However, from the form itself of the integral, the elimination of t will imply that of the radiation parameter u. The calculations and the results will then keep the same form in the case of rest or motion, as long as one takes into account the physical significance of the modulus.

37. – In all of the preceding, I have confined myself to deducing the mathematical consequence of the equation of wave propagation without formulating any new hypothesis or introducing any new principle.

We have been able to state that for the observations that are performed in a moving reference system, it is always the interference ellipsoid that plays the role of directing element.

Time enters only into the radiation parameter, where it generally appears to degree one in the position coordinates, but with a coefficient that can vary a great deal from one phenomenon to another.

Our calculations are not directed towards light phenomena, in particular; they extend to all vibratory phenomena that propagate in an isotropic medium.

The formulas that are obtained are analogous to the ones that one pretends to establish at the basis for the principle of special relativity. We then showed that this principle is useless for the study of optical and electromagnetic phenomena, since the same results can be obtained much more neatly and precisely by the simple analytical study of the wave equation.

On the other hand, since the extension of Einstein's principle beyond that category of phenomena constitutes an abuse that is based upon an ambiguity, the very neat conclusion to which we arrive will be the following one:

The principle of special relativity, in the Einstein sense, sometimes constitutes a pointless redundancy (superfétation) and sometimes an absurdity, according to the domain to which it is applied.

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