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ON THE EQUILIBRIUM

OF

FLEXIBLE, INEXTENSIBLE SURFACES

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CONTENTS

INTRODUCTION

The conditions of equilibrium for a funicular curve – i.e., a flexible and inextensible filament whose elements are subjected to forces that form a continuous system – are discussed in all of the treatises on mechanics. Let ds denote the length of an arbitrary element, let $P\,ds$ denote the force to which it is subjected, let P_n and P_t be the normal and tangential components of R, resp., and finally let T denote the tensions that element feels from the neighboring elements. One finds the two equations:

$$P_t + \frac{dT}{ds} = 0,$$
$$P_n + \frac{T}{\rho} = 0,$$

with no difficulty. $1 / r$ is the curvature of the point considered. Moreover, one has that the osculating plane of the curve passes through the direction of P.

Upon referring the curve to three fixed rectangular axes and calling the components of the force parallel to those three axes X, Y, Z, one will get the following three equations, which will likewise solve the problem:

$$d\left(T\frac{dx}{ds}\right) + X\,ds = 0,$$

$$d\left(T\frac{dy}{ds}\right) + Y\,ds = 0,$$

$$d\left(T\frac{dz}{ds}\right) + Z\,ds = 0.$$

When the forces are given in magnitude, direction, and sense, those three equations will determine the form of the curve and the tension at each point. On the contrary, if one supposes that the form of the curve is given then they will yield two conditions that the forces must satisfy and permit one to calculate the tensions that result from the application of forces that satisfy those two conditions, in addition.

It is natural to seek the generalization of the problem by studying what happens when each of the points of a perfectly flexible and inextensible surface is subjected to forces that have the same order of magnitude as the corresponding elements. In particular, one can demand to know:

1. What are equilibrium conditions for a surface that is subject to well-defined forces?

2. What are the laws by which one develops the efforts of tension between the various elements of a surface in equilibrium?

3. What is the deformation that a given surface will submit to under the action of forces that do not satisfy the equilibrium conditions?

After having done considerable research in order to know the work that might have given rise to those questions, I have acquired the conviction that no one has addressed them further, and that their novelty has made me decide to dedicate the present work to them. Without pretending to exhaust a subject that is as extensive as it is difficult, I will be content if I have succeeded in laying the milestones of a theory that deserves to take its place in science.

I have no need to recall here the beautiful work that has given rise to the geometric properties of surfaces; it will naturally provide me with an ongoing basis. The theory of deformation of surfaces, which has become classical today, thanks to the discoveries of Ossian Bonnet, Bour, Codazzi, etc. has been particularly useful to me.

CHAPTER I

GENERAL CONSIDERATIONS ON THE THEORY OF SURFACES

We believe that we must first establish the fundamental formulas of the theory of surfaces as rapidly as possible. Such a summary of a well-known theory will provide us with the occasion to fix a certain number of notations, as well as to make some remarks that will be useful in what follows, in order to not return to them.

An arbitrary surface can be defined by expressing the Cartesian coordinates x, y, z as functions of two arbitrary parameters λ, μ:

$$x = \varphi_1(\lambda, \mu), \qquad y = \varphi_2(\lambda, \mu), \qquad z = \varphi_3(\lambda, \mu) . \tag{1}$$

The curves λ = const., μ = const. draw a net on the surface that will be orthogonal if one imposes the condition:

$$\frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} + \frac{\partial y}{\partial \lambda}\frac{\partial y}{\partial \mu} + \frac{\partial z}{\partial \lambda}\frac{\partial z}{\partial \mu} = 0,$$

which we write in the abbreviated form:

$$\sum \frac{\partial x}{\partial \lambda}\frac{\partial x}{\partial \mu} = 0. \tag{2}$$

We let L, M denote the metric coefficients that correspond to the two surfaces μ = const., λ = const., resp.; i.e., $L\,d\lambda$, $M\,d\mu$ represent the arc lengths of the two curves that are found between the curves λ, $\lambda + d\lambda$ and μ, $\mu + d\mu$. We denote the radii of geodesic curvature of the two curves by ρ_1, ρ_2, which are supposed to be positive when they point in the same sense as the positive arc lengths $M\,d\mu$, $L\,d\lambda$, resp. R_1, R_2 denote the radii of normal curvature, and T_1, T_2 denote the radii of geodesic torsion. The positive sense of R_1, R_2, T_1, T_2 will be, in a certain sense, chosen arbitrarily on the normal.

Having said that, one has:

$$L^2 = \sum \left(\frac{\partial x}{\partial \lambda}\right)^2, \qquad M^2 = \sum \left(\frac{\partial x}{\partial \mu}\right)^2. \tag{3}$$

On the other hand, some very simple geometric considerations will give:

$$\frac{L^2 M}{\rho_1} = \sum \frac{\partial x}{\partial \mu}\frac{\partial^2 x}{\partial \lambda^2}, \qquad \frac{L^2 M}{\rho_2} = \sum \frac{\partial x}{\partial \lambda}\frac{\partial^2 x}{\partial \mu^2}. \tag{4}$$

Let X, Y, Z be the direction cosines of the normal, which are defined by the equations:

$$LMX = \frac{\partial y}{\partial \lambda}\frac{\partial z}{\partial \mu} - \frac{\partial y}{\partial \mu}\frac{\partial z}{\partial \lambda},$$

$$LMY = \frac{\partial z}{\partial \lambda}\frac{\partial x}{\partial \mu} - \frac{\partial z}{\partial \mu}\frac{\partial x}{\partial \lambda},$$

$$LMZ = \frac{\partial x}{\partial \lambda}\frac{\partial y}{\partial \mu} - \frac{\partial x}{\partial \mu}\frac{\partial y}{\partial \lambda}.$$

One effortlessly finds that:

(5) $$\frac{L^2}{R_1} = \sum X \frac{\partial^2 x}{\partial \lambda^2}, \qquad \frac{LM}{T_1} = \frac{LM}{T_2} = \sum X \frac{\partial^2 x}{\partial \lambda \partial \mu}, \qquad \frac{M^2}{R_2} = \sum X \frac{\partial^2 x}{\partial \mu^2},$$

so $T_1 = T_2$. We let T represent the common value that is given by $\frac{LM}{T} = \sum X \frac{\partial^2 x}{\partial \lambda \partial \mu}$.

In order for that formula to be exact in sign, it will suffice to agree that when T is positive, the positive sense of the normal at the point (λ, μ) will make an acute angle with the positive sense of the tangent to the curve λ = const. at the point $(\lambda, \mu + d\mu)$.

The groups (3), (4), (5) give seven equations that give us L, M, ρ_1, ρ_2, R_1, R_2, T as functions of x, y, z. If one combines them with equation (2) then one will get a system of eight equations, from which, one can eliminate x, y, z, which will lead to five distinct equations between L, M, ρ_1, ρ_2, R_1, R_2, T. We shall construct these five equations.

Equation (2) gives:

$$\sum \frac{\partial x}{\partial \mu}\frac{\partial^2 x}{\partial \lambda^2} + \sum \frac{\partial x}{\partial \lambda}\frac{\partial^2 x}{\partial \mu^2} = 0.$$

As a result:

$$\frac{L^2 M}{\rho_1} = -\sum \frac{\partial x}{\partial \lambda}\frac{\partial^2 x}{\partial \lambda \partial \mu}.$$

Moreover, one infers from the first of equations (3) that:

$$L\frac{\partial L}{\partial \mu} = \sum \frac{\partial x}{\partial \lambda}\frac{\partial^2 x}{\partial \lambda \partial \mu}.$$

Hence:

$$\frac{L^2 M}{\rho_1} = -\,L\frac{\partial L}{\partial \mu},$$

and consequently:

$$\frac{1}{\rho_1} = -\,\frac{\partial L / \partial \mu}{LM};$$

similarly:

$$\frac{1}{\rho_2} = -\frac{\partial L / \partial \lambda}{LM}. \tag{6}$$

The definitions of X, Y, Z lead immediately to the relations:

$$X^2 + Y^2 + Z^2 = 1,$$

$$\sum X \frac{\partial x}{\partial \lambda} = 0, \quad \sum X \frac{\partial x}{\partial \mu} = 0.$$

One deduces from this that:

$$\sum X \frac{\partial x}{\partial \lambda} = 0,$$

$$\sum \frac{\partial x}{\partial \lambda} \frac{\partial X}{\partial \lambda} = -\sum X \frac{\partial^2 x}{\partial \lambda^2} = \frac{-L^2}{R_1},$$

$$\sum \frac{\partial x}{\partial \mu} \frac{\partial X}{\partial \lambda} = -\sum X \frac{\partial^2 x}{\partial \lambda \partial \mu} = \frac{-LM}{T}.$$

If one considers $\frac{\partial X}{\partial \lambda}$, $\frac{\partial Y}{\partial \lambda}$, $\frac{\partial Z}{\partial \lambda}$ to be the unknowns in these three equations, and if one remarks that:

$$\begin{vmatrix} X & Y & Z \\ \frac{\partial x}{\partial \lambda} & \frac{\partial y}{\partial \lambda} & \frac{\partial z}{\partial \lambda} \\ \frac{\partial x}{\partial \mu} & \frac{\partial y}{\partial \mu} & \frac{\partial z}{\partial \mu} \end{vmatrix} = LM$$

then one will get:

$$\frac{\partial X}{\partial \lambda} = -\frac{1}{R_1} \frac{\partial x}{\partial \lambda} - \frac{1}{T} \frac{L}{M} \frac{\partial x}{\partial \mu}.$$

One will likewise have:

$$\frac{\partial X}{\partial \mu} = -\frac{1}{R_2} \frac{\partial x}{\partial \mu} - \frac{1}{T} \frac{M}{L} \frac{\partial x}{\partial \lambda}.$$

Differentiate the value of $\partial X / \partial \lambda$ with respect to μ and then differentiate the value of $\partial X / \partial \mu$ by λ, and equate the two results that one obtains; one will find that:

$$\frac{\partial^2 x}{\partial \lambda \partial \mu} \left(\frac{1}{R_2} - \frac{1}{R_1} \right) - \frac{\partial x}{\partial \lambda} \frac{\partial}{\partial \mu} \left(\frac{1}{R_1} \right) + \frac{\partial x}{\partial \mu} \frac{\partial}{\partial \lambda} \left(\frac{1}{R_2} \right)$$

$$-\frac{1}{T}\left(\frac{L}{M}\frac{\partial^2 x}{\partial\mu^2}-\frac{M}{L}\frac{\partial^2 x}{\partial\lambda^2}\right)-\frac{\partial x}{\partial\mu}\frac{\partial}{\partial\mu}\left(\frac{L}{MT}\right)+\frac{\partial x}{\partial\lambda}\frac{\partial}{\partial\lambda}\left(\frac{M}{LT}\right)=0.$$

Multiply that equation by $\partial x / \partial\lambda$, and combine it with similar equations in y and z. Upon taking into account the relations:

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda\,\partial\mu}=-\frac{L^2 M}{\rho_1},$$

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\mu^2}=\frac{M^2 L}{\rho_2},$$

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda^2}=L\frac{\partial L}{\partial\lambda},$$

moreover, then we will arrive at the equation:

(7)
$$\frac{1}{M}\frac{\partial(1/R_1)}{\partial\mu}-\frac{1}{L}\frac{\partial(1/T)}{\partial\lambda}+\frac{1}{\rho_1}\left(\frac{1}{R_2}-\frac{1}{R_1}\right)+\frac{2}{\rho_2}\frac{1}{T}=0.$$

One will likewise have:

$$\frac{1}{L}\frac{\partial(1/R_2)}{\partial\lambda}-\frac{1}{M}\frac{\partial(1/T)}{\partial\mu}+\frac{1}{\rho_2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{2}{\rho_1}\frac{1}{T}=0.$$

The groups (6) and (7) already provide four of the desired equations. In order to get the last one, calculate $\partial^2 x / \partial\lambda^2$ by means of the equations:

$$\sum X\frac{\partial^2 x}{\partial\lambda^2}=\frac{L^2}{R_1},$$

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda^2}=L\frac{\partial L}{\partial\mu},$$

$$\sum\frac{\partial x}{\partial\mu}\frac{\partial^2 x}{\partial\lambda^2}=\frac{L^2 M}{\rho_1};$$

one gets:

$$\frac{\partial^2 x}{\partial\lambda^2}=\frac{L^2}{R_1}X+\frac{1}{L}\frac{\partial L}{\partial\lambda}\frac{\partial x}{\partial\lambda}+\frac{L^2}{M\,\rho_1}\frac{\partial x}{\partial\mu}.$$

Take the derivative with respect to μ, multiply by $\partial x / \partial\mu$ and add that to the other two analogous expressions; one will find that:

$$\sum \frac{\partial x}{\partial\mu}\frac{\partial^3 x}{\partial\lambda^2\,\partial\mu} = -\frac{L^2}{R_1}\frac{M^2}{R_2} + \frac{M}{L}\frac{\partial L}{\partial\lambda}\frac{\partial M}{\partial\lambda} + \frac{L^2}{\rho_1}\frac{\partial M}{\partial\mu} + \frac{M}{L}\frac{\partial}{\partial\mu}\left(\frac{L^2}{M\,\rho_1}\right).$$

On the other hand, one has the equations:

$$\sum X \frac{\partial^2 x}{\partial\lambda\,\partial\mu} = \frac{LM}{T},$$

$$\sum \frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\lambda\,\partial\mu} = L\frac{\partial L}{\partial\mu},$$

$$\sum \frac{\partial x}{\partial\mu}\frac{\partial^2 x}{\partial\lambda\,\partial\mu} = -\sum \frac{\partial x}{\partial\lambda}\frac{\partial^2 x}{\partial\mu^2} = -\frac{LM^2}{\rho_2},$$

and as a result, upon operating as above:

$$\sum \frac{\partial x}{\partial\mu}\frac{\partial^3 x}{\partial\lambda^2\,\partial\mu} = -\frac{L^2 M^2}{T} + \frac{LM}{\rho_1}\frac{\partial L}{\partial\mu} - \frac{ML}{\rho_1}\frac{\partial M}{\partial\lambda} - M^2\frac{\partial}{\partial\mu}\left(\frac{L}{\rho_2}\right).$$

Upon equating the two values of $\sum \frac{\partial x}{\partial\mu}\frac{\partial^3 x}{\partial\lambda^2\,\partial\mu}$ and making some reductions, one will arrive at the desired equation:

$$\frac{1}{R_1 R_2} - \frac{1}{T^2} + \left(\frac{1}{\rho_1}\right)^2 + \left(\frac{1}{\rho_2}\right)^2 - \frac{1}{M}\frac{\partial(1/\rho_1)}{\partial\mu} - \frac{1}{L}\frac{\partial(1/\rho_2)}{\partial\lambda} = 0.$$

The fundamental relations that exist between the elements of the surface are then:

$$(8)\qquad \left\{\begin{aligned}
&\frac{1}{L}\frac{\partial(1/R_2)}{\partial\lambda} - \frac{1}{M}\frac{\partial(1/T)}{\partial\mu} + \frac{1}{\rho_1}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \frac{2}{\rho_2}\frac{1}{T} = 0,\\
&\frac{1}{M}\frac{\partial(1/R_1)}{\partial\mu} - \frac{1}{L}\frac{\partial(1/T)}{\partial\lambda} + \frac{1}{\rho_1}\left(\frac{1}{R_2} - \frac{1}{R_1}\right) + \frac{2}{\rho_2}\frac{1}{T} = 0,\\
&\frac{1}{T^2} - \frac{1}{R_1 R_2} + \frac{1}{M}\frac{\partial(1/\rho_1)}{\partial\mu} + \frac{1}{L}\frac{\partial(1/\rho_2)}{\partial\lambda} - \frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} = 0,\\
&\frac{1}{\rho_1} = -\frac{\partial L/\partial\mu}{LM}, \qquad \frac{1}{\rho_2} = -\frac{\partial M/\partial\lambda}{LM}.
\end{aligned}\right.$$

These equations show immediately that under the deformation of an inextensible surface, by definition, the coefficients L and M will remain invariable functions of λ and μ, and that the same thing will be true for the geodesic and total curvatures $\frac{1}{R_1 R_2} - \frac{1}{T^2}$. When one proposes to find all of the surfaces that are deduced from each other by a deformation without extension, it will suffice to consider L, M, ρ_1, ρ_2 to be given functions and R_1, R_2, T to be unknowns. The first three equations (8) will then determine those unknowns, and all of the difficulty will come down to the integration of those equations ([1]).

In the rest of this article, we will then be led to envision the infinitely-small deformations of a given surface in a special fashion. If we let l, m, θ denote the unknown variations that result for $1 / R_1$, $1 / R_2$, and $1 / T$ under such a deformation then the equations of the problem will become:

$$(9) \quad \begin{cases} \dfrac{1}{L}\dfrac{\partial m}{\partial \lambda} - \dfrac{1}{M}\dfrac{\partial \theta}{\partial \mu} + \dfrac{l-m}{\rho_2} + \dfrac{2\theta}{\rho_1} = 0, \\ \dfrac{1}{M}\dfrac{\partial l}{\partial \mu} - \dfrac{1}{L}\dfrac{\partial \theta}{\partial \lambda} + \dfrac{m-l}{\rho_1} + \dfrac{2\theta}{\rho_2} = 0, \\ \dfrac{l}{R_2} + \dfrac{m}{R_1} - \dfrac{2\theta}{T} = 0. \end{cases}$$

These equations are much simpler than equations (8), because they are linear in l, m, θ. It can then happen that they are integrable without the same thing being true for equations (8).

If one sets:

$$a = \frac{T}{2R_2}, \qquad b = \frac{T}{2R_1},$$

to abbreviate, then the third equation in (9) will become:

$$\theta = al + bm.$$

When that value of θ is substituted in the first two equations, that will lead to the following system:

$$(10) \quad \begin{cases} -\dfrac{a}{L}\dfrac{\partial l}{\partial \lambda} + \dfrac{1}{M}\dfrac{\partial l}{\partial \mu} - \dfrac{b}{L}\dfrac{\partial m}{\partial \lambda} + \dfrac{m-l}{\rho_1} + \dfrac{2}{\rho_2}(al+bm) - \dfrac{1}{L}\left(l\dfrac{\partial a}{\partial \lambda} + m\dfrac{\partial a}{\partial \lambda}\right) = 0, \\ -\dfrac{a}{M}\dfrac{\partial l}{\partial \mu} + \dfrac{1}{L}\dfrac{\partial m}{\partial \lambda} - \dfrac{b}{M}\dfrac{\partial m}{\partial \mu} + \dfrac{l-m}{\rho_2} + \dfrac{2}{\rho_1}(al+bm) - \dfrac{1}{M}\left(l\dfrac{\partial a}{\partial \mu} + m\dfrac{\partial a}{\partial \mu}\right) = 0. \end{cases}$$

([1]) Ossian Bonnet has proved (Journal de l'École Polytechnique, Cahier **42**, pp. 35) that seven functions L, M, ρ_1, ρ_2, R_1, R_2, T that satisfy equations (8) will always determine one and only surface.

Choose the values of l and m arbitrarily for all points of a curve $f(\lambda, \mu) = 0$. The two equations:

$$\frac{\partial l}{\partial \lambda} d\lambda + \frac{\partial l}{\partial \mu} d\mu - dl = 0,$$

$$\frac{\partial m}{\partial \lambda} d\lambda + \frac{\partial m}{\partial \mu} d\mu - dm = 0,$$

when combined with the previous two, will determine the values of $\frac{\partial l}{\partial \lambda}, \frac{\partial l}{\partial \mu}, \frac{\partial m}{\partial \lambda}, \frac{\partial m}{\partial \mu}$, in general. One will then know the values of the unknowns for the points of a curve that is infinitely close to the first one, and upon proceeding in that way, step-by-step, one will find the deformation of any surface.

However, when the determinant of the four equations is zero, i.e., when:

$$\begin{vmatrix} -\frac{a}{L} & \frac{1}{M} & -\frac{b}{L} & 0 \\ 0 & -\frac{a}{L} & \frac{1}{L} & -\frac{b}{M} \\ d\lambda & d\mu & 0 & 0 \\ 0 & 0 & d\lambda & d\mu \end{vmatrix} = 0,$$

the four equations will be incompatible, unless each of them is not a consequence of the other three. Upon developing the determinant, one will find that:

$$b\, L^2\, d\lambda^2 + LM\, d\lambda\, d\mu + a\, M^2\, d\mu^2 = 0,$$

or

$$\frac{L^2}{R_1} d\lambda^2 + \frac{2LM}{T} d\lambda\, d\mu + \frac{M^2}{R_2} d\mu^2 = 0.$$

That equation is the one for asymptotic lines.

One can then say that:

The asymptotic lines of a surface are the characteristics of its infinitely-small deformations.

For an arbitrary line that is traced on the surface, one can give the deformation that it is subject to arbitrarily; on the contrary, an asymptotic line can be deformed only according to a well-defined law.

One will find that law easily by supposing that the curves λ = const. constitute one of the systems of asymptotic lines. One will then have $1 / R_2 = 0$, and as a result, a will be zero, along with its derivatives (we exclude the case of developable surfaces here, for which $1 / R_2 = 0$ would imply that $1 / T = 0$). Set $a = 0$ in equations (10) and eliminate $\partial m / \partial\lambda$; one will then have:

$$(11)\qquad \frac{1}{M}\frac{\partial l}{\partial\mu} - \frac{b^2}{M}\frac{\partial m}{\partial\mu} + \left(\frac{b}{\rho_2} - \frac{1}{\rho_1}\right) l + \left[\frac{1}{\rho_1} - \frac{b}{\rho_2} + 2b\left(\frac{1}{\rho_2} + \frac{b}{\rho_1}\right) - \frac{1}{L}\frac{\partial b}{\partial\lambda} - \frac{b}{M}\frac{\partial b}{\partial\mu}\right] m = 0.$$

In order to satisfy that equation, one can choose the values of m arbitrarily; l will then be determined by a first-order linear equation. However, in the general case, it is impossible to find how l and m vary when one passes from one characteristic to an infinitely-close one. In order to do that, one must obtain the values of l and m that satisfy equation (11) and one of equations (10); for example:

$$(12)\qquad \frac{1}{L}\frac{\partial m}{\partial\lambda} - \frac{b}{M}\frac{\partial m}{\partial\lambda} + \frac{l - m}{\rho_2} + \left(\frac{2b}{\rho_1} - \frac{1}{M}\frac{\partial b}{\partial\mu}\right) m = 0.$$

Here, we point out an interesting property of the asymptotic line l = const. If one supposes that one has $m = 0$ for that line then θ will also be annulled, since $a = 0$. However, l can vary while remaining subject to only the condition:

$$\frac{1}{M}\frac{\partial l}{\partial\mu} + \left(\frac{b}{\rho_2} - \frac{1}{\rho_1}\right) l = 0.$$

The equations $m = 0$ and $\theta = 0$, when combined with the one that expresses the constancy of the geodesic curvature, say that the asymptotic line considered will experience no deformation. Hence:

One can subject a surface to an infinitely-small deformation such that the figure of a given asymptotic line remains invariable.

That property belongs to only the asymptotic lines. Indeed, the first curvature $1 / R_2$ of an arbitrary line λ = const. has a projection onto the tangent plane that is the geodesic curvature, which will always remain independent of the deformation, since the first curvature can be invariable only if the angle that is defined by the osculating plane and the tangent plane is not modified. Under those conditions, the normal curvature, which is equal to the projection of the first curvature onto the normal plane to the surface, will no longer change. On the other hand, the angle $d\alpha$ between two consecutive osculating planes is equal to the sum of the geodesic torsion $d\beta$ and the increment $d\gamma$ in the angle γ that is formed between the osculating plane and the corresponding tangent plane. When the curve is not deformed, $d\alpha$ and $d\gamma$ will remain invariable. The same thing will then be true for $d\beta$, and consequently, the geodesic torsion $1 / T$. Finally, the constancy of the quantities $1 / R_2$, $1 / T$ will imply that of $1 / R_1$.

One will then have $l = 0$, $m = 0$, $\theta = 0$ for all points of the surface considered, and as a result, from what we have seen, those quantities will be zero for all points on the surface.

When the condition $m = 0$ is applied to an asymptotic line, from the preceding, that condition will be sufficient for that line to remain invariable. Moreover, it expresses the idea that the normal curvature remains zero, and as a result one can say that:

When an asymptotic line has an asymptotic line for its transform, those two lines will be identical.

The proposition, which was proved for an infinitely-small deformation, will obviously extend to an arbitrary finite deformation, provided that the line is asymptotic in all of intermediate states.

Conversely:

If an asymptotic line is not deformed then it will necessarily remain asymptotic.

That is because if it is not asymptotic in the final state then one cannot pass to the initial state with deforming it. That is inadmissible.

If one imposes that the condition that all of the asymptotic lines of a system λ = const. remain asymptotic under the deformation then equation (12), in which one has set $m = 0$, will give $l = 0$, as long as $1 / \rho_2$ is non-zero; i.e., unless, the asymptotic lines are not rectilinear generators. The quantities $1 / R_1$, $1 / R_2$, and as a result, $1 / T$, will then be invariable, and the surface will not be deformable.

The proof is no more difficult in the case of a finite deformation. When the system λ = const. is asymptotic, the total curvature will reduce to $- 1 / T$, and consequently, if the system remains asymptotic then $1 / T$ will be invariable. Furthermore, since $1 / R_2 = 0$, the first of equations (8) will give $1 / R_1$ = const., unless $1 / \rho_2$ is non-zero. Thus:

Two surfaces that can be mapped to each other will coincide when the non-rectilinear asymptotic lines of one of the systems for one of the surfaces has the asymptotic lines of one of the systems for the other surface for its transform.

That theorem is due to Ossian Bonnet (Journal de l'École Polytechnique, Cahier **42**, pp. 44).

We shall not dwell further upon these purely-geometric considerations, which we have presented only because of the cardinal role of asymptotic lines in the problem of the equilibrium of surfaces. In a paper that was read to the Royal Irish Academy on 23 May 1853, Professor Jellett studied the deformation of surfaces from an entirely analogous viewpoint. The report (Proceedings of the Royal Irish Academy, vol. V, pp. 441) contains only the conclusions of that paper, which are the following ones:

I. – CONVEX SURFACES (*oval surfaces*)

If a curve or a portion of a curve that is traced on an inextensible convex surface is kept fixed then all of the surface will likewise remain fixed.

II. – DEVELOPABLE SURFACES

1. *If an arc of a curve that is traced on the surface and belongs to neither the edge of regression nor a rectilinear generator is kept fixed then the entire portion of the surface that is found between the edge of regression and the rectilinear generators that pass through the endpoints of the arc of the curve will likewise remain fixed.*

2. *The edge of regression or a rectilinear generator can generally be made fixed without making any part of the surface fixed.*

3. *The rectilinear generators of a developable surface are rigid.*

Observe here that the latter theorem is true only under certain limits on the deformation, since if one considers a portion of the developable surface that does not contain any part of the edge of regression then one can develop it onto a plane, and then unroll it onto another developable surface. It is obvious that after that deformation, the new rectilinear generators will not generally correspond to the original ones.

III. – CONCAVO-CONVEX SURFACES

Professor Jellett gave the name of *curves of flexure* to the asymptotic lines, which are then real, and stated these three theorems:

1. *If an arc of a curve that is not a curve of flexure is kept fixed on the surface, and if one draws two curves of flexure through each end point of that arc then the quadrilateral that is found between the four curves of flexure thus formed will remain fixed.*

2. *One can fix a curve of flexure without forbidding the deformation of any finite portion of the surface.*

3. *If two arcs of a curve of flexure that start from the same point are kept fixed then the quadrilateral that is found between those two arcs and the other two curves of flexure that are drawn through their second end points will likewise remain fixed.*

The proof of all these theorems is deduced effortlessly from the formulas that were established before. Furthermore, we shall have to prove some propositions of the same type on the subject of the equilibrium state.

When one seeks to treat the case of an infinitely-small deformation, instead of normal curvatures and geodesic torsion, one can consider the displacement that each point of the surface experiences and take the projections of that displacement onto three given rectangular axes to be the unknowns. The first-order differential coefficients satisfy the equations:

$$\sum\left(\frac{\partial x}{\partial \lambda}\right)^2 = L^2,$$

$$\sum\left(\frac{\partial x}{\partial\mu}\right)^2 = M^2,$$

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial x}{\partial\mu} = 0$$

before and after the deformation. If ξ, η, ζ then denote the infinitely-small variations of x, y, z then one will have:

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial\xi}{\partial\lambda} = 0,$$

$$\sum\frac{\partial x}{\partial\mu}\frac{\partial\xi}{\partial\mu} = 0,$$

$$\sum\frac{\partial x}{\partial\lambda}\frac{\partial\xi}{\partial\mu}+\sum\frac{\partial x}{\partial\mu}\frac{\partial\xi}{\partial\lambda} = 0.$$

Make the axes Ox, Oy, Oz coincide with the tangents to the curves μ = const., λ = const., and the normal to the surface at the point (x, y, z), respectively. The three equations reduce to:

$$\frac{\partial\xi}{\partial\lambda} = 0, \qquad \frac{\partial\eta}{\partial\mu} = 0, \qquad L\frac{\partial\xi}{\partial\mu}+M\frac{\partial\eta}{\partial\lambda} = 0.$$

Let h_1, h_2, k be the projections of the displacement (ξ, η, ζ) onto the two tangents and the normal, resp., in a general way. One will have:

$$h_1 = \xi, \qquad h_2 = \eta, \qquad k = \zeta$$

for the present position of the coordinate axes.

Moreover, upon keeping the axes Ox, Oy, Oz fixed and varying the position of the point (x, y, z), one will find, with no difficulty, that:

$$\frac{1}{L}\frac{\partial\xi}{\partial\lambda} = \frac{1}{L}\frac{\partial h_1}{\partial\lambda}-\frac{h_2}{\rho_1}-\frac{k}{R_1}, \qquad \frac{1}{M}\frac{\partial\xi}{\partial\mu} = \frac{1}{M}\frac{\partial h_1}{\partial\mu}+\frac{h_2}{\rho_2}-\frac{k}{T},$$

$$\frac{1}{M}\frac{\partial\eta}{\partial\mu} = \frac{1}{M}\frac{\partial h_2}{\partial\mu}-\frac{h_1}{\rho_2}-\frac{k}{R_2}, \quad \frac{1}{L}\frac{\partial\eta}{\partial\lambda} = \frac{1}{M}\frac{\partial h_2}{\partial\mu}+\frac{h_1}{\rho_1}-\frac{k}{T}.$$

The unknowns h_1, h_2, k must then satisfy the three equations:

$$
(13) \qquad \begin{cases}
\dfrac{1}{L}\dfrac{\partial h_1}{\partial \lambda} = \dfrac{h_2}{\rho_1} + \dfrac{k}{R_1}, \\[2ex]
\dfrac{1}{M}\dfrac{\partial h_2}{\partial \mu} = \dfrac{h_1}{\rho_2} + \dfrac{k}{R_2}, \\[2ex]
\dfrac{1}{M}\dfrac{\partial h_1}{\partial \mu} + \dfrac{1}{L}\dfrac{\partial h_2}{\partial \lambda} = \dfrac{2k}{T} - \dfrac{h_2}{\rho_1} - \dfrac{h_1}{\rho_1}.
\end{cases}
$$

Upon eliminating T, one will get two first-order equations in h_1 and h_2 whose characteristics will again be asymptotic lines of the surface, which one can easily assure oneself.

Equations (13) can serve as the proof of the theorems that we have presented that were concerned with the deformation of surfaces; however, it is would be pointless for us to dwell at length upon that subject.

CHAPTER II

STATIC PROPERTIES OF SURFACES

Consider a surface in equilibrium under the action of given forces that we shall call *external forces*, and imagine that one cuts out an arbitrary closed contour on that surface. The portion of the surface thus-detached will remain in equilibrium when one applies conveniently-chosen forces that are tangent to the surface. In all that follows, we shall refer to them by the name of *forces of tension*, and we shall always suppose that they are referred to a unit of length, just as one refers the pressures in a fluid to a unit of area. The force of tension that is exerted upon an element of the contour is, in general, oblique to that element. It will then have a normal component and a tangential component. One can call the former the *force of extension* or *compression*, according to its sign, and call the latter the *shearing force*. (These terms are borrowed from the theory of the resistance of materials.) Any linear element of the surface is subject to a force of extension and a shearing force, whose sense will reverse when one considers one or the other of the portions of the surface that the linear element separates.

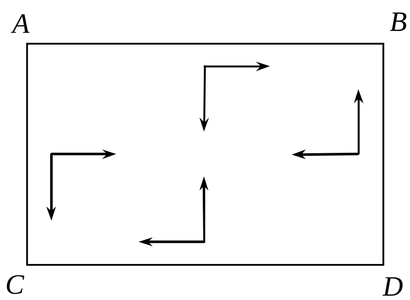

Figure 1.

The surface element that is found between two orthogonal curves (λ, $\lambda + d\lambda$) and (μ, $\mu + d\mu$) projects onto a figure *ABCD* in its tangent plane (Fig. 1) that can coincide with a rectangle upon neglecting the infinitesimals of order higher than one. Within the same limits of approximation, the tensions will project onto true magnitudes; moreover, they will be constants on each of the sides and equal (but with opposite signs) on two parallel sides. If one takes the sum of the moments of all of the forces with respect to the normal that is drawn to the center of the rectangle then one will effortlessly see that the external forces and the extension forces can give only moments of second order, while the shearing forces t and t', one of which is exerted upon *AB* and *CD*, while the other one is exerted upon *AC* and *BD*, will give the resultant moment:

$$(t - t')\, AB \times BD.$$

Equilibrium of the rectangular element is then possible only if $t = t'$. That gives the theorem:

The shearing forces that are developed at a given point on two line elements that cut at a right angle are equal.

The equality $t = t'$ supposes that the forces t and t', which are applied tangentially to the sides *CA* and *CD*, are both directed towards *C*, or even both of them can diverge. We shall assume that for a rectangle whose sides *CD* and *CA* are positive from *C* to *D* and from *C* to *A*, the shearing forces will be positive when they point to *C*. One will see the reason for that convention later one.

We define the sense of the forces of extension that are exerted upon the same rectangle by assuming that on the two sides *CD* and *CA*, they are positive when they have the same sense as the positive directions *CA*, *CD*.

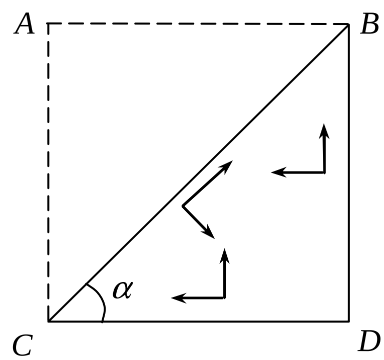

Figure 2.

Now project onto its tangent plane a triangular element that is composed of the orthogonal curves λ, μ, and a third curve that is required only to be at an infinitely-small distance from the point at which the first two meet. Upon neglecting infinitesimals of order higher than one, one can get the right triangle *CBD* (Fig. 2) for the projection, which one can regard as being equal to the one-half of the rectangle *ABCD*. It is easy to see that the forces of tension, which are supposed to be positive *CD* and *DB*, will have the directions that are represented by the arrows. We let t denote the shearing force that is common to *BD* and *CD*, while n_1 and n_2 will denote the extension forces that are exerted on *CD* and *BD*, resp. The side *BC* is likewise subject to tensions. We suppose that the positive sense of the shearing force T is that of *C* to *B* and the sense of the extension force N will be the sense that points into the triangle.

Upon projecting all of the forces onto *BC* and its perpendicular and remarking that the external forces give only second-order terms, one will have:

$$(14) \qquad \left\{ \begin{aligned} N &= n_1 \cos^2\alpha + n_2 \sin^2\alpha + 2t \sin\alpha \cos\alpha, \\ T &= t(\cos^2\alpha - \sin^2\alpha) - (n_1 - n_2)\sin\alpha\cos\alpha. \end{aligned} \right.$$

The second formula shows that T is annulled at each point for two real, rectangular directions OX, OY that are defined by:

$$\tan 2\alpha = \frac{2l}{n_1 - n_2}.$$

Upon making *AC* and *BC* coincide with those two directions, which one can call the *principal directions of tension*, one will have:

$$N = n_1 \cos^2 \alpha + n_2 \sin^2 \alpha,$$

$$T = -(n_1 - n_2) \sin \alpha \cos \alpha .$$

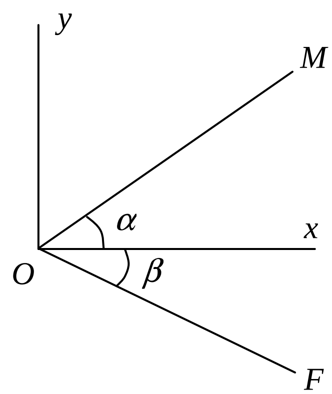

Figure 3.

Let *OM* be the direction α (Fig. 3). Draw a line *OF*, which represents the corresponding tension in sense and magnitude at some arbitrary scale. The coordinates of the point *F* are:

$$x = T \cos \alpha + N \sin \alpha = \quad n_2 \sin \alpha,$$
$$y = T \sin \alpha - N \cos \alpha = -n_1 \cos \alpha .$$

Hence, upon letting β denote the angle *FOX*, one will have:

$$\tan \alpha \tan \beta = -\frac{n_1}{n_2} .$$

When α varies, the point *F* will describe the ellipse:

$$\frac{x^2}{n_2^2} + \frac{y^2}{n_1^2} = 1,$$

but a consideration of that ellipse would have no utility, since it does not depend upon the directions *OF* and *OM* in a simple manner. If we measure out the length $OM = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{n_1 \cos^2 \alpha + n_2 \sin^2 \alpha}}$ on *OM* then the coordinates of the point *M* will be:

$$x = \frac{\cos\alpha}{\sqrt{n_1\cos^2\alpha + n_2\sin^2\alpha}},$$

$$y = \frac{\sin\alpha}{\sqrt{n_1\cos^2\alpha + n_2\sin^2\alpha}},$$

so:

$$n_1\, x^2 + n_2\, y^2 = 1.$$

Upon comparing that equation with the one that gives the product tan α tan β, one will immediately see that *OF* and *OM* are two conjugate directions of the conic thus-obtained. We call that conic the *tension indicatrix*. When referred to two rectangular axes with arbitrary directions, its equation will become:

$$n_1\, x^2 + 2t\, xy + n_2\, y^2 = 1.$$

When the indicatrix is a real ellipse, all of the line elements that are drawn through the point considered will be subject to the part of the surface that has normal efforts of compression. When it is an imaginary ellipse, the normal efforts of compression will be replaced with efforts of traction. When it is a hyperbola, the normal action will produce an effort of compression for certain directions and an effort of traction for some other ones, and in that case, there will exist two real directions for which the normal action reduces to zero.

In certain cases, it can be interesting to know the total intensity of tension that is exerted on a line element. The square of that quantity is given by:

$$N^2 + T^2 = n_1^2\cos^2\alpha + n_2^2\sin^2\alpha.$$

With that, the largest value of the total tension will correspond to the direction of the small axis of the tension indicatrix.

The known properties of the diameters of a conic, when stated for the indicatrix, translate into a series of theorems on the tensions, but it would be pointless for us to dwell upon that. We shall establish only a relation that we shall appeal to in what follows.

Since the two directions α and β are conjugate, if one lets φ denote the angle that they form between them, lets N_1 and N_2 denote their normal tensions, and lets N_1' denote the normal tension in the element whose direction is $\pi / 2 + \alpha$ then one will obviously have:

$$\frac{1}{N_1} + \frac{1}{N_2} = \frac{1}{n_1} + \frac{1}{n_2},$$

$$\frac{\sin^2\varphi}{N_1\, N_2} = \frac{1}{n_1\, n_2},$$

$$N_1 + N_1' = n_1 + n_2 .$$

Upon eliminating n_1 and n_2 , one will infer that:

$$N_1 + N_2 = \sin^2 \varphi (N_1 + N_1'). \tag{15}$$

The considerations that we just discussed show that if one traces out a double series of orthogonal curves λ, μ on a surface then it will suffice to know the extension efforts and the common shearing effort that are exerted upon the elements $L\,d\lambda$, $M\,d\mu$ at each point for the equilibrium state on the surface to be known completely. Let n_1 be the extension effort on $L\,d\lambda$, let n_2 be the extension effort on $M\,d\mu$, and let t be the common shearing effort; n_1, n_2, and t will then be the three functions of λ and μ that must be determined.

Upon considering the surface element whose sides are $L\,d\lambda$, $M\,d\mu$ to be a free solid that is subject to external forces and tension forces that are developed on the contour, one can write the equilibrium equations of an invariable solid. Naturally, the projection axes will be the normal to the surface and the tangents to the coordinate curves. Upon pushing the approximation up to second order in the equations of the projections of the forces and up to third order in those of the moments, one will see that the equations of the moments reduce to those of the projections, and one will arrive effortlessly at the equilibrium conditions. However, the following method leads to that objective in a much-more-rapid fashion:

As we have done before, let h_1, h_2, k denote the projections of the displacements at each point of the surface onto the tangents to the coordinates and the normal after an infinitely-small deformation. One will see immediately that the work done by the forces of tension will be equal to:

$$\left[\frac{\partial}{\partial\mu}(L\,n_1\,n_2 - L\,t\,h_1) + \frac{\partial}{\partial\lambda}(M\,n_2\,h_1 - M\,t\,h_2)\right] d\lambda\,d\mu .$$

Let F be the external force per unit area at the point (λ, μ). Let F_1, F_2 be its components along $d\lambda$, $d\mu$, and let Φ be its normal component. The elementary work done by the external forces that are applied to the element is:

$$(F_1\,h_1 + F_2\,h_2 + \Phi\,k)\,LM\,d\lambda\,d\mu .$$

One must then have:

$$\frac{\partial}{\partial\mu}(L\,n_1\,n_2 - L\,t\,h_1) + \frac{\partial}{\partial\lambda}(M\,n_2\,h_1 - M\,t\,h_2) = (F_1\,h_1 + F_2\,h_2 + \Phi\,k)\,LM$$

for all possible values of h_1, h_2, k , or rather:

$$\frac{1}{M}\left(n_1\frac{\partial h_2}{\partial\mu} + n_1\frac{\partial h_2}{\partial\mu}\right) - \frac{1}{M}\left(t\frac{\partial h_1}{\partial\mu} + h_1\frac{\partial t}{\partial\mu}\right) + \frac{1}{L}\left(n_2\frac{\partial h_1}{\partial\lambda} + h_1\frac{\partial n_2}{\partial\lambda}\right) - \frac{1}{L}\left(t\frac{\partial h_2}{\partial\lambda} + h_2\frac{\partial t}{\partial\lambda}\right)$$

$$+ (n_1\, h_2 - t\, h_1)\ \frac{\partial L / \partial \mu}{LM} + (n_2\, h_1 - t\, h_2)\ \frac{\partial L / \partial \lambda}{LM} = F_1\, h_1 + F_2\, h_2 + \Phi\, k.$$

Replace the derivatives of h_1 and h_2 with their values that one derives from equations (13). Since h_1, h_2, k are capable of taking on an infinitude of independent values, one can then write down that the coefficients of those three quantities are identically zero. That will give:

$$(16) \qquad \left\{ \begin{aligned} &\frac{1}{L}\frac{\partial n_2}{\partial \lambda} - \frac{1}{M}\frac{\partial t}{\partial \mu} + \frac{n_1 - n_2}{\rho_2} + \frac{2t}{\rho_1} = F_1, \\ &\frac{1}{M}\frac{\partial n_1}{\partial \mu} - \frac{1}{L}\frac{\partial t}{\partial \lambda} + \frac{n_2 - n_1}{\rho_1} + \frac{2t}{\rho_2} = F_2, \\ &\qquad \frac{n_1}{R_2} + \frac{n_2}{R_1} - \frac{2t}{T} = \Phi. \end{aligned} \right.$$

These equations are necessary and sufficient for equilibrium, and they give n_1, n_2, t. Since there are as many unknowns as equations, it will follow that F_1, F_2, Φ are not subject to any condition and that consequently:

A surface can exist in equilibrium under the action of arbitrary forces.

That theorem seems paradoxical on first glance, but if one wishes to understand why the equilibrium conditions of a surface reduce to conditions that relate to the boundary then it will suffice to recall that when a curve that is traced upon a surface remains fixed, all of the surface will likewise remain fixed (as long as the curve is not an asymptote). That makes the problem of the funicular curve completely different from the problem that we presently address.

The equations:

$$P_t + \frac{dT}{ds} = 0,$$

$$P_u + \frac{T}{\rho} = 0$$

indeed have a certain analogy with equations (16), but they refer only to an unknown tension whose elimination will lead to the equilibrium condition:

$$P_t = \frac{d(P_n\, \rho)}{ds}.$$

There is no parallel in the case of surfaces.

The boundary conditions can be written explicitly only when one has integrated the equilibrium equations. If one finds that those conditions cannot be satisfied then one must conclude that the equilibrium of the surface does not exist and that it necessarily deforms under the action of the forces that are applied to it. One will then be led to restrict oneself to the third and last problem that was stated in the Introduction, viz.:

What deformation will a given surface experience under the action of forces that do not satisfy the equilibriums conditions?

Here, it goes without saying that one must know how the forces vary when the surface is deformed. In order for the problem to be soluble, one must be able to determine all of the surfaces that can map onto the given surface under the conditions that are imposed upon it. It will then be necessary that one must know how to integrate the equilibrium equations for each of those surfaces and to define the conditions that relate to the boundary. Whosoever would study the theory of the deformation of surfaces and account for the difficulties that are presented will effortlessly admit that the problem that we just posed exceeds the present scope of that science considerably. I have succeeded in obtaining some results along those lines for some very special cases that one will find to be discussed later on. Perhaps one will get some interesting theorems upon considering surfaces that are subject to only very small deformations and being content to use a certain approximation. However, my efforts in this article are directed towards an objective that is easier to achieve: I seek to analyze the equilibrium conditions for surfaces and the laws by which one develops the various efforts of tension. In a word, I have concentrated my attention on the first two problems that were stated in the Introduction.

It is useful to make a remark here in regard to the practical applications that can be made of this theory: The distribution of tensions in a surface in equilibrium is entirely independent of the inextensibility that I have attributed to that surface. That is because when an extensible surface is in equilibrium, one can mentally suppress its extensibility without changing anything. The conditions that relate to the equilibrium of an inextensible surface are then necessary for the equilibrium of that surface when it is made extensible; however, they are generally sufficient.

The left-hand sides of equations (16) are identical to the ones in equations (9), except for the names of the unknowns. It results from this immediately that:

When a surface or a portion of a surface is not subject to any external force, the tensions n_1, n_2, t *that can develop will be proportional to the variations that the curvatures* $1 / R_1$, $1 / R_2$, $1 / T$ *experience under a certain infinitely-small deformation of the surface.*

That fundamental theorem gives the key to the intimate link that exists between the problem of the equilibrium of surfaces and that of their deformation.

There is an important simplification of equations (16), no matter what the external forces are. Set:

$$n_1 = n_1' + \frac{a}{R_1}, \quad n_2 = n_2' + \frac{a}{R_2}, \quad t = t' + \frac{a}{T}.$$

The third equation will become:

$$\frac{n_1'}{R_2}+\frac{n_2'}{R_1}-\frac{2t'}{T}=\Phi-\left(\frac{1}{R_1 R_2}-\frac{1}{T^2}\right).$$

Whenever $\frac{1}{R_1 R_2}-\frac{1}{T^2}$ is non-zero – i.e., whenever the surface is not developable – we can determine the function a by the condition:

$$2a\left(\frac{1}{R_1 R_2}-\frac{1}{T^2}\right)=\Phi.$$

Substitute that value of a in the expression for n_1, n_2, t. Since the first two equations in (16) are linear, their left-hand sides will be annulled when one replaces n_1, n_2, t with $\frac{1}{R_1}$, $\frac{1}{R_2}, \frac{1}{T}$, or by those quantities when they are multiplied by a constant. All that will remain is then:

$$\frac{1}{L}\frac{\partial n_2'}{\partial\lambda}-\frac{1}{M}\frac{\partial t'}{\partial\mu}+\frac{n_1'-n_2'}{\rho_2}+\frac{2t'}{\rho_1}=F_1-\frac{1}{LR_2}\frac{\partial a}{\partial\lambda}+\frac{1}{MT}\frac{\partial a}{\partial\mu}=F_1',$$

$$\frac{1}{M}\frac{\partial n_2'}{\partial\mu}-\frac{1}{L}\frac{\partial t'}{\partial\lambda}+\frac{n_2'-n_1'}{\rho_1}+\frac{2t'}{\rho_2}=F_2-\frac{1}{MR_2}\frac{\partial a}{\partial\mu}+\frac{1}{LT}\frac{\partial a}{\partial\mu}=F_2'.$$

One concludes from this that since a is a well-defined function, as one just saw, and if the tangential forces F_1, F_2 are put into the form:

$$F_1=F_1'+\frac{1}{LR_2}\frac{\partial a}{\partial\lambda}-\frac{1}{MT}\frac{\partial a}{\partial\mu},$$

$$F_2=F_2'+\frac{1}{MR_1}\frac{\partial a}{\partial\mu}-\frac{1}{LT}\frac{\partial a}{\partial\lambda}$$

then the system of external forces can be divided into two parts:

1. The normal force Φ and the tangent forces:

$$\frac{1}{LR_2}\frac{\partial a}{\partial\lambda}-\frac{1}{MT}\frac{\partial a}{\partial\mu},$$

$$\frac{1}{MR_1}\frac{\partial a}{\partial\mu}-\frac{1}{LT}\frac{\partial a}{\partial\lambda}.$$

That system admits the tensions $n_1 = \frac{a}{R_1}$, $n_2 = \frac{a}{R_2}$, $t = \frac{a}{T}$ as particular solutions. We refer to them by the name of the *normal system.*

2. Tangential forces F_1', F_2' with no normal forces. That system, which we refer to by the name of the *tangential system,* will give to some tensions that are determined by the following equations, in which we have dropped the primes, which have become unnecessary:

$$(17)\qquad \left\{\begin{aligned} &\frac{1}{L}\frac{\partial n_2}{\partial \lambda} - \frac{1}{M}\frac{\partial t}{\partial \mu} + \frac{n_1 - n_2}{\rho_2} + \frac{2t}{\rho_1} = F_1, \\ &\frac{1}{M}\frac{\partial n_1}{\partial \mu} - \frac{1}{L}\frac{\partial t}{\partial \lambda} + \frac{n_2 - n_1}{\rho_1} + \frac{2t}{\rho_2} = F_2, \\ &\frac{n_1}{R_2} + \frac{n_2}{R_1} - \frac{2t}{T} = 0. \end{aligned}\right.$$

When one has found three values n_1, n_2, t that define a particular solution of that system, if one wishes to have the general solution of equations (16) then it will suffice to add those particular solutions, term-by-term, to the solutions $\frac{a}{R_1}$, $\frac{a}{R_2}$, $\frac{a}{T}$ of the normal system and the general values that satisfy the equations with a vanishing right-hand side.

The integration of the equations with a vanishing right-hand side constitutes the problem of the deformation of surfaces, when it is simplified by the hypothesis that one is dealing with an infinitely-small deformation. Consequently, the special difficulty in the problem of equilibrium lies in the search for a particular solution that relates to the system of tangential forces.

If one knows two particular solutions (l', m', θ'), (l'', m'', θ'') of equations (9) then one can apply the method of variation of arbitrary constants to the search for a particular solution of equations (17).

Upon setting:

$$n_1 = a\, l' + b\, l'',$$

$$n_2 = a\, m' + b\, m'',$$

$$t = a\, \theta' + b\, \theta'',$$

one will have the two equations:

$$\frac{m'}{L}\frac{\partial a}{\partial \lambda} + \frac{m''}{L}\frac{\partial b}{\partial \lambda} - \frac{\theta'}{M}\frac{\partial a}{\partial \mu} - \frac{\theta''}{M}\frac{\partial b}{\partial \mu} = F_1,$$

$$\frac{l'}{L}\frac{\partial a}{\partial \lambda} + \frac{l''}{M}\frac{\partial b}{\partial \lambda} - \frac{\theta'}{L}\frac{\partial a}{\partial \lambda} - \frac{\theta''}{L}\frac{\partial b}{\partial \lambda} = F_2$$

to determine the functions a and b .

The unknowns enter into these two equations only by their derivatives. That method will lead to an advantageous result when equations (9) admit a solution l'', m'', θ'' such that one will have:

$$\theta''^2 - l''m'' = 0.$$

If one writes:

$$\frac{l''}{\theta''} = \frac{\theta''}{m''} = k$$

then the equations that determine a and b will lead to the following one:

$$\frac{km' - \theta'}{L}\frac{\partial a}{\partial \lambda} + \frac{l' - k\theta'}{M}\frac{\partial a}{\partial \mu} = k\,F_1 + F_2\,,$$

which is an equation with only one unknown whose integration is incomparably simpler than that of the two simultaneous equations.

The expression $\theta''^2 - l''m''$ can be regarded as the invariant of the tension indicatrix upon supposing that the external forces are zero. If one calls the principal tensions l, m then one must have:

$$lm = l''m'' - \theta''^2 = 0,$$

so $l = 0$ and $m = 0$.

For example, let $l = 0$. Moreover, one will have $\theta = 0$ for the principal directions. If one supposes that equations (9) refer to the principal lines of tension then they will reduce to:

$$\frac{1}{L}\frac{\partial m}{\partial \lambda} - \frac{m}{\rho_2} = 0, \qquad \frac{m}{\rho_1} = 0, \qquad \frac{m}{R_1} = 0.$$

In order for these equations to be satisfied without setting $m = 0$ (which would imply that $l'' = m'' = \theta'' = 0$), it is necessary that $1 / \rho_1$ and $1 / R_1$ must both be zero, which would be true only in the case of rectilinear generators.

The solution $\theta''^2 - l''m'' = 0$ cannot exist then, and as a result, the process of integration that we have deduced will be applicable only when we are dealing with a ruled surface.

Let us return to the general system whose equilibrium is expressed by equations (16). Upon substituting some arbitrary functions of λ and μ for n_1, n_2, t, one can obviously find an infinitude of values for F_1, F_2, Φ that make the integration possible. For example, if one calls a constant A and sets:

$$n_1 = \frac{A}{L}, \qquad n_2 = \frac{A}{M}, \qquad t = 0$$

then one will find that:

$$F_1 = \frac{A}{L\,\rho_2}, \qquad F_2 = \frac{A}{M\,\rho_1}, \qquad \Phi = A\left(\frac{1}{LR_1} + \frac{1}{MR_2}\right).$$

It would be puerile to multiply the examples of this kind; nonetheless, the following case deserves to be pointed out.

If one has:

$$F_1 = \frac{1}{L}\frac{\partial\varpi}{\partial\lambda}, \qquad F_2 = \frac{1}{M}\frac{\partial\varpi}{\partial\mu},$$

in which ϖ is an arbitrary function, or what amounts to the same thing:

$$\frac{\partial}{\partial\mu}(L\,F_1) = \frac{\partial}{\partial\lambda}(M\,F_2),$$

then one can set:

$$n_1 = n_1' + \varpi,$$
$$n_2 = n_2' + \varpi,$$

and the equations will reduce to:

$$\frac{1}{L}\frac{\partial n_2}{\partial\lambda} - \frac{1}{M}\frac{\partial t}{\partial\mu} + \frac{n_1 - n_2}{\rho_2} + \frac{2t}{\rho_1} = 0,$$

$$\frac{1}{M}\frac{\partial n_1}{\partial\mu} - \frac{1}{L}\frac{\partial t}{\partial\lambda} + \frac{n_2 - n_1}{\rho_1} + \frac{2t}{\rho_2} = 0,$$

$$\frac{n_1'}{R_2} + \frac{n_2'}{R_1} - \frac{2t}{T} = \Phi - \varpi\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Then set $n_1' = k / R_1 + n_1''$, $n_2' = k / R_2 + n_2''$, $t = k / T + t$, in which k is a constant. The first two equations will not change form, and the right-hand side of the last one will become:

$$\Phi - \varpi\left(\frac{1}{R_1} + \frac{1}{R_2}\right) - 2k\left(\frac{1}{R_1 R_2} - \frac{1}{T^2}\right).$$

If that expression is zero then the three equations will be found to have a vanishing right-hand side, and one will be down to the problem of infinitely-small deformations. The same thing will happen when k is not a constant, but a function that satisfies the two equations:

$$\frac{1}{LR_2}\frac{\partial k}{\partial\lambda} - \frac{1}{MT}\frac{\partial k}{\partial\mu} = 0,$$

$$\frac{1}{LT}\frac{\partial k}{\partial\lambda} - \frac{1}{MR_1}\frac{\partial k}{\partial\mu} = 0\,.$$

The last case is possible only if $\frac{1}{R_1 R_2} - \frac{1}{T^2} = 0$; i.e., if the surface is developable. However, k will then disappear from the expression for the normal force, which will reduce to:

$$\Phi - \varpi\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

It is interesting to look for the conditions that the external forces must fulfill in order for one to constantly have $n_1 = n_2$ and $t = 0$; i.e., for all of the directions that pass through the same point to be subject to identical tensions. By virtue of that hypothesis, equations (16) will reduce to:

$$\frac{1}{L}\frac{\partial n}{\partial \lambda} = F_1 ,$$

$$\frac{1}{M}\frac{\partial n}{\partial \mu} = F_2 ,$$

$$\frac{2n}{R} = \Phi,$$

when one replaces n_1 and n_2 with n and denotes the mean curvature $\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ by $1 / R$.

If one sets $L\, d\lambda = ds_1$, $M\, d\mu = ds_2$ then the desired conditions can be written:

$$2\,F_1 = \frac{\partial(\Phi R)}{\partial s_1},$$

$$2\,F_2 = \frac{\partial(\Phi R)}{\partial s_2}.$$

One will find the third condition, viz.:

$$\frac{\partial F_1}{\partial s_2} - \frac{F_1}{\rho_1} = \frac{\partial F_2}{\partial s_1} - \frac{F_2}{\rho_2},$$

upon eliminating n from the values of F_1 and F_2, which reverts to the first two conditions.

When the external forces are everywhere normal to the surface, the necessary and sufficient condition for one to have $n_1 = n_2$ and $t = 0$ is that ΦR must be constant. If that is true then the value of n will be likewise constant at all points of the surface.

The points of a surface for which $n_1 = n_2$ and $t = 0$ play a role in regard to the tensions that is analogous to the one that is played by umbilics in regard to the curvatures; one can refer to those points by the name of *umbilical equilibrium points.*

In order to complete the analogy, it is necessary that they must generally be isolated on a surface in equilibrium in the manner of umbilics. We will show that this is not the case.

Indeed, consider a surface in equilibrium and refer the equations to the principal lines of tension. In order to do that, it suffices to set $t = 0$, which will give:

$$\frac{1}{L}\frac{\partial n_2}{\partial \lambda} + \frac{n_1 - n_2}{\rho_2} = F_1 ,$$

$$\frac{1}{M}\frac{\partial n_1}{\partial \mu} + \frac{n_2 - n_1}{\rho_1} = F_2 ,$$

$$\frac{n_1}{R_2} + \frac{n_2}{R_1} = \Phi.$$

In order for a point to be umbilical, it is necessary and sufficient that one must have $n_1 = n_2$. In order for an infinitely-close point to be likewise umbilical, it is necessary and sufficient that one can find a direction $d\mu = k\, d\lambda$ such that one has:

$$\frac{\partial n_1}{\partial \lambda} d\lambda + \frac{\partial n_1}{\partial \mu} d\mu = \frac{\partial n_2}{\partial \lambda} d\lambda + \frac{\partial n_2}{\partial \mu} d\mu .$$

Now, it results from the equality $n_1 = n_2$ that:

$$\frac{\partial n_2}{\partial \lambda} = F_1\, L, \qquad \frac{\partial n_1}{\partial \mu} = M\, F_2 .$$

Upon differentiating the third equation with respect to λ and then setting $n_1 = n_2$ and $\frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{R}$, one will find that:

$$\frac{\partial n_1}{\partial \lambda} = R_2 \left(\frac{\partial \Phi}{\partial \lambda} - \frac{2}{R_2}\frac{\partial R}{\partial \lambda} - \frac{F_1 L}{R_1} \right);$$

similarly:

$$\frac{\partial n_2}{\partial \mu} = R_1 \left(\frac{\partial \Phi}{\partial \lambda} - \frac{2}{R_2}\frac{\partial R}{\partial \lambda} - \frac{F_2 M}{R_2} \right).$$

When these values are substituted into the equation:

$$\frac{\partial \mu}{\partial \lambda} = \frac{\dfrac{\partial n_1}{\partial \lambda} - \dfrac{\partial n_2}{\partial \lambda}}{\dfrac{\partial n_2}{\partial \mu} - \dfrac{\partial n_1}{\partial \mu}} ,$$

that will determine a direction (which is always unique, unless it is arbitrary) along which there exists an umbilical point that is infinitely-close to the first one.

As a result, if there exists an umbilical tension point then there will exist an infinitude of them that will form an umbilical tension line on the surface.

When an umbilical line belongs to one of the systems of principal lines, its form will be determined completely from what one knows at one of its points. Indeed, if we suppose that the umbilical is represented by μ = const. then we will have:

$$\frac{1}{L}\frac{\partial n}{\partial \lambda} = F_1 , \qquad n = \frac{\Phi R}{2}$$

for that line, by virtue of the general integrals, so upon replacing $L\,d\lambda$ with ds :

$$\frac{d(\Phi R)}{ds} = 2\,F_1 .$$

At each point of the surface, there always exists one and only one direction for which that condition is fulfilled. Indeed, let AF be the projection of the external force onto the tangent plane at the point A (Fig. 4).

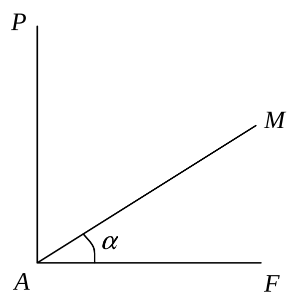

Figure 4.

Let $AM = ds$ be an element that passes through A and makes an angle of $MAF = \alpha$ with AF. Let $d\sigma$, $d\sigma'$ be the projections of that element onto AF and the perpendicular AP to AF. One has:

$$\frac{d(\Phi R)}{ds}ds = \frac{\partial(\Phi R)}{\partial \sigma}d\sigma + \frac{\partial(\Phi R)}{\partial \sigma'}d\sigma' ,$$

or

$$\frac{d(\Phi R)}{ds} = \frac{\partial(\Phi R)}{\partial \sigma}\cos\alpha + \frac{\partial(\Phi R)}{\partial \sigma'}\sin\alpha .$$

Furthermore, the component F_1 of the external force along AM is equal to $F\cos\alpha$. As a result, the condition that was found above can be written:

$$\frac{\partial(\Phi R)}{\partial \sigma}\cos\alpha + \frac{\partial(\Phi R)}{\partial \sigma'}\sin\alpha = F\cos\alpha,$$

or:

$$\tan\alpha = \frac{F - \dfrac{\partial(\Phi R)}{\partial\sigma}}{\dfrac{\partial(\Phi R)}{\partial\sigma'}},$$

which is a direction that is entirely independent of the arbitrary functions that were introduced by the integration.

Whereas the other principal tension lines deform in an arbitrary fashion according to conditions that relate to the boundary, the umbilical line will depend upon just one arbitrary constant. In particular, when there are no normal forces, the umbilical line will always be an orthogonal trajectory of the tangential forces.

Regardless of the external forces, one can repeat a sizable part of the remarks that were made before concerning equations (9) in the context of equations (16). Hence, it is generally possible to choose the values of two of the unknowns arbitrarily for all points of a curve that is traced on the surface, and one can then determine the state of the entire surface. Meanwhile, that proposition cannot be completely true, because without it, if one considers a portion of the surface that is bounded by a closed contour and is subject to arbitrary external forces then one can impose the condition that the extension force and the shearing force must be zero for all points of the contour. One will thus arrive at the conclusion that a portion of the surface that is entirely free will remain in equilibrium for any external forces, which is obviously an absurd result. It is easy to see why the argument is vicious. Trace out a series of closed curves on a portion of the surface, one of which surrounds the others, and begin with the given contour in order to finally reduce it to an isolated point *P*. The law of tensions that the curve obeys will determine the one that the tensions on an infinitely-close curve will obey, and so on, until one arrives at *P*. However, the law of tensions is well-defined for an infinitely-close curve that surrounds the point *P*. When one neglects infinitely-small quantities, it is expressed by formulas (14), in which only three arbitrary constants appear. The arbitrary functions that are introduced by integration must consequently be determined; moreover, they must be chosen in such a fashion that the tensions at the point *P* are not infinite. Consequently, it is not permissible to suppose *a priori* that the contour is subject to no effort of tension.

In the case of surfaces with opposite curvatures, the tensions that are exerted on a closed contour will then satisfy another condition. It is obvious that equations (16), like equations (9), admit the asymptotic lines of the surface for their characteristics, which will be real lines for that type of surface. The two arbitrary functions that are introduced by the integration are each a function of one or the other characteristic parameter. In other words, the general solution can be put into the form:

$$n_1 = f_1\,[\lambda, \mu, \varphi_1(\alpha_1), \varphi_2(\alpha_2)],$$

$$n_2 = f_2\,[\lambda, \mu, \varphi_1(\alpha_1), \varphi_2(\alpha_2)],$$

$$t = f_3\,[\lambda, \mu, \varphi_1(\alpha_1), \varphi_2(\alpha_2)],$$

in which φ_1 and φ_2 denote two arbitrary functions, while α_1, α_2 are two quantities that each remain constant when one displaces along a curve that belongs to one of the two systems of asymptotic lines.

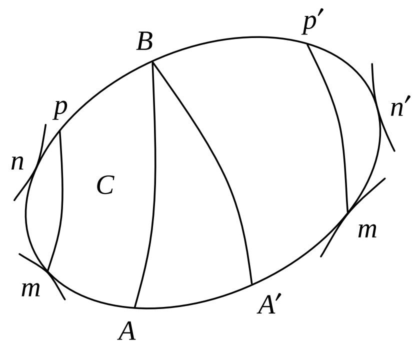

Figure 5.

Consider the closed contour C (Fig. 5), and choose the values of n_1 and n_2 for a point A on the contour arbitrarily. Let AB be the asymptotic line that passes through the point A for which α_2 remains constant. The values of the tensions at the various point of that line will depend upon just one arbitrary function φ_1 (α_1), and upon eliminating it, one will get a relation F_1 (n_1, n_2, t) = 0 between n_1, n_2, t with no arbitrary function. Let B be the second point at which the asymptote meets the contour. Another asymptotic line will pass through the point B that cuts the contour at another point A', and if the tensions at A' are known then one will have a new relation F_2 (n_1, n_2, t) = 0 for the point B. Those two relations will determine n_1, n_2, t when they are combined with:

$$\frac{n_1}{R_2}+\frac{n_2}{R_1}-\frac{2t}{T}=\Phi\ .$$

From this, if one knows the tensions at A and A' then one will likewise know them at B. Let m, n be the contact points of the two asymptotic lines that are tangent to the contour that belong to the same system as AB. Let m', n' be those of the two asymptotic lines from the same system as A' B'. If the points m and m' are chosen in such a fashion that the arc mAA' m' includes the points n and n', and one draws the asymptotes $m'p$, mp' that cut the contour at the points p and p' then one will effortlessly see that knowing the tensions that are exerted on the arc mm will imply the complete knowledge of the tensions that are exerted on the arc pp' and will leave only one tension undetermined on each of the arcs $mn'p'$, $m'np$.

Nothing like that is true for convex surfaces, since their asymptotic lines will be imaginary. Later on, we shall see what does happen in a special example.

CHAPTER III

STUDY OF TANGENTIAL SYSTEMS

In the preceding chapter, we saw how an arbitrary system of forces that is applied to a surface can be decomposed into two systems that were called the *normal system* and the *tangential system*. We proved that one will always have an integral in the former case and that all of the difficulty in the problem consists of integrating the latter one. That is what we shall now address exclusively.

A tangential system is characterized by the absence of the normal components to the external forces. In such a case, one will always have:

$$\frac{n_1}{R_2} + \frac{n_2}{R_1} - \frac{2t}{T} = 0.$$

Upon taking the lines of curvature to be coordinate lines, that equation will become:

$$\frac{n_1}{R_2} + \frac{n_2}{R_1} = 0.$$

That shows that n_1 / n_2 always has the opposite sign to R_1 / R_2, and that consequently:

The tension indicatrix is always hyperbolic for convex surfaces.

The same proposition can be further stated in this fashion:

A convex surface that is subject to tangential forces always presents two systems of lines that are followed uniquely by shearing.

In the case of surfaces with opposite curvatures, the tension indicatrix can be elliptic or hyperbolic, and consequently, the lines that one will be dealing with can be real or imaginary. All that one can conclude from the fact that n_1 / n_2 is positive for lines of curvature is that the acute angle that is defined at a point by the sliding lines when they are real will never include a principal direction of the surface. One explains that immediately upon remarking that two rectangular diameters of a hyperbola can have squares with the same signs only when they are both included in the obtuse angle between the asymptotes.

Consider a surface that is referred to its directions of normal tension and for which $t = 0$. The equation of the tension indicatrix will then be:

$$n_1 x^2 + n_2 y^2 = 1.$$

The angular coefficients m, m' of the two conjugate directions are coupled by the relation:

$$mm' = -\frac{n_1}{n_2}.$$

The equation of the ordinary indicatrix of the surface is:

$$\frac{x^2}{R_1} + \frac{2xy}{T} + \frac{y^2}{R_2} = 1.$$

The angular coefficients μ, μ' of the asymptotic directions are the roots of:

$$\frac{\mu^2}{R_2} + \frac{2\mu}{T} + \frac{1}{R_1} = 0,$$

which gives:

$$\mu\,\mu' = \frac{R_2}{R_1}.$$

Since $\frac{n_1}{R_2} + \frac{n_2}{R_1} = 0$, we can state the following fundamental theorem:

The asymptotic directions of the surface are two conjugate directions of the tension curve.

In other words:

The tensions that act upon an asymptotic line are parallel to the asymptotic lines of the other system.

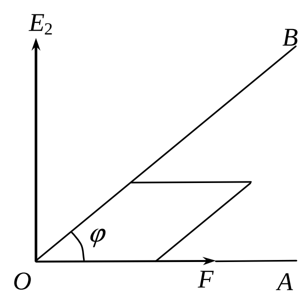

Figure 6.

Due to that property and the ones that were established already for asymptotic lines, it is interesting to look for what the equilibrium equations will become when one refers them to those lines as the coordinate lines. One can establish the equations thus-transformed directly, but one can arrive at them just as well by a change of coordinates. To that end, imagine that the original coordinates are composed of a system of asymptotic lines and orthogonal trajectories. If the asymptotic lines constitute the system μ = const. and if φ denotes the angle between the two asymptotic lines *OA*, *OB* (Fig. 6) then, from the preceding theorem, one will have:

$$t = -n_1 \cot \varphi,$$

if one recalls the sign conventions.

Let n be the normal tension that is exerted on OB. It results from formula (15) that one will have:

$$n + n_1 = \sin^2 \varphi\,(n_1 + n_2)\,.$$

When one puts $L\,d\lambda = ds_1$, $M\,d\mu = ds_2$, to abbreviate, the first two equilibrium equations will become:

$$\frac{\partial n_2}{\partial s_1} - \frac{\partial t}{\partial s_2} + \frac{n_1 - n_2}{\rho_2} + \frac{2t}{\rho_1} = F_1\,,$$

$$\frac{\partial n_1}{\partial s_2} - \frac{\partial t}{\partial s_1} + \frac{n_2 - n_1}{\rho_1} + \frac{2t}{\rho_2} = F_2\,.$$

Let f be an arbitrary function of λ and μ. Let F be what that function will become when one expresses it in terms of the parameters α and β of the asymptotic lines, in such a way that:

$$f(\lambda, \mu) = F(\alpha, \beta).$$

One infers from this that:

$$\frac{\partial f}{\partial \lambda} d\lambda + \frac{\partial f}{\partial \mu} d\mu = \frac{\partial F}{\partial \alpha} d\alpha + \frac{\partial F}{\partial \beta} d\beta\,.$$

Let A, B be the metric coefficients that correspond to α and β. The displacements $d\sigma_1$ and $d\sigma$ that are performed along OA and OB, resp., are expressed by:

$$d\sigma_1 = A\,d\sigma, \qquad d\sigma = B\,d\beta,$$

and one will have, as a result:

$$\frac{\partial f}{\partial \lambda}\frac{ds_1}{L} + \frac{\partial f}{\partial \mu}\frac{ds_2}{M} = \frac{\partial F}{\partial \alpha}\frac{d\sigma_1}{A} + \frac{\partial F}{\partial \beta}\frac{d\beta}{B}\,.$$

or rather:

$$\frac{\partial f}{\partial \lambda} ds_1 + \frac{\partial f}{\partial \mu} ds_2 = \frac{\partial F}{\partial \sigma_1} d\sigma_1 + \frac{\partial F}{\partial \sigma} d\sigma\,.$$

Moreover, Fig. 6 shows that:

$$ds_1 = d\sigma_1 + \cos \varphi\, d\sigma,$$
$$ds_2 = \sin \varphi\, d\sigma,$$

and as a result:

$$\left(\frac{\partial f}{\partial s} - \frac{\partial F}{\partial \sigma_1}\right) d\sigma_1 + \left(\frac{\partial f}{\partial s_1}\cos\varphi + \frac{\partial f}{\partial s_2}\sin\varphi - \frac{\partial F}{\partial \sigma}\right) d\sigma = 0,$$

so since $d\sigma$ and $d\sigma_1$ are arbitrary:

$$\frac{\partial f}{\partial s_1} = \frac{\partial F}{\partial \sigma_1},$$

$$\frac{\partial f}{\partial s_2} = \frac{1}{\sin\varphi}\frac{\partial F}{\partial \sigma} - \cot\varphi \frac{\partial F}{\partial \sigma_1}.$$

Upon setting $f = t = -n_1 \cot \varphi$, one will have:

$$\frac{\partial t}{\partial s_1} = -\cos\varphi \frac{\partial n_1}{\partial \sigma_1} + \frac{n_1}{\sin^2\varphi}\frac{\partial \varphi}{\partial \sigma_1},$$

$$\frac{\partial t}{\partial s_2} = -\frac{\cos\varphi}{\sin^2\varphi}\frac{\partial n_1}{\partial \sigma} + \cot^2\varphi \frac{\partial n_1}{\partial \sigma_1} + \frac{n_1}{\sin^3\varphi}\frac{\partial \varphi}{\partial \sigma} - \frac{n_1 \cos\varphi}{\sin^3\varphi}\frac{\partial \varphi}{\partial \sigma_1}.$$

For $f = n_2 = \dfrac{n}{\sin^2\varphi} + n_1 \cot^2 \varphi$, one gets:

$$\frac{\partial n_2}{\partial s_1} = \frac{1}{\sin^2\varphi}\frac{\partial n}{\partial \sigma_1} + \cot^2\varphi \frac{\partial n_1}{\partial \sigma_1} - 2\frac{\cos\varphi}{\sin^3\varphi}\frac{\partial \varphi}{\partial \sigma_1}(n + n_1).$$

Finally, upon setting $f = n_1$, one will get:

$$\frac{\partial n_1}{\partial s_2} = \frac{1}{\sin\varphi}\frac{\partial n_1}{\partial \sigma} - \cot\varphi \frac{\partial n_1}{\partial \sigma_1}.$$

Upon substituting these values and multiplying everything by $\sin^2 \varphi$, the equilibrium equations will become:

$$\frac{\partial n_1}{\partial \sigma} + \cos\varphi \frac{\partial n_1}{\partial \sigma} - (2n + n_1)\frac{\cos\varphi}{\sin\varphi}\frac{\partial \varphi}{\partial \sigma_1} - \frac{n_1}{\sin\varphi}\frac{\partial \varphi}{\partial \sigma} - \frac{n + n_1 \cos 2\varphi}{\rho_2} - \frac{n_1}{\rho_1}\sin 2\varphi = F_1 \sin^2 \varphi,$$

$$\sin\varphi \frac{\partial n_1}{\partial \sigma} - n_1 \frac{\partial n_1}{\partial \sigma} + \frac{n + n_1 \cos 2\varphi}{\rho_1} - \frac{n_1}{\rho_2}\sin 2\varphi = F_2 \sin^2 \varphi.$$

Let f, f_1 be the components of the external force along OB and OA. One has:

$$F_1 = f_1 + f \cos \varphi,$$

$$F_2 = f \sin \varphi,$$

so one will deduce with no difficulty that:

$$\sin\varphi\frac{\partial n}{\partial\sigma_1}-n\left(2\cos\varphi\frac{\partial\varphi}{\partial\sigma_1}+\frac{\sin\varphi}{\rho_2}+\frac{\cos\varphi}{\rho_1}\right)-n_1\left(\frac{\partial\varphi}{\partial\sigma}-\frac{\sin\varphi}{\rho_2}+\frac{\cos\varphi}{\rho_1}\right)=f_1\sin^3\varphi,$$

$$\sin\varphi\frac{\partial n_1}{\partial\sigma}-n_1\left(\frac{\partial\varphi}{\partial\sigma_1}-\frac{\cos 2\varphi}{\rho_1}+\frac{\sin 2\varphi}{\rho_2}\right)=f\sin^3\varphi\,.$$

Introduce the geodesic curvature $1/\rho$ of the asymptotic line *OB*. In order to calculate it, it will suffice to project the elementary curvilinear triangle that is defined by the two asymptotic lines and the orthogonal trajectory to the point *O* of the lines that belong to the system *OA* onto its tangent plane, upon remarking that the angle between the extreme tangents to one of the sides of that projection is equal to the quotient of the projected arc length with its radius of geodesic curvature, and that the sum of the deviations that the tangent to a point on one of the sides can experience when one make a complete circuit of the triangle is equal to zero. Upon giving a convenient sign to ρ, one will find that:

$$-\frac{1}{\rho}=\frac{\partial\varphi}{\partial\sigma}+\frac{\cos\varphi}{\rho_1}-\frac{\sin\varphi}{\rho_2},$$

and the preceding formulas will become:

$$\sin\varphi\frac{\partial n}{\partial\sigma_1}-n\left[\frac{\partial\varphi}{\partial\sigma}+\frac{1}{\rho}+2\cos\varphi\left(\frac{\partial\varphi}{\partial\sigma_1}+\frac{1}{\rho_1}\right)\right]+\frac{n_1}{\rho}=f_1\sin^3\varphi,$$

$$\sin\varphi\frac{\partial n_1}{\partial\sigma}+\frac{n}{\rho_1}-n_1\left[\frac{\partial\varphi}{\partial\sigma_1}+\frac{1}{\rho_1}+2\cos\varphi\left(\frac{\partial\varphi}{\partial\sigma}+\frac{1}{\rho}\right)\right]=f\sin^3\varphi\,.$$

Upon setting:

$$\frac{\partial\varphi}{\partial\sigma}+\frac{1}{\rho}=\frac{1}{u},$$

$$\frac{\partial\varphi}{\partial\sigma_1}+\frac{1}{\rho_1}=\frac{1}{u_1},$$

for brevity, then one will finally get:

$$(18)\qquad\left\{\begin{aligned}&\sin\varphi\frac{\partial n}{\partial\sigma_1}-n\left(\frac{1}{u}+\frac{2\cos\varphi}{u_1}\right)+\frac{n_1}{\rho}=f_1\sin^3\varphi,\\&\sin\varphi\frac{\partial n_1}{\partial\sigma}-n_1\left(\frac{1}{u_1}+\frac{2\cos\varphi}{u}\right)+\frac{n}{\rho_1}=f\sin^3\varphi.\end{aligned}\right.$$

That is the very simple form that the equilibrium equations will take when one refers them to the asymptotic lines. Furthermore, it should be pointed out that one will arrive at exactly the same equations in regard to the tensions for two arbitrary conjugate lines. However, the asymptotes are the only lines of that type that are known *a priori* and whose orientation does not depend upon either the given forces or the arbitrary functions that are introduced by the integration. It goes without saying that in order to perform that integration, $d\sigma$ and $d\sigma_1$ must be expressed as functions of the asymptotic line parameters.

The property that the asymptotic lines enjoy of being the characteristics of the tensions that develop on the surface is exhibited quite well by equations (18). For example, the value of n_1 can be chosen arbitrarily for all points of the curve along which the arc length σ is measured. However, the second equation will then give the corresponding value of n with no indeterminacy, provided that $1 / \rho_1$ is not zero; i.e., provided that the asymptotes of the other system are not rectilinear generators. (We shall pass over that case, for the moment.) n and n_1 will be determined then, so the first equation will give $\partial n / \partial \sigma_1$; i.e., n will be known for the points of a curve that is infinitely close to the first one, and the second equation will provide the corresponding value of n_1 with the introduction of an arbitrary constant. Upon continuing in that way, one will get the state of the entire surface. However, that will introduce an arbitrary constant whenever one passes from one characteristic to the following one, which amounts to saying that one can assign a series of arbitrary values to the tension n_1 for all points at which one arbitrary curve meets the successive characteristics. If one makes that new curve coincide with a characteristic of the other system then one will get the following theorem:

In order to know the equilibrium state of the surface, it is necessary and sufficient to know one of the tensions in two asymptotic lines that do not belong to the same system.

One can also phrase that by saying:

The most general value of the tension has the form:

$$n = \varphi[\alpha, \beta, \varpi(\alpha), \chi(\beta)], \qquad n_1 = \varphi_1[\alpha, \beta, \varpi(\alpha), \chi(\beta)],$$

in which α *and* β *are the parameters of the asymptotic lines, and* ϖ, χ *are two arbitrary functions.*

Since the knowledge of the tensions that are exerted upon two directions at a point will imply knowledge of the tensions that are exerted in all possible directions, the form that we just established can be applied in any coordinate system, and one can always represent the solutions of the problem by:

$$n_1 = N_1[\alpha, \beta, \varpi(\alpha), \chi(\beta)],$$
$$n_2 = N_2[\alpha, \beta, \varpi(\alpha), \chi(\beta)],$$
$$t = \Theta[\alpha, \beta, \varpi(\alpha), \chi(\beta)].$$

If the surface is convex then α and β will be conjugate imaginary quantities. One must then choose ϖ and χ in such a fashion as to obtain real values for the tensions.

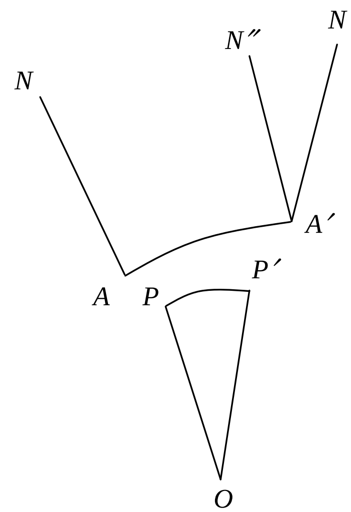

Figure 7

Equations (18) can serve to transform the problem of the equilibrium of surface in a remarkable way. Let *AA′* (Fig. 7) be the element $d\sigma$ of an asymptotic line. Construct the spherical indicatrix of the surface by drawing parallels *OP*, *OP′* to the normals *AN*, *A′N′* through the center *O* of a sphere of radius 1. By virtue of the definition of asymptotic lines, the normal *A′N′* will project onto the plane *NAA′* along a line *A′N″* that is parallel to *AN*. The element *PP′* of the spherical indicatrix is parallel to the plane *N′A′N″*. It will then be perpendicular to *AA′*. As a result:

The spherical indicatrix of an asymptotic line has its tangents perpendicular to those of the asymptotic line.

The tangent planes to the corresponding points on the surface and the sphere are parallel, moreover, so we can add that:

The geodesic contingency angles of the two curves are constantly equal.

Finally, the length $ds = PP'$ of the element of the indication is a measure of the angle *N′A′N″*, which is equal to $d\sigma / T$, where $1 / T$ is the geodesic torsion of the asymptotic line; i.e., the square root of the total curvature (taken positively). Hence:

$$ds = \frac{d\sigma}{T}.$$

Let $d\alpha$ be the contingency angle that is common to the two curves, and let ρ be the radius of geodesic curvature of the spherical indicatrix. One will have:

$$\frac{d\alpha}{ds} = T\frac{d\alpha}{d\sigma} \qquad \text{or} \qquad \frac{1}{\rho} = \frac{T}{\rho}.$$

We now return to equations (18) and remark that the indicatrices of the two asymptotic lines define the same angle α in the sphere as the lines in the surface. We can replace the quantities $1 / d\sigma$, $1 / d\sigma_1$, $1 / \rho$, $1 / \rho_1$ with the quantities that are equal to $\frac{1}{T\,ds}, \frac{1}{T\,ds_1}, \frac{1}{T\,r}, \frac{1}{T\,r_1}$, resp. Consequently, if one lets ν and ν_1 denote the quantities that correspond to u and u_1, resp., on the sphere then the equations will become:

$$\sin\varphi\frac{\partial n}{\partial s_1} - n\left(\frac{1}{\nu} + \frac{2}{\nu_1}\cos\varphi\right) + \frac{n_1}{r} = f_1\,T\sin^3\varphi,$$

$$\sin\varphi\frac{\partial n_1}{\partial s} - n_1\left(\frac{1}{\nu_1} + \frac{2}{\nu}\cos\varphi\right) + \frac{n}{r_1} = f\,T\sin^3\varphi.$$

If the spherical indicatrices of the asymptotic lines are lines of conjugate tension, and if the components of the external forces along the tangents to those curves are f_1T, $f\,T$ then the equilibrium equations will be precisely the ones that we just obtained.

Hence:

When one knows the equilibrium state of a sphere of radius 1 *that is subject to given external forces, one can deduce the equilibrium state of all surfaces for which the asymptotic lines admit lines of conjugate tension in the sphere as spherical indicatrices and which are subject to a tangential force at each point that is perpendicular to the one at the corresponding point on the sphere, where the ratio of the two forces is equal to the square root of the total curvature.*

Observe that the system of forces that are applied to the sphere does not, by any means, need to be tangential. If one is given a surface that is subject to arbitrary tangential forces then one can even perform the transformation that was just pointed out and then seek to determine the normal forces for the sphere such that the transforms of the asymptotic lines can be lines of conjugate tension. The calculation is done in the following manner with no difficulty:

Let:

$$n_1\,x^2 + 2t\,xy + n_2\,y^2 = 1$$

be the indicatrix of the tensions at a point on the sphere. Two conjugate directions are coupled by the relation:

$$n_2\,mm' + t\,(m + m') + n_1 = 0.$$

If that equation is satisfied for two given directions for which $mm' = a$, $m + m' = b$ then one will have:

$$a\,n_2 + bt + n_1 = 0,$$

which is a relation that will permit one to eliminate n_1, n_2, t when it is combined with the three general equations. One will then obtain an equation that determines the normal force Φ. However, that equation is a second-order partial differential equation and cannot have any utility for the solution of the problem, in general. The true interest of the transformation that we have just described results from the statement of the preceding theorem: It consists of the fact that any solution that is found on the sphere can be immediately generalized.

That generalization will often be facilitated by the use of isothermal geographic coordinates, which were invented by Ossian Bonnet [Journal de Liouville (2), t. V] and which provide a very precise instrument for the study of surfaces. One will obtain it upon putting the equation of the tangent plane in the form:

$$X \sin\theta \cos\varphi + Y \sin\theta \sin\varphi + Z \cos\theta = \delta$$

and setting:

$$x = \varphi,$$
$$y = \log\tan\frac{\theta}{2},$$
$$z = -\frac{\delta}{\sin\theta}.$$

The equation of the tangent plane will then become:

$$X \cos x + Y \sin y + Zi \cos iy + z = 0.$$

The geometric interpretation of x, y, z is very simple. If one draws a parallel to each normal to the surface through the origin and takes the point M at which it meets a sphere of radius 1 that has the origin for its center then the stereographic projection M of that point onto the XY-plane will have coordinates with respect to the axes OX, OY that are quantities x', y', which are coupled with x, y by the formulas:

$$x' + i\,y' = e^{y+ix},$$

$$x' - i\,y' = e^{y-ix};$$

i.e., $y + ix$, $y - ix$ are the Napierian logarithms of $x' + i\,y'$, $x' - i\,y'$. The third coordinate z is the distance from the origin to the trace of the tangent plane on the xy-plane. The Cartesian coordinates ξ, η, ζ are coupled with x, y, z by the equations:

$$\xi \sin x - \eta \cos x = p,$$

$$\xi \cos x + \eta \cos x = -z - qi \tan i\,y,$$

$$\zeta \cos i\,y = q.$$

Upon letting p, q, r, s, t denote the partial derivatives of z with respect to x and y, as is customary, and setting:

$$u = r + i \tan i\, y\, q + z,$$
$$v = s,$$
$$w = t + i \tan i\, y\, q,$$

one will find that the differential equation of the asymptotic lines is:

$$u\, dx^2 + 2v\, dx\, dy + w\, dy^2 = 0.$$

The total curvature is equal to $\dfrac{1}{\cos^2 iy\,(uw - v^2)}$, and as a result, the tangential force that is applied to the surface will be equal to the one that is applied to the sphere, divided by $\cos i\, y \sqrt{v^2 - uw}$.

A new transformation will permit one to reduce the study of the equilibrium of the given surface to that of the equilibrium in a plane. In order to do that, it will suffice to consider the stereographic projection that we shall appeal to in order to interpret the coordinates x¸ y, z. That projection does not change angles, and it will reduce the infinitely-small lengths by a ratio that is easy to calculate: That ratio is $(1 + e^{2y}) / 2$. Starting from it, one will see, as before, that both the equations that are applicable to the sphere will be applicable to the plane, provided that one multiplies the forces f_1T, $f\,T$ by $(1 + e^{2y}) / 2 = e^y \cos\ i\, y$.

Here, we are no longer capable of introducing normal forces that would make the transformation of the asymptotic lines be lines of conjugate tension. Indeed, the third equation of equilibrium:

$$\frac{n_1}{R_2} + \frac{n_2}{R_1} - \frac{2t}{T} = \Phi$$

will reduce to $\Phi = 0$ in the case of the plane; i.e.:

A plane can be in equilibrium only if it contains all of the forces that are applied to it.

However, in revenge, when a plane is subject to tangential forces, one needs to satisfy only two equations in the three unknowns n_1, n_2, t, and one can impose a third condition; for example, that a given net of curves defines the lines of conjugate tension. Suppose that this net has a differential equation:

$$U\, d\lambda^2 + 2V\, d\lambda\, d\mu + W\, d\mu^2 = 0$$

when it is referred to arbitrary orthogonal coordinates λ, μ, or, upon setting $\dfrac{M\, d\mu}{L\, d\lambda} = m$:

$$L^2\, W\, m^2 + 2\, VLM\, m + UM^2 = 0.$$

Since the equation of the indicatrix of tensions is:

$$n_1 X^2 + 2t\, XY + n_2 Y^2 = 1,$$

two conjugate directions μ, μ' will be coupled by the relation:

$$n_2\, \mu\mu' + t\,(\mu + \mu') + n_1 = 0.$$

If μ, μ' are roots of the equation in m then one will have:

$$\mu\mu' = \frac{UM^2}{WL^2}, \qquad \mu + \mu' = -\frac{2V}{W}\frac{L}{M},$$

and thus, the condition:

$$n_2\, UM^2 - 2VLM\, t + n_1\, WL^2 = 0.$$

When this equation is combined with:

$$\frac{1}{L}\frac{\partial n_2}{\partial \lambda} - \frac{1}{M}\frac{\partial t}{\partial \mu} + \frac{n_1 - n_2}{\rho_2} + \frac{2t}{\rho_1} = F_1,$$

$$\frac{1}{M}\frac{\partial n_1}{\partial \mu} - \frac{1}{L}\frac{\partial t}{\partial \lambda} + \frac{n_2 - n_1}{\rho_1} + \frac{2t}{\rho_2} = F_2,$$

that will determine the values of n_1, n_2, t that are appropriate to the problem.

Upon replacing λ, μ with the isothermal coordinates x, y, we will have to set:

$$L = M = \frac{1}{\cos iy},$$

$$\frac{1}{\rho_1} = 0, \qquad \frac{1}{\rho_2} = -\cos iy;$$

U, V, W will become u, v, w for the transforms of the asymptotic lines. With that, the equilibrium equations will take the form:

$$\frac{\partial n_2}{\partial x} - \frac{\partial t}{\partial y} + n_2 - n_1 = \frac{F_1}{\cos iy},$$

$$\frac{\partial n_1}{\partial y} - \frac{\partial t}{\partial x} - 2t \qquad = \frac{F_2}{\cos iy},$$

$$n_2\, u - 2vt + n_1\, w = 0.$$

The quantities F_1, F_2 that figure in the right-hand sides are the components in the plane of the external force along the two directions y = const., x = const., resp. Upon letting α, β denote the angles that are forces by the first of those direction and the transforms of the asymptotic lines then one will have:

$$F_1 = T\,e^y \cos iy\,(f_1 \cos\alpha + f_2 \cos\beta),$$

$$F_2 = T\,e^y \cos iy\,(f_1 \sin\alpha - f_2 \sin\beta),$$

where $\tan\alpha$ and $\tan\beta$ are the roots of the second-degree equation:

$$w\,m^2 + 2v\,m + u = 0.$$

The transformation is then achieved with no difficulty here. Upon performing the inverse transformation, if one knows the equilibrium conditions for a portion of the plane that is subject to given forces then one can always deduce equilibrium equations for an infinitude of surfaces that are located in manner that one must determine.

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CHAPTER IV

APPLICATIONS

The objective of this chapter is to apply the general theories that were just presented to a certain number of particular cases.

The simplest of all surfaces is the plane. Its static properties differ radically from those of all other surfaces and can be summarized in the following two propositions, which we have already had occasion to prove:

I. *A plane can be in equilibrium only if it contains all of the forces that are applied to it.*

II. *The equilibrium of a plane (when it is possible) is defined by a system of two equation in three unknowns, which will permit one impose an arbitrary third condition.*

In Cartesian coordinates, the equilibrium equations are:

$$\frac{\partial n_2}{\partial x} - \frac{\partial t}{\partial y} = F_1 ,$$

$$\frac{\partial n_1}{\partial y} - \frac{\partial t}{\partial x} = F_2 .$$

If one imposes the condition t = const. then they will reduce to:

$$\frac{\partial n_2}{\partial x} = F_1 , \qquad \frac{\partial n_1}{\partial y} = F_2 ,$$

which are equations whose integration will come down to simple quadratures; n_1 refers to an arbitrary function of x, and n_2 refers to an arbitrary function of y.

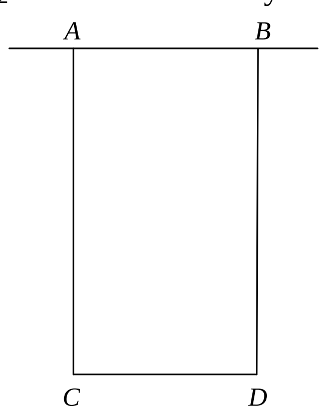

Figure 8.

For example, let $ABCD$ be a vertical rectangle (Fig. 8) whose horizontal side AB is kept fixed, while the other three sides are free, and its weight per unit area is a constant P and is supported by its lower part with a weight of $\varpi \times CD$ that is distributed uniformly along CD. Upon taking CD, CA to be the x and y axes, resp., and setting $AC = h$, $CD = l$, one will have:

$$\frac{\partial n_2}{\partial x} = 0, \qquad \frac{\partial n_1}{\partial y} = P,$$

under the hypothesis that $t = 0$, so:

$$n_2 = \text{funct. } y, \quad n_1 = Py + \text{funct. } x.$$

n_2, which will be annulled for $x = 0$ and $x = l$ for any y, is identically zero. The arbitrary function that n_1 is referred to is determined by setting $y = 0$, which will give $n_1 = \varpi$ for any x. The equilibrium state is then:

$$n_1 = Py + \varpi, \qquad n_2 = 0, \qquad t = 0.$$

If the side BD, rather than being free, is subject to a variable effort of extension that is expressed by the condition $n_2 = Y$ (Y being a function of y) then one no longer make the hypothesis $t = 0$; however, one can suppose that $t = F(y)$, where F is a conveniently-chose function of y. Indeed, one will then have:

$$\frac{\partial n_2}{\partial x} - F'(y) = 0,$$

so

$$n_2 = F'(y)\, x + \text{funct. } y.$$

The second function of y will zero, since n_2 will then be subject to being annulled for $x = 0$. Upon setting $x = l$, one will have:

$$l\, F'(y) = Y,$$

which will determine F. The value of n_1 will not be modified.

In polar coordinates ρ, ω, the equilibrium equations in the plane are:

$$\frac{\partial n_2}{\partial \rho} - \frac{1}{\rho}\frac{\partial t}{\partial \omega} - \frac{n_1 - n_2}{\rho} = F_1 ,$$

$$\frac{1}{\rho}\frac{\partial n_2}{\partial \omega} - \frac{\partial t}{\partial \rho} - \frac{2t}{\rho} = F_2 .$$

We apply these formulas to the case of a circle that is subject to external forces F that are everywhere directed along the radius and constant for each value of ρ.

Upon supposing $t = 0$, one will have:

$$\frac{1}{\rho}\frac{\partial n_1}{\partial \omega} = 0,$$

$$\frac{\partial n_2}{\partial \rho} - \frac{n_1 - n_2}{\rho} = F,$$

for an arbitrary point.

The first equation shows that n_1 depends upon only ρ. Let $n_1 = d\chi / d\rho = \chi'(\rho)$. The second equation will give:

$$\frac{d}{d\rho}(n_2\,\rho) = \chi'(\rho) + F\rho,$$

so

$$n_2\,\rho = \chi(\rho) + \int_0^\rho F\rho\,d\rho + \Omega,$$

in which Ω is a function of ω. One will infer from this that:

$$n_2 = \frac{\chi(\rho)}{\rho} + \frac{1}{\rho}\int_0^\rho F\rho\,d\rho + \frac{\Omega}{\rho}.$$

In order for n_2 to remain finite when one approaches the center, it is first necessary that Ω should be identically zero. Moreover, it is necessary that $\chi(\rho)$ should tend to zero. If that is true then the limit of n_2 will be $\chi'(\rho)$. It will then be equal to that of n_1. Without that equality, t cannot be zero for any of the directions in the neighborhood of the center. In summary, the equilibrium state is represented by:

$$n_1 = \chi'(\rho),$$

$$n_2 = \frac{\chi(\rho)}{\rho} + \frac{1}{\rho}\int_0^\rho F\rho\,d\rho,$$

$$t = 0.$$

The arbitrary function $\chi(\rho)$ is required only to be annulled for $\rho = 0$ and to take a well-defined value when ρ becomes equal to the radius of the circle.

If, instead of a circle, one considers that circular zone that is found between two circumferences of radius ρ and ρ', on which normal efforts that are constant per unit length are exerted then the same formulas will be applicable. In that case, the function χ does not need to become zero for $\rho = 0$, but it must take well-defined values for the limiting radii ρ and ρ'.

We shall give an example of the transformations that were at issue in the preceding chapter for the very simple case that was just studied.

The tension indicatrix has the equation:

$$\chi'(\rho)\, x^2 + \frac{\chi(\rho) + \int F\rho\, d\rho}{\rho}\, y^2 = 1.$$

The relation between two conjugate directions m, m' is then:

$$m\, m'\, [\chi(\rho) + \int F\rho\, d\rho\,] + \rho\, \chi'(\rho) = 0.$$

Introduce isothermal coordinates and set:

$$\rho = e^y$$

for them.

Let $\varpi(y)$ be what $\chi(\rho)$ becomes, and let $\varphi(y)$ be what $\int F\rho\, d\rho$ becomes. One has:

$$\rho\, \chi'(\rho) = \frac{d\varpi}{dy} = \varpi'(y)\,.$$

Hence:

$$m\, m'\, [\varpi(y) + \varphi(y)] + \varpi'(y) = 0.$$

In order for the transforms of the asymptotic lines of a surface to satisfy that equation, one must have:

$$\frac{u}{w} = -\frac{\varpi'(y)}{\varpi(y) + \varphi(y)}\,,$$

or, upon setting $\dfrac{\varpi'(y)}{\varpi(y) + \varphi(y)} = -K$:

$$\frac{u}{w} = K.$$

Suppose, to simplify, that one has $v = 0$, in addition (which amounts to saying that the transforms are inclined the same with respect to the polar radius). If one refers to the definitions of u, v, w then one will first find that:

$$\frac{\partial^2 z}{\partial x\, \partial y} = 0,$$

so

$$z = X + Y.$$

X and Y represent two functions, one of which is a function of x and the other of which is a function of y, resp. Moreover, one has:

$$r + i \tan i\, y\, q + z - K\, (t + i \tan i\, y\, q) = 0$$

or rather:

$$X'' + i \tan i\, y\, Y' + X + Y - K\, (Y'' + i \tan i\, y\, Y') = 0.$$

Since K is necessarily a function of only y, that equation will be possible only if one has:

$$X + X = C,$$
$$KY'' + (K - 1)\, i \tan i\, y\, Y' - Y = C,$$

when one lets C denote a constant.

The latter equation can be integrated in certain special cases. For example, if one has $K = -1$ then it will become:

$$KY'' + 2i \tan i\, y\, Y' + Y + C = 0,$$

and if h and g are constants, one will find that:

$$Y = h\,(\cos i\, y + i\, y \sin i\, y) + g \sin i\, y - C\,;$$

X is equal to $A \cos x + B \sin x + C$.

As a result:

$$z = h\,(\cos i\, y + i\, y \sin i\, y) + g \sin i\, y + A \cos x + B \sin x.$$

According to a remark by Ossian Bonnet, one can always make the terms in $\cos x$, $\sin x$, and $\sin i\, y$ disappear by a simple change of origin. What will then remain is:

$$z = h\,(\cos i\, y + i\, y \sin i\, y)\,.$$

One will infer from this that:

$$q = -h\, y \cos i\, y,$$

$$z + i \tan i\, y\, q = h \cos i\, y,$$

and upon passing to the Cartesian coordinates:

$$\xi^2 + \eta^2 = (z + i \tan i\, y\, q)^2 = h^2 \cos^2 i\, y,$$

$$z = \frac{q}{\cos iy} = -hy\,.$$

Those equations represent a surface of revolution whose meridian is given by:

$$\xi = h \cos i\, y,$$

$$\zeta = -h\, y,$$

or

$$\xi = \frac{h}{2}(e^{\zeta/h} + e^{-\zeta/h})\,.$$

It is a catenary whose directrix coincides with the axis of revolution. The surface that is generated in that fashion has been studied by several geometers, and Bour gave it the name of *alysseid*. It is the only surface of revolution that is minimal; in addition, it enjoys the property of being mappable to a ruled helicoid.

The semi-angle of the asymptotic has the tangent:

$$\sqrt{\frac{\varpi'(y)}{\varpi(y)+\varphi(y)}} = \sqrt{-K} = 1.$$

These lines are real then and intersect at a right angle, which is a property that is characteristic of minimal surfaces.

Since $\dfrac{\varpi'(y)}{\varpi(y)+\varphi(y)} = 1$, one has:

$$\varpi'(y) = \varpi(y) + \varphi(y),$$

or rather:

$$\rho\,\varpi'(y) = \chi(\rho) + \int F\,\rho\,d\rho,$$

$$\chi'(y) = \frac{\chi(\rho)+\int F\,\rho\,d\rho}{\rho};$$

it results from this that $n_1 = n_2$, and since $t = 0$, each point of the surface will be in a state of umbilical equilibrium. Hence:

If a portion of an alysseide that is subjected to external forces that are tangent to the parallel and constant along each of them terminates with a parallel to the elements upon which a constant tension is exerted then all of the points will be in a state of umbilical equilibrium.

Naturally, the law by which the tension varies when one passes from one parallel to another depends upon F. One should not forget that F is the force applied to the plane. In order to get the corresponding force on the surface, it suffices to recall that its components along the asymptotic lines are equal to the analogous components for the plane, multiplied by $\dfrac{1}{T\,e^y\cos iy}$. Since the angles between the asymptotic lines and the coordinate lines do not vary under the transformation, the force applied to the surface will be directed tangentially to the parallel and will be equal to $\dfrac{F}{T\,e^y\cos iy}$. T is the inverse of the square root of the curvature; its value for the alysseide is $h\cos^2 i\,y$. The applied force is then:

$$\frac{F}{T\,e^y\cos iy}.$$

When the external force is constantly zero, one will have $F = 0$. The equation that determines c will then reduce to:

$$\rho\,\chi'(\rho) - \chi(\rho) = 0,$$

so

$$\frac{\chi(\rho)}{\rho} = \text{const.} = a,$$

and as a result $\chi'(\rho) = a$. In that case, the tensions n_1, n_2 are equal and constant for all points of the surface.

Developable surfaces

From what we know already, the developable surfaces are the only ones for which one cannot make the normal component of the external forces disappear.

Take the rectilinear generators to be the coordinates μ = const. One will see immediately that:

$$\frac{1}{R_1} = 0, \qquad \frac{1}{\rho_1} = 0, \qquad \frac{1}{T} = 0\,.$$

Since $\dfrac{\partial L}{\partial \mu} = -\dfrac{LM}{\rho_1} = 0$, one will have L = funct. λ, and one can set $L = 1$ with a convenient choice of λ. The general equations of the theory of surfaces then reduce to:

$$\frac{\partial M}{\partial \lambda} = -\frac{M}{\rho_2},$$

$$\frac{\partial(1/R_2)}{\partial \lambda} = \frac{1}{R_2\,\rho_2},$$

$$\frac{\partial(1/\rho_2)}{\partial \lambda} = \frac{1}{\rho_2^2}\,.$$

The last equation can be written:

$$\frac{1}{\rho_2^2}\left(1 + \frac{\partial \rho_2}{\partial \lambda}\right) = 0.$$

If $1 / \rho_2 = 0$, M, and R_2 are functions of only μ. They will then be constants all along the same generator; it is easy to see that this is possible only for cylinders. In the other cases, one will have:

$$\frac{\partial \rho_2}{\partial \lambda} + 1 = 0,$$

so

$$\rho_2 = a - \lambda\,.$$

a is a function of μ that represents the distance from the point for which $\lambda = 0$ to the point of contact with the edge of regression. One then deduces that:

$$M = -\frac{1}{b}(a - \lambda),$$

$$R_2 = + R\left(\frac{a-\lambda}{a}\right),$$

upon letting b and R denote two other functions of μ.

As a result, the equilibrium equations become:

$$(\lambda - a)\frac{\partial n_2}{\partial \lambda} - b\frac{\partial t}{\partial \mu} + n_2 - n_1 = F_1\,(\lambda - a),$$

$$b\,(\lambda - a)\frac{\partial n_1}{\partial \mu} - (\lambda - a)^2\frac{\partial t}{\partial \lambda} + 2\,(\lambda - a)\,t = F_2\,(\lambda - a)^2,$$

$$n_1 = - R\,\Phi\left(\frac{\lambda - a}{a}\right),$$

or, upon setting:

$$n = (\lambda - a)\,n_2\,,$$

$$\theta = -(\lambda - a)^2\,t,$$

$$\frac{\Phi R}{a} = -\Psi,$$

one will get:

$$\frac{\partial n}{\partial \lambda} = -\frac{b}{(\lambda - a)^2}\frac{\partial \theta}{\partial \mu} + (F_1 + \Psi)\,(\lambda - a) + \frac{b}{(\lambda - a)^2}\frac{\partial \theta}{\partial \mu},$$

$$\frac{\partial \theta}{\partial \lambda} = \left(F_2 - b\frac{\partial \Psi}{\partial \mu}\right)(\lambda - a)^2 + b\,\Psi\,(\lambda - a)\frac{\partial a}{\partial \mu},$$

$$n_1 = \Psi\,(\lambda - a).$$

The third equation gives n_1 immediately. The second one determines θ by means of an equation that one integrates by a simple quadrature upon considering μ to be constant. The value of θ refers to an arbitrary function of μ. One will then get n by means of the first equation, which one likewise integrates by quadrature and which introduces a new arbitrary function of μ.

When the portion of the surface considered refers to an arc of the edge of regression, the values that pertain to that edge will be obtained by making $\lambda - a$ tend to zero. Upon writing out that this is true, one will get two equations that determine the two arbitrary functions of m for all points that are found between the tangents that are drawn through the two endpoints of the arc of regression. If those two conditions are satisfied then one will have:

$$n_2 = \lim \frac{n}{\lambda - a} = \frac{\partial n}{\partial \lambda},$$

$$t = \lim -\frac{\theta}{(\lambda - a)^2} = -\frac{1}{2(\lambda - a)}\frac{\partial \theta}{\partial \lambda} = -\frac{b\Psi}{2}\frac{\partial a}{\partial \mu}.$$

Since $\partial n / \partial \lambda$ can be written:

$$(F_1 + \Psi)\,(\lambda - a) + b\frac{\partial t}{\partial \mu},$$

all that will remain is:

$$n_2 = b\frac{\partial t}{\partial \mu}.$$

Choose μ in such a fashion that $d\mu$ is the angle between the two consecutive generators. In that case, b will be equal to 1, and $\partial a / \partial \mu$ represents the radius r of the first curvature of the edge of regression. One will then find the system of tensions:

$$t = -\frac{\Psi r}{2} = \frac{\Phi}{2a} Rr,$$

$$n_2 = \frac{\partial t}{\partial \mu},$$

$$n_1 = 0.$$

It should be remarked that a and b will remain invariable when one deforms the surface. As a result, when F_1, F_2, Ψ (or F_1, F_1, and ΦR, if one prefers) keep constant values at each point, the equilibrium equations will not be modified. In particular, when $\Phi = 0$, one can deform the surface while keeping the same tangential forces without changing the equilibrium equations. For any Φ, one can continue the deformation up to the point that R becomes infinite while keeping ΦR constant; i.e., until that portion of the given surface is mapped to a plane. The equilibrium conditions of developable surfaces are then found to be identified with those of the plane. The only exceptional case is the case that was envisioned above in which the portion of the surface considered contains a part of the edge of regression, since then surface is not truly developable in the neighborhood of that edge.

Whenever R is finite, n_1 will be annulled at the same time as Φ. As a result:

When a developable surface is subject to tangential external forces, it will produce no effort of extension along the generators.

We have skipped over the case of the cylinder. That surface is characterized by $1 / \rho_2 = 0$, which implies that:

$$M = \text{funct.}\,(\mu), \qquad R_2 = \text{funct.}\,(\mu).$$

One can suppose that $M = 1$, and the equilibrium equations will then reduce to:

$$\frac{\partial n_2}{\partial \lambda} - \frac{\partial t}{\partial \mu} = F_1 ,$$

$$\frac{\partial n_1}{\partial \mu} - \frac{\partial t}{\partial \lambda} = F_2 ,$$

$$n_1 = R_2\, \Phi,$$

which are equations whose integration once more comes down to quadratures.

The equation $n_1 = R_2\, \Phi$ is completely analogous to the equation $T = - P_n\, \rho$, which corresponds to the equilibrium of a funicular curve. If one imposes the condition that $t = 0$ then the second equation will reduce to $\partial T / \partial s = F_2$, and will then be equivalent to the equation $\partial T / \partial s = - P_t$ for funicular equilibrium. In that case, the first equation will reduce to $\partial n_2 / \partial \lambda = F_1$. It will determine the variation of the third tension n_2 along a generator of the cylinder.

The cone reverts to the general case of developable surfaces. No matter what the directrix is, the quantity a will be constant. The quantity b is also constant and can be supposed to be equal to 1. That amounts to saying that by virtue of the formula $M = \lambda / b$, m will represent the length of the variable arc on the cone that is intercepted by the sphere of radius 1 that is described with its center at the summit of the cone when one starts from a fixed origin.

As a special case, consider the lateral surface of a frustum of a cone whose bases are entirely free and to which one applies arbitrary forces. Upon setting a = const., $b = 1$ and introducing two new functions f_1, f_2 that are defined by:

$$(\lambda - a)^2 \left(F_2 - \frac{\partial \Psi}{\partial \mu} \right) = \frac{\partial f_2}{\partial \lambda},$$

$$(F_1 + \Psi)(\lambda - a) - \frac{1}{(\lambda - a)^2} \frac{\partial f_2}{\partial \mu} = \frac{\partial f_1}{\partial \lambda},$$

one will first get:

$$\frac{\partial \theta}{\partial \lambda} = \frac{\partial f_2}{\partial \lambda},$$

by virtue of the established equations, so $\theta = f_2 + \varpi(\mu)$, in which ϖ denotes an arbitrary function, and then:

$$\frac{\partial n}{\partial\lambda} = (F_1 + \Psi)(\lambda - a) - \frac{1}{(\lambda-a)^2}\left[\frac{\partial f_2}{\partial\mu} + \varpi'(\mu)\right],$$

so

$$n = f_1 + \varpi'(\mu) + \chi(\mu),$$

in which χ is an arbitrary function.

From this, the tensions n_1, n_2, t have the values:

$$n_1 = -\Phi\, R\,\frac{\lambda - a}{a},$$

$$n_2 = \frac{1}{\lambda - a}(f_1 + \chi) + \frac{1}{(\lambda-a)^2}\varpi',$$

$$t = -\frac{1}{(\lambda-a)^2}(f_2 + \varpi).$$

It remains to determine $\varpi(\mu)$ and $\chi(\mu)$ in such a fashion that the boundary conditions will be fulfilled. In order to do that, we suppose that we have taken the base of the cone to be its two curves of intersection with the two spheres of radius $a + c$, $a - c$ that are described with their centers at the cone's summit. Those two curves correspond to the values $\pm c$ of λ.

From that, n_2 and t must be annulled for any μ when one sets $\lambda = \pm c$. In order to continue the calculations to their ultimate conclusion, we shall suppose that $F_1 + \Psi$ and $F_2 - \partial\Psi / \partial\mu$ are independent of λ. Upon setting:

$$F_1 + \Psi = 2\mu_1,$$

$$F_2 - \frac{\partial\Psi}{\partial\mu} = 3\mu_2,$$

we will have:

$$f_1 = (\lambda - a)^2 (\mu_1 - \tfrac{1}{2}\mu_2'),$$

$$f_2 = (\lambda - a)^2\, \mu_2.$$

It is pointless to add new arbitrary functions, which would do double duty with ϖ and χ. As a result, the values of n_2 and t will be:

$$n_2 = \frac{1}{(\lambda-a)^2}[\varpi' + \chi(\lambda - a) + (\mu_1 - \tfrac{1}{2}\mu_2')(\lambda - a)^3],$$

$$t = -\frac{1}{(\lambda-a)^2}[\varpi + (\lambda - a)^3\mu_2],$$

and one can conclude immediately that if n and t are zero for any μ then the quantities ϖ, χ, μ_1, μ_2 will all be identically zero for $\lambda = + c$ and $\lambda = - c$.

Equilibrium will be possible then only if one has $\mu_1 = 0$, $\mu_2 = 0$, or rather:

$$F_1 + \Psi = 0, \qquad F_2 - \frac{\partial \Psi}{\partial \mu} = 0.$$

For the points that are situated on the mean curve $\lambda = 0$, those equations can be written as:

$$a\,F_1 + \Phi R = 0, \qquad a\,F_2 - \frac{\partial \Phi R}{\partial \mu} = 0.$$

If the forces that are applied to the frustum of the cone do not verify those two conditions then the surface will necessarily be deformed. It will preserve the same geometric character while it is deforming, and the same equilibrium equations will continue to be applicable. If one then supposes that F_1, F_2 , Φ do not vary during the deformation then the condition (viz., $a\,F_1 + \Phi R = 0$) will determine R, and a result, the equilibrium figure of the cone. When R has been determined in that way, equilibrium cannot be realized unless the second condition (viz., $a\,F_2 - \frac{\partial \Phi R}{\partial \mu} = 0$) is fulfilled.

The equation $F_2 - \frac{1}{a}\frac{\partial \Phi R}{\partial \mu} = 0$ is exactly the one that one obtains by applying the normal force Φ and the tangential force F_2 to the mean curve, when it is considered to be a funicular curve, and looking for the condition for it to not deform.

When the equilibrium figure is a cone of revolution, the equation $F_1 + \Phi R = 0$ expresses the idea that the sum of the projections of the external forces that are applied to an arbitrary point onto the axis of the cone is equal to zero. Indeed, if one lets α denote the semi-angle of the summit of the cone then the radius of curvature of the normal section to the edge that is made at a distance a from the summit will be $R = a \tan \alpha$, and as a result, the equation that one must address can be written as $F_1 \cos \alpha + \Phi \sin \alpha = 0$.

Skew surfaces

Upon taking the coordinates lines to be the rectilinear generators (μ = const.) and their orthogonal trajectories (λ = const.), one will have:

$$\frac{1}{R_1} = 0, \qquad \frac{1}{\rho_1} = 0,$$

from which, one will first deduce that $\partial L / \partial \mu = 0$, which will permit one to set $L = 1$, as before. The general equations of the theory of surfaces will then become:

$$\frac{1}{\rho_2} = -\frac{\partial M/\partial\lambda}{M},$$

$$\frac{\partial(1/R_t)}{\partial\lambda} - \frac{1}{M}\frac{\partial(1/T)}{\partial\mu} - \frac{1}{\rho_2}\frac{1}{R_2} = 0,$$

$$-\frac{\partial(1/T)}{\partial\lambda} + \frac{2}{\rho_2}\frac{1}{T} = 0,$$

$$\frac{1}{T^2} + \frac{\partial(1/\rho_2)}{\partial\lambda} - \frac{1}{\rho_2^2} = 0.$$

One infers from the last two that:

$$\frac{\partial}{\partial\lambda}\left(\frac{i}{T} + \frac{1}{\rho_2}\right) - \left(\frac{i}{T} + \frac{1}{\rho_2}\right)^2 = 0,$$

$$\frac{\partial}{\partial\lambda}\left(\frac{i}{T} - \frac{1}{\rho_2}\right) - \left(\frac{i}{T} - \frac{1}{\rho_2}\right)^2 = 0,$$

and as a result, upon letting μ_1 and μ_2 denote two functions of m :

$$\frac{1}{\dfrac{i}{T} + \dfrac{1}{\rho_2}} = \mu_1 - \lambda,$$

$$\frac{1}{\dfrac{i}{T} - \dfrac{1}{\rho_2}} = -\mu_2 + \lambda,$$

or rather:

$$\frac{2i}{T} = \frac{1}{\mu_1 - \lambda} - \frac{1}{\mu_2 - \lambda} = \frac{\mu_2 - \mu_1}{(\mu_1 - \lambda)(\mu_2 - \lambda)},$$

$$\frac{2}{\rho_2} = \frac{1}{\mu_1 - \lambda} + \frac{1}{\mu_2 - \lambda} = \frac{\mu_1 + \mu_2 - 2\lambda}{(\mu_1 - \lambda)(\mu_2 - \lambda)}.$$

Upon letting μ_3 denote a third function of μ, the first two general equations will give, in turn:

$$M^2 = (\mu_1 - \lambda)(\mu_2 - \lambda)$$

and

$$2\frac{\partial(1/R_2)}{\partial\lambda}-\frac{1}{R_2}\left(\frac{1}{\mu_1-\lambda}+\frac{1}{\mu_2-\lambda}\right)-i\sqrt{\frac{\mu_3}{(\mu_1-\lambda)(\mu_2-\lambda)}}\left[\frac{\mu_1'}{(\mu_1-\lambda)^2}-\frac{\mu_2'}{(\mu_2-\lambda)^2}\right]=0,$$

or

$$2\sqrt{(\mu_1-\lambda)(\mu_2-\lambda)}\,\frac{\partial(1/R_2)}{\partial\lambda}+\frac{1}{R_2}\frac{2\lambda-\mu_1-\mu_2}{\sqrt{(\mu_1-\lambda)(\mu_2-\lambda)}}-i\sqrt{\mu_3}\left[\frac{\mu_1'}{(\mu_1-\lambda)^2}-\frac{\mu_2'}{(\mu_2-\lambda)^2}\right]=0.$$

Upon integrating this and letting μ_4 denote a fourth function of μ, one will get:

$$2\sqrt{(\mu_1-\lambda)(\mu_2-\lambda)}\,\frac{1}{R_2}+i\sqrt{\mu_3}\left(\frac{\mu_1'}{\mu_1-\lambda}-\frac{\mu_2'}{\mu_2-\lambda}\right)=\mu_4\,.$$

M must be real for all real values of λ. That will be possible only if the roots μ_1, μ_2 of the trinomial $(\mu_1 - \lambda)(\mu_2 - \lambda)$ are conjugate imaginary quantities and μ_3 is positive. We then set:

$$\mu_1 = \alpha - \beta\, i,$$

$$\mu_2 = \alpha + \beta\, i,$$

and we set $\mu_3 = 1$, which is permissible. Upon replacing μ_4 with 2γ, we will find that:

$$(19)\qquad\left\{\begin{aligned}
\frac{1}{T}&=\frac{\beta}{(\lambda-\alpha)^2+\rho^2},\\
\frac{1}{\rho_2}&=-\frac{\lambda-\alpha}{(\lambda-\alpha)^2+\beta^2},\\
\frac{1}{R_2}&=\frac{\gamma}{\sqrt{(\lambda-\alpha)^2+\beta^2}}+\frac{(\lambda-\alpha)\beta'+\alpha'\beta}{[(\lambda-\alpha)^2+\beta^2]^{3/2}},\\
M&=\sqrt{(\lambda-\alpha)^2+\beta^2}.
\end{aligned}\right.$$

Some very simple geometric considerations that were pointed out by Bour in his theory of the deformation of surfaces (Journal de l'École Polytechnique, Cahier **34**, pp. 33) will allow us to interpret the parameters α and β. For a well-defined generator, α is the distance from the point $\lambda = 0$ to the central point and β is the ratio of the shortest distance between two infinitely-close generators to the angle between those generators. We add that γ can be interpreted just as simply. Indeed, we can write:

$$\frac{M\,d\mu}{R_2}=\gamma\,d\mu+d\arctan\frac{\beta}{\lambda-\alpha},$$

in which the differentials are taken while keeping λ constant. Now, the left-hand side expresses the contingency angle between the section of the surface and the plane that is perpendicular to the generator at the point considered. That contingency angle is obviously composed of:

1. The angle between two infinitely-close central planes.
2. The variation of the angle that is defined by the tangent plane to the surface and the corresponding central plane. That angle will have the tangent:

$$\frac{\beta\, d\mu}{(\lambda-\alpha)\, d\mu} = \frac{\beta}{\lambda-\alpha}.$$

Having said that, it will suffice to consider the right-hand side of the equality above in order to see that its last term represents precisely the second part of the contingency angle, and that as a result, its first term will represent the first part. $\gamma\, d\mu$ will then be the angle between two infinitely-close central planes, and one can say that:

The parameter γ is the quotient of the angle between two infinitely-close central planes with the angle between the corresponding generators.

If one constructs a spherical indicatrix of the skew surface by drawing parallels to the generators through a fixed point and takes their intersection with the sphere of radius 1 that has that fixed point for its center then one can further say that:

The parameter γ is the geodesic curvature of the spherical indicatrix

By means of the expressions (19), the general equilibrium equations will become:

$$\frac{\partial n_2}{\partial\lambda} - \frac{1}{\sqrt{(\lambda-\alpha)^2+\beta^2}}\frac{\partial t}{\partial\mu} + \frac{\lambda-\alpha}{\sqrt{(\lambda-\alpha)^2+\beta^2}}(n_2-n_1) = F_1\,,$$

$$-\frac{\partial t}{\partial\lambda} + \frac{1}{\sqrt{(\lambda-\alpha)^2+\beta^2}}\frac{\partial n_1}{\partial\mu} - \frac{\lambda-\alpha}{(\lambda-\alpha)^2+\beta^2}\,2t = F_2\,,$$

$$n_1\left[\gamma + \frac{(\lambda-\alpha)\beta' + \alpha'\beta}{(\lambda-\alpha)^2+\beta^2}\right] - \frac{2\beta t}{\sqrt{(\lambda-\alpha)^2+\beta^2}} = \Phi\sqrt{(\lambda-\alpha)^2+\beta^2}$$

in the case of skew surfaces.

The last one gives n_1 as a function of t, and upon substituting that value in the preceding equation, one will get t by means of a linear first-order partial differential equation. t will then include an arbitrary function of a certain quantity (which is obviously the parameter of the non-rectilinear asymptotic lines), in its expression: We can call those lines the *asymptotic lines of the second system*. Finally, the first equation will give n_2 by quadrature, with an arbitrary function of μ; i.e., of the parameter of the rectilinear generators. In order get back to the case of developable surfaces, it will suffice to set $\beta = 0$.

When the external forces are zero, the equation in t will have the form:

$$A\frac{\partial t}{\partial \lambda} + B\frac{\partial t}{\partial \mu} = Ct,$$

in which A, B, C depend upon λ and μ uniquely. If one lets $f(\lambda, \mu) = u$ represent the integral of the equation:

$$\frac{d\lambda}{A} = \frac{d\mu}{B},$$

or, in other words, if one lets u denote the parameter of the asymptotes of the second system, and if $t = \theta(\lambda, \mu)$ is a non-zero particular solution of the equation in t, moreover, then one will effortlessly see that the general solution will have the form:

$$t = \theta \cdot \varpi(u),$$

Which is an expression in which ϖ denotes an arbitrary function.

If results from this that if t is required to be annulled for all points along a segment of the generator then the corresponding values of $\varpi(u)$ will be zero. Since $\varpi(u)$ is constant along an arbitrary asymptote of the second system, it will follow that:

When t is zero for all points along a segment of the generator, if one draws non-rectilinear asymptotic lines through the endpoints of the segment then t will remain zero for any portion of the surface that is found between the two asymptotic lines.

The relation $\frac{n_1}{R_2} - \frac{2t}{T} = 0$ shows that if $1 / R_2$ is not zero (which is the general case) then the same theorem can be stated for the tension n_1.

Geometrically, one can conclude a property of ruled surfaces from that whose statement I have not encountered anywhere. In order to do that, it will suffice to recall that in the absence of any external force, the values of n_1, n_2, t will be proportional to the infinitely-small variations that $\frac{1}{R_1}, \frac{1}{R_2}, \frac{1}{T}$ can be subjected to. Saying that n_1 is zero for a portion of the ruled surface amounts to saying that $1 / R_1$ does not vary, and will consequently remain zero. Hence:

When an arbitrary generator remains rectilinear on a certain length under the infinitely-small deformation of a ruled surface, all of the generators will likewise remain rectilinear for the portion of that surface that is bounded by the same two asymptotic lines of the second system.

The theorem extends to an arbitrary finite deformation, but the two asymptotes that bound the band considered will then be progressively modified under the deformation, and one will need a special discussion in order to find the portions of the generator that remain rectilinear in each case. That difficulty will disappear when a generator remains rectilinear along all of its length, and one can state the following property:

When a generator of a ruled surface remains rectilinear for a certain deformation of the surface, all of the generators will likewise remain rectilinear.

That theorem can be regarded as the converse of a theorem that was established by Ossian Bonnet (Journal de l'École Polytechnique, Cahier **42**) and that we state thus:

If a ruled surface remains ruled after deformation then any arbitrary generator will remain rectilinear.

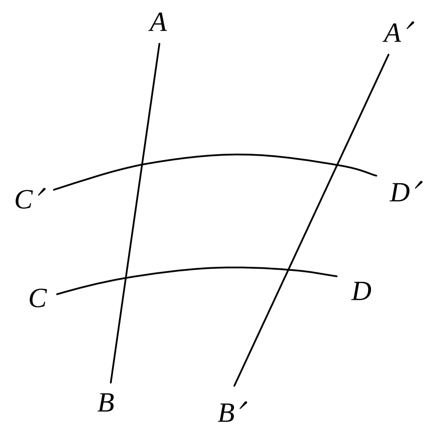

Figure 9.

The theorem of Ossian Bonnet can be deduced with no calculations, moreover, on the basis of what we just stated. In order to do that, consider a ruled surface whose rectilinear generators are *AB*, *A′B′*, … (Fig. 9). Assume that a system of geodesic lines *CD*, *C′D′*, … can become rectilinear as a result of a certain deformation.

Imagine a third deformation under which we keep the generator *AB* fixed and let the radii of normal curvature of the line *CD* grow until it becomes infinite. *CD* will then be a straight line, and since the surface Σ thus-obtained can be reduced to the one that admits *CD*, *C′D′*, … as its rectilinear generators by a convenient deformation, our theorem will show that the lines *CD*, *C′D′*, … are all rectilinear on the surface Σ. However, *AB*, and consequently, all analogous lines, will likewise remain straight. Hence:

If the two systems AB, AB,… and CD, CD, … do not coincide then the surface Σ *possesses a double system of rectilinear generators. It will then be a second-degree surface.*

The surfaces that can be mapped onto ones of degree two are then the only ones that can remain ruled without the original generators all remaining rectilinear. That is the only exception to the theorem of Ossian Bonnet, who also pointed out that exception himself.

When the surface has a director plane, γ will be zero, and if one supposes, moreover, that one has reduced the system of forces to a tangential system then the third equilibrium equation will take the form:

$$n_1 \frac{\partial}{\partial \mu}\left(\arctan \frac{\beta}{\gamma - \alpha}\right) = \frac{2\beta t}{\sqrt{(\lambda - \alpha)^2 + \beta^2}} .$$

In particular, consider the square-threaded screw surface. That surface is characterized by the constancy of α and β. One can arrange that α is zero. Under those conditions, the last equation will give $t = 0$ immediately. The other two will reduce to:

$$\frac{\partial n_2}{\partial \lambda} + \frac{\lambda}{\lambda^2 + \beta^2}(n_2 - n_1) = F_1 ,$$

$$\frac{\partial n_1}{\partial \mu} = F_2 \sqrt{\lambda^2 + \beta^2} .$$

Upon setting $F_2 = \partial f_2 / \partial \mu$, so $f_2 = \int F_2 \, d\mu$, and supposing that integral is taken from a well-defined lower limit (for example, from $\mu = 0$), one will get:

$$n_1 = f_2 \sqrt{\lambda^2 + \beta^2} + \varpi(\lambda);$$

hence:

$$\frac{\partial n_2}{\partial \lambda} + n_2 \frac{\lambda}{\lambda^2 + \beta^2} = F_1 + \frac{n_1 \lambda}{\lambda^2 + \beta^2} ,$$

or, upon multiplying by $\sqrt{\lambda^2 + \beta^2}$:

$$\frac{\partial}{\partial \lambda}\left(n_2 \sqrt{\lambda^2 + \beta^2}\right) = F_1 \sqrt{\lambda^2 + \beta^2} + \varpi(\lambda) \frac{\lambda}{\sqrt{\lambda^2 + \beta^2}} + f_2 \lambda.$$

Furthermore, set:

$$F_1 \sqrt{\lambda^2 + \beta^2} + \varpi(\lambda) \frac{\lambda}{\sqrt{\lambda^2 + \beta^2}} + f_2 \lambda = \frac{\partial f_1}{\partial \lambda} ;$$

one will get:

$$n_1 \sqrt{\lambda^2 + \beta^2} = f_1 + \chi(\mu) .$$

The integral that gives f_1 is supposed to take a well-defined value of λ; for example, $\lambda = 0$.

The two arbitrary functions then include λ, in one case, and μ, in the other, which one could have expected upon remarking that the asymptotic lines of the square-threaded screw surface are rectilinear generators and orthogonal trajectories. The value of n_1 can be written:

$$n_1 = \varpi(\lambda) + \int F_2 M \, d\mu .$$

When one displaces along one of the asymptotes of the second system (λ = const.), the variation of the extension force that is exerted between two consecutive generators will be equal to the integral of the projections of the corresponding external forces onto the tangents to the asymptotes.

If the surface is bounded by an asymptotic curve and that curve forms a *free boundary* (i.e., it is not subject to any extension or sliding force) then one must have $n_2 = 0$ for $\lambda = a$, for any μ (a is the parameter of the curve considered). That condition will determine $\chi(\mu)$ when ϖ is known, because it can be written:

$$\chi(\mu) = - f_1(\lambda, \mu)_{(\lambda = a)} .$$

In order to determine $\varpi(\lambda)$, it suffices to know the law of variation of n_2 along a generator. For example, suppose that $\varpi(\lambda)$ has the form $A\lambda$, in which A is a constant and assume, moreover, that the external forces F_1 and F_2 are zero. For any generator, one will have $n_1 = \varpi(\lambda) = A\lambda$; the equation:

$$\frac{\partial f_1}{\partial \lambda} = \varpi(\lambda) \frac{\lambda}{\sqrt{\lambda^2+\beta^2}} = A \frac{\lambda^2}{\sqrt{\lambda^2+\beta^2}}$$

will then give:

$$f_1 = A[\lambda\sqrt{\lambda^2+\beta^2} - \beta^2 \log(\lambda + \sqrt{\lambda^2+\beta^2})],$$

and $\chi(\mu)$ will reduce to the constant:

$$-A[\alpha\sqrt{\alpha^2+\beta^2} - \beta^2 \log(\alpha + \sqrt{\alpha^2+\beta^2})].$$

For the surfaces that can be mapped to the square-threaded screw surface, and within the limits in which the generators remain rectilinear, α and β will be the same as they are for the screw surface; however, γ is an arbitrary function of μ.

$$\frac{\partial n_2}{\partial \lambda} - \frac{1}{\sqrt{\lambda^2+\beta^2}} \frac{\partial t}{\partial \mu} + \frac{\lambda}{\sqrt{\lambda^2+\beta^2}} (n_2 - n_1) = F_1 ,$$

$$-\frac{\partial t}{\partial \lambda} + \frac{1}{\sqrt{\lambda^2+\beta^2}} \frac{\partial n_1}{\partial \mu} - \frac{\lambda}{\sqrt{\lambda^2+\beta^2}} 2t = F_2 ,$$

$$n_1\,\gamma - \frac{\lambda}{\sqrt{\lambda^2+\beta^2}} = \Phi\sqrt{\lambda^2+\beta^2}\,.$$

We first look for the integrals of those equations when they have vanishing right-hand sides. Upon replacing $\sqrt{\lambda^2+\beta^2}$ with M, we will have:

$$n_1 = \frac{2\beta}{M\gamma}t,$$

$$\frac{\partial n_1}{\partial\mu} = \frac{2\beta}{M\gamma}\left(\frac{1}{\gamma}\frac{\partial t}{\partial\mu} - \frac{t}{\gamma^2}\frac{\partial\gamma}{\partial\mu}\right),$$

and as a result:

$$-\frac{\partial t}{\partial\lambda} + \frac{2\beta}{\gamma M^2}\frac{\partial t}{\partial\mu} - \frac{2}{\gamma M^2}\left(\lambda + \frac{\beta}{\gamma^2}\frac{d\gamma}{d\mu}\right)t = 0.$$

In order to integrate that equation, we define the simultaneous equations:

$$\frac{d\lambda}{-1} = \frac{d\mu}{\dfrac{2\beta}{\gamma M^2}} = \frac{dt}{\dfrac{2t}{M^2}\left(\lambda + \dfrac{\beta}{\gamma^2}\dfrac{d\gamma}{d\mu}\right)},$$

so

$$\frac{dt}{t} + \frac{2\,d\lambda}{M^2}\left(\lambda + \frac{\beta}{\gamma^2}\frac{d\gamma}{d\mu}\right) = 0,$$

or rather:

$$\gamma d\mu + \frac{2\beta}{M^2}d\lambda = 0,$$

$$\frac{dt}{t} + \frac{2\lambda\,d\lambda}{M^2} - \frac{d\gamma}{\gamma} = 0,$$

so:

$$\int \gamma\,d\mu + 2\arctan\frac{\lambda}{\beta} = \text{const.}$$

Upon letting ϖ denote an arbitrary function, the general integral will then be:

$$t = \frac{\gamma}{M^2}\varpi\left(\int\gamma\,d\mu + 2\arctan\frac{\beta}{\lambda}\right);$$

as a result:

$$n_1 = \frac{2\beta}{M^3}\varpi\,.$$

Finally, n_2 is given by the first equilibrium equation, from which, one infers that:

$$M\frac{\partial n_2}{\partial \lambda}+\frac{\lambda n_2}{M}=\frac{1}{M^4}(2\beta\varpi+\gamma'\varpi M^2+\gamma^2\varpi' M^2),$$

or rather:

$$n_2=\frac{1}{M}\int\frac{d\lambda}{M^4}(2\beta\varpi+\gamma'\varpi M^2+\gamma^2\varpi' M^2)+\text{funct. }\mu\,.$$

The values of n_1, n_2, t have the remarkable property that they are obtained in an explicit form without making any hypothesis on the function γ.

In general, one cannot arrive at a similar result then the right-hand sides are arbitrary. Impose the condition that the complete equations admit a particular system of solutions for which n_1 is zero. That condition will be obtained by writing out that the last two equations are compatible; i.e., upon eliminating t from:

$$\frac{\partial t}{\partial \lambda}+\frac{2\lambda}{\lambda^2+\beta^2}t=-F_2\,,$$

$$2\beta t=-\Phi M^2.$$

One will then find that:

$$M^2\frac{\partial \Phi}{\partial \lambda}+4\lambda\Phi=2F_2\,\beta\,.$$

One will have:

$$t=-\frac{\Phi M^2}{2\beta t}\,,\qquad\qquad n_1=0,$$

and

$$\frac{\partial n_2}{\partial \lambda}+\frac{\lambda n_2}{M^2}=F_1-\frac{1}{2M}\frac{\partial}{\partial \mu}\left(\frac{\Phi M^2}{\beta}\right),$$

so

$$n_2=\frac{1}{M}\int\left[MF_1-\frac{1}{2}\frac{\partial}{\partial \mu}\left(\frac{\Phi M^2}{\beta}\right)\right]d\lambda\,.$$

Hence, when the condition that was written above is fulfilled, the equilibrium of surfaces that can be mapped to the square-thread screw surface will be expressed by:

$$n_1=\frac{2\beta}{M^3}\varpi\,,$$

$$t=-\frac{\Phi M^2}{2\beta}+\frac{\gamma}{M^2}\varpi\,,$$

$$n_1 = \frac{1}{M}\int_0^\lambda \left[MF_1 - \frac{1}{2}\frac{\partial}{\partial\mu}\left(\frac{\Phi M^2}{\beta}\right)\right] d\lambda + \frac{1}{M}\int_0^\lambda \frac{d\lambda}{M^4}\,(2\beta\varpi + \gamma\varpi M^2 + \gamma^2\varpi M^2) + \chi,$$

which are expressions in which χ and ϖ are two arbitrary functions, the one being a function of μ, while the other is a function of $\int \gamma\, d\mu + 2 \arctan \lambda / \beta$.

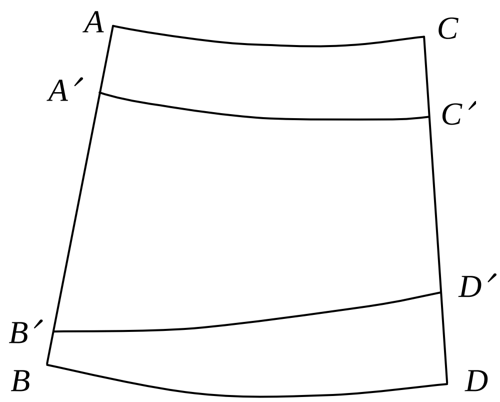

Figure 10.

Having said that, imagine that a portion of the square-threaded screw surface is bounded by the quadrilateral that is defined by two generators *AB*, *CD*, and two orthogonal trajectories *AC*, *BD* (Fig. 10), that the line *AB* is kept fixed, and that the other three sides remain free, and finally, that this portion of the surface is subjected to forces that satisfy the condition that was stated above.

From a theorem that was established before, the generator *CD* will remain straight under each deformation along a certain part *C′D′* of its length. When the surface is in the equilibrium state, one will have $n_1 = 0$ for all points of *CD*. For all points of the part that is common to *CD* and *C′D′*, it will result that $\varpi = 0$, since the preceding equations are applicable. Upon drawing the asymptotes of the second system that pass through the endpoints of that common part, one will bound a region for which ϖ is zero in its entire extent. For all points of the part that is common to *CD* and *C′D′*, one will have $t = 0$, in addition. Since ϖ is already equal to zero, the value of t shows that Φ must be zero or γ must be infinite.

If the asymptotes *A′C′*, *B′D′* that are drawn through *C′* and *D′* are inside of *ABCD* then our equations will cease to be applicable to the portions *ACA′C′*, *BDB′D′*, and *CC′*, *DD′* can be deformed in an arbitrary fashion, moreover. If the lines *A′C′*, *B′D′* are entirely external to *ABCD* then our equations will be applicable to the entire extent of *ABCD*, and the generator *CD* will remain entirely rectilinear. In that case (which we shall now envision exclusively), ϖ is zero for all points of the portion of the surface considered, and as a result, for the entire length of *CD*, one will have $\Phi = 0$ or $\gamma = \infty$, which amounts to saying that the equilibrium will be possible under the conditions that we supposed have been imposed only if the surface is subjected to forces along *CD* that are exclusively tangential, or even if the spherical indicatrix of the surface presents a regression at the corresponding point.

When Φ is zero for all points of *CD*, the component F_2 of the external force is likewise zero, by virtue of the relation:

$$F_2 = \frac{M^2}{2\beta}\frac{\partial\Phi}{\partial\lambda} + 2\frac{\lambda}{\beta}\Phi\,.$$

In this case, the generator can be subject to only forces that are directed along its length.

Along the curves *AC*, *BD*, one must have $n_2 = 0$ and $t = 0$. Since ϖ is supposed to be zero, the condition $t = 0$ will once more imply that $\Phi = 0$ or $\gamma = \infty$. However, it is impossible for γ to be infinite for all points of a finite arc of a curve that is not a segment of the generator. One must then have $\Phi = 0$. n_2 will be given by the formula:

$$n_2 = \frac{1}{M}\int_0^\lambda \left[MF_1 - \frac{1}{2}\frac{\partial}{\partial\mu}\left(\frac{\Phi M^2}{\beta}\right)\right] d\lambda + \chi\,.$$

That tension must be annulled for any μ for two constant values λ_1 and λ_2 of λ, and since χ is a function of only μ, one will be led to the condition:

$$\frac{1}{M_1}\int_0^\lambda \left[MF_1 - \frac{1}{2}\frac{\partial}{\partial\mu}\left(\frac{\Phi M^2}{\beta}\right)\right] d\lambda = \frac{1}{M_2}\int_0^\lambda \left[MF_1 - \frac{1}{2}\frac{\partial}{\partial\mu}\left(\frac{\Phi M^2}{\beta}\right)\right] d\lambda,$$

in which M_1 and M_2 are the values of M for $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

That relation must be true for all values of μ that correspond to the generators that are found between *AB* and *CD*.

If F_1 and Φ do not depend upon γ then that relation cannot be fulfilled, in general, and consequently equilibrium will be impossible under the conditions that we have supposed. However, if F_1 and Φ depend upon γ then we will be dealing with an equation in γ that determines that unknown, and consequently, the equilibrium form.

For example, consider a system of normal forces. In order for F_2 to be zero, one must have:

$$M^2\frac{\partial\Phi}{\partial\lambda} + 4\lambda\,\Phi = 0,$$

which one can write as:

$$(\lambda^2 + \beta^2)^2\frac{\partial\Phi}{\partial\lambda} + 4\lambda\,(\lambda^2 + \beta^2)\,\Phi = 0,$$

or

$$\Phi\,(\lambda^2 + \beta^2) = \text{funct. }\mu\,.$$

Suppose that the function of μ that is found in the right-hand side is $\int \gamma\, d\mu$, where the limits are taken in such a fashion that this integral is annulled for the generator *CD*. Upon setting $F_1 = 0$, the condition that is imposed upon γ will be:

$$\frac{1}{M_1}\int_0^{\lambda_1}\frac{M^2}{\beta}\frac{\partial\Phi}{\partial\mu}d\lambda = \frac{1}{M_2}\int_0^{\lambda_2}\frac{M^2}{\beta}\frac{\partial\Phi}{\partial\mu}d\lambda$$

or

$$\frac{\gamma}{M_1}\int_0^{\lambda_1}\frac{d\lambda}{M^2} = \frac{\gamma}{M_2}\int_0^{\lambda_1}\frac{d\lambda}{M^2} ,$$

$$\frac{\gamma}{M_1}\arctan\frac{\lambda_1}{\beta} = \frac{\gamma}{M_2}\arctan\frac{\lambda_2}{\beta} ,$$

and since the coefficients of γ in the two sides are unequal, in general, it will result that γ = 0; i.e., the *surface of equilibrium will be the square-threaded screw surface.*

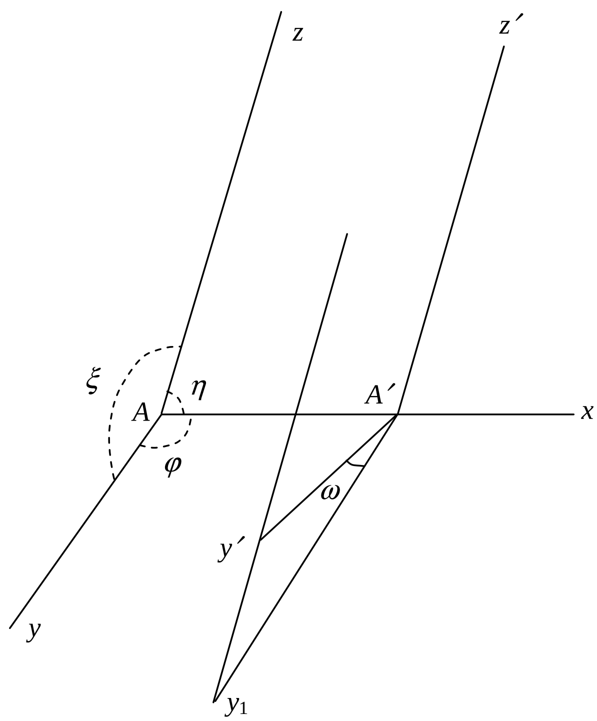

Figure 11.

The second-degree surfaces can be treated like ruled surfaces with real or imaginary generators, and consequently, one can be certain *a priori* that the separation of the unknowns can always be carried out for those surfaces. The problem even comes down to the integration of two total differential equations that are linear and first-order, because if one refers the surface to its generators then one will find immediately upon applying formulas (18), in which one must set $1 / \rho = 1 / \rho_1 = 0$, that:

$$\sin\varphi\,\frac{\partial n}{\partial\sigma_1} - n\left(\frac{\partial\varphi}{\partial\sigma} + 2\cos\varphi\frac{\partial\varphi}{\partial\sigma_1}\right) = f_1\sin^3\varphi,$$

$$\sin\varphi\,\frac{\partial n_1}{\partial\sigma} - n_1\left(\frac{\partial\varphi}{\partial\sigma_1} + 2\cos\varphi\frac{\partial\varphi}{\partial\sigma}\right) = f\sin^3\varphi\,.$$

We shall study the case of the hyperbolic paraboloid in particular, since the calculations and formulas will take on great simplicity for it.

When one refers a hyperbolic paraboloid to three oblique axes that are defined by the rectilinear generators *Ax*, *Ay* (Fig. 11), which cross at a point *A* on the surface, and the parallel *Az* to the axis, the equation of the paraboloid will take the form:

$$xy = kz.$$

We give *k* the name of the *parameter of the surface* at the point *A*.

Let ξ, η be the angles *yAz*, *xAz*, and let φ be the angle *yAx*. Take an infinitely-small length $AA' = \varepsilon$ on *Ax*; the generator of the system *Ay* that passes through A' will have the equations:

$$x = \varepsilon,$$

$$z = \frac{\varepsilon}{k}\,y\,.$$

It is situated in the plane zAy_1 that is parallel to *zAy*. The triangle $y'A'y_1$, whose side $y'y_1$ is parallel to *Az*, gives:

$$\frac{\sin\omega}{\sin(\xi-\omega)} = \frac{y'y_1}{A'y_1} = \frac{\varepsilon}{k},$$

so, since ω is infinitely small:

$$\omega = \frac{\varepsilon}{k}\sin\xi.$$

If one refers the surface to the new axes $A'x$, $A'y$, $A'z$ then one will have:

$$x = x' + \varepsilon = x' + \frac{k}{\sin\xi}\,\omega,$$

$$\frac{y}{y'} = \frac{A'y_1}{A'y'} = \frac{\sin(\xi-\omega)}{\sin\xi}, \qquad y = y'(1 - \omega\cot\xi),$$

$$z = z' + y'y_1 = z' + y\,\frac{\varepsilon}{k} = z' + \frac{\omega\,y'}{\sin\xi}(1 - \omega\cot\xi)\,.$$

With that, the equation of the surface will become:

$$\left(x' + \frac{k}{\sin\xi}\,\omega\right)y'(1 - \omega\cot\xi) = kz' + k\,\frac{\omega\,y'}{\sin\xi}(1 - \omega\cot\xi)$$

or

$$x'y'(1 - \omega \cot \xi) = k z'.$$

Thus, if k' is the new value of k then one will have:

$$k' = \frac{k}{1 - \omega \cot \xi} = k(1 + \omega \cot \xi),$$

or rather, upon setting $k' - k = dk$ and $\omega = - d\xi$:

$$\frac{dk}{k} = - \cot \xi \, d\xi,$$

which will give:

$$\log k = - \log \sin \xi + \text{const.}$$

The constant is a function of η, since that angle will remain constant when one displaces along the generator Ax. Since nothing will distinguish the two generators, one will likewise find that:

$$\log k = - \log \sin \eta + \text{const.}$$

As a result, one will generally have:

$$\log k = - \log \sin \xi - \log \sin \eta + \text{const.},$$

so

$$k = \frac{k_0}{\sin \xi \sin \eta}.$$

k_0 is the value of k for $\xi = \eta = \pi / 2$; i.e., for the summit of the paraboloid.

The displacement along Ax is expressed as a function of ξ and η by:

$$d\sigma = \varepsilon = k \frac{\omega}{\sin \xi} = - \frac{k_0 \, d\xi}{\sin^2 \xi \sin \eta}.$$

Similarly:

$$d\sigma_1 = - \frac{k_0 \, d\eta}{\sin^2 \xi \sin \eta}.$$

If one sets:

$$\frac{d\xi}{\sin \xi} = dx, \qquad \frac{d\eta}{\sin \eta} = dy$$

then the preceding formulas will become:

$$k = \quad k_0 \cos ix \cos iy,$$

$$d\sigma = - k_0 \cos ix \cos iy \, dx,$$

$$d\sigma_1 = -k_0 \cos ix \cos iy\, dy\,.$$

The angle φ in the trihedron $Axyz$ is given by:

$$\cos \varphi = \cos \xi \cos \eta + \sin \xi \sin \eta \cos A,$$

in which A is the dihedron along Ax; i.e., the angle between the director planes. That formula can be written:

$$\cos \varphi = -\tan i\, x \tan i\, y + \frac{\cos A}{\cos i\, x \cos i\, y}\,.$$

Now suppose that a hyperbolic paraboloid is subject to a system of tangential forces, and apply formulas (18) to it, in which we set $1 / \rho = 1 / \rho_1 = 0$, and that we divide by $\sin^3 \varphi$:

$$\frac{1}{\sin^2 \varphi}\frac{\partial n}{\partial \sigma_1} - 2n\frac{\cos \varphi}{\sin^3 \varphi}\frac{\partial \varphi}{\partial \sigma_1} - \frac{n}{\sin^3 \varphi}\frac{\partial \varphi}{\partial \sigma} = f_1\,,$$

$$\frac{1}{\sin^2 \varphi}\frac{\partial n_1}{\partial \sigma} - 2n_1\frac{\cos \varphi}{\sin^3 \varphi}\frac{\partial \varphi}{\partial \sigma} - \frac{n}{\sin^3 \varphi}\frac{\partial \varphi}{\partial \sigma_1} = f\,.$$

Consider only the first equation and set $n / \sin^2 \varphi = N$. That equation will become:

$$\frac{\partial N}{\partial \sigma_1} - \frac{N}{\sin \varphi}\frac{\partial \varphi}{\partial \sigma} = f_1\,,$$

or rather, upon replacing $d\varphi / \sin \varphi$ with du :

$$\frac{\partial N}{\partial \sigma_1} - N\frac{\partial u}{\partial \sigma} = f_1\,,$$

or also, upon replacing $d\sigma$ and $d\sigma_1$ with their values:

$$\frac{\partial N}{\partial y} - N\frac{\partial u}{\partial x} = -f_1\, k_0 \cos i\, x \cos i\, y\,.$$

The equation that determines φ will lead to the following one:

$$-\sin \varphi\,\frac{\partial \varphi}{\partial x} = -i \operatorname{tin} i\, y\,(1 + \tan^2 ix) + i\frac{\cos A \sin ix}{\cos^2 ix \cos iy}\,,$$

so

$$\frac{\partial u}{\partial x}=\frac{\partial\varphi/\partial x}{\sin\varphi}=\frac{i\tan iy\,(1+\tan^2 ix)-i\,\dfrac{\cos A\sin ix}{\cos^2 ix\cos iy}}{1-\left(\tan ix\tan iy-\dfrac{\cos A}{\cos ix\cos iy}\right)^2}.$$

or

$$\frac{\partial u}{\partial x}=\frac{i\,(\sin iy\cos iy-\cos A\sin ix\cos iy)}{\cos^2 ix\cos^2 iy-(\sin ix\sin iy-\cos A)^2}.$$

Upon setting:

$$v = \log\,[\cos^2 i\,x\cos^2 i\,y-(\sin i\,x\sin i\,y-\cos A)^2],$$

that equation will become:

$$\frac{\partial u}{\partial x}=-\,\frac{\partial v}{\partial y}.$$

Upon multiplying everything by e^v, the equation in N will be:

$$e^v\frac{\partial N}{\partial y}+Ne^v\frac{\partial v}{\partial y}=-f_1\,k_0\cos i\,x\cos i\,y\,e^v,$$

so

$$N\,e^v=-\,k_0\cos ix\int_0^y f_1\cos iy\,e^v\,dy+X,$$

with the condition that:

$$e^{2v}=\cos^2 i\,x\cos^2 i\,y-(\sin i\,x\sin i\,y-\cos A)^2;$$

N_1 will be given by an analogous formula, and the question is thus found to have been solved completely.

Upon replacing e^v with its equivalent $\cos i\,x\cos i\,y\sin\varphi$ and N by $n/\sin^2\varphi$, one can once more write:

$$\frac{n}{\sin\varphi}=-\,k_0\,\frac{\cos ix}{\cos iy}\int_0^y f_1\cos^2 i\,x\sin\varphi\,dy+X.$$

Finally, upon recalling that the resultant r of the tensions that are exerted upon an asymptote is directed along the other asymptote and as a result equal to $n/\sin\varphi$, and upon letting F_1 denote the component $f_1\sin\varphi$ of the external forces along the normal to the asymptotic line x = const., one will obtain:

$$r=r_0-k_0\,\frac{\cos ix}{\cos iy}\int_0^y F_1\,\cos^2 i\,y\,dy.$$

When F_1 is a function of only x, one will have:

$$r = r_0 - k_0 \frac{F_1 \cos ix}{4i \cos iy} (2iy + \sin 2i\, y).$$

When F_1 has the form $\varpi(x) / \cos iy$, or rather (what amounts to the same thing), $\chi(x) \sin \eta$, where ϖ, and χ are arbitrary functions, one will have:

$$r = r_0 - k_0 \frac{\cos ix}{\cos iy} \varpi(x) \sin i\, y = r_0 + \chi(\xi)\, k_0 \frac{\cos \eta}{\cos \xi}.$$

In particular, if $\chi(\xi) = A / k_0 \sin \xi$, in which A is a constant, then one will find that:

$$r = r_0 + A \cos \eta.$$

In this case, F_1 will be equal to $(A / k_0) \sin \xi \sin \eta = A / k$; it will then be inversely proportional to the parameter.

When $F_1 = 0$, all that will remain is $r = r_0$, which one can state by saying that:

If the tangential external forces that are applied to the various points of a generator G are directed along the generators of another system then they will all experience the same tension, which is directed along G to the points where they meet G.

Surfaces of revolution

Upon taking the lines λ = const. to be the parallels and the lines μ = const. to be the meridians, one will have immediately:

$$\frac{1}{\rho_1} = 0, \qquad \frac{1}{T} = 0.$$

Furthermore, all of the other parameters are functions of only λ. One can set $L = 1$, and the general equations of the theory of surfaces will then reduce to:

$$\frac{d(1/R_2)}{d\lambda} + \frac{1}{\rho_2}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = 0,$$

$$\frac{1}{R_1 R_2} + \frac{d(1/\rho_2)}{d\lambda} - \frac{1}{\rho_2^2} = 0,$$

$$\frac{1}{\rho_2} = -\frac{dM / d\lambda}{M}.$$

One will infer that:

$$\frac{1}{\rho_2}\frac{d(1/R_2)}{d\lambda}-\frac{1}{R_2}\frac{d(1/\rho_2)}{d\lambda}+\frac{1}{R_1}\left(\frac{1}{\rho_2^2}+\frac{1}{R_2}\right)=0,$$

so

$$\arctan\left(\frac{\rho_2}{R_2}\right)+\int\frac{d\lambda}{R_1}=\text{const.},$$

which is a relation that is easy to predict, since one will see geometrically that ρ_2 is the length of the tangent to the meridian between the contact point and the axis, that R_2 is the length of the normal, and that arctan ρ_2 / R_2 is consequently the angle between the normal and the axis, which is an angle whose differential must be equal (up to sign) to the contingency angle $d\lambda / R_1$ of the meridian. Set:

$$\arctan\frac{\rho_2}{R_2}=-\theta,$$

so

$$\rho_2 = -R_2 \tan\theta$$

and

$$\frac{1}{R_1}=\frac{d\theta}{d\lambda}.$$

Upon replacing $1 / R_2$ with $-\tan\theta / \rho_2$ and multiplying by ρ_2^2, the second general equations can be written:

$$\cos\theta\,\frac{d\rho_2}{d\lambda}-\sin\theta\,\frac{d\theta}{d\lambda}+\cos\theta=0\,;$$

as a result:

$$\rho_2=-\frac{1}{\cos\theta}\int\cos\theta\,d\lambda$$

and

$$R_2=\frac{1}{\sin\theta}\int\cos\theta\,d\lambda\,.$$

The quantity M is obtained from the equation:

$$\frac{dM}{M}=-\frac{d\lambda}{\rho_2}=-\frac{d\lambda\cos\theta}{\rho_2\cos\theta}=\frac{d(\rho_2\cos\theta)}{\rho_2\cos\theta},$$

so

$$M=f(\mu)\,\rho_2\cos\theta.$$

One can take $M = -\rho_2 \cos\theta$. It is the radius of the parallel.

The preceding formulas reduce the equilibrium equations of the surfaces of revolution that are subject to tangential forces to:

$$\frac{\partial n_2}{\partial\lambda} - \frac{1}{\int\cos\theta\,d\lambda}\frac{\partial t}{\partial\mu} - (n_1 - n_2)\frac{\cos\theta}{\int\cos\theta\,d\lambda} = F_1 ,$$

$$-\frac{\partial t}{\partial\mu} - \frac{1}{\int\cos\theta\,d\lambda}\frac{\partial n_1}{\partial\mu} - 2t\frac{\cos\theta}{\int\cos\theta\,d\lambda} = F_2 ,$$

$$\frac{\sin\theta}{\int\cos\theta\,d\lambda}n_1 + \frac{d\theta}{d\lambda}n_2 = 0.$$

If one sets $\cos\theta = d\Lambda / d\lambda = \Lambda'$ then those equations can be written:

$$\Lambda\frac{\partial n_2}{\partial\lambda} + \frac{\partial t}{\partial\mu} - (n_1 - n_2)\Lambda' = F_1\,\Lambda,$$

$$\Lambda\frac{\partial t}{\partial\lambda} + \frac{\partial n_1}{\partial\mu} + 2t\Lambda' = -F_2\,\Lambda,$$

$$\frac{n_1}{\Lambda} - \frac{\Lambda''}{1-\Lambda'^2}n_2 = 0,$$

or rather, upon eliminating n_1 and suppressing the index of n_2, which has become useless, one will get:

$$\Lambda\frac{\partial n}{\partial\lambda} + \frac{\partial t}{\partial\mu} + n\frac{\Lambda'(1-\Lambda'^2 - \Lambda\Lambda'')}{1-\Lambda'^2} = F_1\,\Lambda,$$

$$\Lambda\frac{\partial t}{\partial\lambda} + \frac{\Lambda\Lambda''}{1-\Lambda'^2}\frac{\partial n}{\partial\mu} + 2t\Lambda' = -F_2\,\Lambda .$$

If one sets:

$$n\Lambda\sqrt{1-\Lambda'^2} = N,$$

$$t\,\Lambda^2 = T$$

then those equations will finally become:

$$\frac{\partial N}{\partial\lambda} + \frac{\sqrt{1-\Lambda'^2}}{\Lambda^2}\frac{\partial T}{\partial\mu} = F_1\,\Lambda\sqrt{1-\Lambda'^2} ,$$

$$\frac{\partial T}{\partial\lambda} + \frac{\Lambda\Lambda''}{(1-\Lambda'^2)^{3/2}}\frac{\partial N}{\partial\mu} = -F_2\,\Lambda^2.$$

Consider these equations with a vanishing right-hand side, and replace the coefficients $\partial T / \partial\mu$, $\partial N / \partial\mu$ with L_1 and L_2 , resp., to abbreviate:

$$\frac{\partial N}{\partial\lambda} + L_1 \frac{\partial T}{\partial\mu} = 0,$$

$$\frac{\partial T}{\partial\lambda} + L_2 \frac{\partial N}{\partial\mu} = 0.$$

If one sets:

$$\frac{\partial N}{\partial\lambda} d\lambda + \frac{\partial N}{\partial\mu} d\mu = dN,$$

$$\frac{\partial T}{\partial\lambda} d\lambda + \frac{\partial T}{\partial\mu} d\mu = dT,$$

in addition, then one will obtain differential equations that pertain to the characteristics upon writing that the preceding four equations will give indeterminate values for $\frac{\partial N}{\partial\lambda}, \frac{\partial N}{\partial\mu}, \frac{\partial T}{\partial\lambda}, \frac{\partial T}{\partial\mu}$. One will then find that:

$$L_1\, L_2\, d\lambda^2 + d\mu^2 = 0,$$

$$L_2\, d\lambda\, dN - d\mu\, dT = 0,$$

or rather:

$$\frac{d\mu}{d\lambda} = \pm\sqrt{-L_1 L_2}\,,$$

$$\sqrt{L_2}\, dN = \pm\sqrt{-L_1}\, dT\,.$$

We limit ourselves to studying the case in which the second equation is integrable, which will happen only when L_2 / L_1 is constant. We first look for the surfaces of revolution that enjoy that property.

One has:

$$\frac{L_2}{L_1} = \frac{\Lambda^3\Lambda''}{(1-\Lambda'^2)^2} = \text{const.} = -\frac{1}{c},$$

and upon integrating:

$$\frac{1}{\Lambda^2} - \frac{c}{1-\Lambda'^2} = \text{const.} = b,$$

$$\Lambda' = \sqrt{\frac{(b+c)\Lambda^2 - 1}{b\Lambda^2 - 1}}\,.$$

The quadrature that remains to be performed depends upon elliptic functions. However, one can geometrically interpret the relation that exists between Λ and λ. In order to do that, one remarks that if one replaces Λ' with $\cos\theta$ then that will make:

$$\frac{1}{\Lambda^2} = b + \frac{c}{\sin^2\theta} \qquad \text{or} \qquad \Lambda = \frac{\sin\theta}{\sqrt{b\sin^2\theta + c}},$$

and as a result:

$$\Lambda' = \cos\theta = \frac{\sqrt{b\sin^2\theta+c}\cos\theta - \dfrac{b\sin^2\theta\cos\theta}{\sqrt{b\sin^2\theta+c}}}{b\sin^2\theta+c}\frac{d\theta}{d\lambda} = \frac{c\cos\theta}{(b\sin^2\theta+c)^{3/2}}\frac{d\theta}{d\lambda},$$

$$d\lambda = \frac{c\,d\theta}{(b\sin^2\theta+c)^{3/2}}.$$

Recall that λ is the arc length of the meridian, and θ is the angle between the normal and the axis.

Now, if one considers the ellipse $\frac{X^2}{A^2}+\frac{Y^2}{B^2} = 1$ and lets θ denote the angle between the normal and the y-axis then one will find the following formulas with no difficulty:

$$dX = \frac{A^2B^2\cos\theta\,d\theta}{(A^2\sin^2\theta+B^2\cos^2\theta)^{3/2}}, \qquad dY = -\frac{A^2B^2\sin\theta\,d\theta}{(A^2\sin^2\theta+B^2\cos^2\theta)^{3/2}},$$

so

$$d\lambda = \sqrt{dX^2+dY^2} = \frac{A^2B^2\,d\theta}{[(A^2-B^2)\sin^2\theta+B^2]^{3/2}} = \frac{(B^2/A^2)\,d\theta}{(A^2\sin^2\theta+B^2\cos^2\theta)^{3/2}}.$$

That value for $d\lambda$ will become identical to the one that we found above if we set:

$$\frac{B^2}{A^2} = c, \qquad \frac{A^2-B^2}{A^2} = b.$$

If c is negative then the ellipse will be replaced by a hyperbola. If $b + c = 0$ then it will result that $1 / \Lambda^2 = 0$, which gives a parabola. The integrability case that we shall examine is then that of all second-degree surfaces of revolution. Upon re-establishing the right-hand sides, the equations that one must integrate will be:

$$\frac{\partial N}{\partial\lambda} + L_1\frac{\partial T}{\partial\mu} = F_1\,\Lambda\sqrt{1-\Lambda'^2},$$

$$\frac{\partial T}{\partial\lambda} + L_2\frac{\partial N}{\partial\mu} = -F_2\,\Lambda^2.$$

Replace L_1 by its value $-c\,L_2$ and add the corresponding sides of the two equations after multiplying the second one by $\sqrt{-c}$. That will give:

$$\frac{\partial N}{\partial \lambda}+\sqrt{-c}\,\frac{\partial T}{\partial \lambda}+L_2\sqrt{-c}\left(\frac{\partial N}{\partial \mu}+\sqrt{-c}\,\frac{\partial T}{\partial \mu}\right)$$

$$= F_1\,\Lambda\sqrt{1-\Lambda'^2}-\sqrt{-c}\,F_2\,\Lambda^2.$$

Upon setting $N + T\sqrt{-c} = \varpi$, and replacing the right-hand side with F, that equation will take the form:

$$\frac{\partial \varpi}{\partial \lambda}+L_2\sqrt{-c}\,\frac{\partial \varpi}{\partial \mu}=F\,;$$

it is integrated by means of the simultaneous equations:

$$d\lambda=\frac{d\mu}{L_2\sqrt{-c}}=\frac{d\varpi}{F}.$$

Upon replacing $\sqrt{-c}$ with $-\sqrt{-c}$ and setting $N - T\sqrt{-c} = \chi$, one will get the second system:

$$d\lambda=-\frac{d\mu}{L_2\sqrt{-c}}=\frac{d\chi}{F'}.$$

If one knows ϖ and χ then one will know N and T, and as a result, the problem will be solved. The equations $d\mu = \pm L_2\sqrt{-c}\,d\lambda$ are the differential equations of the asymptotic lines.

When F_1 , F_2 , and as a result, F, F' are functions of only λ, one will have immediately:

$$\varpi=\int F\,d\lambda+\text{funct.}\,(\alpha),\qquad \chi=\int F'\,d\lambda+\text{funct.}\,(\beta),$$

if one lets α, β denote the parameters of the asymptotic lines.

If one would like to have the relations that characterize the asymptotic lines in an explicit form then it will suffice to change the variables by setting:

$$dz=\sqrt{1-\Lambda'^2}\,d\lambda=\sin\theta\,d\lambda.$$

When one is given the geometric significance of λ and θ, one will see effortlessly that dz is nothing but the distance between the planes of two infinitely-close parallels. With that new variable, one will have:

$$\Lambda = \int \cos\theta \, d\lambda = \int \cot\theta \, dz .$$

Now:

$$d\lambda = \frac{c\,d\theta}{(b\sin^2\theta + c)^{3/2}},$$

which will give:

$$dz = \frac{c\tan\theta(1+\tan^2\theta)\,d\theta}{[c+(b+c)\tan^2\theta]^{3/2}},$$

$$z = -\frac{c}{b+c}\frac{1}{\sqrt{c+(b+c)\tan^2\theta}},$$

$$\cot\theta = \frac{(b+c)^{3/2}\,z}{\sqrt{c^2 - c\,(b+c)^2\,z^2}},$$

if one measures z upon starting from the equator.

Hence:

$$\Lambda = (b+c)^{3/2}\int\frac{z\,dz}{\sqrt{c^2 - c\,(b+c)^2\,z^2}} = \frac{1}{c\sqrt{b+c}}\sqrt{c^2 - c\,(b+c)^2\,z^2} .$$

The relation:

$$d\mu = L_2\,\sqrt{-c}\,d\lambda,$$

or

$$d\mu = \frac{L_1}{\sqrt{-c}}\,d\lambda = \frac{\sqrt{1-\Lambda'^2}}{\Lambda^2\sqrt{-c}}\,d\lambda,$$

will then become:

$$d\mu = \frac{c\,(b+c)\,dz}{\sqrt{-c}\,[c^2 - c\,(b+c)^2\,z^2]}.$$

One infers from this that:

$$\mu - \arctan\left[\frac{(b+c)\,z}{\sqrt{-c}}\right] = \text{const.}$$

The quantities that we have called α and β are then:

$$\mu \pm \arctan\left[\frac{(b+c)\,z}{\sqrt{-c}}\right].$$

In the case of the ellipsoid, c is positive, and α, β are conjugate imaginary; Λ and $\sqrt{1-\Lambda'^2}$ are real. In order for the values of n and t :

$$n = \frac{\varpi + \chi}{2\Lambda\sqrt{1-\Lambda'^2}},$$

$$t = \frac{\varpi - \chi}{2\sqrt{-c}\Lambda^2}$$

to be real, it is necessary and sufficient that ϖ and χ should be conjugate imaginary quantities.

We shall push the application of the method to its limit by studying the very simple problem that takes the form:

Determine the equilibrium conditions for an ellipsoid of revolution that contains a fluid that exerts a constant normal pressure on all points of its surface.

The first thing to do is to decompose the forces into a normal system and a tangential system. In order to find the former, it will suffice to set:

$$\frac{2a}{R_1\,R_2} = \Phi, \qquad \frac{1}{L\,R_2}\frac{\partial a}{\partial \lambda} = F_1, \qquad \frac{1}{M\,R_1}\frac{\partial a}{\partial \mu} = F_2,$$

upon calling the normal force Φ.

The first equation determines α as a function of only λ. The second one gives the component F_1 of the external force along the tangent to the meridian. The third one shows that the component F_2 of the external force along the parallel is equal to zero.

We have found that:

$$\frac{1}{R_1} = \frac{d\theta}{d\lambda} = \frac{\sin\theta\, d\theta}{dz},$$

$$\frac{1}{R_1} = \frac{\sin\theta}{\Lambda}.$$

Hence:

$$\frac{1}{R_1\,R_2} = \frac{\sin^2\theta}{\Lambda}\frac{d\theta}{dz}.$$

One has, moreover:

$$\cot\,\theta = \frac{(b+c)^{3/2}\,z}{\sqrt{c^2 - c\,(b+c)^2\,z^2}},$$

so

$$\sin^2\,\theta = \frac{1}{1+\cot^2\theta} = \frac{c^2 - c\,(b+c)^2\,z}{c^2 + b\,(b+c)^2\,z^2}.$$

The value of Λ is:

$$\Lambda = \frac{\sqrt{c^2 - c\,(b+c)^2\,z^2}}{c\sqrt{b+c}}.$$

Finally:

$$\sin\theta\,\frac{d\theta}{dz} = \frac{d\theta}{d\lambda} = c^2\,\frac{(b+c)^{3/2}}{[c^2 + b\,(b+c)^2\,z^2]^{3/2}}.$$

Consequently, one has:

$$\frac{1}{R_1\,R_2} = c^2\,\frac{(b+c)^2}{[c^2 + b\,(b+c)^2\,z^2]^2}$$

and

$$a = \frac{\Phi}{2}\,\frac{[c^2 + b\,(b+c)^2\,z^2]^2}{c^2\,(b+c)^2}.$$

Hence:

$$\frac{da}{dz} = \frac{2\Phi b}{c^3}\,z\,[c^2 + b\,(b+c)^2\,z^2],$$

and since:

$$F_1 = \frac{1}{R_2}\,\frac{da}{d\lambda} = \frac{\sin^2\theta}{\Lambda}\,\frac{da}{dz},$$

one will get:

$$F_1\,\Lambda = \frac{2\Phi b\,z}{c^3}\,[c^2 + b\,(b+c)^2\,z^2],$$

upon replacing $\sin^2\theta$ and da / dz by their values.

Upon setting:

$$B^2 = \frac{c}{(b+c)^2}, \qquad \frac{2\Phi b\,(b+c)^2}{c^2} = P,$$

one can further write:

$$F_1\,\Lambda = Pz\,(B^2 - z^2).$$

With that, the forces that are applied to the ellipsoid are composed of:

1. The normal system that is defined by the normal force Φ and the tangential force $F_1 = Pz\,(B^2 - z^2) / \Lambda$, which is directed along the tangents to the meridians.

2. The tangential system that is defined by the force $F_1' = -Pz\,(B^2 - z^2) / \Lambda$, which is tangent to the meridians.

The first system gives rise to the effort of extension $n_1 = a / R_1 = \Phi\,R_1 / 2$ at each point, which is normal to the meridian, and the effort of extension $n_2 = a / R_2 = \Phi\,R_1 / 2$, which is normal to the parallel; these efforts must be added to the ones that result from the second system.

The equation in ϖ that was written previously will then become:

$$\frac{\partial\varpi}{\partial\lambda}+\frac{\sqrt{1-\Lambda'^2}}{\Lambda^2\sqrt{-c}}\frac{\partial\varpi}{\partial\mu}= F_1'\Lambda\sqrt{1-\Lambda_2'}$$

here, or, upon replacing $d\lambda$ with $dz / \sin\theta = dz / \sqrt{1-\Lambda_2'}$ and Λ^2 with $\frac{B^2-z^2}{(b+c)B^2}$:

$$\frac{\partial\varpi}{\partial\lambda}+i\frac{B}{B^2-z^2}\frac{\partial\varpi}{\partial\mu}= - Pz\,(B^2-z^2).$$

The corresponding simultaneous equations are:

$$dz = \frac{d\mu}{iB/(B^2-z^2)} = \frac{d\varpi}{-Pz\,(B^2-z^2)}.$$

It will have:

$$\mu - i\log\sqrt{\frac{B+z}{B-z}} = \text{const.}$$

$$\varpi - \frac{P}{4}(B^2-z^2)^2 = \text{const.}$$

for its integrals. That will give the general value of ϖ:

$$\varpi = \frac{P}{4}(B^2-z^2)^2 + \varphi_1\left(\mu - i\log\sqrt{\frac{B+z}{B-z}}\right).$$

One will likewise find that:

$$\chi = \frac{P}{4}(B^2-z^2)^2 + \varphi_2\left(\mu + i\log\sqrt{\frac{B+z}{B-z}}\right),$$

and then:

$$n = \frac{\varpi+\chi}{2\Lambda\sqrt{1-\Lambda'^2}} = \frac{(\varpi+\chi)B^2\sqrt{b+c}\sqrt{1+\frac{b}{c}\frac{z^2}{B^2}}}{2(B^2-z^2)},$$

$$t = \frac{\varpi-\chi}{2\sqrt{-c}\,\Lambda^2} = \frac{(\varpi-\chi)(b+c)B^2}{2ic\,(B^2-z^2)},$$

or even, if one replaces b and c with their values as functions of the axes of the ellipse:

$$b = \frac{A^2 - B^2}{A^4}, \quad c = \frac{B^2}{A^4},$$

$$n = \frac{(\varpi + \chi)\dfrac{B^2}{A}\sqrt{1 + \dfrac{A^2 - B^2}{B^4} z^2}}{2(B^2 - z^2)},$$

$$t = \frac{(\varpi - \chi)B}{2i(B^2 - z^2)}.$$

Those values are applicable no matter what portion of the ellipsoid that one considers. If one is dealing with the entire ellipsoid then they must remain finite when z tends to $\pm B$. For that to be true, it is necessary that ϖ and χ must both tend to zero for any μ. Upon considering the expressions that were found for those two quantities and setting:

$$u = \mu + i \log \sqrt{\frac{B+z}{B-z}},$$

$$v = \mu + i \log \sqrt{\frac{B+z}{B-z}},$$

one must have:

$$\varphi_1 (u)_{v=\infty} = \varphi_2 (u)_{v=\infty} = 0$$

for any μ.

Furthermore, n and t, and consequently $\varphi_1 (u)$ and $\varphi_2 (v)$, must remain finite, continuous and well-defined for any u and v. When one describes an absolutely arbitrary closed contour on the surface, one must find the same values for $\varphi_1 (u)$ and $\varphi_2 (v)$ upon returning to the starting point.

From the known properties of functions of imaginary variables, a function such as $\varphi_1(u)$ that remains finite, continuous, and well-defined for all possible values of the value that it contains must necessarily reduce to a constant. In the present case, that constant can be nothing but zero, and we will have simply:

$$\varpi = \chi = \frac{P}{4} (B^2 - z^2)^2.$$

It results immediately from this that $t = 0$; i.e., that:

The meridians and parallels are the lines of principal tension of the surface.

One then finds:

$$n_2 = n = -\frac{P}{4}\frac{B^2}{A}(B^2 - z^2)\sqrt{1 + \frac{A^2 - B^2}{B^4} z^2}$$

$$= \frac{\Phi}{4}\frac{(B^2 - z^2)}{AB^2}\sqrt{1+\frac{A^2-B^2}{B^4}z^2}\,(A^2 - B^2)$$

for the effort of extension that is exerted upon the parallels.

The effort of extension n_1 that is exerted on the meridians is obtained by multiplying n times:

$$\frac{\Lambda\Lambda''}{1-\Lambda'^2} = -\frac{1-\Lambda'^2}{\Lambda^2 c} = \frac{-c\,(b+c)}{c^2 + b\,(b+c)^2 z^2} = \frac{-A^2/B^2}{1+\dfrac{A^2-B^2}{B^4}z^2};$$

one will then find that:

$$n_1 = -\frac{PA}{4}\frac{B^2 - z^2}{\sqrt{1+\dfrac{A^2-B^2}{B^4}z^2}} = -\frac{\Phi}{2}\frac{A(A^2-B^2)}{B^4}\frac{B^2-z^2}{\sqrt{1+\dfrac{A^2-B^2}{B^4}z^2}}\,.$$

In order to get values of n_1 and n_2 that are appropriate to the problem, one must add the quantities that produce the normal system (i.e., $\Phi R_1 / 2$, $\Phi R_1 / 2$) to the quantities n_1 and n_2 thus-determined. Now, the formulas that were established before give:

$$R_1 = \frac{B^2}{A}\left(1+\frac{A^2-B^2}{B^4}z^2\right)^{3/2},$$

$$R_2 = A\left(1+\frac{A^2-B^2}{B^4}z^2\right)^{3/2}.$$

As a result, the definitive values of n_1 and n_2 will be:

$$n_1 = \frac{\Phi}{2}A\frac{2-\dfrac{A^2}{B^2}+2\dfrac{A^2-B^2}{B^4}z^2}{\sqrt{1+\dfrac{A^2-B^2}{B^4}z^2}},$$

$$n_2 = \frac{\Phi}{2}A\sqrt{1+\frac{A^2-B^2}{B^4}z^2}\,.$$

It is easy to verify that n_1 and n_2 satisfy the condition:

$$\frac{n_1}{R_2}+\frac{n_2}{R_1} = \Phi,$$

as they must.

One can obtain a more important verification of the lengthy deductions that have led us to the result by considering the ellipsoidal cap that is bounded by the parallel z = const. and writing out that the sum of the projections onto the z-axis of the tensions that are normal to that parallel is equal to the sum of the components with respect to the same axis of the pressures that are exerted on the cap. That condition will determine n_2 immediately, and one will recover the value to which we were led.

The value of n_1 can be written:

$$n_1 = n_2 - \frac{\Phi}{2}\frac{A(A^2-B^2)}{B^4}\frac{B^2-z^2}{\sqrt{1+\frac{A^2-B^2}{B^4}z^2}}.$$

The difference $n_1 - n_2$ will have a constant sign then, which will be positive if the ellipsoid is elongated and negative if it is flattened. It will be annulled for the two poles $z = \pm B$, which will be umbilical equilibrium points, as a result. Contrary to what happens in general, those points do not link up with a line of points that enjoy the same property. That amounts to saying that in the present case, the differential equation of the line that we are dealing with will reduce to $dz = 0$.

Upon setting $n_1 = 0$, one will get the equation:

$$\frac{z^2}{B^2} = \frac{A^2-2B^2}{2(A^2-B^2)},$$

which can be satisfied only if A is greater than $B\sqrt{2}$. When that condition is satisfied, there will always be two parallels for which the tension that is perpendicular to the meridians is zero. In the zone that is found between those parallels, the two principal tensions will have opposite signs, and one knows that there will then exist two directions at each point whose elements do work solely by shearing.

Now consider an arbitrary portion of the surface in place of the complete ellipsoid. One can no longer say that $\varphi_1(u)$ and $\varphi_2(v)$ are constant, because u and v will no longer pass through all possible values. Since $\varphi_1(u)$ must be a periodic function of μ that has a period of 2π or a divisor of 2π, we can take the variable to be the quantity:

$$e^{iu} = e^{i\mu-\log\sqrt{\frac{B+z}{B-z}}} = \sqrt{\frac{B-z}{B+z}}\,e^{i\mu},$$

instead of u, without introducing any indeterminacy.

Upon setting $z = B\cos\omega$, one will have:

$$e^{iu} = \tan\frac{\omega}{2}e^{i\mu} \qquad \text{and} \qquad \varphi_1(u) = \psi\left(\tan\frac{\omega}{2}e^{i\mu}\right).$$

If considers an ellipsoidal cap that has its center at the pole $\omega = 0$ and is found entirely within the portion of the given surface then since the expression ψ must remain finite, continuous, and well-defined for all points on that cap, it must be developable into a series in increasing integer powers of tan $(\omega / 2)\, e^{iu}$. Within the limits of the cap thus-defined, we will have:

$$\varphi_1 = A_0 + A_1 \tan\frac{\omega}{2} e^{i\mu} + A_2 \tan^2\frac{\omega}{2} e^{i\mu} .$$

As before, one will see that φ_1 must be annulled for any μ and for $z = B$; i.e., for tan $\omega/2 = 0$. For the neighboring points of $z = B$, φ_1 must have the same order as $B^2 - z^2 = B^2 \sin^2 \omega$. That demands that A_0 and A_1 must be zero. Upon setting:

$$A_2 = a_2 + i\, b_2 , \qquad A_2 = a_2 + i\, b_2 , \qquad \ldots,$$

one will then have:

$$\begin{aligned}\varphi_1 (u) &= (a_2 + i\, b_2) \tan^2 \frac{\omega}{2} (\cos 2\mu + i \sin 2\mu) \\ &+ (a_3 + i\, b_3) \tan^2 \frac{\omega}{2} (\cos 3\mu + i \sin 3\mu) \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

Likewise:

$$\varphi_2 (v) = (a_2 - i\, b_2) \tan^2 \frac{\omega}{2} (\cos 2\mu - i \sin 2\mu) + \ldots$$

As a result, in order to get n_2 , one must add the expression:

$$\frac{B^2}{A}\frac{\varphi_1(u)+\varphi_2(v)}{2(B^2-z^2)}\sqrt{1+\frac{A^2-B^2}{B^4}z}$$

$$= \frac{\sqrt{A^2\cos^2\omega + B^2 \sin^2\omega}}{2AB\cos^4\omega/2}[(a_2 \cos 2\mu - b_2 \sin 2\mu) + \tan \frac{\omega}{2}(a_3 \cos 3\mu - b_3 \sin 3\mu) + \ldots]$$

to the value that was found before; n_1 will be consequently modified, and t_1 will not be equal to zero, but to:

$$B\frac{\varphi_1(u)-\varphi_2(v)}{2i(B^2-z^2)}$$
$$= \frac{1}{2B\cos^4\omega/2}\ [(a_2 \cos 2\mu + b_2 \sin 2\mu) + \tan\frac{\omega}{2}(a_3 \cos 3\mu + b_3 \sin 3\mu) + \ldots].$$

The inspection of these formulas shows immediately why the arbitrary functions must be annulled when the surface is closed: It is because they must remain finite for any μ when ω tends to π, and consequently, cos $(\omega / 2)$ tends to zero.

The indeterminacy that is introduced into the equilibrium conditions by the arbitrary functions should not surprise us: It is completely analogous to what one encounters when one seeks the reactions that are experienced by an invariable solid that has more than three support points.

Minimal surfaces

The minimal surfaces are characterized by the property that their asymptotic lines are orthogonal. One can then apply the general equations of the theory of surfaces to those lines, and since one will then have $1 / R_1 = 1 / R_2 = 0$, the equations that one must deal with will reduce to:

$$\frac{1}{\rho_1} = -\frac{\partial L / \partial \mu}{LM}, \qquad \frac{1}{\rho_2} = -\frac{\partial M / \partial \lambda}{LM},$$

$$-\frac{1}{M}\frac{\partial(1/T)}{\partial \mu} + \frac{2}{\rho_1}\frac{1}{T} = 0,$$

$$-\frac{1}{L}\frac{\partial(1/T)}{\partial \lambda} + \frac{2}{\rho_2}\frac{1}{T} = 0,$$

$$\frac{1}{T^2} + \frac{1}{M}\frac{\partial(1/\rho_1)}{\partial \mu} + \frac{1}{L}\frac{\partial(1/\rho_2)}{\partial \lambda} - \frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} = 0.$$

One infers from this that:

$$\frac{1}{M}\frac{\partial(1/T)}{\partial \mu} + \frac{2}{LM}\frac{\partial L}{\partial \mu}\frac{1}{T} = 0$$

or rather:

$$\frac{\partial}{\partial \mu}\left(\frac{L^2}{T}\right) = 0.$$

L^2 / T is then a function of only λ, and upon choosing that variable conveniently, one can arrange that $L^2 / T = 1$, in some way; similarly, $M^2 / T = 1$. Upon setting $T = 1 / \theta^2$, one will have:

$$L = M = \frac{1}{\theta}.$$

One will then find that:

$$\frac{1}{\rho_1} = -\theta^2 \frac{\partial(1/\theta)}{\partial \mu} = \frac{\partial \theta}{\partial \mu},$$

$$\frac{1}{\rho_2} = \frac{\partial \theta}{\partial \lambda},$$

and finally, that:

$$\theta^4 + \theta\left(\frac{\partial^2\theta}{\partial\lambda^2}+\frac{\partial^2\theta}{\partial\mu^2}\right)-\left(\frac{\partial\theta}{\partial\lambda}\right)^2-\left(\frac{\partial\theta}{\partial\mu}\right)^2 = 0,$$

which one can write as:

$$\theta^2 + \frac{\partial^2}{\partial\lambda^2}(\log\theta)+\frac{\partial^2}{\partial\mu^2}(\log\theta) = 0$$

upon dividing by θ^2, or even:

$$2\,\frac{\partial^2(\log\theta^2)}{\partial x\,\partial y} = -\,\theta^2,$$

when one sets:

$$dx = d\lambda + i\,d\mu, \qquad dy = d\lambda - i\,d\mu.$$

If one replaces θ^2 with $4u$ then that equation will become:

$$\frac{\partial^2(\log u)}{\partial x\,\partial y} = -\,2u,$$

The general integral, which is due to Liouville, is:

$$u = -\,\frac{X'Y'}{(X+Y)^2},$$

in which X denotes an arbitrary function of x, and Y denotes an arbitrary function of y, and X', Y' are the derivatives of those two functions. θ^2 will then be equal to $-\dfrac{4X'Y'}{(X+Y)^2}$.

The line element of the surface:

$$ds^2 = L^2\,d\lambda^2 + M^2\,d\mu^2$$

will consequently be:

$$ds^2 = \frac{1}{\theta^2}(d\lambda^2 + d\mu^2) = -\,\frac{(X+Y)^2}{4X'Y'}\,dx\,dy\,.$$

If one sets:

$$\frac{dx}{2X'} = dx_1, \qquad \frac{dy}{2Y'} = dy_1\,,$$

and if one lets X_1, Y_1 denote the functions X and Y when they are expressed in terms of x_1, y_1, resp., then one will get the following formula, which was contributed by Ossian Bonnet [Journal de Liouville (2), t. V]:

$$ds^2 = (X_1 + Y_1)^2\,dx_1\,dy_1\,.$$

Having said that, consider a minimal surface that is subject to a system of tangential forces and look for the equilibrium conditions that refer to the asymptotic lines. Since those lines are, in a fashion, lines of conjugate tension, they must be lines of principal tension in the present case. One will then have $t = 0$.

Moreover, $1 / R_1 = 1 / R_2 = 0$. The last equilibrium equation will then disappear entirely. The first two of them reduce to:

$$\frac{1}{L}\frac{\partial n_2}{\partial\lambda}+\frac{n_1-n_2}{\rho_2} = F_1\,,$$

$$\frac{1}{M}\frac{\partial n_1}{\partial\mu}+\frac{n_2-n_1}{\rho_1} = F_2\,,$$

or rather:

$$\theta\frac{\partial n_2}{\partial\lambda}+(n_1-n_2)\frac{\partial\theta}{\partial\lambda} = F_1\,,$$

$$\theta\frac{\partial n_1}{\partial\mu}+(n_2-n_1)\frac{\partial\theta}{\partial\mu} = F_2\,,$$

so

$$\theta\frac{\partial^2 n_2}{\partial\lambda\,\partial\mu}+\frac{\partial n_2}{\partial\lambda}\frac{\partial\theta}{\partial\mu}+(n_1-n_2)\frac{\partial^2\theta}{\partial\lambda\,\partial\mu}+\frac{\partial\theta}{\partial\lambda}\frac{\partial(n_1-n_2)}{\partial\mu} = \frac{\partial F_1}{\partial\mu}\,,$$

or even, upon replacing $\partial n_2 / \partial\lambda$ with its value and setting $n_1 - n_2 = u$:

$$\theta^2\frac{\partial^2 n_2}{\partial\lambda\,\partial\mu}+\frac{\partial\theta}{\partial\mu}\left(F_1-u\frac{\partial\theta}{\partial\lambda}\right)+u\,\theta\frac{\partial^2\theta}{\partial\lambda\,\partial\mu}+\theta\frac{\partial\theta}{\partial\lambda}\frac{\partial u}{\partial\mu} = \frac{\partial F_1}{\partial\mu}\theta\,;$$

similarly, one will have:

$$\theta^2\frac{\partial^2 n_1}{\partial\lambda\,\partial\mu}+\frac{\partial\theta}{\partial\lambda}\left(F_2+u\frac{\partial\theta}{\partial\lambda}\right)-u\,\theta\frac{\partial^2\theta}{\partial\lambda\,\partial\mu}+\theta\frac{\partial\theta}{\partial\mu}\frac{\partial u}{\partial\lambda} = \frac{\partial F_2}{\partial\lambda}\theta\,,$$

and upon subtracting, one will find that:

$$\theta^2\frac{\partial^2 u}{\partial\lambda\,\partial\mu}+2u\left(\frac{\partial\theta}{\partial\lambda}\frac{\partial\theta}{\partial\mu}-\theta\frac{\partial\theta}{\partial\lambda\,\partial\mu}\right)-\theta\frac{\partial\theta}{\partial\mu}\frac{\partial u}{\partial\lambda}-\theta\frac{\partial\theta}{\partial\lambda}\frac{\partial u}{\partial\mu}$$

$$= \theta\left(\frac{\partial F_2}{\partial\lambda}-\frac{\partial F_1}{\partial\mu}\right)+F_1\frac{\partial\theta}{\partial\mu}-F_2\frac{\partial\theta}{\partial\lambda}\,.$$

Now set $u = v\,\theta$, so:

$$\frac{\theta \dfrac{\partial u}{\partial \lambda} - u \dfrac{\partial \theta}{\partial \lambda}}{\theta^2} = \frac{\partial v}{\partial \lambda},$$

$$\frac{\theta^2 \dfrac{\partial^2 u}{\partial \lambda \partial \mu} + u\left(2\dfrac{\partial \theta}{\partial \lambda}\dfrac{\partial \theta}{\partial \mu} - \theta \dfrac{\partial^2 \theta}{\partial \lambda \partial \mu}\right) - \theta \dfrac{\partial \theta}{\partial \mu}\dfrac{\partial u}{\partial \lambda} - \theta \dfrac{\partial u}{\partial \mu}\dfrac{\partial \theta}{\partial \lambda}}{\theta^3} = \frac{\partial^2 v}{\partial \lambda \partial \mu}.$$

From those formulas, the equation that gives v can be written:

$$\theta \frac{\partial^2 v}{\partial \lambda \partial \mu} - v \frac{\partial^2 \theta}{\partial \lambda \partial \mu} = \frac{\partial f_2}{\partial \lambda} - \frac{\partial f_1}{\partial \mu},$$

when one sets $F_1 / \theta = f_1$, $F_2 / \theta = f_2$. The knowledge of v, n_1, and n_2 will be given by just one quadrature.

The function θ is defined by the relation:

$$\theta^2 = -\frac{4X'Y'}{(X+Y)^2}.$$

However, we have to express that quantity as a function of λ and μ. In order to do that, we remark that if one sets:

$$\log (X + Y) = \varphi$$

then one will have:

$$\theta^2 = -4\frac{\partial \varphi}{\partial x}\frac{\partial \varphi}{\partial y},$$

or, since $x = \lambda + i\,\mu$, $y = \lambda - i\,\mu$:

$$\theta^2 = -\left[\left(\frac{\partial \varphi}{\partial \lambda}\right)^2 + \left(\frac{\partial \varphi}{\partial \mu}\right)^2\right],$$

and since that is equal to $1 / L$, it must unavoidably be real; as a result, $\left(\dfrac{\partial \varphi}{\partial \lambda}\right)^2 + \left(\dfrac{\partial \varphi}{\partial \mu}\right)^2$ must be real and negative.

φ is a function of λ and μ that one can always put into the form:

$$\varphi = P(\lambda, \mu) + i\,Q(\lambda, \mu),$$

in which P and Q are two real functions. One will then have:

$$-\theta^2 = \left(\frac{\partial \varphi}{\partial \lambda}\right)^2 + \left(\frac{\partial \varphi}{\partial \mu}\right)^2 = \left(\frac{\partial P}{\partial \lambda}\right)^2 + \left(\frac{\partial P}{\partial \mu}\right)^2 - \left(\frac{\partial Q}{\partial \lambda}\right)^2 - \left(\frac{\partial Q}{\partial \mu}\right)^2 + 2i\left(\frac{\partial P}{\partial \lambda}\frac{\partial Q}{\partial \lambda} + \frac{\partial P}{\partial \mu}\frac{\partial Q}{\partial \mu}\right).$$

In order for θ^2 to be real, it is necessary that the condition:

$$\frac{\partial P}{\partial \lambda}\frac{\partial Q}{\partial \lambda}+\frac{\partial P}{\partial \mu}\frac{\partial Q}{\partial \mu}=0$$

must be fulfilled. What will then remain is:

$$\theta^2=\left(\frac{\partial Q}{\partial \lambda}\right)^2+\left(\frac{\partial Q}{\partial \mu}\right)^2-\left(\frac{\partial P}{\partial \lambda}\right)^2-\left(\frac{\partial P}{\partial \mu}\right)^2.$$

On the other hand, since e^φ is equal to $X + Y$, it must have the form:

$$\text{funct. }(\lambda + i\,\mu) + \text{funct. }(\lambda - i\,\mu),$$

which will imply the condition:

$$\frac{\partial^2}{\partial \lambda^2}(e^\varphi)+\frac{\partial^2}{\partial \mu^2}(e^\varphi)=0,$$

or rather:

$$\frac{\partial^2 \varphi}{\partial \lambda^2}+\frac{\partial^2 \varphi}{\partial \mu^2}+\left(\frac{\partial \varphi}{\partial \lambda}\right)^2+\left(\frac{\partial \varphi}{\partial \mu}\right)^2=0,$$

or, upon developing this:

$$\frac{\partial^2 P}{\partial \lambda^2}+\frac{\partial^2 P}{\partial \mu^2}+\left(\frac{\partial P}{\partial \lambda}\right)^2+\left(\frac{\partial P}{\partial \mu}\right)^2-\left(\frac{\partial Q}{\partial \lambda}\right)^2-\left(\frac{\partial Q}{\partial \mu}\right)^2+i\left(\frac{\partial^2 Q}{\partial \lambda^2}+\frac{\partial^2 Q}{\partial \mu^2}\right)=0,$$

hence:

$$\frac{\partial^2 Q}{\partial \lambda^2}+\frac{\partial^2 Q}{\partial \mu^2}=0,$$

$$\frac{\partial^2 P}{\partial \lambda^2}+\frac{\partial^2 P}{\partial \mu^2}+\left(\frac{\partial P}{\partial \lambda}\right)^2+\left(\frac{\partial P}{\partial \mu}\right)^2=\left(\frac{\partial Q}{\partial \lambda}\right)^2+\left(\frac{\partial Q}{\partial \mu}\right)^2.$$

Let ϖ be a function of λ and μ, such that one will have:

$$\frac{\partial \varpi}{\partial \lambda}=\frac{\partial Q}{\partial \mu},\qquad \frac{\partial \varpi}{\partial \mu}=-\frac{\partial Q}{\partial \lambda}.$$

The function ϖ exists, since $\dfrac{\partial^2 Q}{\partial \lambda^2}+\dfrac{\partial^2 Q}{\partial \mu^2}=0$.

The relation $\dfrac{\partial P}{\partial \lambda}\dfrac{\partial Q}{\partial \lambda}+\dfrac{\partial P}{\partial \mu}\dfrac{\partial Q}{\partial \mu}=0$ will become:

$$\frac{\partial P}{\partial \lambda}\frac{\partial \varpi}{\partial \mu} - \frac{\partial P}{\partial \mu}\frac{\partial \varpi}{\partial \lambda} = 0,$$

which obviously demands that P should be a function of ϖ. It will then result that:

$$\frac{\partial^2 P}{\partial \lambda^2} = \frac{\partial P}{\partial \varpi}\frac{\partial^2 \varpi}{\partial \lambda^2} + \frac{\partial^2 P}{\partial \mu^2}\left(\frac{\partial \varpi}{\partial \lambda}\right)^2,$$

$$\frac{\partial^2 P}{\partial \mu^2} = \frac{\partial P}{\partial \varpi}\frac{\partial^2 \varpi}{\partial \mu^2} + \frac{\partial^2 P}{\partial \varpi^2}\left(\frac{\partial \varpi}{\partial \mu}\right)^2,$$

and as a result, upon remarking that $\frac{\partial^2 \varpi}{\partial \lambda^2} + \frac{\partial^2 \varpi}{\partial \mu^2} = 0$:

$$\theta^2 = \frac{\partial^2 P}{\partial \lambda^2} + \frac{\partial^2 P}{\partial \mu^2} = \frac{\partial^2 P}{\partial \varpi^2}\left[\left(\frac{\partial \varpi}{\partial \lambda}\right)^2 + \left(\frac{\partial \varpi}{\partial \mu}\right)^2\right].$$

P is not an arbitrary function of ϖ, because the condition:

$$\frac{\partial^2 P}{\partial \lambda^2} + \frac{\partial^2 P}{\partial \mu^2} + \left(\frac{\partial P}{\partial \lambda}\right)^2 + \left(\frac{\partial P}{\partial \mu}\right)^2 = \left(\frac{\partial Q}{\partial \lambda}\right)^2 + \left(\frac{\partial Q}{\partial \mu}\right)^2$$

can be written:

$$\left[\frac{\partial^2 P}{\partial \varpi^2} + \left(\frac{\partial P}{\partial \varpi}\right)^2\right]\left[\left(\frac{\partial \varpi}{\partial \lambda}\right)^2 + \left(\frac{\partial \varpi}{\partial \mu}\right)^2\right] = \left(\frac{\partial \varpi}{\partial \lambda}\right)^2 + \left(\frac{\partial \varpi}{\partial \mu}\right)^2$$

or

$$\frac{\partial^2 P}{\partial \varpi^2} + \left(\frac{\partial P}{\partial \varpi}\right)^2 = 1.$$

Upon including the constant that is introduced by integration over ϖ, one will infer from this that:

$$\frac{\partial P}{\partial \varpi} = \frac{e^{2\varpi} - 1}{e^{2\varpi} + 1},$$

and

$$\frac{\partial^2 P}{\partial \varpi^2} = \frac{4e^{2\varpi}}{(e^{2\varpi} + 1)^2} = \frac{4}{(e^{\varpi} + e^{-\varpi})^2} = \frac{1}{\cos^2 i\varpi}.$$

Hence, one will finally have:

If one lets ϖ *denote an isothermal function of* λ *and* μ *– i.e., an arbitrary solution of the equation* $\frac{\partial^2\varpi}{\partial\lambda^2}+\frac{\partial^2\varpi}{\partial\mu^2}$ = 0 *– then the most general value of* θ *will be:*

$$\theta = \frac{1}{\cos i\varpi}\sqrt{\left(\frac{\partial\varpi}{\partial\lambda}\right)^2+\left(\frac{\partial\varpi}{\partial\mu}\right)^2}\,.$$

When ϖ is a function of only λ, it can be isothermal only if it reduces to $A\lambda$, where A is a constant. One will then have $\theta = \frac{A}{\cos iA\lambda}$, so $\frac{\partial^2\theta}{\partial\lambda\,\partial\mu}$ = 0, and the equation in v will reduce to:

$$\frac{\partial^2 v}{\partial\lambda\,\partial\mu} = \frac{\cos iA\lambda}{A}\left(\frac{\partial f_2}{\partial\lambda}-\frac{\partial f_1}{\partial\mu}\right).$$

v can then be obtained by means of two successive quadratures. The hypothesis ϖ = funct. (λ) gives $\frac{1}{\rho_1} = \frac{\partial\theta}{\partial\mu}$ = 0; i.e., the minimal surface is ruled. One knows that the only minimal surface that enjoys that property is the square-threaded screw surface: We then revert to a case that was studied before.

When the quantities f_1, f_2 are partial derivatives of the same function (which will happen any time that the interval between two consecutive orthogonal trajectories of the external forces varies from one point to the other along one of those trajectories in inverse proportion to $\sqrt{f_1^2+f_2^2} = F / \theta$), one will have $\frac{\partial f_2}{\partial\lambda}-\frac{\partial f_1}{\partial\mu}$ = 0. As a result, the equation in v will become:

(20) $$\frac{1}{v}\frac{\partial^2 v}{\partial\lambda\,\partial\mu} = \frac{1}{\theta}\frac{\partial^2\theta}{\partial\lambda\,\partial\mu}\,.$$

In that remarkable form, one will immediately perceive the solution v / θ = const. Consequently, for an arbitrary minimal surface that is subject to tangential forces that satisfy the condition $\frac{\partial f_2}{\partial\lambda}-\frac{\partial f_1}{\partial\mu}$ = 0, one will always have to determine a particular state of equilibrium. It will suffice to choose the boundary conditions suitably.

Minimal surface of revolution

In the case of the minimal surface of revolution, θ will remain constant for each parallel; i.e., for each line such that $\lambda - \mu$ is constant. If one sets:

$$\lambda + \mu = \alpha,$$

$$\lambda - \mu = \beta,$$

and if one takes α, β to be new variables then θ will depend upon only α. The isothermal function ϖ reduces to α, and θ will have the value:

$$\theta = \frac{\sqrt{2}}{\cos i\alpha}.$$

One will infer from this that:

$$\frac{\partial\theta}{\partial\lambda} = \frac{\partial\theta}{\partial\mu} = \frac{\partial\theta}{\partial\alpha} = i\,\theta \tan i\alpha,$$

and the equilibrium equations, which are initially supposed to have vanishing right-hand sides, namely:

$$\theta\frac{\partial n_2}{\partial\lambda} + (n_1 - n_2)\frac{\partial\theta}{\partial\lambda} = 0,$$

$$\theta\frac{\partial n_1}{\partial\mu} + (n_2 - n_1)\frac{\partial\theta}{\partial\mu} = 0,$$

will give:

$$\frac{\partial n_2}{\partial\lambda} = -\frac{\partial n_1}{\partial\mu}.$$

As a result, there will exist a function P such that:

$$n_1 = \frac{\partial P}{\partial\lambda}, \qquad n_2 = -\frac{\partial P}{\partial\mu},$$

and that function will be defined by the equation:

$$\frac{\partial^2 P}{\partial\lambda\,\partial\mu} - \left(\frac{\partial P}{\partial\lambda} + \frac{\partial P}{\partial\mu}\right)\frac{d\theta/d\alpha}{\theta} = 0$$

or rather:

$$\frac{\partial^2 P}{\partial\alpha^2} - \frac{\partial^2 P}{\partial\beta^2} - 2i\tan i\alpha\,\frac{\partial P}{\partial\alpha} = 0.$$

Set:

$$x = i\tan i\alpha,$$
$$y = \beta;$$

x and y are two integrals of the equations:

$$\frac{\partial^2 P}{\partial\alpha^2} - 2i\tan i\alpha\,\frac{\partial P}{\partial\alpha} = 0, \qquad \frac{\partial^2 P}{\partial\beta^2} = 0.$$

If one takes x and y to be independent variables then one will find that:

$$\frac{\partial^2 P}{\partial x^2}\left(\frac{dx}{d\alpha}\right)-\frac{\partial^2 P}{\partial y^2}=0 \; ;$$

however:

$$\frac{dx}{d\alpha}=-(1+\tan^2 i\alpha)=-(1-x^2).$$

The equation that the solution of the problem depends upon can then be written:

$$\frac{\partial^2 z}{\partial y^2}=(1-x^2)^2\,\frac{\partial^2 z}{\partial x^2}$$

upon replacing P with z.

In order for perform the integration, we begin by investigating whether there exist solutions of the form:

$$z = XY,$$

in which X and Y are functions of only x, in the former case, and only y, in the latter. Upon making the substitution, one will find that:

$$X\,Y'' = (1-x^2)^2\,Y\,X'',$$

which is possible only if one has:

$$Y'' = a^2\,Y,$$

$$X'' = \frac{a^2}{(1-x^2)^2}X,$$

upon calling a constant α.

The equation in Y is integrated with no difficulty. The equation in X falls within one of the general types of linear equations that were studied by J. Tannery (Annales de l'École Normale, 1875). We follow the method that was pointed out in order to find a solution. Set:

$$X = (1-x^2)^p\,u,$$

in which u is a new unknown, and p is an undetermined coefficient. Upon dividing everything by $(1-x^2)^{p-2}$, the equation to be solved will become:

$$(1-x^2)^2\,u'' - 4px\,(1-x^2)\,u' - 2p\,u + 4p\,(p-1)\,x^2\,u - a^2\,u = 0.$$

If we set:

$$4p\,(p-1) = a^2$$

then the left-hand side will be divisible by x^2, and what will remain is:

$$(1-x^2)\,u'' - 4p\,x\,u' - 2p\,(2p-1)\,u = 0.$$

One changes the variable by writing:

$$2t = x + 1,$$

so

$$1 - x^2 = 4t\,(1 - t).$$

One will get:

$$t\,(t-1)\frac{d^2u}{dt^2} + 2p\,(2t-1)\frac{du}{dt} + 2p\,(2p-1)\,u = 0.$$

Upon introducing three new parameters α, β, γ such that one has:

$$\gamma = 2p,$$
$$\alpha + \beta + 1 = 4p,$$
$$\alpha\beta = 2p\,(2p-1),$$

the equation will finally take the form:

$$(t^2 - t)\,\frac{d^2u}{dt^2} - [\gamma - (\alpha + \beta + 1)\,t]\,\frac{du}{dt} + \alpha\beta\,u = 0.$$

This an equation that Gauss studied, and it admits the hypergeometric series:

$$F\,(\alpha, \beta, \gamma, t) = 1 + \frac{\alpha\beta}{1\cdot\gamma}t + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\cdot\gamma\cdot(\gamma+1)}t^2 + \ldots$$

for a solution.

In order for that series to be convergent, it is necessary that t must be less than 1. That condition is always fulfilled when a, and therefore p, is real, because for all real values of a, the quantity:

$$x = \frac{e^a - e^{-a}}{e^a + e^{-a}}$$

will remain between + 1 and – 1, and as a result, t will vary between 1 and 0. In addition, it is generally necessary that γ must not be a negative integer. However, that restriction is pointless here, since one immediately recognizes that α and β are roots of the second-degree equation:

$$m^2 + (1 - 4p)\,m + 2p\,(2p-1) = 0,$$

which are roots that are equal to $2p$ and $2p - 1$, or γ and $\gamma - 1$. Upon taking $\beta = \gamma$, for example, the hypergeometric series will reduce to:

$$1 + (\gamma - 1)\,t + \frac{(\gamma-1)\gamma}{1\cdot 2}t^2,$$

or rather, to:

$$(1 - t)^{1-\gamma}.$$

From that, the value of X is:

$$X = (1 - x^2)^p \left(\frac{1-x}{2}\right)^{1-2p} = \frac{1-x}{2^{2p-1}}\left(\frac{1+x}{1-x}\right)^p .$$

Upon taking Y to have the value $C\, e^{ay}$, in which C denotes an arbitrary constant, and dropping the factor 2^{2p-1}, one will get the following value for z :

$$z = C\, e^{ay} (1 - x) \left(\frac{1+x}{1-x}\right)^p .$$

In order to deduce the value of the unknown tension n_1, it will suffice to remark that:

$$n_1 = \frac{\partial z}{\partial \lambda} = \frac{\partial z}{\partial \alpha} + \frac{\partial z}{\partial \beta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial z}{\partial y} = \frac{\partial z}{\partial y} - (1 - x^2)\, \frac{\partial z}{\partial x} .$$

Upon performing the calculation, one will get:

$$n_1 = C\, e^{ay}\, \frac{(1+x)^p}{(1-x)^{p-1}} (a - 2p + 1 + x) .$$

The corresponding value of n_2 is:

$$n_2 = C\, e^{ay}\, \frac{(1+x)^p}{(1-x)^{p-1}} (a - 2p - 1 - x) .$$

In order to verify these results, we investigate how n_1 varies when we displace along the curve λ = const. or $d\alpha + d\beta = 0$, or (what amounts to the same thing):

$$\frac{d\alpha}{dx} + \frac{d\beta}{dx} = -\frac{1}{1-x^2} + y' = 0.$$

Upon considering y to be a function of x that is defined by that equation and taking the derivative of n_1 with respect to x, one will find that:

$$dn_1 = dx\, C\, e^{ay}\, (1 + x)^{p-1} (1 - x)^{-p} (2p - 1 - x)\, 2x,$$

or rather:

$$dn_1 = \frac{x\, dx}{1-x^2} (n_1 - n_2),$$

or furthermore, upon replacing $\frac{dx}{1-x^2}$ with $-d\alpha$ and x by $i \tan i\alpha$:

$$dn_1 = i \tan i\alpha \, d\alpha \, (n_1 - n_2) .$$

Since $d\lambda = 0$, by hypothesis, which entails that $d\alpha = d\mu$, one can write:

$$\frac{\partial n_1}{\partial \mu} = i \tan i\alpha \, (n_1 - n_2) = \frac{\partial \theta / \partial \mu}{\theta} (n_1 - n_2).$$

This is the first of the equations that n_1 and n_2 must satisfy. The second one will be obtained in an analogous fashion.

Let p, p' be the two roots of the equation:

$$4p\,(p - 1) = a^2.$$

One can substitute one or the other roots p in the values of n_1 and n_2 that were found above. One will then get two systems of solutions, which will permit one to form the complete solution:

$$n_1 = C\,e^{ay} \frac{(1+x)^p}{(1-x)^{p-1}} (a - 2p + 1 + x) + C' e^{ay} \frac{(1+x)^{p'}}{(1-x)^{p'-1}} (a - 2p' + 1 + x),$$

$$n_2 = C\,e^{ay} \frac{(1+x)^p}{(1-x)^{p-1}} (a + 2p - 1 - x) + C' e^{ay} \frac{(1+x)^{p'}}{(1-x)^{p'-1}} (a + 2p' - 1 - x) .$$

Since $p + p' = 1$, one can also write:

$$n_1 = C\,e^{ay} \frac{(1+x)^p}{(1-x)^{p-1}} (a - 2p + 1 + x) + C' e^{ay} \frac{(1+x)^{p}}{(1-x)^{p-1}} (a + 2p - 1 + x),$$

$$n_2 = C\,e^{ay} \frac{(1+x)^p}{(1-x)^{p-1}} (a + 2p - 1 - x) + C' e^{ay} \frac{(1+x)^{p}}{(1-x)^{p-1}} (a - 2p + 1 - x) .$$

Finally, one will deduce the general integrals from this upon considering C and C' to be two arbitrary functions of a, $\varphi(a)$, $\psi(a)$, and performing integrals that are defined with respect to a between the two well-defined limits a_1 and a_2 :

$$n_1 = (1 + x) \int_{a_1}^{a_2} e^{ay} \left(\frac{1+x}{1-x}\right)^{p-1} (a - 2p + 1 + x)\ \varphi(a)\ da$$

$$+ (1 - x) \int_{a_1}^{a_2} e^{ay} \left(\frac{1+x}{1-x}\right)^{p-1} (a + 2p - 1 + x)\ \psi(a)\ da .$$

$$n_2 = (1 + x) \int_{a_1}^{a_2} e^{ay} \left(\frac{1+x}{1-x}\right)^{p-1} (a + 2p - 1 - x)\ \varphi(a)\ da$$

$$+ (1 - x) \int_{a_1}^{a_2} e^{ay} \left(\frac{1+x}{1-x}\right)^{p-1} (a - 2p + 1 - x)\ \psi(a)\ da$$

In order to solve the general problem of equilibrium of the minimal surface of revolution, it remains for one to find a particular solution of the equations with non-vanishing right-hand sides. One will arrive at it by once more taking the values of n_1 and n_2 that include two arbitrary constants C and C' and applying the method of variation of arbitrary constants. One will then be led to two equations of the form:

$$A\frac{\partial C}{\partial \lambda} + A'\frac{\partial C'}{\partial \lambda} = F_1\ ,$$

$$A\frac{\partial C}{\partial \mu} + A'\frac{\partial C'}{\partial \mu} = F_2\ .$$

These equations can be solved only in particular cases. For example, when F_1 / A is a function of only λ and F_1 / A' is a function of only μ, one can suppose $\partial C / \partial \mu = 0$ and $\partial C' / \partial \mu = 0$; C and C' will then be given by simple quadratures.

When $\frac{\partial}{\partial \mu}\left(\frac{F_1}{A}\right) = \frac{\partial}{\partial \lambda}\left(\frac{F_2}{A}\right)$, one can suppose that $C' = 0$ and get C by a quadrature. If $\frac{\partial}{\partial \mu}\left(\frac{F_1}{A'}\right) = \frac{\partial}{\partial \lambda}\left(\frac{F_2}{A'}\right)$ then one will suppose, on the contrary, that $C = 0$ and one gets C' by a quadrature. Finally, whenever $\frac{\partial}{\partial \mu}\left(\frac{F_1}{\theta}\right) = \frac{\partial}{\partial \lambda}\left(\frac{F_2}{\theta}\right)$, one will immediately procure a particular solution without resorting to the variation of arbitrary constants by the method that was indicated previously.

Besides the cases that were just studied, the general integration of equation (20) seems impracticable to me, no matter what hypothesis that one imposes upon the function θ. Moutard has studied equations of the form:

$$\frac{1}{v}\frac{\partial^2 v}{\partial \lambda\, \partial \mu} = f(\lambda, \mu)$$

in Cahier XLV of the *Journal de l'École Polytechnique* and determined the conditions that the right-hand side must fulfill in order for there to be an explicit general integral. I do not believe that one can find values of θ that belong to the minimal surfaces and at the same time give one of the integrability cases that Moutard established for $\frac{1}{\theta}\frac{\partial^2 \theta}{\partial \lambda\, \partial \mu}$.

Nonetheless, I shall increase the number of applications. The preceding discussion seems to me to suffice to elucidate the general theory that was presented in the first chapters. On the subject of those applications, there is good reason to point out that the cases in which I have succeeded in finding explicit general integrals of the problem all refer to ruled surfaces. It was easy to predict, by inspecting equations (18), that these surfaces will be the easiest to treat by far, because when one sets $1 / \rho = 0$, the unknown n_1 will disappear from the first equation.

If one would like to extend this research and find some other surfaces for which one can obtain the general solution then one must eliminate the unknown n_1 from equations (18), with vanishing right-hand sides, which will lead to an equation of the form:

$$\frac{\partial^2 n}{\partial\alpha\,\partial\beta} + A\frac{\partial n}{\partial\alpha} + B\frac{\partial n}{\partial\beta} + Cn = 0,$$

in which α and β denote the parameters of the asymptotic lines. One then imposes various hypotheses upon the values of A, B, C that will permit one to integrate that equation. If one would like, at the same time, to determine the surfaces that correspond to each hypothesis then all that would remain to be done would be to apply systems of forces to them such that their introduction in the right-hand side would not prevent one from performing the integration. However, that field of study is much too vast for me to attempt to enter into it today.