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CANONICAL EQUATIONS

Application to the study of the equilibrium of flexible filaments and brachistochrone curves

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Translated D. H. Delphenich

One supposes that the number of variables that define the positions of the points of a material system has been reduced to the minimum number, while taking into account the constraints, and one calls that minimum number of variables $q_1, q_2, q_3, \ldots, q_k$, while $q_1', \ldots, q_k'$ are their derivatives with respect to time. In addition, if one lets T denote the total *vis viva* of the system, while $Q_1, Q_2, \ldots$ denote some well-defined functions of the variables then one will know that the equations of motion can be put into the following form, which is due to Lagrange:

$$(1) \qquad \frac{d}{dt}\frac{dT}{dq_1'} - \frac{dT}{dq_1} = Q_1, \qquad \ldots, \qquad \frac{d}{dt}\frac{dT}{dq_k'} - \frac{dT}{dq_k} = Q_k .$$

Upon transforming those equations, which are k in number and which have second order, one can put them into another form that was pointed out by Hamilton and that one calls the *canonical form.*

In all of what follows, we will suppose that the principle of *vis viva* is valid and that there exists a force function U ; i.e., a function such that the quantities $Q_1, Q_2, \ldots$, represent the partial derivatives of U with respect to the variables $q_1, q_2, \ldots$

If one sets:

$$(2) \qquad \frac{dq_1}{dt} = q_1', \qquad \frac{dq_2}{dt} = q_2', \ldots$$

and regards $q_1', q_2', \ldots$ as unknowns, while combining equations (2) with equations (1), then the latter will be of first order, and one will have a system of $2k$ simultaneous first-order equations.

([1]) Read at the session on 18 June 1885.

Instead of taking the q_1 , q_2 , ..., q_1', q_2', … to be the variables, take q_1 , q_2 , ... and $p_1 = \frac{dT}{dq_1'}$, $p_2 = \frac{dT}{dq_2'}$, …, $p_k = \frac{dT}{dq_k'}$.

We examine what the system of equations (1) and (2) will become after changing the variables.

Equations (1) will take the form $\frac{dp_1}{dt} - \left(\frac{dT}{dq_1}\right) = Q_1$, …, when one lets $\left(\frac{dT}{dq_1}\right)$ denote what the derivative $\frac{dT}{dq_1}$ will become when one replaces the variables q_1 , q_2 , ..., q_1', q_2', … with the variables q_1 , q_2 , ..., p_1 , p_2 , … in the expression for T.

Now, T is a homogeneous function of second degree in the variables q', because if one supposes that the points are first represented by rectangular coordinate axes then one will have:

$$T = \tfrac{1}{2}\sum m_i\,(x_i'^2 + y_i'^2 + z_i'^2),$$

and by hypothesis, if the constraints are independent of time then one will have:

$$x_i' = \frac{dx_i}{dq_1}q_1' + \frac{dx_i}{dq_2}q_2' + \ldots,$$

and similarly:

$$y_i' = \frac{dy_i}{dq_1}q_1' + \frac{dy_i}{dq_2}q_2' + \ldots$$

Hence, from the substitution of variables x, y, z and x', y', z' as functions of the variables q and q', T will become a linear function that is homogeneous of second order. One will then get a homogeneous function from the theorem:

$$2T = q_1'\frac{dT}{dq_1'} + q_2'\frac{dT}{dq_2'} + \ldots + q_k'\frac{dT}{dq_k'}.$$

Upon subtracting T from both sides, this can be written:

$$T = p_1\,q_1' + p_2\,q_2' + \ldots - T.$$

Now, the T that is found in the left-hand side is a function of all the variables q_1 , q_2 , ..., q_1', q_2', …, as well as q_1 , q_2 , ..., q_k .

If one takes the total variation of T while considering it to be a function of all those variables then upon suppressing the terms that cancel, like $p_1\,\delta q_1'$ and $-\frac{dT}{dq_1'}\delta q_1'$, one will have:

$$\delta T = q_1' \, \delta p_1 + q_2' \, \delta p_2 + \ldots - \frac{dT}{dq_1} \delta q_1 - \frac{dT}{dq_2} \delta q_2 - \ldots,$$

from which, one deduces that:

$$\left(\frac{dT}{dq_1} \right) = - \frac{dT}{dq_1}, \ldots$$

and

$$\frac{dT}{dp_1} = q_1' = \frac{dq_1}{dt}, \ldots$$

One remarks that the two expressions for dT / dq_1 are equal and of opposite sign. Introduce that hypothesis into equations (1), and also replace Q_i with $dU / dq_1, \ldots$ One can replace systems (1) and (2) with the following one:

(3) $$\frac{dp_1}{dt} = \frac{dU}{dq_1} - \frac{dT}{dq_1}, \qquad \ldots, \frac{dp_k}{dt} = \frac{dU}{dq_k} - \frac{dT}{dq_k},$$

(4) $$\frac{dT}{dp_1} = q_1' = \frac{dq_1}{dt}, \qquad \ldots, \frac{dT}{dp_k} = \frac{dq_k}{dt}.$$

Finally, if one sets $U - T = H$ and remarks that U does not contain the variables $p_1, p_2, \ldots$ then the system will take the following form:

(A) $$\frac{dp_1}{dt} = \frac{dH}{dq_1}, \quad \frac{dp_2}{dt} = \frac{dH}{dq_2}, \ldots$$

(B) $$\frac{dq_1}{dt} = -\frac{dH}{dp_1}, \quad \frac{dq_2}{dt} = -\frac{dH}{dp_2}, \ldots$$

Integrating the canonical equations

One calls any equation:

$$\varphi = \alpha$$

an *integral* of equations (A), (B), in which φ is a function of $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k$, and t, α is an arbitrary constant, φ does not include α, and that function φ is such that its total derivative with respect to t reduces to zero, when one can eliminate the derivatives of the functions p and q with respect to time by means of equations (A) and (B).

Complete solution of the canonical equations. – The complete solution of those equations is composed of $2k$ distinct integrals that include $2k$ arbitrary constants.

Remark: Since the variable t does not enter into H, one can eliminate it by writing the canonical equations in the form:

$$\frac{dp_1}{\frac{dH}{dq_1}} = \frac{dp_2}{\frac{dH}{dq_2}} = \ldots \frac{dp_k}{\frac{dH}{dq_k}} = \frac{dq_1}{-\frac{dH}{dp_1}} = \ldots = \frac{dq_k}{-\frac{dH}{dp_k}} .$$

Upon integrating that system completely, one will have the expressions for $2k - 1$ variables as functions of each other – for example, p_1 and $2k - 1$ arbitrary constants a_1, a_2, …, a_{2k-1} . If one substitutes these values in the equation:

$$\frac{dp_1}{dt} = \frac{dH}{dq_1}$$

then one can infer that:

$$dt = \frac{dp_1}{\frac{dH}{dq_1}} ,$$

so:

$$t + a_{2k} = \int \frac{dp_1}{\frac{dH}{dq_1}} ,$$

in such a way that one can represent the complete integral of the system by the following equations, which are supposed to be solved for the constants:

$$\varphi_1 = a_1 , \qquad \varphi_2 = a_2 , \ \ldots, \ \varphi_{2k-1} = a_1 , \ \varphi_{2k} = a_{2k} + t = \int \frac{dp_1}{\frac{dH}{dq_1}} ,$$

φ_1 , φ_2 , …, φ_{2k} do not contain t explicitly.

Condition equation that one of the functions φ must satisfy

From the definition of an integral, it is necessary that the total derivative of φ with respect to t must be identically zero when one takes the canonical equations into account. Now, one has:

$$\frac{d\varphi}{dt} = \frac{d\varphi}{dp_1}\frac{dp_1}{dt} + \ldots + \frac{d\varphi}{dp_k}\frac{dp_k}{dt}$$
$$+ \frac{d\varphi}{dq_1}\frac{dq_1}{dt} + \ldots + \frac{d\varphi}{dq_k}\frac{dq_k}{dt} ,$$

and upon replacing the dp / dt and dq / dt with their values:

$$\frac{d\varphi}{dt} = \frac{d\varphi}{dp_1}\frac{dH}{dq_1} + \frac{d\varphi}{dp_2}\frac{dH}{dq_2} + \ldots + \frac{d\varphi}{dp_k}\frac{dH}{dq_k} - \frac{d\varphi}{dq_1}\frac{dH}{dp_1} - \frac{d\varphi}{dq_2}\frac{dH}{dp_2} - \ldots - \frac{d\varphi}{dq_k}\frac{dH}{dp_k}$$

$$= (\varphi, H), \quad \text{if one adopts Poisson's notation.}$$

One sees that this equation is satisfied for $\varphi = H$; i.e., that H = const. is an integral, which one knows in advance.

FUNDAMENTAL THEOREM. – *If one knows* $k - 1$ *integrals:*

$$\varphi_1 = a_1,\ \varphi_2 = a_2,\ \ldots,\ \varphi_{k-1} = a_{k-1}\,,$$

in addition to the integral of vis viva $H = h$, *in which* $\varphi_1, \varphi_2, \ldots, \varphi_{k-1}$ *do not contain time* t *explicitly and are such that, in addition, for i or* $m = 1, 2, \ldots, k - 1$, *the values of* $p_1, p_2, \ldots, p_k$ *that one deduces must satisfy the relations* $\frac{dp_1}{dq_m} = \frac{dp_m}{dq_1}$, *or rather* $(\varphi_1, \varphi_m) = 0$, *then one will get the k remaining integrals of the equations that were posed:*

1. *Upon integrating the differential expression:*

$$p_1\,dq_1 + p_2\,dq_2 + \ldots + p_k\,dq_k + H\,dt,$$

after one has replaced $p_1, p_2, \ldots, p_k$ *with their values that are inferred from the equations* $H = h$, $\varphi_1 = a_1$, $\varphi_2 = a_2$, $\ldots$, $\varphi_{k-1} = = a_{k-1}$.

2. *Upon equating the derivatives of the function thus found with respect to* $a_1, a_2, \ldots, a_{k-1}, h$ *to constants.*

By hypothesis, the values of p_i satisfy the integrability conditions, so the expression:

$$p_1\,dq_1 + p_2\,dq_2 + \ldots + p_k\,dq_k$$

will be the exact differential of a certain function *V*. I say, moreover, that one will also have $\frac{dH}{dq_i} = \frac{dp_i}{dt}$.

Indeed, when one replaces $p_1, p_2, \ldots, p_k$ in H with their values as functions of the $q_1, \ldots, q_k$, H reduces to h, so dH / dq_i is identically zero, and since p_i does not contain t, one will also have $dp_i / dt = 0$, so the preceding expression is indeed the total differential of a certain function Ω, such that:

$$d\Omega = dV + h\,dt, \qquad \Omega = V + ht.$$

I now say that the k remaining integrals will be:

$$(5)\qquad \begin{cases} \dfrac{d\Omega}{da_1} = \dfrac{dV}{da_1} = b_1, \\ \dfrac{d\Omega}{da_2} = \dfrac{dV}{da_2} = b_2, \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ \dfrac{d\Omega}{da_{k-1}} = \dfrac{dV}{da_{k-1}} = b_{k-1}, \\ \dfrac{d\Omega}{dh} = \dfrac{dV}{dh} + t = \tau, \end{cases}$$

in which $b_1, b_2, \dots, b_{k-1}, \tau$ are new constants.

In order to prove the second part of the theorem, it will suffice to see that, from the definition of the integral, one will have:

$$\frac{d}{dt}\frac{dV}{da} = 0$$

identically for $a = a_1, a_2, \dots, a_{k-1}, h$.

dV / da contains t only implicitly by the intermediary of the variables q_i ; one will then have:

$$\frac{d}{dt}\frac{dV}{da} = \frac{d\,\dfrac{dV}{dq_1}}{da}\frac{dq_1}{dt} + \frac{d\,\dfrac{dV}{dq_2}}{da}\frac{dq_2}{dt} + \dots + \frac{d\,\dfrac{dV}{dq_k}}{da}\frac{dq_k}{dt}.$$

Now, from the way that V was defined, one will obviously have:

$$(6)\qquad p_1 = \frac{dV}{dq_1}, \qquad \dots, \qquad p_k = \frac{dV}{dq_k},$$

so, upon substituting this in the equation above and taking the canonical equations (A) and (B) into account, one will get:

$$\begin{aligned} \frac{d}{dt}\frac{dV}{da} &= -\frac{dH}{dp_1}\frac{dp_1}{da} - \frac{dH}{dp_2}\frac{dp_2}{da} - \frac{dH}{dp_k}\frac{dp_k}{da} \\ &= -\frac{d(H)}{da}, \end{aligned}$$

in which (H) denotes what H will become after one has replaced $p_1, p_2, \dots, p_k$ with their values $dV / dq_1, \dots, dV / dq_k$, or rather, what amounts to the same thing, their values that one infers from the k first integrals. However, after that substitution, H will become equal to h, so if one gives one of the values $a_1, a_2, \dots, a_{k-1}$ to a, one will have $d(H) / da =$

0. If one sets $a = h$ then $d(H) / da = dh / dh = 1$, in this case, and $\frac{d}{dt}\frac{dV}{dh} = -1$. If one differentiates the last of equations (5) with respect to t then that will give:

$$\frac{d}{dt}\frac{dV}{dh} + 1 = -1 + 1 = 0.$$

Hence, the k equations (5) will represent k integrals of equations (A) and (B) and complete the integrals of the system.

Remark I. – The values of p_i that are provided by equations (6) are obviously the same as the ones that are given by the equations $\varphi_1 = a_1, \ldots, \varphi_{k-1} = a_{k-1}$, $H = h$, and that results from the way that V was formed. Hence, one can say that the complete solution of equations (A) and (B) is represented by equations (5) and (6).

Remark II. – If the function V is known then if one replaces the p_i in the equation $H = f(p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_k) = h$ with their values that one infers from equations (6) then that equation will become an identity. Hence, the function V is such that it will make the equation:

$$f\left(\frac{dV}{dq_1}, \frac{dV}{dq_2}, \ldots, \frac{dV}{dq_k}, q_1, q_2, \ldots, q_k\right) = h$$

into an identity, and since it contains k arbitrary constants, it will be a *complete integral of that first-order partial differential equation.*

JACOBI's THEOREM. – *Conversely, any complete integral – i.e., one that satisfies the preceding partial differential equation and contains k arbitrary constants, including the constant h of the vis viva integral – will enjoy the same property as the function V. It will provide the complete solutions of equations* (A) *and* (B). *Those solutions are represented by equations* (5) *and* (6).

I shall first say that one can deduce equations (B) from the equation $H = f = h$, (7), and equations (5). Indeed, if one differentiates (7) with respect to $a_1, \ldots, a_{k-1}$, h and (7) with respect to t then one will have, on the one hand, since the a are included in H only by the intermediary of the p:

$$\frac{dh}{da_1} = \frac{dh}{da_2} = \ldots = \frac{dh}{da_{k-1}} = 0 \quad \text{and} \quad \frac{dh}{dh} = 1,$$

so:

$$(a)\quad \begin{cases} \dfrac{dH}{dp_1}\dfrac{dp_1}{da_1}+\dfrac{dH}{dp_2}\dfrac{dp_2}{da_1}+\cdots+\dfrac{dH}{dp_k}\dfrac{dp_k}{da_1}=0, \\ \dfrac{dH}{dp_1}\dfrac{dp_1}{da_2}+\dfrac{dH}{dp_2}\dfrac{dp_2}{da_2}+\cdots+\dfrac{dH}{dp_k}\dfrac{dp_k}{da_2}=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dH}{dp_1}\dfrac{dp_1}{da_{k-1}}+\dfrac{dH}{dp_2}\dfrac{dp_2}{da_{k-1}}+\cdots+\dfrac{dH}{dp_k}\dfrac{dp_k}{da_{k-1}}=0, \\ \dfrac{dH}{dp_1}\dfrac{dp_1}{dh}+\dfrac{dH}{dp_2}\dfrac{dp_2}{dh}+\cdots+\dfrac{dH}{dp_k}\dfrac{dp_k}{dh}=1, \end{cases}$$

and on the other hand, if one differentiates dV / da_i with respect to t then one will have:

$$\frac{d\dfrac{dV}{da_1}}{dq_1}\frac{dq_1}{dt}+\frac{d\dfrac{dV}{da_1}}{dq_2}\frac{dq_2}{dt}+\cdots+\frac{d\dfrac{dV}{da_1}}{dq_k}\frac{dq_k}{dt}=0,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{d\dfrac{dV}{da_{k-1}}}{dq_1}\frac{dq_1}{dt}+\dots\qquad+\frac{d\dfrac{dV}{da_{k-1}}}{dq_k}\frac{dq_k}{dt}=0,$$

$$\frac{d\dfrac{dV}{dh}}{dq_1}\frac{dq_1}{dt}+\dots\qquad+\frac{d\dfrac{dV}{dh}}{dq_k}\frac{dq_k}{dt}=-1,$$

because dV / da_i is, in general, a function of $q_1, q_2, \dots, q_k$. Upon inverting the order of differentiation, while taking equations (6), one will have:

$$(b)\quad \begin{cases} \dfrac{dp_1}{da_1}\dfrac{dq_1}{dt}+\dfrac{dp_2}{da_1}\dfrac{dq_2}{dt}+\cdots+\dfrac{dp_k}{da_1}\dfrac{da_k}{dt}=0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dp_1}{da_{k-1}}\dfrac{dq_1}{dt}+\dfrac{dp_2}{da_{k-1}}\dfrac{dq_2}{dt}+\cdots+\dfrac{dp_k}{da_{k-1}}\dfrac{da_k}{dt}=0, \\ \dfrac{dp_1}{dh}\dfrac{dq_1}{dt}+\dfrac{dp_2}{dh}\dfrac{dq_2}{dt}+\cdots+\dfrac{dp_k}{dh}\dfrac{da_k}{dt}=-1, \end{cases}$$

Now, if one considers dH / dp_1, dH / dp_2, … to be the unknowns in equations (*a*) and considers $- dq_1 / dt$, $- dq_2 / dt$, … to be the unknowns in equations (*b*) then one will see that the coefficients of the unknowns are the same, so one will have $\frac{dH}{dp_1} = -\frac{dq_1}{dt}$. These are equations (B).

One can deduce equations (A) from the equation $H = h$ and equations (6). Differentiate $H = h$ with respect to q_1, q_2, …:

$$(a') \qquad \left(\frac{dH}{dq_1}\right) = \frac{dH}{dq_1} + \frac{dH}{dp_1}\frac{dp_1}{dq_1} + \frac{dH}{dp_2}\frac{dp_2}{dq_1} + \ldots + \frac{dH}{dp_k}\frac{dp_k}{dq_1} = 0,$$

while letting $\left(\frac{dH}{dq_1}\right)$ denote the total derivative of H with respect to q_1. Differentiate the first of equations (6) with respect to t:

$$\frac{dp_1}{dt} = \frac{d\,\frac{dV}{dq_1}}{dp_1}\frac{dq_1}{dt} + \frac{d\,\frac{dV}{dq_1}}{dp_2}\frac{dq_2}{dt} + \ldots + \frac{d\,\frac{dV}{dq_i}}{dq_k}\frac{dq_k}{dt}.$$

Since t enters into dV / dq_i only by the intermediary of the variables q_1, q_2, …, upon inverting the order of differentiation and taking equations (6) into account, one will have:

$$(b') \qquad \frac{dp_1}{dt} = \frac{dp_1}{dq_1}\frac{dq_1}{dt} + \frac{dp_2}{dq_1}\frac{dq_2}{dt} + \ldots + \frac{dp_k}{dq_1}\frac{dq_k}{dt}.$$

Now, if one takes equations (B) into account, since they are satisfied, then one will infer from equation (*a*) that:

$$\frac{dH}{dq_1} = \frac{dp_1}{dq_1}\frac{dq_1}{dt} + \frac{dp_2}{dq_1}\frac{dq_2}{dt} + \ldots + \frac{dp_k}{dq_1}\frac{dq_k}{dt},$$

so:

$$\frac{dp_1}{dt} = \frac{dH}{dq_1},$$

and similarly:

$$\frac{dp_2}{dt} = \frac{dH}{dq_2}, \ldots$$

These are equations (A) precisely.

Jacobi's theorem is paramount in this theory. Indeed, it permits one to write down the integrals of a dynamical problem immediately without performing any other operations than simple differentiation when one knows any complete integral of the partial differential equation $H = h$. In what follows, we will see that in a large number of cases, one can find a complete integral of that equation immediately, and consequently, *integrate the equations of the problem by inspection*, as Bour so cleverly phrased it.

In order to write down the partial differential equation upon which the solution of the problem depends, it will suffice to know the force function U and the *vis viva* as functions of the variables q, when they are reduced to their minimum number, and their derivatives with respect to time. One introduces new variables p into T such $p_i = dT / dq_i'$ in place of the variables q, and one finally replaces p_1, p_2, … with dV / dq_1, dV / dq_2, … in equation $U - T = h$.

ON THE EQUILIBRIUM OF FLEXIBLE, INEXTENSIBLE FILAMENTS

In a note that appeared in the *Comptes rendus de l'Académie des Sciences*, Appell showed that one could reduce the equations of equilibrium for a flexible filament to canonical form, and consequently, apply the theorems of Hamilton and Jacobi to the integration of those equations. One can present that reduction in a slightly different and slightly simpler form that will provide the value of the arc length, expressed by means of one quadrature, at the same time that it provide the equations of the funicular curve.

As one knows, the equations of equilibrium of a filament are:

$$d\left(T\frac{dx}{ds}\right) + X\,ds = 0,$$

(1) $$d\left(T\frac{dy}{ds}\right) + Y\,ds = 0,$$

$$d\left(T\frac{dz}{ds}\right) + Z\,ds = 0.$$

Let:

$$x = q_1, \qquad y = q_2, \qquad z = q_3.$$

Set:

(2) $$T\frac{dx}{ds} = p_1, \qquad T\frac{dy}{ds} = p_2, \qquad T\frac{dz}{ds} = p_3,$$

so

(3) $$T^2 = p_1^2 + p_2^2 + p_3^2.$$

Suppose that one has a force function U – i.e., a function such that:

$$-dU = X\,dx + Y\,dy + Z\,dz.$$

One will have:

$$-\frac{dU}{dx} = -\frac{dU}{dq_1} = X,$$

$$-\frac{dU}{dy} = -\frac{dU}{dq_2} = Y,$$

$$-\frac{dU}{dz} = -\frac{dU}{dq_3} = Z\,.$$

One infers from equations (1) that:

$$dT = -\,(X\,dx + Y\,dy + z\,dz) = dU\,,$$

so:

$$-\,T + U = h = \text{constant.}$$

Set:

(4) $$U - T = H$$

and remark that U will be a function of only the variables q_1, q_2, q_3, T, and that the variables p_1, p_2, p_3, and in turn H, will be functions of the variables p and q.

With that, equations (1) will take the form:

$$\frac{dp_1}{ds} = \frac{dU}{dq_1}\,, \qquad \frac{dp_2}{ds} = \frac{dU}{dq_2}\,, \qquad \frac{dp_3}{ds} = \frac{dU}{dq_3}\,,$$

or rather, since $dU / dq_1 = dH / dq_1$:

(A) $$\left\{\begin{aligned} \frac{dp_1}{ds} &= \frac{dH}{dq_1}, \\ \frac{dp_2}{ds} &= \frac{dH}{dq_2}, \\ \frac{dp_3}{ds} &= \frac{dH}{dq_3}, \end{aligned}\right.$$

and equations (2), which define the variables p, can be written, upon taking into account equation (3), which gives $\frac{dT}{dp_1} = \frac{p_1}{T}$:

$$\begin{aligned} \frac{dq_1}{ds} &= \frac{dT}{dp_1}, \\ \frac{dq_2}{ds} &= \frac{dT}{dp_2}, \\ &\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots \end{aligned}$$

However, from equation (4), $\frac{dT}{dp_1} = -\frac{dH}{dp_1}$, so:

$$\text{(B)} \qquad \left\{ \begin{aligned} \frac{dq_1}{ds} &= -\frac{dH}{dp_1}, \\ \frac{dq_2}{ds} &= -\frac{dH}{dp_2}, \\ \frac{dq_3}{ds} &= -\frac{dH}{dp_3}. \end{aligned} \right.$$

Equations (A) and (B) form a system of six simultaneous first-order differential equations that can replace the system (1). They are all presented in canonical form.

One can remark that these equations have the same form as the equations of motion of a unique material point, with the difference that the variable t, which represents time in the latter equations, is replaced here with the variable s, which represents the length of the arc in such a fashion that these six equations will determine the six variables q_1, q_2, q_3, p_1, p_2, p_3 as functions of s in the present case. However, of the three equations that determine q_1, q_2, q_3 as functions of s, two of them will be equations of the curve, and the third one must be a consequence of the first two, by virtue of the relation $ds^2 = dq_1^2 + dq_2^2 + dq_3^2$.

Indeed, one sees that only two of equations (B) are distinct, because if one adds their corresponding sides after squaring them then the first sum will be equal to unity, and similarly for the second sum, since:

$$\frac{dH^2}{dp_1^2} + \frac{dH^2}{dp_2^2} + \frac{dH^2}{dp_3^2} = \frac{dT^2}{dp_1^2} + \frac{dT^2}{dp_2^2} + \frac{dT^2}{dp_3^2} = \frac{p_1^2 + p_2^2 + p_3^2}{T^2} = 1.$$

The system of canonical equations is then equivalent to a system of five equations, in reality. Now, the integration of the canonical equations will introduce six arbitrary constants. One can take one of those constants to be equal to unity.

One will determine the five constants from the initial givens – for example, by expressing the idea that the extremities are fixed and the length of the filament is given.

As one knows, the integration of equations (A) and (B) can be converted into the search for a complete integral of a certain first-order partial differential equation:

$$\text{(5)} \qquad \frac{dV^2}{dq_1^2} + \frac{dV^2}{dq_2^2} + \frac{dV^2}{dq_3^2} = (U - h)^2.$$

When one has found the value of V that satisfies that equation and contains two new constants g, f, in addition to the constant h, one will get the solution of the problem by means of the following equations:

$$\text{(6)} \qquad \frac{dV}{df} = \alpha, \qquad \frac{dV}{dg} = \beta, \qquad \frac{dV}{dh} - s = \gamma,$$

in which α, β, γ represent three new constants. In addition to equations (6), the six integral of the canonical system are:

$$
(7) \qquad
\begin{aligned}
p_1 &= \frac{dV}{dq_1} = T\frac{dq_1}{ds}, \\
p_2 &= \frac{dV}{dq_2} = T\frac{dq_2}{ds}, \\
p_3 &= \frac{dV}{dq_3} = T\frac{dq_3}{ds}.
\end{aligned}
$$

The preceding method shows that one can find the arc length *ds* by a simple quadrature.

Application to the case in which the filament is constrained to remain on a given surface in the absence of friction.

The partial differential equation upon which the solution of the problem depends is the same as the one that one will obtain by studying the motion of a material point in the case where the *vis viva* is represented by $(U - h)^2$.

Now, we have seen that in this case, if one refers the position of the moving point to a system of curvilinear coordinates u, v, w, which are the parameters of three orthogonal surfaces, and if the square ds^2 of the distance between two infinitely-close points is:

$$ds^2 = f^2\, du^2 + g^2\, dv^2 + k^2\, dw^2,$$

in which f, g, k are given functions of u, v, w, then the left-hand side of equation (5) will take the form:

$$\frac{1}{f^2}\frac{d^2V}{du^2} + \frac{1}{g^2}\frac{d^2V}{dv^2} + \frac{1}{k^2}\frac{d^2V}{dw^2},$$

and the partial differential equation will become:

$$\frac{1}{f^2}\frac{d^2V}{du^2} + \frac{1}{g^2}\frac{d^2V}{dv^2} + \frac{1}{k^2}\frac{d^2V}{dw^2} = (U - h)^2.$$

In the case where one studies the motion of a point, or even in the case where one lays the filament on the surface w = const., the preceding equation will become:

$$\frac{1}{f^2}\frac{d^2V}{du^2} + \frac{1}{g^2}\frac{d^2V}{dv^2} = (U - h)^2.$$

One will arrive at this formula, moreover, by the usual formulas for the transformation of rectilinear coordinates into curvilinear coordinates.

Case of a filament that rests upon a sphere.

Upon taking ordinary spherical coordinates, one will have:

$$ds^2 = r^2\, d\theta^2 + r^2 \sin^2 \theta\, d\psi^2.$$

The partial differential equation will then be:

$$\frac{1}{r^2}\frac{dV^2}{d\theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{dV^2}{d\psi^2} = (U - h)^2.$$

Suppose that U is a function of only θ. One will satisfy that equation by setting $V = V_0 + V_\psi$, in which V_0 and V_ψ are two functions, one of which is a function of only θ and the other of which is a function of only ψ, and they are determined by the two equations:

$$\sin^2 \theta - r^2 \sin^2 \theta\, (U - h)^2 + k^2 = 0,$$

$$\frac{dV}{d\psi} = k,$$

in which k denotes an arbitrary constant. One infers from this that:

$$V_\psi = k\, \psi, \qquad V_0 = \int \frac{d\theta}{\sin\theta}\sqrt{r^2 \sin^2\theta\,(U-h)^2 - k^2}\,,$$

so:

$$V = k\psi + \int \frac{d\theta}{\sin\theta}\sqrt{r^2 \sin^2\theta\,(U-h)^2 - k^2}\,.$$

The integrals are:

$$\frac{dV}{dk} = \beta, \qquad \frac{dV}{dh} - s = \gamma,$$

The first of them will be the equation of the curve in spherical coordinates, while the second one will give s by a quadrature. That will be another form for the equation of the curve.

Example: *Case in which the forces reduce to weight.*

$$dU = m\, dz, \qquad U = \mu\, r \cos\, \theta,$$

so one will have:

$$V = k\psi + \int \frac{d\theta}{\sin\theta}\sqrt{r^2 \sin^2\theta\,(\mu\cos\theta - h)^2 - k^2}\,.$$

The integrals are:

$$\psi - \int \frac{k\,d\theta}{\sin\theta\sqrt{r^2\sin^2\theta(r\mu\cos\theta - h)^2 - k^2}} = \beta,$$

$$s = -r^2 \int \frac{\sin\theta(r\mu\cos\theta - h)\,d\theta}{\sqrt{r^2\sin^2\theta(r\mu\cos\theta - h)^2 - k^2}} - \gamma.$$

One has:

$$T = U - h = \mu\, r \cos\,\theta - h.$$

The preceding integrals are elliptic integrals.

The preceding formulas permit one to find the law that governs the force when the trajectory is given on a surface. Indeed, take one of them (say, the first one) in the differential form:

$$d\psi - \frac{k\,d\theta}{\sin\theta\sqrt{r^2\sin^2\theta(U-h)^2 - k^2}} = 0.$$

One infers the value of U from this:

$$(U-h)^2 = \frac{k^2}{r^2\sin^2\theta}\left(1 + \frac{d\theta^2}{\sin^2\theta\,d\psi^2}\right).$$

Application. – *What must the value of the force F that is tangent to the meridian at each point be in order for the filament on the sphere to take the form of a loxodrome?*

Let φ be the angle that the curve makes with the meridian, so one will have:

$$\tan\,\gamma = \frac{\sin\theta\,d\psi}{d\theta}.$$

Upon substituting this, one will then have:

$$(U-h)^2 = \frac{k^2}{r^2\sin^2\theta\sin^2\varphi}, \qquad U - h = \frac{k}{r\sin\theta\sin\varphi}.$$

Now, the elementary work is represented by dU and also by $Fr\,d\theta$. One will then have:

$$-\,dU = \frac{+k\cos\theta\,d\theta}{r\sin^2\theta\sin\varphi} = Fr\,d\theta,$$

so:

$$F = \frac{C\cos\theta}{\sin^2\theta}.$$

Equilibrium of a filament that rests upon a surface of revolution

One can suppose that the meridian of the surface is determined by an equation between the radius of the parallel r and the inclination θ of the meridian above the plane of the parallel at each of its points. If one lets ψ denote the angle between the meridian and a fixed meridian then one will find the following expression for the length ds :

$$ds^2 = r^2\, d\psi^2 + \frac{dr^2}{\cos^2\theta}.$$

Suppose that the equation of the meridian is taken in the form $\theta = F(r)$, so the partial differential equation can be put into the following form:

$$\frac{dV^2}{d\psi^2} + r^2\cos^2\theta\,\frac{dV^2}{dr^2} = r^2\,(U-h)^2,$$

and if U depends upon only r, one will see that one can write down the integrals of the problem immediately.

In the second place, suppose that the equation of the meridian is put into the form $r = f(\theta)$, so the partial differential equation will be:

$$\frac{1}{r^2}\frac{dV^2}{d\psi^2} + \frac{\cos^2\theta}{[f'(\theta)]^2}\frac{dV^2}{d\theta^2} = (U-h)^2.$$

One will find the known result that relates to the sphere upon setting $r = a\sin\theta$.

Upon following a path that is analogous to the one that was followed before in the preceding examples, one will find that:

$$V = k\psi + \int \frac{r'd\theta}{\cos\theta}\sqrt{(U-h)^2 - \frac{k^2}{r^2}}, \qquad r' = f'(\theta).$$

The two integrals will be:

$$\beta = \psi - \int \frac{kr'd\theta}{r^2\cos\theta\sqrt{(U-h)^2 - \dfrac{k^2}{r^2}}}, \qquad s = -\int \frac{r'(U-h)\,d\theta}{\cos\theta\sqrt{(U-h)^2 - \dfrac{k^2}{r^2}}} + \gamma.$$

Application. – Upon taking the first equation in the following form:

$$d\psi = \frac{kr'd\theta}{r^2\cos\theta\sqrt{(U-h)^2 - \dfrac{k^2}{r^2}}},$$

one will deduce:

$$(U-h)^2 = \frac{k^2}{r^2}\left(1+\frac{dr^2}{r^2\cos^2\theta\,d\psi^2}\right),$$

which will give the value of the force as a function of the elements of the trajectory. Let i be the inclination of the trajectory above the meridian, so one will have:

$$\tan i = \frac{r\,d\psi\cos\theta}{dr},$$

so:

$$(U-h)^2 = \frac{k^2}{r^2\sin^2 i},$$

and

$$U-h = \frac{k}{r\sin i}.$$

One will conclude from this that if $U = 0$ then one will have $r \sin i$ = constant, which is the well-known equation for geodesics that are traced on a surface of revolution.

One will remark that if one demands to know what would be the nature of the force that acts in the meridian plane and is capable of making the filament take the form of a loxodrome then that force will be independent of the meridian of the surface of revolution, because the force function depends upon only r, since $\sin i$ is constant. That result was pointed out by Aoust.

Equilibrium of a filament that rests upon an arbitrary ruled helicoid

The ruled helicoid is the surface that is generated by a line that turns around an axis in such a fashion that each of its points describes a helix that likewise has that given axis for their common axis.

If one considers a point of the generator then that point will always remain at the same distance from the axis, since it describes a helix on a cylinder that has the axis of the surface for its axis. In particular, the foot of the common perpendicular to the axis and the line that is described by a helix that is located on a cylinder, such that the moving line is constantly tangent to it, and in turn, the helicoidal surface is also likewise tangent to it; that is the *nucleus* of the surface.

Take the planes of projection to be a plane perpendicular to the axis (horizontal plane) and a plane that is parallel to the axis (vertical plane) that cuts the first one along a line LT. Let O be the projection of the axis and let OP be the shortest distance between the line and the axis. Take the axis to be the Z-axis itself, take the X-axis to be a parallel, and take the Y-axis to be a perpendicular to the land line (*ligne de terre*).

Let a point of the line project horizontally at M and vertically at M'. Suppose that one starts from a position of the line that is tangent to A at the circumference $OA = R$, in which R is the radius of the central nucleus. One draws the line from its initial position to the present position by a rotation ω around that axis and a shift. The three constants that determine the surface are the radius of the nucleus R, the step that is common to all helices h', and the constant angle b that the line makes with the axis.

Take the coordinates of the horizontal projection of the point to be the angle $AOP = \omega$ and the length $MP = \rho$. In the initial position, the height of the point over the horizontal plane is $\rho \cot b$; let the shift by δ. One will have $\frac{\delta}{h'} = \frac{\omega}{2\pi}$, so $\delta = \frac{h'\omega}{2\pi}$. Hence, the height of the point above the horizontal plane in the second position will be $\frac{h'\omega}{2\pi} + \rho \cot b$, and one will have:

$$x = R \cos \omega - \rho \sin \omega,$$
$$y = R \sin \omega + \rho \cos \omega,$$
$$z = \frac{h'\omega}{2\pi} + \rho \cot b .$$

One infers from this that:

$$ds^2 = \left(R^2 + \rho^2 + \frac{h'^2}{4\pi^2}\right) d\omega^2 + (1 + \cot^2 b)\, d\rho^2 + 2\left(R + \frac{h'}{2\pi}\cot b\right) d\rho \cdot d\omega$$

$$= (m^2 + \rho^2)\, d\omega^2 + 2c\, d\rho\, d\omega + n^2\, d\rho^2,$$

in which one has set:

$$m^2 = R^2 + \frac{h'^2}{4\pi^2}, \qquad n^2 = 1 + \cot^2 b, \qquad c = R + \frac{h' \cot b}{2\pi}.$$

Upon applying the Jacobi method, one will find that the solution to the problem of the equilibrium of a filament that lies on that surface will depend upon the following equation:

$$(m^2 + \rho^2)\, \frac{dV^2}{d\rho^2} + n^2\, \frac{dV^2}{d\omega^2} - 4c\frac{dV}{d\rho}\frac{dV}{d\omega} = (U - h)^2\, \{n^2\, (m^2 + \rho^2) - c^2\}.$$

One will get a complete solution of that equation upon taking $V = V_\omega + V_\rho$ (with U a function of ρ), in which V_ω and V_ρ are integrals of the following two equations:

(1) $$\frac{dV_\omega}{d\omega} = g, \qquad V_\omega = g\, \omega, \qquad g \text{ constant},$$

(2) $$(m^2 + \rho^2)\, \frac{dV_\rho^2}{d\rho^2} + n^2 g^2 - 4cg\, \frac{dV_\rho}{d\rho} - (U - h)^2\{n^2\, (m^2 + \rho^2) - c^2\} = 0,$$

$$V_\rho = \int d\rho\, \frac{2cg \pm \sqrt{4c^2 g^2 - (m^2 + \rho^2)\{n^2 g^2 - (U - h)^2 [n^2 (m^2 + \rho^2) - c^2]\}}}{m^2 + \rho^2}.$$

The two integrals of the problem will be:

$$\frac{dV}{dg} = \beta, \qquad \frac{dV}{dh} - s = \gamma.$$

The problem is found to reduce to quadratures whenever U is a function of only ρ or a function of the distance r from a point on the surface to the axis, by virtue of the relation $r^2 = R^2 + \rho^2$.

Special cases:

1.	$\cot b = 0$, $b = 90^{\circ}$	helicoid with a director plane
2.	$\cot b = 0$, $R = 0$	surface with a square-threaded screw
3.	$R = 0$	surface with a triangular-threaded screw
4.	$h' = 0$	helicoid of revolution
5.	$h' = 0$, $R = 0$	cone of revolution
6.	$h' = 0$, $1 + \tan^2 b = 0$	sphere
7.	$\dfrac{h'}{2\pi R}\cot b$	developable helicoid
8.	$\cot b = \infty$, $\rho = 0$	right cylinder

Upon supposing that $U = 0$ in the preceding formulas, one will get the equation for geodesic lines on the most general ruled helicoidal surfaces.

The equation:

$$\frac{dV}{dg} = \beta \qquad d\,\frac{dV}{dg} = 0$$

will become:

$$\left[(m^2+\rho^2)\frac{d\omega}{d\rho} + 2c\right]^2 = \frac{g^2[n^2(m^2+\rho^2) - 4c]^2}{4c^2g^2 - (m^2+\rho^2)\{n^2g^2 - n^2h^2(m^2+\rho^2) + h^2c^2\}}$$

in this case.

One will get an elliptic integral in the general case.

APPLICATION OF JACOBI'S METHOD TO THE STUDY OF BRACHISTOCHRONE CURVES

The search for brachistochrone curves comes down to the search for the minimum of the integral that is defined by $\int \frac{ds}{v}$. Suppose that the integral of the *vis viva* exists – namely, $mv^2 = 2U$, in which U is the force function – and that the point is constrained to remain on a surface such that the expression for the distance between two infinitely-close points will be:

(1) $$ds^2 = f^2\, du^2 + g^2\, dv^2,$$

in which u and v are two variable parameters that define the position of a point on the surface.

Set:

$$\frac{ds}{v} = ds_1\,,$$

so

(2) $$ds_1^2 = \frac{mf^2}{2U}du^2 + \frac{mg^2}{2U}dv^2 = f_1^2\, du^2 + g_1^2\, dv^2\,,$$

in which f_1 and g_1 are given functions of u and v. One will be reduced to the search for the minimum of the integral $\int ds_1$. However, equations (1) and (2) each define a class of surfaces that can be mapped to each other, and the search for the brachistochrones on the surfaces (1) is found to come down to the search for geodesic lines on the surfaces (2).

Now, one knows that the solution to the problem of geodesic lines on the surfaces (2) depends upon a knowledge of a complete integral of the Jacobi equation:

$$\frac{1}{f_1^2}\frac{dV^2}{du^2} + \frac{1}{g_1^2}\frac{dV^2}{dv^2} = 2h,$$

or rather:

(3) $$\frac{1}{f^2}\frac{dV^2}{du^2} + \frac{1}{g^2}\frac{dV^2}{dv^2} = \frac{mh}{U}\,.$$

If U is constant then one will find the geodesic lines on the surface, which should be obvious *a priori*.

Application to the case of surfaces of revolution

One has:

$$ds^2 = \frac{dr^2}{\cos^2\theta} + r^2\, d\varphi^2,$$

in which φ denotes the azimuth of an arbitrary meridian and θ is the angle between the tangent to the meridian at a point and the radius of the corresponding parallel; θ is a function of r. The preceding formula is deduced from the general formula (1) by setting $u = \rho$, $v = \varphi$, $g = r$, $f = 1 / \cos\theta$. We suppose that U is a function of only r. In that case, the partial differential equation will become:

$$\cos^2\theta\,\frac{dV^2}{dr^2} + \frac{1}{r^2}\frac{dV^2}{d\varphi^2} = \frac{mh}{U}\,.$$

One will effortlessly find the following complete integral:

$$V = \alpha\,\varphi + \int dr \sqrt{\frac{mh}{U\cos^2\theta} - \frac{\alpha^2}{r^2\cos^2\theta}}\,,$$

in which α represents a constant.

If one lets β and τ denote two new constants then the integrals of the problem will be:

$$\beta = \varphi - \alpha \int \frac{dr}{r^2\cos^2\theta\sqrt{\dfrac{mh}{U\cos^2\theta} - \dfrac{\alpha^2}{r^2\cos^2\theta}}}\,, \qquad \frac{dV}{dh} - t = \tau.$$

Upon calling the angle between the curve and the meridian i and remarking that one has:

$$dr = \cos\theta\cos i\, ds, \qquad r\, d\rho = \sin i\, ds,$$

one will easily infer from the first one that:

$$\frac{\alpha^2}{\sin^2 i} = \frac{mr^2 h}{U}\,,$$

and since $mv^2 = 2U$:

$$r \sin i = \frac{\alpha v}{\sqrt{2h}}\,.$$

This is the equation for the brachistochrones on a surface of revolution that was found by Roger (*Thesis*).

Corollary. – Let P be the force that acts upon a moving body. The corresponding elementary work that is done is represented, one the one hand, by $(dU / dr)\, dr$, and on the other hand, by $P\, ds \cos i$; hence:

$$P\, ds \cos i = \frac{dU}{dr}\, dr = \frac{2h}{\alpha^2}\, r \sin i\;\, d\,(r \cos i)$$

and

$$P \sin i = \frac{2h}{\alpha^2} \cdot \frac{r\sin^2 i}{\cos i} \cdot \frac{d(r\sin i)}{ds}\,.$$

However, if one denotes the angle of geodesic contingency by l_g then one will have:

$$l_g = \frac{d(r\sin i)}{r\sin i} \quad \text{and} \qquad \rho_g = \frac{ds}{l_g}\,,$$

in which ρ_g is the radius of geodesic curvature. One will then have:

$$P \sin i = \frac{2h}{\alpha^2} \cdot \frac{r \sin^2 i}{\rho_g} = \frac{mv^2}{\rho_g}.$$

That formula leads to a theorem that is analogous to Euler's theorem for planar brachistochrones, and which consists of saying that the normal component to the force is equal to the centrifugal force. In the case that concerns us, the radius of curvature of the curve is replaced by the radius of geodesic curvature.

If one agrees to call the quantity mv^2 / ρ_g the *geodesic* centrifugal force then one can state the following theorem:

THEOREM. – *For the brachistochrones that are traced on a surface of revolution, the component of the force along the perpendicular to the tangent that is drawn in the tangent place is equal to the geodesic centrifugal force.*

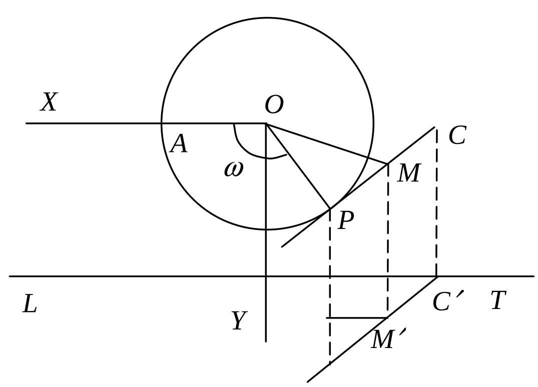
The Jacobi method then permits one to determine the brachistochrone curves in all cases where one can find a complete integral of the partial differential equation (3). In particular, one can apply it to the curves that are traced on an ellipsoid and on arbitrary ruled helicoid, and upon observing the complete analogy that exists between that problem and the problem of the equilibrium of a filament that lies on those surfaces.

Absolute brachistochrones. – The Jacobi method is also applicable in the case where the brachistochrone is not constrained to be found in a given surface. Upon always taking curvilinear coordinates u, v, w, one will have:

$$ds^2 = f^2\, du^2 + g^2\, dv^2 + k^2\, dw^2,$$

and one will easily prove that the solution to the question will depend upon one's knowledge of a complete integral of the following partial differential equation:

$$\frac{1}{f^2}\frac{dV^2}{du^2} + \frac{1}{g^2}\frac{dV^2}{dv^2} + \frac{1}{k^2}\frac{dV^2}{dw^2} = \frac{mh}{U}.$$



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