"Über die Einfluß der Eigenrotation der Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinschen Gravitationstheorie," Zeit. Phys. **19** (1918), 156-163.

On the influence of the proper rotation of a central body on the motion of the planets and the moon, according to Einstein's theory of gravitation.

By J. Lense and H. Thirring

Translated by D. H. Delphenich

In a paper that appeared recently ([1]), one us computed the field that is found inside of a rotating hollow sphere approximately in Einstein's theory of gravitation. That example seemed to be interesting primarily in the context of answering the question of whether the rotation of distant masses actually produces a gravitational field that is equivalent to a "centrifugal field" in Einstein's theory of gravitation. In another respect, it is also interesting to perform the same easily-performed integration of the field equations for a rotating solid sphere. As long as one stands on the basis of Newton's theory, one can replace the field in the space that is outside of a sphere of constant mass density (which is at rest or rotating) with the field of a material point of equal mass precisely. Moreover, in Einstein's theory, the field of a sphere at rest is equivalent to that of a mass point ([2]), as an incompressible fluid, but that will no longer be true for rotating spheres. As we will show in what follows, additional terms will then appear that will correspond to the centrifugal and Coriolis forces. Now, since the planets move in the field of a Sun that rotates around itself, and the moon moves in the field of a planet that rotates around itself, it does not seem out of the question at the outset to obtain a new astronomical confirmation of Einstein's theory by observing the perturbations that the additional terms yield. The numerical computations that are performed in what follows will produce perturbations of the orbital elements of the planets that lie beyond the limits of observability. However, they will yield relatively large perturbations for the moons of Jupiter that might, in fact, lie within the limits of measurement.

§ 1. The computation of $g_{\mu\nu}$ for the field of a rotating solid sphere.

Notations:

l	the radius of a sphere
M	its mass
ω	its angular velocity

([1]) Hans Thirring, this Zeit., **10** (1918), 33; referred to as *loc. cit.* in what follows.
([2]) K. Schwarzschild, Berl. Ber. (1916), 424.

x', y', z'	the rectangular coordinates of a point in the integration space
x, y, z	the coordinates of the origin
k	the gravitational constant
ρ_0	the naturally-measured spatial density of matter

The computation proceeds in a manner that is completely analogous to what was done in the paper that was cited in the introduction: Einstein's approximate method of integration ([1]) is used, except that this time, in the construction of the energy tensor for matter, the velocity of the mass that creates the field is regarded as small enough in comparison to 1 (viz., the speed of light) that one can neglect the squares and products of the velocity components. (As a result, the difference between the present treatment and the example that was treated in the previous paper is that centrifugal force terms, which are proportional to ω^2, will go away, and the Coriolis terms will remain.) In hindsight, neglecting these terms is justified completely, since $l\omega$ will be very small for the Sun and all of the planets when it is measured in any system of measurement for which the speed of light is 1. For that reason, we will consider the field at a great distance from the boundary surface of the ball in the case that will be treated here. If r stands for the distance from the origin to the center of the sphere, r' stands for the distance from the center to the integration element, and R stands for the distance from the origin to the integration element then we will develop $1 / R$ into a series in r' / r, which we will truncate with the quadratic terms.

We shall now go on to the approximate solution that was given by Einstein, exactly as we did in *loc. cit.* ([2]):

$$g_{\mu\nu} = -\,\delta_{\mu\nu} + \gamma_{\mu\nu}\,, \qquad \delta_{\mu\nu} = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu, \end{cases}$$

$$(1) \qquad \gamma_{\mu\nu} = \gamma'_{\mu\nu} - \tfrac{1}{2}\delta_{\mu\nu}\sum_\alpha \gamma'_{\alpha\alpha}\,,$$

$$\gamma'_{\mu\nu} = -\frac{k}{2\pi}\int \frac{T_{\mu\nu}(x', y', z', t - R)}{R}\, dV_0\,.$$

We then construct the energy tensor for stress-free matter:

$$(2) \qquad T_{\mu\nu} = T^{\mu\nu} = \rho_0 \frac{dx_\mu}{ds}\frac{dx_\nu}{ds} = \rho_0 \frac{dx_\mu}{dx_4}\frac{dx_\nu}{dx_4}\left(\frac{dx_4}{ds}\right)^2,$$

with the following expressions for the velocity components:

$$\frac{dx_1}{dx_4} = -\,i\,\frac{dx'}{dt} = ir'\omega \sin\vartheta' \sin\varphi',$$

([1]) A. Einstein, Berl. Ber. (1916), 688.
([2]) The factor $\delta_{\mu\nu}$ was obviously omitted from the corresponding eq. (2) in *loc. cit.*.

$$(3)\qquad \frac{dx_2}{dx_4} = -\,i\,\frac{dy'}{dt} = -\,ir'\omega \sin\,\vartheta'\sin\,\varphi',$$
$$\frac{dx_3}{dx_4} = 0$$

(r', ϑ', φ' are the polar coordinates of a point of the ball; rotation takes place around the Z-axis), and upon neglecting the terms in ω^2, we will get:

$$(4)\qquad T_{\mu\nu} = \rho_0 \left(\frac{dx_4}{ds}\right)^2 \begin{bmatrix} 0 & 0 & 0 & ir'\omega\sin\vartheta'\sin\varphi' \\ 0 & 0 & 0 & -ir'\omega\sin\vartheta'\cos\varphi' \\ 0 & 0 & 0 & 0 \\ ir'\omega\sin\vartheta'\sin\varphi' & -ir'\omega\sin\vartheta'\cos\varphi' & 0 & 1 \end{bmatrix}.$$

According to equations (7) and (8) of *loc. cit.*, we will have to set:

$$(5)\qquad dV_0 = i\,\frac{dx_4}{ds}\,r'^2\,dr'\sin\,\vartheta'\,d\vartheta'\,d\varphi'.$$

In order to express $1 / R$ in terms of the integration variables, we choose the coordinate system in such a way that its origin lies in the ZX-plane. With the introduction of polar coordinates, we will then have:

$$x = r\sin\,\vartheta, \qquad y = 0, \quad z = r\cos\,\vartheta,$$

and we will get:

$$R^2 = (r'\sin\,\vartheta'\cos\,\vartheta' - r\sin\,\vartheta)^2 + (r'^2\sin^2\,\vartheta'\sin^2\,\vartheta' + (r'\cos\,\vartheta'\cos\,\vartheta' - r\cos\,\vartheta)^2$$

$$= r^2\left[1 - \frac{2r'}{r}(\sin\vartheta'\cos\varphi'\sin\vartheta + \cos\vartheta'\cos\vartheta) + \frac{r'^2}{r^2}\right].$$

We develop this into a binomial series and truncate it after the second term:

$$(6)\qquad \frac{1}{R} = \frac{1}{r}\left\{1 + \frac{r'}{r}(\sin\vartheta'\cos\varphi'\sin\vartheta + \cos\vartheta'\cos\vartheta)\right.$$
$$\left. -\,\frac{1}{2}\frac{r'^2}{r^2} + \frac{3}{2}\frac{r'^2}{r^2}(\sin\vartheta'\cos\vartheta'\sin\vartheta + \cos\vartheta'\cos\vartheta)^2\right\}.$$

We further denote the expression in the curly brackets by K and write:

$$(6a)\qquad \frac{1}{R} = \frac{K}{r}.$$

If we now introduce (4), (5), and (6a) into the last of equations (1) then we will get:

$$
(7)\quad
\begin{aligned}
\gamma'_{44} &= -\frac{i\kappa}{2\pi}\frac{\rho_0}{r}\int_0^l r'^3 dr' \int_0^{2\pi} d\varphi' \int_0^{\pi} d\vartheta' \left(\frac{dx_4}{ds}\right)^3 \sin\vartheta' K\,,\\
\gamma'_{14} &= \frac{\kappa}{2\pi}\frac{\rho_0}{r}\,\omega\int_0^l r'^3 dr' \int_0^{2\pi} d\varphi' \int_0^{\pi} d\vartheta' \left(\frac{dx_4}{ds}\right)^3 \sin\vartheta' \sin\varphi' K\,,\\
\gamma'_{24} &= -\frac{\kappa}{2\pi}\frac{\rho_0}{r}\,\omega\int_0^l r'^3 dr' \int_0^{2\pi} d\varphi' \int_0^{\pi} d\vartheta' \left(\frac{dx_4}{ds}\right)^3 \sin\vartheta' \cos\varphi' K\,,\\
\gamma'_{11} &= \gamma'_{22} = \gamma'_{33} = \gamma'_{12} = \gamma'_{13} = \gamma'_{23} = \gamma'_{34} = 0.
\end{aligned}
$$

By neglecting the terms in ω^2 and assuming the first viewpoint, the approximation will yield:

$$\left(\frac{dx_4}{ds}\right)^3 = i \qquad \text{[cf., eq. (11), } \textit{loc. cit.}\text{]}.$$

If one introduces this value for $\left(\frac{dx_4}{ds}\right)^3$, as well as the expressions for K from (6) and (6a), into (7) then upon evaluating the integrals, one will get:

$$
(8)\quad
\begin{aligned}
\gamma'_{44} &= \frac{\kappa}{2\pi}\frac{M}{r}\,,\\
\gamma'_{14} &= 0,\\
\gamma'_{24} &= -i\,\frac{\kappa}{2\pi}\frac{M}{r}\frac{1}{5r}\,\omega\, l \sin\vartheta,\\
\gamma'_{11} &= \gamma'_{22} = \gamma'_{33} = \gamma'_{12} = \gamma'_{13} = \gamma'_{23} = \gamma'_{34} = 0.
\end{aligned}
$$

According to (1), when one once more introduces rectangular coordinates and uses the Newtonian gravitational constant $k = \kappa / 8\pi$, in place of Einstein's, it will then follow from this that:

$$g_{11} = g_{22} = g_{33} = -1 - \frac{2kM}{r}\,,$$

$$g_{44} = -1 + \frac{2kM}{r}\,,$$

(9)

$$g_{24} = -i\,\frac{4\kappa M}{r}\frac{l\kappa}{5r^2}\,\omega\,l,$$

$$g_{12} = g_{13} = g_{23} = g_{14} = g_{34} = 0.$$

If one now makes the special choice of coordinate system in which the origin falls in the *ZX*-plane by means of a rotation of the system then one will ultimately get the following coefficient matrix:

$$\text{(10)} \qquad g_{\mu\nu} = \begin{bmatrix} -1-\frac{2kM}{r} & 0 & 0 & i\frac{4kM}{5r}\frac{ly}{r^2}\omega l \\ 0 & -1-\frac{2kM}{r} & 0 & -i\frac{4kM}{5r}\frac{lx}{r^2}\omega l \\ 0 & 0 & -1-\frac{2kM}{r} & 0 \\ i\frac{4kM}{5r}\frac{ly}{r^2}\omega l & -i\frac{4kM}{5r}\frac{lx}{r^2}\omega l & 0 & -1+\frac{2kM}{r} \end{bmatrix}$$

§ 2. The equations of motion of a mass-point in the field of a rotating solid sphere.

In what follows, the equations of motion of a mass-point in the field of a rotating solid sphere will be presented, in which we will assume that its speed is so small that we can neglect the squares and products of its velocity components in comparison to 1. Thus, it shall be emphasized from the outset that all we have to do here is to find the perturbational terms to the planetary motion that originate in the rotation of the central body. In order to obtain a sufficiently exact solution to the planetary problem in the sense of Einstein's theory, one must add the terms that lead to the known motion of the perihelion to the perturbing terms that were computed ([1]). However, if the terms that originate in the proper rotation of the central body already come from the first approximation of Einstein's theory then since the aforementioned perturbation of the perihelion was first obtained from the second approximation, it would certainly not be unreasonable to consider the former and neglect the latter. The reason that one cannot do that comes from the following consideration: Any additional terms that make the further-developed force expression differ from the Newtonian one will be proportional to $\omega l\, v$, where v represents the velocity of the planet (moon, resp.), while ωl represents the velocity of a point on the equator of a central body. Now, for the Sun-planet system as well as for the planet-moon systems that come under consideration, we will have the inequality:

$$\text{(11)} \qquad v > \omega l.$$

Thus, when we include the terms in $\omega\, l\, v$ in our calculation, we must also properly consider any terms in the equations of motion that involve the squares and products of the

([1]) A. Einstein, Berl. Ber. (1913), pp. 831.

velocity components of the mass-points. However, if we do that then we can no longer compute in the first approximation alone, since any terms that combine with the Newtonian terms in the second approximation will compare to them like $\alpha / r : 1$ ($\alpha = 2kM$). The square of the velocity of a planet likewise has an order of magnitude of α / r. The consideration of the quadratic terms in velocity will thus logically imply that one must consider the terms that arise from the second approximation. It will then follow that the calculations that were employed here will have no meaning in and of themselves, due to the validity of the inequality (11). However, we can use them in practice if we realize that all of the perturbations that come under consideration here are small enough that one can regard them as linear with respect to each other. One will then arrive at the desired result of an orbital calculation that includes all relativistic effects when one starts with Einstein's calculations in the equations of motion that he gave for the precession of the perihelion of Mercury as a basis and adds the perturbing terms that are computed in what follows to them.

As was shown in *loc. cit.*, by the use of the aforementioned omission of terms and the coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = it$, the general equations of motion:

$$\frac{d^2 x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu} \frac{dx_\mu}{ds}\frac{dx_\nu}{ds}$$

will go over into:

$$\frac{d^2 x_\tau}{ds^2} = 2i\left(\Gamma^\tau_{14}\frac{dx_1}{dt} + \Gamma^\tau_{24}\frac{dx_2}{dt} + \Gamma^\tau_{34}\frac{dx_3}{dt}\right) - \Gamma^\tau_{44}. \tag{12}$$

From the initial viewpoint of the stationary field approximation, the 16 quantities $\Gamma^\tau_{\sigma 4}$ that appear here will read like:

$$\Gamma^1_{14} = 0, \quad \Gamma^1_{24} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_1}\right), \quad \Gamma^1_{34} = \frac{1}{2}\left(\frac{\partial g_{14}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_1}\right), \quad \Gamma^1_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_1},$$

$$\Gamma^2_{14} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_2}\right), \quad \Gamma^2_{24} = 0, \quad \Gamma^2_{34} = \frac{1}{2}\left(\frac{\partial g_{24}}{\partial x_3} - \frac{\partial g_{34}}{\partial x_2}\right), \quad \Gamma^2_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_2},$$

$$\Gamma^3_{14} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_1} - \frac{\partial g_{14}}{\partial x_3}\right), \quad \Gamma^3_{24} = \frac{1}{2}\left(\frac{\partial g_{34}}{\partial x_2} - \frac{\partial g_{24}}{\partial x_3}\right), \quad \Gamma^3_{34} = 0, \quad \Gamma^3_{44} = -\frac{1}{2}\frac{\partial g_{44}}{\partial x_3},$$

$$\Gamma^4_{14} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_1}, \quad \Gamma^4_{24} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_2}, \quad \Gamma^4_{34} = \frac{1}{2}\frac{\partial g_{44}}{\partial x_3}, \quad \Gamma^4_{44} = 0.$$

For our field, which is given by equation (10), this table will go to:

$$0 \qquad -i\frac{2kM}{5r^2}\frac{\omega l^2}{r}\frac{x^2+y^2+z^2}{r^2} \qquad -i\frac{6kM}{5r^2}\frac{\omega l^2}{r}\frac{yz}{r^2} \qquad \frac{kM}{r^2}\frac{x}{r}$$

$$
\begin{array}{cccc}
+\,i\dfrac{2kM}{5r^2}\dfrac{\omega l^2}{r}\dfrac{x^2+y^2+z^2}{r^2} & 0 & +\,i\dfrac{6kM}{5r^2}\dfrac{\omega l^2}{r}\dfrac{xz}{r^2} & \dfrac{kM}{r^2}\dfrac{y}{r} \\[2ex]
-\,i\dfrac{6kM}{5r^2}\dfrac{\omega l^2}{r}\dfrac{yz}{r^2} & -\,i\dfrac{6kM}{5r^2}\dfrac{\omega l^2}{r}\dfrac{xz}{r^2} & 0 & \dfrac{kM}{r^2}\dfrac{z}{r} \\[2ex]
-\dfrac{kM}{r^2}\dfrac{x}{r} & -\dfrac{kM}{r^2}\dfrac{y}{r} & -\dfrac{kM}{r^2}\dfrac{z}{r} & 0
\end{array}
\tag{14}
$$

If we substitute these values for the $\Gamma^{\tau}_{\sigma 4}$ in (12) then we will get the desired equations of motion:

$$
\begin{aligned}
\ddot{x} &= \frac{kM}{r^2}\frac{\omega l^2}{r}\left[\frac{4}{5}\frac{x^2+y^2+z^2}{r^2}\dot{y}+\frac{12}{5}\frac{yz}{r^2}\dot{z}\right]-\frac{kM}{r^2}\frac{x}{r}, \\
\ddot{y} &= -\frac{kM}{r^2}\frac{\omega l^2}{r}\left[\frac{4}{5}\frac{x^2+y^2+z^2}{r^2}\dot{x}+\frac{12}{5}\frac{z}{r^2}\dot{z}\right]-\frac{kM}{r^2}\frac{y}{r}, \\
\ddot{z} &= \frac{kM}{r^2}\frac{\omega l^2}{r}\frac{12}{5}\frac{z}{r}\frac{x\dot{y}-y\dot{x}}{r} \qquad -\frac{kM}{r^2}\frac{z}{r}.
\end{aligned}
\tag{15}
$$

The last terms on the right-hand side represent the Newtonian force; as explained above, one must replace them with the force components that follow from Einstein's work with Mercury. The first term on the right-hand side is the perturbing term that is of interest to us, since it arises from the proper rotation of the central body.

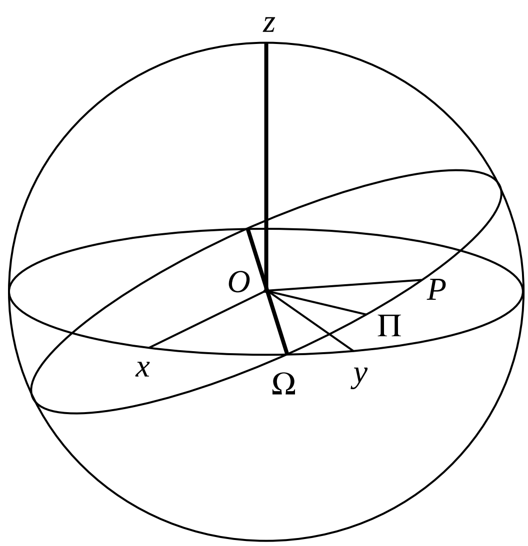
§ 3. The computation of the aforementioned perturbations that are due to the proper rotation of the central body.

The perturbing terms that appear in equations (15) are seen to be the components of the perturbing force that originates in the proper rotation of the central body. We decompose them into three other mutually-orthogonal components *S*, *T*, *W*, where *S* can be the radial component, *T*, the transversal, and *W*¸ the orthogonal one (i.e., the one that is normal to the planetary orbital plane), and introduce the following customary astronomical nomenclature:

a	semi-major axis
e	eccentricity
$p = a(1 - e^2)$	semi-parameter
$i = \angle\, y\Omega\Pi$,	inclination
$\Omega = \angle\, XO\Omega$	longitude of the ascending node

ϖ = broken $\angle\, XO\Pi$	longitude of periapsis
L_0	mean longitude of the epoch = mean longitude of the planet or satellite at the time $t = 0$ (likewise a broken angle that is measured from the X-axis)
$v = \angle\, \Pi OP$	true anomaly
$u = \angle\, \mathrm{W}OP = v + \varpi$	argument of latitude
U	period of the planet or satellite, in days
$n = \dfrac{2\pi}{U} = \sqrt{\dfrac{kM}{a^3}}$	mean daily motion
$C = r^2 \dot{v} = na^2\sqrt{1-e^2}$	twice the areal velocity

Furthermore, in order to abbreviate, we will set the constant K that appears in equations (15) equal to $4kM\omega l^2 / 5$.



Π and P mean the positions of the periapsides of the planets and satellites, when projected from the center O of the central body onto the sphere.

We now have:

$$x = r\,(\cos u \cos \Omega - \sin u \sin \Omega \cos i)$$
$$y = r\,(\cos u \sin \Omega + \sin u \cos \Omega \cos i)$$
$$z = r \sin u \cos i,$$

$$r = \frac{P}{1 + e\cos v},$$

$$x\dot{y} - y\dot{x} = C \cos i,$$

$S = \quad X(\cos u \cos \Omega - \sin u \sin \Omega \cos i) + Y(\cos u \sin \Omega + \sin u \cos \Omega \cos i) + Z \sin u \sin i$
$T = -X(\sin u \cos \Omega + \cos u \sin \Omega \cos i) + Y(\sin u \sin \Omega - \cos u \cos \Omega \cos i) + Z \cos u \sin i$
$W = X \sin \Omega \sin i - Y \cos \Omega \sin i + Z \cos i$.

If one inserts the values of X, Y, Z that are provided by equations (15) into these formulas for S, T, W then by using the given relations and notations, one will obtain, after some lengthy calculations:

$$S = \frac{KC\cos i}{r^4},$$

$$(16)\qquad T = -\frac{K\dot{r}\cos i}{r^4} = -\frac{KCe\cos i\sin v}{pr^3},$$

$$W = \frac{K\sin i}{r^4}(2C\sin u + r\dot{r}\cos u) = \frac{KC\sin i}{r^4}\left(\frac{re\sin v\cos u}{P} + 2\sin u\right).$$

The change in the orbital elements under the perturbing force is given by the equations:

$$\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}}\left(Se\sin v + T\frac{P}{r}\right),$$

$$\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na}\left(S\sin v + T\left(e + \frac{r+a}{a}\cos v\right)\right),$$

$$\frac{di}{dt} = \frac{1}{C}Wr\cos u,$$

$$\frac{d\Omega}{dt} = \frac{1}{C\sin i}Wr\sin u,$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{1-e^2}}{nae}\left(-S\cos v + T\left(1 + \frac{r}{P}\sin v\right)\sin v\right) + 2\sin^2\frac{i}{2}\frac{d\Omega}{dt},$$

$$\frac{dL_0}{dt} = -\frac{2}{na^2}Sr + \frac{e^2}{1+\sqrt{1-e^2}}\frac{d\varpi}{dt} + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d\Omega}{dt},$$

which can be represented in the following form when one replaces the values (16):

$$\frac{da}{dt} = 0,$$

$$\frac{de}{dt} = \frac{K\cos i}{Ca}\sin v\,\dot{v}\,,$$

$$\frac{di}{dt} = \frac{K\sin i}{Cp}\cos u\,[e\sin v\cos u + 2(1+e\cos v)\sin u]\,\dot{v}\,,$$

$$\frac{d\Omega}{dt} = \frac{K}{Cp}\sin u\,[e\sin v\cos u + 2(1+e\cos v)\sin u]\,\dot{v}\,,$$

$$\frac{d\varpi}{dt} = -\frac{K\cos i}{Ca}\left(2+\frac{1+e^2}{e}\cos v\right)\dot{v} + 2\sin^2\frac{i}{2}\frac{d\Omega}{dt}\,,$$

$$\frac{dL_0}{dt} = -\frac{2K\cos i}{na^2P}\left(1+e\cos v\right)\dot{v} + \frac{e^2}{1+\sqrt{1-e^2}}\frac{d\varpi}{dt} + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\frac{d\Omega}{dt}\,.$$

In the spirit of perturbation theory, we consider the orbital element that appears on the right-hand side with the infinitesimally small factor K to be constant and integrate over just v, while observing that $u = v + \varpi - \Omega$. Thus, we compute the first-order perturbations. If we introduce $K_1 = K / na^2$ then we will get:

$$\Delta a = 0,$$

$$\Delta e = -\frac{K_1\cos i}{\sqrt{1-e^2}}\cos v,$$

$$\Delta i = -\frac{K_1\cos i}{2\sqrt{1-e^2}}(\cos 2u + 2e\cos v\cos^2 u),$$

$$\Delta\Omega = \frac{K_1}{(1-e^2)^{3/2}}[v - \tfrac{1}{2}\sin 2u + e\,(\sin v - \tfrac{1}{2}\sin 2u\cos v)],$$

$$\Delta\varpi = -\frac{K_1\cos i}{(1-e^2)^{3/2}}\left(2v+\frac{1+e^2}{e}\sin v\right) + 2\sin^2\frac{i}{2}\Delta\Omega\,,$$

$$\Delta L_0 = -\frac{2K_1\cos i}{1-e^2}\left(v+e\sin v\right) + \frac{e^2}{1+\sqrt{1-e^2}}\Delta\varpi + 2\sqrt{1-e^2}\sin^2\frac{i}{2}\Delta\Omega\,.$$

The interesting result follows from this that the perturbation of the semi-major axis vanishes precisely. Whereas only periodic terms arise in Δe and Δi, secular terms will also appear in the remaining elements, namely, since $v = nt$ + period:

(17)
$$\Delta\Omega = \frac{K_1}{(1-e^2)^{3/2}}\,nt,$$
$$\Delta\varpi = \Delta L_0 = -\frac{2K_1}{(1-e^2)^{3/2}}\left(1-3\sin^2\frac{i}{2}\right)nt.$$

§ 4. Numerical results.

Numerical analysis shows that these secular perturbations will remain below the threshold of observability over the span of a century for the Sun-planets system, since they will reach a maximum of 0.01″ (for the perihelion of Mercury). The situation is different for the planet-moons systems: Larger numbers will appear in that case. For the sake of numerical calculations, it is better to transform formulas (17). We shall use the following notations:

l	Radius of the planet in cm.
τ	Rotational duration of the planet in days
a	Semi-major axis of the satellite orbit in cm.
a_1	" planetary orbit in cm.
U	Period of the satellite in days
U_1	" planet "
J	Number of days in a year
ε	Velocity of light in cm sec^{-1}

The following formula, which results from (17):

(18)
$$2\Delta\Omega = -\Delta\varpi = -\Delta L_0 = \frac{\pi^2 J l^2}{9c^2 \tau U^2}$$

will give us the aforementioned perturbation of the satellite elements that is due to the rotation of the planets in arc seconds per century. We have set $e^2 = i^2 = 0$ in it, since that would be permitted to the desired degree of accuracy for the moons under consideration.

In the spirit of § 2, the perturbations that were discussed by Einstein in his research on Mercury will then remain additive as a contribution that originates in the direct action of the planet and a contribution that originates in perturbing force of the Sun. The former contribution is given by:

(19)
$$\Delta\Omega = 0, \qquad \Delta\varpi = \Delta L_0 = \frac{5\pi^2 J}{24c^2}\frac{a^2}{U^2(1-e^2)},$$

and the latter ([1]) by:

([1]) W. de Sitter, "Planetary motion and the motion of the Moon according to Einstein's theory," Amsterdam Proc. **16** (1916). The orbital plane is subsequently used for the *XY*-plane in formula (20). In de Sitter's treatment, the formula (38) for $\delta\varpi$ on pp. 379 is missing a factor of 1/4.

(20) $$4\Delta\Omega = \Delta\varpi = \Delta L_0 = \frac{5\pi^2 J}{12c^2}\frac{a_1^2}{U_1^3},$$

all of which are in arc seconds per century. Both the eccentricity and the inclination of the planetary and satellite orbital planes were neglected in the latter, which is justified by the infinitesimal magnitude of these terms, as is shown by Table I:

Table I.

	$\Delta\Omega$	$\Delta\varpi = \Delta L_0$
Earth moon	+ 1.9″	+ 7.7″
Both moons of Mars	+ 0.7″	+ 2.7″

The values are much smaller for all of the remaining moons.

The perturbations that are due to the proper rotations of the planets are included in Table II.

Table II.

	Jupiter			Saturn				
	V	I	II	1	2	3	4	5
$\Delta\Omega$	+ 1′ 53″	+ 9″	+ 2″	+ 20″	+ 10″	+ 5″	+ 2″	+ 1″
$\Delta\varpi = \Delta L_0$	– 3′ 46″	– 18″	– 4″	– 41″	– 19″	– 10″	– 5″	– 2″

The numbers are less than 0.5″ for all of the other satellites.

The largest terms are analogous to the ones that relate to Einstein's precession of the perihelion of Mercury [formula (19)], as Table III shows:

Table III.
($\Delta\Omega = 0$)

	$\Delta\varpi = \Delta L_0$		$\Delta\varpi = \Delta L_0$
Mars 1	22″	Jupiter I	4′ 28″
2	2	II	1 24
Saturn 1	5′ 46″	III	26
2	3 03	IV	6
3	1 47	V	36 37
4	59	Uranus 1	22
5	25	2	10
6	3	3	3
7	2	4	1
10	2	Neptune moon	5

They are less than 0.5″ for all of the moons that were not entered.

If we would now wish to add together all three types of terms in order to obtain the total relativistic influence then we would have to consider the following: The correction to the Newtonian laws that was treated by Einstein's Mercury research was caused by a perturbing force along the radius vector whose components in the cited reference were:

$$S = -\frac{3n^2a^3C}{2c^2}\frac{\dot{v}}{r^2}, \qquad T = W = 0;$$

hence, it is independent of the choice of coordinate system. In what follows, the corresponding perturbations [formula (19) and Table III] can be referred to an arbitrary *XY*-plane. The variations of the elements that are included in formulas (20) that arise from the perturbing force of the Sun and that deviate from the classical form, as we have already mentioned, are referred to the orbital plane of the planets, and thus, the numbers that computed for them in Table I, as well, while everything in Table II, which includes the perturbing terms that originate in the rotation of the plane, is referred to the choice of coordinate system that was made in the present treatment regarding the equatorial plane of the central body.

The total relativistic influence is then summarized in Table IV: Only the terms (19) and (20) appear for the Earth moon and both moons of Mars, so the reference plane will then be the orbital plane of the planets. On the other hand, the plane of the central body in question is used for the satellites of Jupiter and Saturn, since once more only the terms (19) and (20) will appear. The perturbation of the moon of Uranus and the moon of Neptune include only the term (19), so the reference planes can be chosen arbitrarily.

Table IV.

		$\Delta\Omega$	$\Delta\varpi = \Delta L_0$	Δt
Earth moon		2″	8″	13.9″
Mars	1. Phobos	1	25	0.5
	2. Deimos	1	5	0.4
Jupiter	I	9	4′ 10″	29.5
	II	2	1 20	18.9
	III	0	26	12.5
	IV	0	6	7.1
	V	1′ 53″	32 51	1^m 5.4″
Saturn	1. Mimas	20	5 05	19.2
	2. Enceladus	10	2 44	15.0
	3. Thetys	5	1 37	12.2
	4. Dione	2	54	9.2
	5. Rhea	1	23	6.9
	6. Titan	0	3	3.3
	7. Hyperion	0	2	2.7
	10. Themis	0	2	2.9
Uranus	1. Ariel	0	22	3.7
	2. Umbriel	0	10	2.7

	3. Titania	0	3	1.5
	4. Oberon	0	1	1.0
Neptune moon		0	5	2.1

Let us say this about the column that is labeled Δt: The secular perturbations to the mean longitude produce a variation in the mean daily motion; i.e., in the time that is elapsed between two events (e.g., the eclipses of the moons of Jupiter), which adds a certain correction to the case in which there are no relativistic influences. This correction is given in the last column of Table IV for a span of one hundred years, and is obtained from the following formula:

$$\Delta t = U\,\Delta L_0\,.$$

Summary

The perturbing terms for the planetary and moon orbits that originate in Einstein's theory of the proper rotation of a central body are smaller than the ones that come from the second approximation and lead to the precession of the perihelion of Mercury. We do not encounter these terms for the planetary orbits, but they must be introduced for the computation of the orbits of the moons of Jupiter and Saturn. The secular perturbations that originate upon considering the total relativistic influence were computed for the moons of the outer planets. Whether or not they will individually (e.g., for the fifth moon of Jupiter) attain a magnitude that is sufficient to permit a proof of the theory for the perturbations of the moon orbits lies beyond the limits of precision for existing observations.

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