# The physical reality of some normal Bianchi spaces 

By T. LEVI-Civita

Translated by D. H. Delphenich

Einstein's gravitational equations were integrated rigorously by Schwarzschild $\left({ }^{1}\right)$ in a case of fundamental importance (viz., symmetry around a center) that included the Newtonian attraction of the Sun (with corrections that were derived from general relativity), which accounted for the secular precession of the perihelia of the planets exactly, and notably, the perihelion of Mercury: That celebrated result had previously been obtained by Einstein by means of an approximate integration (which was perfectly adequate for the numerical evaluation of the discrepancy).

I shall indicate a new case of integration that is not devoid of physical interest and is based upon my deduction of the gravitational equations that are already reduced to a form (spatiallyinvariant) that agrees with the static case that was the subject of my preceding communication $\left({ }^{2}\right)$.

Conceptually, this is what is treated: Suppose that a uniform (and constant with respect to time) electric or magnetic field exists in the vacuum. One wishes to know if and how such a field will influence the geometric nature of the ambient space. One finds that the space does not remain Euclidian (as it would be in the absence of the field), but becomes a normal Bianchi manifold ( ${ }^{3}$ ) with two principal curvatures equal to zero, while the third one (which corresponds to the normal section (giaciture) to the field lines) is positive and proportional to the square of the field intensity. Naturally, the proportionality factor is very small. When the non-zero curvature is $1 / R^{2}$, it will result, e.g., that for a magnetic field of 25000 Gauss, $R$ has the order of a tenth of a Syria meter (siriametro) (one Syria meter = one million times the distance between the Earth and the Sun). Despite that, it is not out of the question that some consequences (e.g., the way that the velocity of light varies along the force lines) will become amenable to the observations of cosmic physics.

One will be led to a final type of rigorous solution that is even more elementary (no. 5) when one assumes that the space has constant curvature and that purely-normal forces are exerted. That type is connected with a question that is much-debated in stellar statistics, and which attracted the attention of Einstein $\left({ }^{4}\right)$.

[^0]
## 1. - STATIONARY ELECTROMAGNETIC FIELDS.

Let:

$$
\begin{equation*}
d s^{2}=\sum_{i, k=1}^{3} a_{i k} d x_{i} d x_{k} \tag{1}
\end{equation*}
$$

be the expression in general coordinates $x_{1}, x_{2}, x_{3}$ for the square of the line element of the physical space in a region that one supposes to be the basis for electromagnetic phenomena. According to Einstein's theory, those phenomena influence the metric character of space, so the $d s^{2}$ will not be rigorously Euclidian, in general. However, under static conditions, it follows that the ordinary electromagnetic picture is valid when referred to the metric (1), even in the aforementioned theory.

We will address the elementary case in which the field consists of just one of the two forces: electric or magnetic. Let $X_{i}\left(x_{1}, x_{2}, x_{3}\right)(i=1,2,3)$ denote the system of covariant coordinates of that force. Its components (in general, non-orthogonal) along the trihedron of the coordinate lines will agree with $X_{i} / \sqrt{a_{i i}}$. Also denote the reciprocal elements by:

$$
X^{(i)}=\sum_{i, k=1}^{3} a^{(i j)} X_{j} \quad(i=1,2,3),
$$

and set:

$$
\begin{equation*}
8 \pi u=\sum_{i, k=1}^{3} X^{(i)} X_{i} \tag{2}
\end{equation*}
$$

The measurement is intended to refer to the absolute electrostatic system, in the Gauss-Hertz sense $\left(^{1}\right.$ ) (in which the dielectric constant and the magnetic permeability are considered to be pure numbers that are unity for the vacuum). With that, inside of an unpolarizable medium (air or vacuum), $u$ will represent the energy density that is due to the force $X_{i}$, while the relative Maxwellian tensor of the forces is still defined by the (covariant system):

$$
\begin{equation*}
T_{i k}=u a_{i k}-\frac{1}{4 \pi} X_{i} X_{k} \quad(i, k=1,2,3) \tag{3}
\end{equation*}
$$

The individual specific forces (which are treated as pressures) prove to have the ratios $T_{i k} / \sqrt{a_{i i} a_{k k}}$, whose significance for a given pair of indices is that they are the orthogonal components along the line $x_{k}$ of the force that is exerted upon a surface element that is normal to the line $x_{i}$ (or vice versa).

Since the field in question is supposed to be essentially stationary, we must assume that the $X_{i}$ are derived from a potential $\varphi\left(X_{i}=-\partial \varphi / \partial x_{i}\right)$, which will be excluded from our considerations when we treat a magnetic force in those regions of space in which there are possibly currents.
( ${ }^{1}$ Cf., e.g., Abraham, Theorie der Elektrizität, v. I, § 61, $4^{\text {th }}$ ed., Leipzig, Teubner, 1912.

In particular, assume that $\varphi=-C x_{1}$, with $C$ constant. We will have:

$$
X_{1}=C, \quad X_{2}=X_{3}=0,
$$

which corresponds to a force whose intensity $|C|$ points along the line $x_{1}$.
If one supposes, in addition, that the coordinate lines are orthogonal, or that $d s^{2}$ has the form:

$$
H_{1}^{2} d x_{1}^{2}+H_{2}^{2} d x_{2}^{2}+H_{3}^{2} d x_{3}^{2}
$$

then one will have from (2) and (3) that:

$$
\begin{gather*}
u=\frac{C^{2}}{8 \pi H_{1}^{2}}, \\
\left\{\begin{array}{c}
\frac{T_{11}}{H_{1}^{2}}=-u, \quad \frac{T_{22}}{H_{2}^{2}}=\frac{T_{33}}{H_{3}^{2}}=u, \\
T_{i k}=0, \quad(i \neq k),
\end{array}\right.
\end{gather*}
$$

which reflects the characteristic distribution of the Maxwellian force, which are tensions along the normal elements and pressures along the elementary parallels to the lines of force and have the common intensity $u$.

## 2. - BIANCHI (B) SPACES.

Bianchi has called spaces normal when the three congruences that are composed of the principal lines of curvature prove to be normal (to the other families of surfaces), and he characterized all of the normal spaces by the three constant principal curvatures. Among them, apart from the classical case of three equal curvatures (i.e., a space of constant curvature), there is only one type that exhibits positive mean curvature $\mathcal{M}$, which is called type $(B)\left({ }^{1}\right)$.

Two principal curvatures are zero in them, and the third one is positive, which is then denoted by $\mathcal{M}$. Set:

$$
\mathcal{M}=\frac{1}{R^{2}} \quad(\text { with } R>0)
$$

and one can attribute the following expression to the square of the line element:

$$
\begin{equation*}
d x_{1}^{2}+d x_{2}^{2}+\sin ^{2} \frac{x_{2}}{R} d x_{3}^{2} \tag{B}
\end{equation*}
$$

( ${ }^{1}$ ) Bianchi, loc. cit., page 68.
when it is referred to the triply-orthogonal system whose coordinate lines form the principal congruences.

Assuming that, when one recalls that, in general, whenever the principal congruences are normal, when one lets $\omega_{i}$ denote the three principal curvatures and lets $H_{i}^{2}$ denote the coefficients of the $d s^{2}$ in the orthogonal form that corresponds to the aforementioned congruence, one will have the canonical expressions for Ricci's $\alpha_{i k}\left({ }^{1}\right)$ :

$$
\alpha_{i k}=0 \quad(i \neq k), \quad \alpha_{i i}=\omega_{i} H_{i}^{2} \quad(i=1,2,3) .
$$

For the $d s^{2}$ in (B), which reflects the curvatures $\omega_{1}=1 / R^{2}, \omega_{2}=\omega_{3}=0$, one will get, in particular:

$$
\begin{equation*}
\alpha_{i k}=0 \quad(i \neq k), \quad \alpha_{11}=\frac{1}{R^{2}}, \quad \alpha_{22}=\alpha_{33}=0 . \tag{4}
\end{equation*}
$$

It will be convenient to imagine that a (B) space is represented in ordinary Euclidian space by interpreting the $x_{1}, x_{2}, x_{3} / R$ as cylindrical coordinates: Ordinarily, they are $z, \rho, \vartheta$, and the meaning of the last one is obvious. They establish a bijective correspondence between the points of the two spaces, although their two metrics are different. One can compare the expression (B) for $d s^{2}$ for the representative space with the Euclidian expression in cylindrical coordinates:

$$
d x_{1}^{2}+d x_{2}^{2}+\left(\frac{x_{2}}{R}\right)^{2} d x_{3}^{2}
$$

The two forms tend to coincide for very large $R$; more precisely, the discrepancy becomes negligible when the ratio $x^{2} / R$ is sufficiently small since it can coincide with the trace of the arc. In any case, the lines $x_{1}$ and $x_{2}$ (which are lines parallel to the $z$-axis and lines that will meet normally in the representative space) are geodetics for the metric (B), as well.

## 3. - MAGNETIC OR ELECTROSTATIC PRODUCTIONH OF A (B) SPACE.

Suppose that in some region of the ambient space that is devoid of ponderable matter one produces a uniform field; for example, a magnetic one, as it might be realized conveniently inside of a solenoid that is traversed with constant current. One must expect (assuming Einstein's general relativity) that the space that is occupied by the field is not rigorously Euclidian, since the modification of the geometric structure of the space can imply, in turn, a (tenuous) distortion of the lines of force until complete equilibrium is reestablished. One the addresses the determination

[^1]of the nature of the space and the final arrangement of the phenomenon when equilibrium is reached.

Naturally, the solution to the problem must be deduced from the general equations of Einsteinian statics, in which one attributes the determinations of the energy density and the force that correspond to the specified case.

To begin with, here are the static equations (in the spatially-invariant form that is in $\S \mathbf{2}$ of the preceding note):

$$
\begin{equation*}
\mathcal{M}=\kappa u, \tag{I}
\end{equation*}
$$

$$
\begin{equation*}
\alpha_{i k}+\frac{V_{i k}}{V}-\frac{\Delta V_{i k}}{V} a_{i k}=-\kappa T_{i k} \quad(i, k=1,2,3) \tag{II}
\end{equation*}
$$

in which $u$ is the energy density (which was denoted by $T_{00} / V^{2}$ in the preceding communication), $V$ is the velocity of the propagation of light, $V_{i k}$ and $\Delta_{2} V$ denote the second covariant derivative and second-order parameter, resp., when referred to the $d s^{2}$ of the ambient space. The $\alpha_{i k}$ are the corresponding Ricci symbols, $\mathcal{M}=\sum_{i, k=1}^{3} a^{(i k)} \alpha_{i k}$ is the mean curvature, and finally, the $T_{i k}$ constitute the force tensor, and the constant:

$$
\begin{equation*}
\kappa=\frac{8 \pi f}{c^{4}} \tag{5}
\end{equation*}
$$

in which $f$ is the universal gravitation constant and $c$ is the speed of light in vacuo, in the absence of perturbing actions.

We say that in the regime that was defined, the basis for our uniform field is a (B) space whose lines of force (which are reasonably straight) constitute the principal (geodetic) congruence $x_{1}$, which corresponds to non-zero curvature.

To prove that, it is enough to verify that (I), (II) will still be satisfied when:

1. One introduces the expressions for the $a_{i k}, \alpha_{i k}$ that they acquire for the metric (B).
2. One attributes the expressions $\left(2^{\prime}\right)$ and ( $3^{\prime}$ ), with the values $1,1, \sin ^{2}\left(x^{2} / R\right)$ for $H_{1}^{2}, H_{2}^{2}$, $H_{3}^{2}$, resp., to the energy density $u$ and the forces $T_{i k}$ (which are provided by the field exclusively under the assumed absence of matter).
3. One determines the function $V$ opportunely.

Based upon (2'), in which one sets $H_{1}=1$, so $\mathcal{M}$ will then coincide with $1 / R^{2}$, (I) will then yield:

$$
\begin{equation*}
\frac{1}{R^{2}}=\kappa u=\frac{\kappa C^{2}}{8 \pi} \tag{6}
\end{equation*}
$$

and one will then determine the curvature of the space normal to the lines of force (which are the only ones that remain non-zero) as a function of the field intensity. With the desired value (5) for $\kappa$, one will get:
(6')

$$
R=\frac{c^{2}}{\sqrt{f C}}
$$

It follows from the general expression for the second covariant derivative:

$$
V_{i k}=\frac{\partial^{2} V}{\partial x_{i} \partial x_{k}}-\sum_{l=1}^{3}\left\{\begin{array}{c}
i k \\
l
\end{array}\right\} \frac{\partial V}{\partial x_{l}}
$$

that the sum will be annulled for the fundamental form (B) and for a function of only $x_{1}$, such that the second covariant derivatives will not differ from the ordinary ones, and they must all be zero, with the exception of $V_{11}$, which reduces to $V^{\prime \prime}$ (the prime indicates derivation with respect to the argument $x_{1}$ ). One has [with the values of $a^{(i k)}$ that correspond to (B)]:

$$
\Delta_{2} V=\sum_{i, k=1}^{3} a^{(i k)} V_{i k}=V^{\prime \prime}
$$

With that, those of (II) that correspond to distinct indices $i, k$ prove to be simple identities. Based upon ( $3^{\prime}$ ) and (4'), the other three, or:

$$
\alpha_{i i}+\frac{V_{i i}}{V}-\frac{\Delta V}{V} H_{i}^{2}=-\kappa T_{i i} \quad(i, k=1,2,3)
$$

give:

$$
\frac{1}{R^{2}}=\kappa u \quad(i=1), \quad \frac{V^{\prime \prime}}{V}=\kappa u \quad(i=2,3) .
$$

The first one coincides with (6), and when one introduces the value $1 / R^{2}$ in place of $\kappa u$ and integrates, the second one will yield:

$$
\begin{equation*}
V=c_{1} e^{\alpha_{1} / R}+c_{2} e^{-\alpha_{1} / R} \quad\left(c_{1}, c_{2} \text { constants }\right) \tag{7}
\end{equation*}
$$

## 4. - ORDER OF MAGNITUDE OF $R$.

Magnetic fields. - The intensity $C$ that can be achieved in practice might amount to a few tens of thousands of Gauss; take 25000 as an estimate. Also express $c$ and $f$ in CGS units, so ( $6^{\prime}$ ) will give $R$ in centimeters. Now, $c=3 \times 10^{10}, f=6.6 \times 10^{-8}$, so one will have $R=\frac{3}{2} \times 10^{20} \mathrm{~cm}=\frac{3}{2} \times$ $10^{15} \mathrm{~km}$, upon rounding. If one notes that the distance between the Sun and the Earth is $\frac{3}{2} \times 10^{8}$ km then one reach the conclusion that for a field of 25000 Gauss, the radius of curvature will be ten million times the distance from the Earth and the Sun, or ten Syria meters. It varies inversely to the intensity of the field in a region but remains beyond any current experimental possibility of reducing it to dimensions that are observable in a laboratory. However, one should not rule out the possibility that other predictions of the theory - e.g., the exponential variation of the speed $V$ of light along the force lines that results from (7) - might become observable in cosmic physics.

Electrostatic fields. - For the numerical evaluation of $R$, formula ( $6^{\prime}$ ) continues to apply, provided that the intensity $C$ of the field is expressed in electrostatic units. Let $C_{v}$ denote the intensity in question, when expressed in volts per centimeter. Let $10^{8} C_{v}$ be its measure in CGS electromagnetic units. Therefore, $C=\frac{1}{c} 10^{8} C_{v}=\frac{1}{300} C_{v}$ is the number that must be introduced into ( $6^{\prime}$ ) in order to get $R$ in cm , as above.

When one simply attributes a value to $C_{v}$ that is amongst the other ones that were achieved so far - say, $5 \times 10^{5}$ (which can be justified by imagining that one treats fields in vacuo, so one will not be preoccupied with disruptive static) - then one will have the value $\frac{5}{3} 10^{3}$ for $C$, which is just one-fifteenth of the one that was considered in the preceding example of a magnetic field. The radius $R$ will prove to be 15 times larger.

## 5. - PARTICULAR SOLUTIONS UNDER THE HYPOTHESIS THAT SPACE ASSUMES A CONSTANT CURVATURE $K$.

Above all, one has the fundamental geometric relations:

$$
\left\{\begin{array}{l}
\alpha_{i k}=K a_{i k},  \tag{8}\\
\mathcal{M}=3 K,
\end{array} \quad(i, k=1,2,3),\right.
$$

with which, (I) will become:

$$
\begin{equation*}
3 K=\kappa u . \tag{9}
\end{equation*}
$$

We infer that $K \geq 0$, so we come back to the general observation in the preceding note that under static conditions, the mean curvature $\mathcal{M}$ will always be positive or zero. (9) will then show
that $u$ is necessarily constant, along with $K$, or that the medium must present a uniform distribution of energy.

Under the hypothesis that the distribution of forces is also uniform and that they are exerted normally, one will also have:

$$
\begin{equation*}
T_{i k}=p a_{i k} \quad(i, k=1,2,3), \tag{10}
\end{equation*}
$$

with $p$ a positive or negative constant according to whether the normal force that is found to exist on an element of the medium has the character of a pressure or a tension, resp.

Taking (9) and (10) into account, (II) will become:

$$
\begin{equation*}
\frac{V_{i k}}{V}+\left(K+\kappa p-\frac{\Delta_{2} V}{V}\right) a_{i k}=0 \quad(i, k=1,2,3) \tag{11}
\end{equation*}
$$

which can be satisfied in ways:

1. $V$ constant. - In this case, it is necessary and sufficient to add the condition to (9) that:

$$
\begin{equation*}
K+\kappa p=0 . \tag{12}
\end{equation*}
$$

A comparison will give $p=-\frac{1}{3} u$, and one will be led to the following statement:
Inside of a homogeneous medium that is uniformly stretched with a traction of $\frac{1}{3} u$ ( $u$ is the energy density), space will assume constant positive curvature $K=\frac{\kappa}{3} u$, and the speed of light will remain constant.

Above all, one must notice that such a medium cannot be composed of ordinary matter in either the fluid state or the solid state. It cannot be in the fluid state because the internal forces in that state always have the character of pressures, and it cannot be in the solid state because the order of magnitude of the traction $u / 3$ is much greater than the rupture limit. One will get a numerical estimate immediately when one imagines that if one lets $\mu$ denote the material density in a possibly solid medium in the assumed condition then one would have roughly $u=c^{2} \mu$, and one would then be dealing with a traction of $\frac{1}{3} c^{2} \mu=3 \times 10^{20}$ dynes per $\mathrm{cm}^{2}$.
2. $V$ variable. - When (11) is multiplied by $a^{(i k)}$ and summed over the two indices $i$ and $k$, it will initially follow that:

$$
\frac{\Delta_{2} V}{V}+3\left(K-\frac{\Delta_{2} V}{V}+\kappa p\right)=0
$$

or

$$
\frac{\Delta_{2} V}{V}=K^{*} a_{i k}=0 \quad(i, k=1,2,3)
$$

in which we have set:

$$
K^{*}=K-\frac{1}{2}(3 K+\kappa p),
$$

for brevity.
It is easy to see that equations (13) prove to be effectively compatible for non-constant $V$, so they constitute a complete system with respect to that $V$, when considered to be an unknown variable, but even then only when $K^{*}=K\left({ }^{1}\right)$.

That gives:

$$
\begin{equation*}
3 K+\kappa p=0, \tag{14}
\end{equation*}
$$

and when that is associated with (9), it will yield $p=-u$ and give rise to the same qualitative considerations that were made a few moments ago.

In order to integrate (13), one must take $d s^{2}$ (which has constant curvature $K$, by hypothesis) to have the typical form $\left({ }^{2}\right)$ :
( ${ }^{1}$ ) In order to establish that, it is convenient to recall [Ricci and Levi-Civita, Math. Ann. 54 (1900), pp. 143] that the second covariant derivative of any simple system $V_{i}$ verifies the relations:

$$
V_{i k l}-V_{i l k}=\sum_{j, h=1}^{3} a_{k l, j i} a^{(j k)} V_{h} \quad(i, k, l=1,2,3) .
$$

Appeal to the systems (E), and on one side replace the Riemann symbols with Ricci's $\alpha^{(p q)}$ according to the formula:

$$
a_{h l, j k}=\sum_{p, q=1}^{3} a^{(p q)} \varepsilon_{p h l} \varepsilon_{q j i} .
$$

On the other side, take into account that in the present case $\alpha^{(p q)}=K a^{(p q)}$, so one can also set:

$$
\alpha^{(p q)}=\frac{K}{2} \sum_{v, \rho, \sigma, \tau=1}^{3} a_{v \rho} a_{\rho \tau} \varepsilon^{(p \nu \rho)} \varepsilon^{(q \sigma \tau)}
$$

Keeping in mind the identity:

$$
\sum_{p=1}^{3} \varepsilon^{(p v \rho)} \varepsilon_{(q \sigma \tau)}=\varepsilon_{v k} \varepsilon_{\rho l}-\varepsilon_{v l} \varepsilon_{\rho k} \quad(v, \rho, k, l=1,2,3)
$$

which can also be presented in the form:

$$
\sum_{q=1}^{3} \varepsilon^{(q \sigma \tau)} \varepsilon_{(q j i)}=\varepsilon_{\rho \varphi k} \varepsilon_{\tau l}-\varepsilon_{\sigma l l} \varepsilon_{\tau j} \quad(\sigma, \tau, j, i=1,2,3)
$$

in which, of course, the $\varepsilon$ with two indices equal zero or one according to whether those indices are distinct of coincide, resp., and it will result that:

$$
V_{i k l}-V_{i l k}=\frac{K}{2} \sum_{j, h, v, \rho, \sigma, \tau=1}^{3} a_{v \rho} a_{\rho \tau} a^{(j h)} V_{h}\left(\varepsilon_{v k} \varepsilon_{\rho l}-\varepsilon_{v l} \varepsilon_{\rho k}\right)\left(\varepsilon_{\sigma j} \varepsilon_{\tau i}-\varepsilon_{\sigma i} \varepsilon_{\tau j}\right)=K\left(a_{i l} V_{k}-a_{i l} V_{k}\right)^{2} .
$$

Under the hypothesis that the $V_{i}$ are the derivatives of a function $V$ that verifies (13), one will infer from (13) itself, after multiplying by $V$ and covariant differentiating, that:

$$
V_{i k l}=-K^{*} a_{i k} V_{l}
$$

and when that is introduced into the preceding, it will give rise to the integrability conditions:

$$
\left(K-K^{*}\right)\left(a_{i l} V_{k}-a_{i l} V_{k}\right)=0
$$

for any triple of indices $i, k, l$.
Hence, by hypothesis, $V$ will be an effective function, and one of its derivatives - say, e.g., $V_{h}$ - will be non-zero. Fix that value of $k$ in the equations that were just established and a value of $l$ that is different from $k$. Then multiply by $a^{(i k)}$ and sum over the index $i$. That will give:

$$
\left(K-K^{*}\right) V_{h}=0,
$$

so one has $K-K^{*}=0$ precisely.
Q. E. D.
( ${ }^{2}$ ) Bianchi, Lezioni di geometria differenziale, vol. I, Pisa, Spoerri, 1902, page 345.

$$
\begin{equation*}
\frac{1}{\psi^{2}}\left(d x_{1}^{2}+d x_{2}^{2}+d x_{3}^{2}\right) \tag{15}
\end{equation*}
$$

with

$$
\begin{equation*}
\psi=1+\frac{1}{4} K \cdot\left(d x_{1}^{2}+d x_{2}^{2}+d x_{3}^{2}\right) \tag{16}
\end{equation*}
$$

One gets the explicit expressions for the covariant derivatives $V_{i k}$ directly (from the defining formulas):

$$
V_{i k}=\frac{\partial^{2} V}{\partial x_{i} \partial x_{k}}+\frac{1}{\psi}\left(\frac{\partial \psi}{\partial x_{k}} \frac{\partial V}{\partial x_{i}}+\frac{\partial \psi}{\partial x_{i}} \frac{\partial V}{\partial x_{k}}\right)-\frac{\varepsilon_{i k}}{\psi} \sum_{l=1}^{3} \frac{\partial \psi}{\partial x_{l}} \frac{\partial V}{\partial x_{l}},
$$

with the usual meaning for the $\varepsilon_{i k}$ (viz., 0 for $i \neq k, 1$ for $i=k$ ).
Upon substituting that in (13) and recalling the form (15) of $d s^{2}$, as well as (16), one will first have:

$$
\frac{\partial^{2} V}{\partial x_{i} \partial x_{k}}=0 \quad(i \neq k)
$$

so it seems that:

$$
W=\psi V
$$

must be a separable variable (viz., the sum of three functions, one of which is a function of only $x_{1}$, one is a function of only $x_{2}$, and one is a function of only $x_{3}$ ).

The remaining (13), in which, of course, one sets $K^{*}=K$, will give:

$$
\psi^{2} \frac{\partial^{2} V}{\partial x_{i}^{2}}+\psi \frac{\partial \psi}{\partial x_{i}} \frac{\partial V}{\partial x_{i}}-\psi \sum_{l=1}^{3} \frac{\partial \psi}{\partial x_{l}} \frac{\partial V}{\partial x_{l}}+K V=0
$$

or

$$
\psi \frac{\partial^{2} W}{\partial x_{i}^{2}}-\frac{\partial^{2} \psi}{\partial x_{i}^{2}} W-\sum_{l=1}^{3} \frac{\partial \psi}{\partial x_{l}} \frac{\partial W}{\partial x_{l}}+\left\{K+\sum_{l=1}^{3}\left(\frac{\partial \psi}{\partial x_{l}}\right)^{2}\right\} \frac{W}{\psi}=0 .
$$

By virtue of (16):

$$
\frac{\partial^{2} \psi}{\partial x_{i}^{2}}=\frac{1}{2} K, \quad K+\sum_{l=1}^{3}\left(\frac{\partial \psi}{\partial x_{l}}\right)^{2}=K \psi
$$

so, by definition, the auxiliary variable $W$ (with separated variables) is found to be subject to the three conditions:

$$
\psi \frac{\partial^{2} W}{\partial x_{i}^{2}}-\sum_{l=1}^{3} \frac{\partial \psi}{\partial x_{i}} \frac{\partial \psi}{\partial x_{l}}+\frac{1}{2} K W=0 \quad(i=1,2,3)
$$

with $\psi=1+\frac{1}{4} K \cdot\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right)$. When one is given that form for $\psi$, any $\frac{\partial^{2} W}{\partial x_{i}^{2}}$ will reduce to a constant, and it will result immediately that the most general solution is:

$$
W=b_{0}\left\{\frac{1}{4} K \cdot\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right)-1\right\}+b_{1} x_{1}+b_{2} x_{2}+b_{3} x_{3},
$$

in which the $b$ denotes an arbitrary constant. With that expression for $W$ :

$$
V=\frac{1}{\psi} W
$$

will constitute the general integral of (13) accordingly.

## 6. - ADDITIONAL TERMS RECENTLY PROPOSED BY EINSTEIN.

Statistical reflections on the asymptotic distribution of matter in the stellar universe induced Einstein $\left({ }^{1}\right)$ to test the introduction of a small correction term (which is perfectly compatible with the postulates of general relativity) into his fundamental equations. They were (when referred to the quadri-dimensional $d s^{2}$ ):

$$
\begin{equation*}
G_{i k}-\frac{1}{2} G g_{i k}=-\kappa T_{i k} \quad(i, k=0,1,2,3), \tag{E}
\end{equation*}
$$

and must be modified as follows:

$$
G_{i k}-\left(\frac{1}{2} G+\lambda\right) g_{i k}=-\kappa T_{i k}
$$

in which $\lambda$ denotes a positive universal constant.
Under static conditions, the quaternary form:

$$
\sum_{i, k=0}^{3} g_{i k} d x_{i} d x_{k}
$$

will reduce to:

$$
V^{2} d x_{0}^{2}-d s^{2}=V^{2} d x_{0}^{2}-\sum_{i, k=1}^{3} a_{i k} d x_{i} d x_{k}
$$

and it is worthwhile to exhibit the spatial metric.

[^2]If one accepts $\left(\mathrm{E}^{\prime}\right)$, in place of $(\mathrm{E})$, then one will have the following static equations in place of (I) and (II) ${ }^{1}$ ):

$$
\begin{gather*}
\mathcal{M}-\lambda=\kappa u  \tag{I'}\\
\alpha_{i k}+\frac{V_{i k}}{V}-\left(\frac{\Delta_{2} V}{V}+\lambda\right) a_{i k}=-\kappa T_{i k} \quad(i, k=1,2,3) .
\end{gather*}
$$

Since $\lambda>0$, ( $\left.I^{\prime}\right)$ will show that the complementary term insures that the general property of physical space that it cannot assume negative mean curvature under static conditions (that we observed in $\S \mathbf{1}$ of the preceding note) is verified a fortiori (excluding the limiting case $\mathcal{M}=0$ ).

If one then introduces the special hypotheses of no. $\mathbf{5}$, while supposing that one treats a space of constant curvature $K$ that is subject to normal forces with which (8) and (10) are valid then (I') and (II') will assume the appearance:

$$
\begin{gather*}
3 K-\lambda=\kappa u  \tag{9'}\\
\frac{V_{i k}}{V}+\left(K+\kappa p-\frac{\Delta_{2} V}{V}+\lambda\right) a_{i k}=0 \quad(i, k=1,2,3) . \tag{11'}
\end{gather*}
$$

They obviously correspond to (9) and (11) in the preceding section when one identifies $\lambda=0$ in them. The discussion proceeds in the same way, with the advantage that the presence of the constant $\lambda$ leaves a certain margin for positive values of $p$.

In particular, let us occupy ourselves with solutions for which $V$ is constant. We must associate:

$$
\begin{equation*}
K+\kappa p=\lambda \tag{12'}
\end{equation*}
$$

with ( $9^{\prime}$ ), since when we eliminate $K$, we will get:

$$
\begin{equation*}
3 \kappa p=2 \lambda-\kappa u . \tag{17}
\end{equation*}
$$

For $p=0$, one has Einstein's particular solution:

$$
\bar{u}=\frac{2 \lambda}{\kappa}, \quad \bar{K}=\lambda
$$

[^3]which characterizes the mean distribution $\bar{u}$ of energy (and therefore of matter) in all of space, which supposes that it is (except for local divergences) endowed with constant curvature and filled with incoherent matter, between which the particles exert no molecular forces.
(17) shows that one can generalize Einstein's solution by assigning the value $u$ (constant and $\geq 0$ ) at will. With that, one will preserve all of the uniformity in the geometric and mechanical characteristics, but not the absence of normal forces. They are exerted as pressures for $u<\bar{u}$ and tensions for $u>\bar{u}$. Reasonable inductions on the behavior of matter, no matter how diffuse, will lead one to exclude the second possibility: $\bar{u}$ therefore presents itself as an upper limit on the mean density of energy that is attributable to the stellar universe. Because of both the absence of forces and its very character, Einstein's solution undoubtedly presents the greatest speculative interest.


[^0]:    $\left({ }^{1}\right)$ "Ueber das Gravitationsfeld eines Masspunktes nach der Einstein'schen Theorie," Sitz. Kgl. Preuss. Akad. Wiss. (1916), 189-196. Also see the recent paper by Hilbert "Die Grundlagen der Physik (zweite Mitteilung) Nachr. der Kön. Ges. der Wiss. zu Göttingen (1917).
    $\left(^{2}\right)$ "Statica einsteiniana," in this volume of the Rendiconti, pp. 458-470.
    $\left({ }^{3}\right)$ "Sugli spazi normali a tre dimensionali colla curvature principali constanti," in these Rendiconti 25 ( $1^{\text {st }}$ sem. 1916), 59-68.
    $\left({ }^{4}\right)$ "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie," Sitz. der Kgl. Preuss. Akad. Wiss (1917), 142-152.

[^1]:    $\left({ }^{1}\right)$ In three-dimensional spaces, those $\alpha_{i k}$ can advantageously substitute for the four-index Riemann symbols. I have already had occasion to point that out in § $\mathbf{2}$ of the preceding note.

[^2]:    $\left({ }^{1}\right)$ That hypothesis of a quasi-uniform distribution of matter in the world suggested some interesting specifications to Almansi that were positive and formal and could be classified in the Newtonian picture. Cf., "Le equazioni fondamentali della Dinamica e la legge di gravitazione" in the Memorie of this Academy 9 (1913), 473-502.

[^3]:    $\left({ }^{1}\right)$ The passage from $\left(\mathrm{E}^{\prime}\right)$ to $\left(\mathrm{I}^{\prime}\right)$, ( $\left.\mathrm{II}^{\prime}\right)$ is accomplished exactly it was in the original form for the passage from (E) to (I), (II) (§§ $\mathbf{1 - 2}$ in the prec. note that was cited many times already). All that is necessary is to take into account the additional term in $\lambda$.

