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Einsteinian statics

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I propose to study that particular case of the Einstein gravitational equations that corresponds to static phenomena, namely (always from the standpoint of general relativity), the motion of a material point in a static field under the hypothesis that the modification of the field that is produced by the point is negligible.

In §§ **1-2**, I shall endow the aforementioned equations with a form that is invariant with respect to the ds^2 of the ambient space. With that, the metric nature of that space will be connected directly with the phenomena of equilibrium that are based in it, while in the general Einstein form (which is valid for phenomena that vary with both position and the instant), the measures of space and time are fused together into a quadri-dimensional form.

An immediate consequence of the equations thus-transformed is that in the static regime, the mean curvature of the physical space is necessarily *positive or zero.*

We then pass on (§§ **3-4**) to the equations of motion of a material point, as well as simply highlight the particular properties that are due to the static regime of the field. The variational equation of the trajectory figures in that, which is essentially the expression for the principle of minimal action that is appropriate to the case. The transformation by which one succeeds in eliminating time is also perfectly applicable to the usual mechanics of holonomic systems and allows one to pass (assuming conservative forces) from Hamilton's principle to that of minimal action with a great spontaneity and simplicity that is not present in the classical procedure ([1]).

1. – The Einstein equations in the static case.

When one treats static phenomena, the quaternary differential form ds'^2 that encompasses the measures of space and time is presented in the form:

(1) $$\sum_{i,k=0}^{3} g_{ik}\, dx_i\, dx_k = V^2\, dx_0^2 - ds^2.$$

([1]) Cf., e.g., the following treatises: Appell, *Traité de mecanique rationelle*, tome 2, 3rd ed., Gauthier-Villars, Paris, 1911, pp. 483-487. Maggi, *Principii di stereodinamica*, Hoepli, Milan, 1908, §§ 102-103. Whittaker, *Analytical Dynamics*, Cambridge University Press, 1904, secs. 99-100.

in which x_0 represents time, and:

$$ds^2 = \sum_{i,k=1}^{3} a_{ik}\, dx_i\, dx_k \tag{2}$$

is the square of the line element in the ambient physical space. The coefficients a_{ik} , like V, must be considered to be functions of only x_1, x_2, x_3 . V is interpreted as the speed of light and is therefore considered to be essentially positive.

When one gives the symbols the obvious significance, one will have:

$$\left\{\begin{array}{c} g_{ik} = -a_{ik}, \quad g_{0i} = 0, \quad g_{00} = V^2, \\ g = -aV^2, \\ g^{(ik)} = -a^{(ik)}, \quad g^{(0i)} = 0, \quad g^{(00)} = \dfrac{1}{V^2}, \\ (i, k = 1, 2, 3). \end{array}\right. \tag{3}$$

We agree to label the Christoffel and Riemann symbols that relate to the quaternary form (1) with a prime, while reserving the usual notation without a prime for the analogous symbols that relate to (2).

On the basis of (3), one immediately gets from the defining formulas that:

$$\left\{\begin{array}{l} \begin{Bmatrix} i\,k \\ l \end{Bmatrix}' = \begin{Bmatrix} i\,k \\ l \end{Bmatrix}, \\ \begin{Bmatrix} i\,k \\ 0 \end{Bmatrix}' = \begin{Bmatrix} 0\,i \\ k \end{Bmatrix}' = \begin{Bmatrix} 0\,0 \\ 0 \end{Bmatrix}' = 0, \\ \begin{Bmatrix} i\,0 \\ 0 \end{Bmatrix}' = \dfrac{V_i}{V}, \\ \begin{Bmatrix} 0\,0 \\ i \end{Bmatrix}' = V\,V^{(i)}, \end{array}\right. \tag{4}$$

in which i, k, l can assume the values 1, 2, 3, $V_i = \partial V / \partial x_i$, and $V^{(i)} = \sum_{j=1}^{3} a_{ij} V^j$ is the reciprocal system to ds^2 .

The following doubly-covariant system has fundamental importance in Einstein's theory:

$$G'_{ik} = \sum_{h=0}^{3} \{ih, hk\}' = \sum_{h=0}^{3} \left[\frac{\partial}{\partial x_k} \begin{Bmatrix} i\,h \\ h \end{Bmatrix}' - \frac{\partial}{\partial x_h} \begin{Bmatrix} i\,k \\ h \end{Bmatrix}' \right] + \sum_{h,i=0}^{3} \left[\begin{Bmatrix} i\,h \\ l \end{Bmatrix}' \begin{Bmatrix} k\,l \\ h \end{Bmatrix}' - \begin{Bmatrix} i\,k \\ l \end{Bmatrix}' \begin{Bmatrix} l\,h \\ h \end{Bmatrix}' \right] \tag{5}$$

$$(i, k = 0, 1, 2, 3) .$$

If one introduces the analogous system:

(6) $$G_{ik} = \sum_{h=1}^{3} \{ih, hk\} \qquad (i, k = 1, 2, 3),$$

which relates to the ternary form (2), then after reductions, one will find that:

(7) $$\begin{cases} G'_{ik} = G_{ik} + \dfrac{V_{ik}}{V}, & \\ G'_{0i} = 0, & (i, k = 1, 2, 3), \\ G'_{00} = -V\,\Delta_2 V, & \end{cases}$$

in which the V_{ik} represent second covariant derivatives, and Δ_2 is the second-order differential parameter that refers to the spatial ds^2 (2).

Based upon those formulas and (3), one will have:

$$G' = \sum_{i,k=0}^{3} g^{(ik)}\, G'_{ik} = g^{(00)}\, G'_{00} - \sum_{i,k=1}^{3} a^{(ik)}\, G'_{ik} = -\frac{\Delta_2 V}{V} - \sum_{i,k=1}^{3} a^{(ik)}\, G_{ik} - \frac{\Delta_2 V}{V}$$

for the linear invariant of the system G'_{ik}. If one then sets:

(8) $$-2\mathcal{M} = \sum_{i,k=1}^{3} a^{(ik)}\, G_{ik},$$

in which $\mathcal{M}$ represents the *mean curvature* of the ambient space, as will be verified in § **3**, it will result that:

(9) $$\tfrac{1}{2} G' = \mathcal{M} - \frac{\Delta_2 V}{V},$$

which provides the static expression for the invariant G'.

Given the above, recall ([1]) that the gravitational equations are:

$$G'_{ik} - \tfrac{1}{2} G' g_{ik} = -\kappa T_{ik} \qquad (i, k = 0, 1, 2, 3),$$

in which κ is constant and T_{ik} denotes the energy tensor.

Under static conditions, the T_{ik} are, like everything else, independent of time x_0. In addition, the $T_{i0} = T_{0i}$ are annulled, and (after previously dividing by $-\sqrt{-g_{00}\, g_{ii}} = -V\sqrt{a_{ii}}$) they will

([1]) Cf., e.g., the paper "Sulla espressione analitica spettante al tensore gravitationale nella teoria di Einstein," in this volume of the Rendiconti, pp. 381-391.

represent the components of the energy flux. That is because when one recalls (3) and (7), three of the desired equations will reduce to just identities, and the remaining seven (which correspond to the non-zero indices) will be:

$$G_{ik} + \mathcal{M} a_{ik} + \frac{V_{ik}}{V} - \frac{\Delta_2 V}{V} a_{ik} = -\kappa T_{ik} \qquad (i, k = 1, 2, 3), \tag{10}$$

as well as (for $i = k = 0$):

$$-V\,\Delta_2 V - \tfrac{1}{2} G' g_{00} = -\kappa T_{00},$$

or, when one keeps (9) in mind:

$$\mathcal{M} = -\kappa \frac{T_{00}}{V^2}. \tag{I}$$

As is the nature of things, the seven equations (10) and (I) reduce Einsteinian statics to the three dimensions of the ambient space. It has an invariant form with respect to the metric on that space, and with the nomenclature of the absolute differential calculus, it acts as the fundamental form that relates to ds^2. In addition, the two (invariant) functions V and T_{00} appear as elements that are associated with the fundamental form, along with the doubly-covariant system T_{ik} ($i, k = 1, 2, 3$). That latter characterizes the distribution of forces, while T_{00} / V^2 can be interpreted as the energy density [cf., § **3** of my paper, *loc. cit.*], and V represents the speed of light, as was said before.

In regard to the energy density, it must be said that, at least within the scope of the phenomena that are well-known nowadays (such as electromagnetic materials in the broad sense), there are no examples of negative energy density ([1]), so one can keep the right-hand side of (I) ≥ 0. That leads to this geometric corollary:

The mean curvature $\mathcal{M}$ (which is the sum of the three principle curvatures) *that one determines in the physical space as an effect of purely-static phenomena is positive or zero in any case.*

2. – Ricci's α_{ik}. The definitive form for the equations of statics.

For three-dimensional manifolds, the Riemann symbols (of the first kind) $a_{ij,\,hk}$ ($i, j, h, k = 1, 2, 3$) essentially reduce to the following schema $a_{i+1\,j+2,\,h+1\,k+2}$ (with the convention that one regards two indices as equivalent when they differ by 3), and one can opportunely replace them with the ratios:

$$\alpha^{(ik)} = \frac{a_{i+1\,i+2,\,k+1\,k+2}}{a} \qquad (i, k = 1, 2, 3),$$

([1]) Indeed, if matter at rest is distributed with a density of μ at a given position then it will carry an energy of material origin $V^2\mu$, which will dominate over all the other possible contributions at great distances (under ordinary conditions). On the other hand, the contribution to the energy density that is of electromagnetic origin happens to be ≥ 0, because even in the absence of matter, the energy density does not seem to be capable of taking on negative values.

(11) which Ricci introduced and which constitute a symmetric, doubly-contravariant system, as he showed (and as one can verify materially in an obvious way).

Naturally, one intends α_{ik} to mean the reciprocal covariant system, by virtue of which, the $\alpha^{(ik)}$ can be expressed in the form:

$$\alpha^{(ik)} = \sum_{j,h=1}^{3} a^{(ij)}\, a^{(hk)}\, \alpha_{jh}\,. \tag{12}$$

We would like to establish the relations that link the α_{ik} to the G_{ik} (1).

Proceeding by the direct route, one can start from (6) and replace the symbols $\{ih, hk\}$ in the right-hand side with the ones of the first kind, and write:

$$G_{ik} = \sum_{j,h=1}^{3} a^{(jh)}\, a_{ij,hk}\,. \tag{6′}$$

Develop the right-hand side, while giving j the values i, $i + 1$, $i + 2$ and giving h the values k, $l + 1$, $k + 2$. When one takes into account the identity:

$$a_{ij,\,hk} = a_{ji,\,hk} = -\,a_{ij,\,kh}\,,$$

along with (11), one will have:

$$G_{ik} = a\,\{-\,a^{(i+1\;k+1)}\,\alpha^{(i+1\;k+1)} + a^{(i+1\;k+2)}\,\alpha^{(i+2\;k+1)} + a^{(i+2\;k+1)}\,\alpha^{(i+1\;k+2)} - a^{(i+2\;k+2)}\,\alpha^{(i+1\;k+1)}\}\,,$$

or based upon (12):

$$G_{ik} = a\sum_{j,h=1}^{3}\alpha_{jh}\{-a^{(i+1k+1)}a^{(i+2\,j)}a^{(k+2h)} + a^{(i+1k+2)}a^{(i+2\,j)}a^{(k+1h)} + a^{(i+2k+1)}a^{(i+1\,j)}a^{(k+2h)} - a^{(i+2k+2)}a^{(i+1\,j)}a^{(k+1h)}\}.$$

In the summation, one should group the first term with the second and the third with the fourth, while attributing the values i, $i + 1$, $i + 2$ to h .

Since the algebraic complement to $a^{(ik)}$ in the determinant of that quantity is equal to a_{ik} / a, it will result that:

$$G_{ik} = \sum_{j,h=1}^{3}\{\alpha_{jh}\, a_{i+2k}\, a^{(i+2\,j)} - \alpha_{ji+2}\, a_{ik}\, a^{(i+2\,j)} + \alpha_{ji}\, a_{i+1k}\, a^{(i+1\,j)} - \alpha_{j\,i+1}\, a_{ik}\, a^{(i+1\,j)}\},$$

and upon adding and removing $\alpha_{ji}\, a_{ik}\, a^{(ik)}$ (inside the summation), that can be written more simply as:

$$G_{ik} = \alpha_{ik} - \sum_{l,j=1}^{3}\alpha_{lj}\, a^{(lj)}.$$

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(1) These relations were already pointed out by Ricci in the paper “Direzioni e invarianti principali in una varietà qualunque,” Atti del. R. Istituto Veneto, **63** (1904), pp. 1235.

Multiply this by $a^{(ik)}$ and sum over the two indices i and k. Taking into account that $\sum_{i,k=1}^{3} a^{(ik)} a_{ik}$ = 3, a comparison with (8) will give:

$$\mathcal{M} = \sum_{i,k=1}^{3} a^{(ik)} \alpha_{ik}, \tag{13}$$

so the relations that were obtained between the G_{ik} and the α_{ik} will assume the form:

$$\alpha_{ik} = G_{ik} + \mathcal{M}\, a_{ik} \qquad (i, k = 1, 2, 3), \tag{14}$$

which is more convenient to our purpose ([1]).

(13) justifies the meaning of $\mathcal{M}$ as the mean curvature of the manifold. Indeed, the principal curvatures are, by definition, the (necessarily real) roots ω_1, ω_2, ω_3 of the cubic equation ([2]):

$$\| \alpha_{ik} - \omega\, a_{ik} \| = 0 ,$$

and the right-hand side of (13) is precisely the sum of those roots (viz., the coefficient of ω^2 divided by $-a$, where $-a$ is the coefficient of ω^3 in the left-hand side of that cubic equation).

([1]) One can arrive at this more elegantly by recalling the systems E that belong to our ternary ds^2 [cf., Ricci and Levi-Civita, *Méthode de calcul différentiel absolus et leurs applications*," Math. Ann. **54** (1900), pp. 135. Ricci, "Sulle superficie geodetiche…," in these Rendiconti **12** 1st semester (1903), pp. 410]. In the first place, by means of the covariant E, one can write (12) in the form:

$$a_{ij,hk} = \sum_{p,q=1}^{3} \alpha^{(pq)} \varepsilon_{pij}\, \varepsilon_{qhk},$$

with which (6′) will assume the form:

$$G_{ik} = \sum_{j,h,p,q=1}^{3} \alpha^{(jh)} \alpha^{(pq)} \varepsilon_{pij}\, \varepsilon_{qhk} = -\sum_{j,h,p,q=1}^{3} \alpha^{(jh)} \alpha^{(pq)} \varepsilon_{pij}\, \varepsilon_{qkh}.$$

On the other hand, by means of the contravariant E, the definition of $a^{(jh)}$ will translate into the formula:

$$a^{(jh)} = \tfrac{1}{2} \sum_{\nu,\rho,\sigma,\tau=1}^{3} \varepsilon^{(\nu\rho j)} \varepsilon^{(\sigma\tau h)} a_{\nu\sigma}\, a_{\rho\tau}.$$

Substitute that in G_{ik}, while keeping in mind the identity (whose proof is immediate):

$$\sum_{j=1}^{3} \varepsilon_{pij} \varepsilon^{(\nu\rho j)} = \varepsilon_{p\nu} \varepsilon_{i\rho} - \varepsilon_{p\rho} \varepsilon_{i\nu} \qquad (p, i, \nu, \rho = 1, 2, 3),$$

along with the other one (which differs only by the notation of the indices):

$$\sum_{h=1}^{3} \varepsilon_{qkh} \varepsilon^{(\sigma\tau j)} = \varepsilon_{q\sigma} \varepsilon_{h\tau} - \varepsilon_{q\tau} \varepsilon_{k\sigma} \qquad (q, k, \sigma, \tau = 1, 2, 3),$$

in which the ε with two indices represent zero when the indices are distinct and unity when they coincide, as usual.

One then notes that the expression (13) for $\mathcal{M}$ will now be equivalent to:

$$\mathcal{M} = \sum_{p,q=1}^{3} a_{pq} \alpha^{(pq)},$$

which gives (14) precisely.

([2]) Ricci and Levi-Civita, *loc. cit.*, pp. 163.

One introduces the α_{ik} into the gravitational equations (10) by means of (14). If one also writes down (I) then one will have, by definition, the system:

(I) $$\mathcal{M} = \kappa \frac{T_{00}}{V^2},$$

(II) $$\alpha_{ik} + \frac{V_{ik}}{V} - \frac{\Delta_2 V}{V} a_{ik} = -\kappa T_{ik} \qquad (i, k = 1, 2, 3),$$

in which the mean curvature $\mathcal{M}$ has the expression (13).

A noteworthy consequence of (II) happens when one multiplies it by $a^{(ik)}$ and sums over the two indices. When one recalls (13) and (I), one will get:

(14) $$\frac{\Delta_2 V}{V} = \tfrac{1}{2}\kappa \left(\mathcal{F} + \frac{T_{00}}{V^2} \right),$$

in which

(15) $$\mathcal{F} = \sum_{i,k=1}^{3} a^{(ik)} T_{ik}$$

obviously represents the linear invariant of the system of forces with respect to our ds^2 (in the ambient space). Incidentally, that invariant must not be confused with the scalar of the quadrilateral tensor:

$$T = \sum_{i,k=0}^{3} g^{(ik)} T_{ik},$$

which is, however, due to the expression:

$$T = \frac{T_{00}}{V^2} - \mathcal{F},$$

according to (3).

3. – Motion of a material point. Trajectories. Equivalence of geodetics and conservative paths of the usual type.

According to Einstein's general theory, no matter what the quadri-dimensional ds' might be, the equations of motion of a material point are included in the variational principle:

(16) $$\delta \int ds' = 0 .$$

Under static conditions, ds' will have the form (1), such that when one sets:

(17) $$x_0 = t, \qquad \frac{dx_i}{dt} = \dot{x}_i \qquad (i = 1, 2, 3), \qquad \frac{ds^2}{dt^2} = v^2,$$

in order to make it easier to compare with the usual notations, the preceding can be written ([1]):

$$\delta\int\sqrt{V^2 - v^2}\, dt = 0.$$

In order to remain in the real and regular domain, one must exclude the motions in which the velocity exceeds the critical value V and set (with the arithmetic value of the radical) ([2]):

$$L = \begin{cases} \sqrt{V^2 - v^2} & \text{for} \quad v < V, \\ \sqrt{v^2 - V^2} & \text{for} \quad v > V, \end{cases} \tag{18}$$

and thus get the variational equation of motion:

$$\delta\int L\, dt = 0, \tag{16}$$

which is valid in both cases.

It is well-known that this has the Lagrange differential equations as a consequence:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \qquad (i = 1, 2, 3).$$

One will note that in (16), and therefore in its equivalent (16′), t must also be subjected to variation and treated in the same way as the spatial coordinates, and, in conformity with the latter, one must suppose that δt is zero at the extremes (of the integration interval). (After the usual integration by parts), that will give rise to a fourth equation:

$$\frac{d}{dt}\left(\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L\right) + \frac{\partial L}{\partial t} = 0,$$

which is therefore a consequence of the first three.

When $\partial L / \partial t = 0$, in particular, as in the present case, it can be identified with the typical integral of the Lagrangian system (the *vis viva*, when L has the form of a quadratic form in the velocity, which is true in ordinary mechanics):

([1]) This was proposed before in that form and discussed by Abraham, although constrained by the hypothesis that ds^2 was Euclidian. Cf., in particular, "Le equazioni di Lagrange nella nuova meccanica," Ann. di Mat. tomo XX (which was dedicated to the memory of Lagrange) (1913), pp. 29-39.

([2]) One can also limit oneself to that first case, which is physically interesting only for an effective material point. I feel that is it entirely preferable (when the occasion presents itself) to treat that question completely.

In the recent paper by Hilbert, "Die Grundlagen der Physik (part two)," Nach. der. K. Ges. der Wiss. zu Göttingen (1917), in which the postulates of general relativity are specified, one will find a qualitative specification in the context of the motion of material points that is equivalent to precisely $v < V$ under static conditions.

(19) $$\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L = V_0 \qquad (V_0 \text{ constant}).$$

One can eliminate *dt* from (16′) by exploiting that equation, *with all due formality*, and obtain a formula that is also variational and corresponds to the principle of minimum action and includes the equations of the trajectory. Here is how one proceeds:

One first assumes that the variations δx_i, δt in (16′) are kept arbitrary, but zero at the limits. One then notes that when one regards V_0 as a constant that is fixed beforehand (so $\delta V_0 = 0$), one can replace (16′) with the formula:

(16″) $$\delta \int (L + V_0)\, dt = 0,$$

which is essentially equivalent to it, because it gives rise to the same Lagrange equations. It then has the advantage over (16′) that there is no longer any need to impose the condition on δt that it must be annulled at the extremes [provided that one intends that the constant on the right-hand side of the integral (19) has the prearranged value V_0]. That follows directly from the observation that when one puts δ under the sign, one will essentially have:

$$\int \delta\, dt \left[-\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L + V_0\right]$$

as the contribution that is provided by the variation of *t*, which will be annulled by virtue of (19).

That being the case, it becomes legitimate to consider the x_i, t in (16″) and subject their variations to (19), rather than let them be independent. In truth, nothing prevents one *a priori* from introducing constraints at will in either (16′) or (16″), *provided only that one still respects the limit conditions for the* δx_i, δt. How does (19) behave in that regard? One can say that its variation provides $\delta\, dt$ (or what amounts to the same thing, $d\, \delta t$) in terms of the δx_i, which remain arbitrary (but functions of *t*), except that they are annulled at the limits. Consistent with that, the δt will result from a quadrature, and therefore one can make it zero at *one* of the extremes of the integration interval, but not generally at the other one. That is because the introduction of the constraint (19) is perfectly legitimate in (16″), which does not require the annulment of δt at the limits; however, that is not true for (16′).

In order to clarify that concept, let us make the calculations explicit.

When (19) is multiplied by *L*, it can be written:

$$\tfrac{1}{2}\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i - L^2 = V_0\, L.$$

By virtue of (18):

$$L^2 = \pm (V^2 - v^2),$$

whose sign must be chosen in such a way that L^2 will be positive. One then deduces (when v^2 is homogeneous in the $\dot{x}_i$ of degree two and V^2 is a function of only the x_i):

$$\tfrac{1}{2}\sum_{i=1}^{3}\frac{\partial L}{\partial \dot{x}_i}\dot{x}_i = \mp\tfrac{1}{2}\sum_{i=1}^{3}\frac{\partial v^2}{\partial \dot{x}_i}\dot{x}_i = \mp v^2,$$

in which one adopts the upper or lower sign according to whether the one sign or the other is true for L^2, resp. With that same convention, the preceding [i.e., essentially the integral (19)] can be written:

(19′) $$V^2 = V_0\,L.$$

That is because when one recalls (18), one will see that *the constant* V_0 *is necessarily negative, and its absolute value is* > V *for motions that ensue with a velocity* $v < V$. *However, it will be positive and can assume arbitrary values between* 0 *and* ∞ *under the opposite hypothesis.*

When (19′) is divided by V_0 and squared, one will have:

$$v^2 = V^2 \mp \frac{V^4}{V_0^2},$$

with the usual choice of sign that makes either the upper or lower sign valid in all of the formulas simultaneously.

Since $v = ds / dt$ (with ds and dt positive), one will ultimately get:

$$dt = \frac{ds}{v}\left(1 \mp \frac{V^2}{V_0^2}\right)^{-1/2}.$$

On the other hand, (19′) gives:

$$L + V_0 = V_0\left(1 \mp \frac{V^2}{V_0^2}\right)^{-1/2}.$$

If one substitutes that in (16), suppresses the constant factor V_0, and sets:

(20) $$2U = \frac{1}{V^2}\left(1 \mp \frac{V^2}{V_0^2}\right) = \frac{1}{V^2} \mp \frac{1}{V_0^2}$$

then what will result is *the comprehensive equation of the trajectory:*

(21) $$\delta\int\sqrt{2U}\,ds = 0.$$

That appears to coincide (when one intends that to mean, for each value of the constant V_0 that appears implicitly in U) *with the geodetic of a space with a line element* $\sqrt{2U}\,ds$ *, or also* ([1]) *with a congruence of trajectories for a conservative problem in ordinary mechanics in the physical space of the line element ds*. The congruence is characterized as follows: Its *vis viva* is $\frac{1}{2}\frac{ds^2}{dt^{*2}}$, where t^* denotes an auxiliary variable that serves as time. Its force function is c^4U, with c arbitrary. Its total energy is $\frac{1}{2}\frac{ds^2}{dt^{*2}} - c^4\,U = 0$. Keeping in mind the expression (20) for U, one can also say that the force function is $\frac{c^4}{2V^2}$ and the total energy is $\pm\frac{c^4}{2V_0^2}$.

4. – Limiting cases. Optical interpretation.

1. *Newtonian attraction*. – Under the hypothesis that the quadri-dimension form $ds'^2 = V^2\,dt^2 - ds^2$ is very close to the Euclidian type, one can set:

$$V = c\,(1+\gamma), \quad ds^2 = \sum_{i,k=1}^{3}(\varepsilon_{ik} + \gamma_{ik})\,dx_i\,dx_k, \tag{22}$$

in which c is constant (viz., the speed of light in the absence of any perturbing circumstances), *and the* γ *are all pure numbers that must be treated as first-order quantities.*

According to (18), the Lagrangian function for motion that is endowed with a velocity $v < V$ is:

$$\sqrt{V^2 - v^2},$$

or, when one multiplies by $-c$ (which is permissible, since it will not change the equations of motion), recalls the first of (22), and neglects the γ^2:

$$-c^2\sqrt{1+2\gamma-\frac{v^2}{c^2}}.$$

Suppose (as one can do for the motions of ponderable bodies, as a rule) that one can also neglect the square of the ratio v^2 / c^2. When one develops the radical and drops the inessential additive constant $-c^2$, the given Lagrangian function will assume the form:

$$L = \tfrac{1}{2}v^2 - c^2\,\gamma.$$

According to (21), one must intend that $v^2 = ds^2 / dt^2$ should mean:

([1]) Cf., e.g., the previously-cited *Traité de mécanique rationelle* by Appell, no. 487.

$$\sum_{i,k=1}^{3} (\varepsilon_{ik} + \gamma_{ik})\,\dot{x}_i\,\dot{x}_k = \sum_{i=1}^{3} \dot{x}_i^2 + \sum_{i,k=1}^{3} (\varepsilon_{ik} + \gamma_{ik})\,\dot{x}_i\,\dot{x}_k ,$$

but it will quickly become clear that the γ_{ik} can certainly be equal to zero, since they contribute to the equations of motion only to second-order.

We are therefore brought back (in the first approximation) to the ordinary mechanics of a material point in Euclidian space under the action of a unit potential (which is mean to be a force per unit mass):

$$-c^2\,\gamma .$$

If we expect the expression $c\,(1 + \gamma)$ for V then (14), with the agreed-upon approximation, will become:

$$\Delta_2 \gamma = \tfrac{1}{2}\kappa \left(\mathcal{F} + \frac{T_{00}}{V^2} \right),$$

which reduces, in substance, to the Poisson-Laplace equation that characterizes the Newtonian potential. Indeed, at great distances, the intrinsic energy inside of ponderable bodies will dominate all other forms, since the energy density is roughly equal to $c^2\,\mu$ (where μ is the density of matter), and $\mathcal{F}$ will prove to be negligible in comparison to $c^2\,\mu$. In empty space ($\mu = 0$), the entire sum $\mathcal{F} + T_{00} / V^2$ will be negligible in comparison to the order of magnitude of the values that it has inside of matter. One can then set:

$$\Delta_2\,\gamma = \tfrac{1}{2}\,\kappa\,c^2\,\mu$$

in all of space.

Since $\Delta_2\,\gamma$ can be referred to the Euclidian ds^2 (at least to second-order terms) and:

$$\kappa = \frac{8\pi f}{c^4} \qquad (f = \text{constant of attraction}),$$

one will effectively recover the Poisson-Laplace equation for the potential $-c^2\,\gamma$.

It was by precisely that argument that Einstein fixed the numerical value of his universal constant κ.

2. $|\,V_0\,|$ *very large*. *Comparison with light rays*. – If v assumes a value that is very close to V in the course of motion then the corresponding value of L will be very small, and therefore on the basis of (19′), the constant V_0 must stay very large in absolute value (and negative or positive according to whether $v < V$ or $v > V$, resp.).

Suppose that V^2 / V_0^2 is negligible compared to unity, so one will have from (20) that:

$$2U = \frac{1}{V^2} ,$$

so that from the equivalence theorem in the preceding paragraph, the trajectories will coincide with the geodetics of the line element ds / V, which form a congruence that is due to the force function $c^4 / 2V^2$.

The former result gives rise to an interesting optical approximation that was observed before along a different path by Caldonazzo ([1]) in the context of Abraham's theory. In order to arrive at it, it is enough to recall that one has attributed the meaning to V of the speed of light in our space (which is the realm of static phenomena) with the line element ds. Preserving Fermat's principle, even in the new mechanics, the course of light rays will still be contained in the formula:

$$\delta \int \frac{ds}{V} = 0.$$

As was pointed out before, (20) will reduce to that formula (i.e., to the geodetics of the line element ds / V) for $| V_0 |$ very large. Therefore, the trajectories of material points will tend to coincide with light rays as V_0 increases indefinitely, or what amounts to the same thing, when the velocity of motion converges to the velocity of light.

3. V_0 very small. – According to (19′) and (18), that case can occur only when $v > V$ and very large. (20) shows that $\sqrt{2U}$ is roughly constant then, so the equation (21) of the trajectory will reduce to:

$$\delta \int ds = 0 .$$

One can infer from this that (as in ordinary mechanics, when the accelerating action of the force is negligible in comparison to the inertia) the trajectories in a gravitational field will tend to become geodetics as the velocity increases indefinitely.

([1]) "Traiettorie dei raggi luminosi e dei punti materiali nel campo gravitationale," Nuovo Cimento (5) **5** (1913), 267-300.