"Sulla riducibilità della equazioni elettrodinamiche di Helmholtz alla forma Hertziana," Il Nuovo Cim. (4) **6** (1897), 93-108.

On the reducibility of Helmholtz's equations of electrodynamics to the Hertzian form

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The mathematical relations between the electric and magnetic forces in the free ether that are attributed to Clerk Maxwell were extended by Hertz (with some requisite modifications) to an arbitrary field in a manner that would embrace the entire class of electromagnetic phenomena.

It is known that Maxwell had deduced his equations from some rather complicated hypotheses and by reasoning that was not always correct. In order to not incur similar inconveniences, it seemed better to Hertz to forego any *a priori* justification (at least in the case of bodies at rest) and limit himself to achieving perfect agreement between the mathematical representation and almost everything that had been studied experimentally up until then.

From a positive viewpoint, one can demand nothing else ([1]), as long as many electromagnetic phenomena (e.g., stationary ones) do not possess prior systematic treatments, but likewise conform to experiments and are seductive for not only their logical rigor, but also the transparent simplicity of their fundamental hypotheses.

The noteworthy historical and methodological value of that classical theory keeps one from remaining silent now about materialistic speculations. It will then become more desirable to collect all of the manifestations of electromagnetism into the same location. To that end, we propose to give some disparate physical and eminently-mathematical hypotheses from which, although some of them might imply unacceptable physical consequences ([2]), others will arrive at a mature

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([1]) However, *even from the strictly-physical standpoint*, the possibility remains of reducing the phenomena in question to other simpler or more intuitive ones, such as, e.g., dynamical phenomena. One such possibility that is probably linked with the psychological need to reduce everything to causal laws gave rise to many mechanical theories and interpretations of electromagnetism that are all possible abstractly, but are in reality only attempts that could be, or perhaps one day might be, sanctioned or rejected. If one overlooks their mathematical importance then they will still have a certain value as models for the time being. In regard to all of that, one can consult:

H. Ebert, "Zur Theorie der magnetischen und elektrischen Erscheinungen," and "Ueber die Bewegungsformen, welche den elektromagnetischen Erscheinungen zu Grunde gelegt werden können," (Wiedemann's Annalen, Bd. 51 and 52).

Besides containing the exposition of a new theory, those papers of Ebert also contain an accurate bibliography of the argument.

([2]) See, e.g., Helmholtz's critique of Weber's law in many articles (*Wissenschaftliche Abhandlungen*, Bd. 1, pp. 537-687).

discussion due to the pleasing harmony in their deductions and their lack of contradiction with the facts up to now.

What is without a doubt particularly noteworthy about the classical theory of electrodynamics, as it was established and amply developed by Helmholtz ([1]), is that it was based upon F. Neumann's laws of the potential. Whether it coincides, in substance, with that of Hertz, or at what point and in what way does it differ from it, has not been examined as of yet ([2]), but since that investigation does not seem trivial to me, I would like to include it in the letters to this periodical.

Meanwhile, let me present the result that was obtained and some considerations that are intimately connected with it:

Helmholt's electrodynamical theory (which corresponds to F. Neumann's law of the potential) *leads to Hertz's equation as long as one assumes that the action at a distance* (whether of electrostatic or electrodynamical origin) *propagates with a finite velocity* ([3]).

More precisely, we shall introduce the hypothesis that in an indefinite medium that is generally homogeneous ([4]) and isotropic, the speed of propagation will be uniform and expressed by $\frac{1}{A\sqrt{\varepsilon\mu}}$, in which $1 / A$ is the speed of light in the ether, while ε and μ represent the dielectric constants and magnetization of the medium, respectively. For brevity, we shall sometimes denote that hypothesis by the letter (I).

One sees immediately that in order to attribute a mathematical form to it, it is enough to replace the elementary potential:

$$\frac{\Omega(\xi,\eta,\zeta,t)}{r} \qquad (r = \sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}\,)$$

([1]) *Loc. cit.*, pp. 537-820, but since we must concern ourselves with bodies at rest, it is enough to refer almost exclusively to the first of the electrodynamical papers ("Ueber die Bewegungsgleichungen der Elektricität für ruhende Körper," *ibidem*, pp. 545-628.)

([2]) Helmholtz showed that Maxwell's equations (and therefore that of Hertz) would reappear as a limiting case of the ones that he established. However, that passage to the limit completely distorts the original viewpoint, and it does not lend itself to a comparison with the relations that govern those physical quantities in the two cases. One attempt in that spirit was made by Hertz in his article "Ueber die Beziehungen zwischen den Maxwell'schen elektrodynamischen Grundgleichungen und den Grundgleichungen der gegnerischen Elektrodynamik" (*Ges. Werke*, Bd. 1, pp. 295-319), in which one finds a presentation of the corrections by means of which one can pass from the ordinary equations of electrodynamics to Maxwell's in the case of the ether. Nonetheless, one cannot keep silent about the fact that the hypotheses of that illustrious figure of history are physically infelicitous and are in obvious contradiction with the subsequent progress in mathematics. Cf., C. Neumann, *Allgemeine Untersuchungen ueber das Newton'sche Princip der Fernwirkungen*, Leipzig, 1896, Chapter Eight, **§ 5**.

([3]) One should be forewarned that the idea of attributing a finite speed of propagation to the action at a distance is anything but new: Both Gauss and Riemann believed that one should always find it at the foundations of electrodynamics: As for the qualitative existence of the phenomenon (or something else whose experimental consequences are equivalent), an indisputable proof of it can be found in Hertz's famous experiments with rays of electrical force.

([4]) We intend that to mean a medium whose homogeneity has exceptions only on the surface.

of a mass or the component of the current Ω that exists at the point (ξ, η, ζ) at the instant t with the potential $\Omega(\xi,\eta,\zeta,t-A\sqrt{\varepsilon\,\mu}\cdot r)/r$. The two expressions will coincide when Ω does not depend upon t, i.e., for all phenomena of a stationary character. In any case, in order for $A\sqrt{\varepsilon\,\mu}$ to be a very small quantity, the hypothesis (I) will have only a corrective value with respect to Helmholtz's theory, and the divergences from it will be negligible as long as the domain in which the electromagnetic actions take place and are observed is sufficiently restricted with respect to the speed of light. It will then result that to the extent that Hertz's equations effectively correspond to reality, they will deviate very little from Helmholtz's original equations, at least within certain limits.

The hypothesis (I) will make them coincide with the Hertzian equations identically, or more precisely, it will make them become *integrals* (viz., functionals) of them.

The integral relations, which are thus proposed incidentally, are not as simple as Hertz's equation, but they obviously say more than the latter and can also sometimes render a useful service from the mathematical viewpoint.

Another circumstance that deserves to be pointed out is made probable by our study, viz., that *among the elementary laws of electrodynamical induction that were proposed by Ampère, Faraday, Grassmann, F. Neumann, W. Weber, C. Neumann* ([1]), *the most reliable of them, even for open circuits, is* [in the case of bodies at rest, and taking into account the corrective hypothesis (I) that was cited above] *F. Neumann's potential law* ([2]), *which is the only one* ([3]) *that allows one to arrive at Hertz's equations*.

Here, I am myself limited to the consideration of electromagnetic actions in an indefinite medium that is generally homogeneous, isotropic, and at rest because since the discrepancy between classical and Hertzian electrodynamics was removed from its landscape, one will see with no difficulty how one can arrive at complete agreement between them by proceeding in an analogous manner.

For example, for a moving medium, according to the path that was followed by Prof. Volterra ([4]), it is enough for one to be able to transform Helmholtz's equations into general coordinates.

However, that shall be postponed to another occasion. I would then have to enter into more minute details in order to set the scene for the treatment of a general electromagnetic field, after which it would be possible for me (by invoking the principle of the conservation of energy) to investigate the law of ponderomotive action.

([1]) Cf., **Helmholtz**, *Wiss. Abh.*, Bd. I, pp. 774-790.

In an important paper [Leipziger Berichte (1896)], C. Neumann devoted a deep analysis to the mathematical form of the elementary laws of electrodynamics and presented a summary of some results that were of great interest. When one relates them with the present work, an argument will follow that seems to favor a certain hypothesis "epsilon" that the cited author seems to prefer without also attributing an essential value to it.

([2]) In regard to that, recall that Helmholtz constantly appealed to F. Neumann's potential law, and he did not disguise his personal conviction that it should prevail as a definition.

([3]) To be honest, I did not show explicitly that *only* F. Neumann's law will allow one to arrive at Hertz's equations, but I did show that if one assumes any other starting point than the aforementioned laws, one can see how to apply the same procedure to it that was employed for F. Neumann's law. In that way, one would obtain equations that would *contradict* those of Hertz.

([4]) "Sopra le equazioni di Hertz" [in this journal, **29** (1891)].

For the moment, it would seem convenient for me to exclude any complication, in the hopes that the simplicity of the development will justify the kind attention of the studious.

1. – Let S be an indefinite medium that is generally homogeneous, isotropic, and at rest, and in which electrodynamical phenomena take place. Say that σ is, in general, a surface that is situated in some way, although it is meant to be fixed in S, and along which the homogeneity in S can break down.

The elements that determine the physical state of the medium (when one considers the phenomenon of the motion of electricity in its most-general aspect, but overlooks ones of a different nature) are the electrostatic density and the components of the current at each point (ξ, η, ζ) and at each instant t. Suppose that the distribution of static electricity and current is generally over a volume, but a two-dimensional electrostatic distribution can also exist on σ. Let E (ξ, η, ζ, t), u (ξ, η, ζ, t), v (ξ, η, ζ, t), w (ξ, η, ζ, t) denote the electrostatic density in the volume and the components of the current at a generic point (ξ, η, ζ) of S and at the instant t. Let e (ξ, η, ζ, t) denote the surface density at a point on the surface σ and at the instant t.

The functions E, u, v, w are regarded as zero at infinity to order greater than two, and for any value of t, they will be finite, continuous, and differentiable in all of space S, with the exception of the surface σ, across which, it can be subject to a discontinuity. The surface density e is meant to be finite, continuous, and differentiable with respect to time over any surface σ.

Let p', E', u', v', w', f'; p'', E'', u'', v'', w'', f'' be the two directions of the normal, the limiting values of E, u, v, w, and more generally, an arbitrary function f, on one side of σ and the other ([1]). Let $\alpha' = -\alpha''$, $\beta' = -\beta''$, $\gamma' = -\gamma''$ be the direction cosines of p' and p'', resp.

To complete the analytical determination of the phenomenon, which is considered to isolated, it is enough to regard the fundamental relations (viz., continuity equations) as:

(1) $$-\frac{\partial E}{\partial t} = \frac{\partial u}{\partial \xi} + \frac{\partial v}{\partial \eta} + \frac{\partial w}{\partial \zeta},$$

(2) $$-\frac{\partial e}{\partial t} = u'\alpha' + v'\beta' + w'\beta' + u''\alpha'' + v''\beta'' + w''\gamma''.$$

2. – Having done that, we shall now address the problem of relating the motion of the electricity with other physical concepts, and in the first place, if we let ε and μ denote the dielectric constant and magnetization of the medium S then, along with the true density E, e and the true currents u, v, w, we shall also consider the quantities:

([1]) In order to not become prolix, we shall reason only under the assumption that the surface σ possesses a tangent plane at each point. Nonetheless, we intend that the condition should be satisfied only in general, since that would be sufficient to legitimize the operations of calculus that must be performed.

(3) $$\mathbf{E} = \frac{E}{\varepsilon}, \quad \mathbf{e} = \frac{e}{\varepsilon},$$

(4) $$\mathbf{u} = \mu\, u, \qquad \mathbf{v} = \mu\, v, \qquad \mathbf{w} = \mu\, w,$$

which are called, as is known, the density of *free* electricity and the components of the *free* current, resp., because it is precisely those quantities (and not the corresponding italicized ones) that will remain *free* (or become free, respectively) to exert any action at a distance. According to the usual theory of instantaneous propagation, the potentials that relate to them will be:

$$\int_S \frac{\mathbf{E}}{r} dS, \qquad \int_\sigma \frac{\mathbf{e}}{r} d\sigma, \qquad \int_S \frac{\mathbf{u}}{r} dS, \qquad \text{etc.}$$

(with $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ and $dS = d\xi\, d\eta\, d\zeta$).

If one assumes the hypothesis (I), moreover, that the speed of propagation is $\dfrac{1}{A\sqrt{\varepsilon\,\mu}}$ then one will have the potentials:

(5) $$\mathbf{F}(x, y, z, t) = \int_S \frac{\mathbf{E}(\xi,\eta,\zeta,t - Ar\sqrt{\varepsilon\,\mu})}{r} dS + \int_\sigma \frac{\mathbf{e}(\xi,\eta,\zeta,t - Ar\sqrt{\varepsilon\,\mu})}{r} d\sigma,$$

(6) $$\begin{cases} \mathbf{U}(x,y,z,t) = \displaystyle\int_S \frac{\mathbf{u}(\xi,\eta,\zeta,t - Ar\sqrt{\varepsilon\,\mu})}{r} dS, \\ \mathbf{V}(x,y,z,t) = \displaystyle\int_S \frac{\mathbf{v}(\xi,\eta,\zeta,t - Ar\sqrt{\varepsilon\,\mu})}{r} dS, \\ \mathbf{W}(x,y,z,t) = \displaystyle\int_S \frac{\mathbf{w}(\xi,\eta,\zeta,t - Ar\sqrt{\varepsilon\,\mu})}{r} dS. \end{cases}$$

With respect to the analytic nature of the functions **F**, **U**, **V**, **W**, we observe the following ([1]):

The function **F** and its first derivatives are finite and continuous in all of space, with the exception of the surface σ, where **F** and its tangential derivatives will remain continuous, while the normal derivatives will present the discontinuity:

(5.a) $$\frac{\partial \mathbf{F}'}{\partial p'} + \frac{\partial \mathbf{F}''}{\partial p''} = -\,4\pi\,\mathbf{e}\,.$$

The equation:

([1]) **Volterra**, "Sul principio di Huygens," (Volumes 31, 32, and 33 of this journal).

(5′) $$\underset{2}{\Delta}\mathbf{F} - A^2\varepsilon\,\mu\frac{\partial^2\mathbf{F}}{\partial t^2} = -\,4\pi\,\mathbf{E}\,.$$

The functions **U**, **V**, **W** are everywhere finite and continuous, along with their first derivatives, and they will ordinarily satisfy the equations:

(6′) $$\begin{cases} \underset{2}{\Delta}\mathbf{U} - A^2\varepsilon\,\mu\dfrac{\partial^2\mathbf{U}}{\partial t^2} = -4\pi\,\mathbf{u}, \\ \underset{2}{\Delta}\mathbf{V} - A^2\varepsilon\,\mu\dfrac{\partial^2\mathbf{V}}{\partial t^2} = -4\pi\,\mathbf{v}, \\ \underset{2}{\Delta}\mathbf{W} - A^2\varepsilon\,\mu\dfrac{\partial^2\mathbf{W}}{\partial t^2} = -4\pi\,\mathbf{w}. \end{cases}$$

In addition to those properties, which due to the form of (5) and any one of (6) exclusively, one can establish a very important relation between the four functions **F**, **U**, **V**, **W**.

To that end, note that due to (3), (4), (1) and (2) will give:

(1′) $$-\,\varepsilon\,\mu\frac{\partial\mathbf{E}}{\partial t} = \frac{\partial\mathbf{u}}{\partial\xi} + \frac{\partial\mathbf{v}}{\partial\eta} + \frac{\partial\mathbf{w}}{\partial\zeta}\,,$$

(2′) $$-\,\varepsilon\,\mu\frac{\partial\mathbf{e}}{\partial t} = \mathbf{u}'\alpha' + \mathbf{v}'\beta' + \mathbf{w}'\beta' + \mathbf{u}''\alpha'' + \mathbf{v}''\beta'' + \mathbf{w}''\gamma''\,,$$

or when one changes t into $t - Ar\sqrt{\varepsilon\,\mu}$, and letting $\overline{f}$ denote, for brevity, what a function f of t will become when one replaces t with $t - Ar\sqrt{\varepsilon\,\mu}$:

(1″) $$-\,\varepsilon\,\mu\frac{\partial\overline{\mathbf{E}}}{\partial t} = \frac{\partial\overline{\mathbf{u}}}{\partial\xi} + \frac{\partial\overline{\mathbf{v}}}{\partial\eta} + \frac{\partial\overline{\mathbf{w}}}{\partial\zeta} + A\sqrt{\varepsilon\,\mu}\left\{\frac{\partial\overline{\mathbf{u}}}{\partial t}\frac{\partial r}{\partial\xi} + \frac{\partial\overline{\mathbf{v}}}{\partial t}\frac{\partial r}{\partial\eta} + \frac{\partial\overline{\mathbf{w}}}{\partial t}\frac{\partial r}{\partial\zeta}\right\},$$

(2″) $$-\,\varepsilon\,\mu\frac{\partial\overline{\mathbf{e}}}{\partial t} = \overline{\mathbf{u}}'\alpha' + \overline{\mathbf{v}}'\beta' + \overline{\mathbf{w}}'\beta' + \overline{\mathbf{u}}''\alpha'' + \overline{\mathbf{v}}''\beta'' + \overline{\mathbf{w}}''\gamma''\,.$$

On the other hand:

$$\frac{\partial\mathbf{U}}{\partial x} + \frac{\partial\mathbf{V}}{\partial y} + \frac{\partial\mathbf{W}}{\partial z} = \int_S \left\{ \frac{\partial\dfrac{\overline{\mathbf{u}}}{r}}{\partial x} + \frac{\partial\dfrac{\overline{\mathbf{v}}}{r}}{\partial y} + \frac{\partial\dfrac{\overline{\mathbf{w}}}{r}}{\partial z} \right\} dS$$

$$= -\int_S \left\{ \bar{\mathbf{u}} \frac{\partial \frac{\mathbf{1}}{r}}{\partial x} + \bar{\mathbf{v}} \frac{\partial \frac{1}{r}}{\partial y} + \bar{\mathbf{w}} \frac{\partial \frac{1}{r}}{\partial z} \right\} dS + A\sqrt{\varepsilon\,\mu} \int_S \frac{1}{r} \left\{ \frac{\partial \bar{\mathbf{u}}}{\partial t}\frac{\partial r}{\partial \xi} + \frac{\partial \bar{\mathbf{v}}}{\partial t}\frac{\partial r}{\partial \eta} + \frac{\partial \bar{\mathbf{w}}}{\partial t}\frac{\partial r}{\partial \zeta} \right\} dS$$

$$= \int_S \left[\frac{\partial \bar{\mathbf{u}}}{\partial \xi} + \frac{\partial \bar{\mathbf{v}}}{\partial \eta} + \frac{\partial \bar{\mathbf{w}}}{\partial \zeta} + A\sqrt{\varepsilon\,\mu} \left\{ \frac{\partial \mathbf{u}}{\partial t}\frac{\partial r}{\partial \xi} + \frac{\partial \mathbf{v}}{\partial t}\frac{\partial r}{\partial \eta} + \frac{\partial \mathbf{w}}{\partial t}\frac{\partial r}{\partial \zeta} \right\} \right] \frac{dS}{r}$$
$$+ \int_\sigma [\bar{\mathbf{u}}'\alpha' + \bar{\mathbf{v}}'\beta' + \bar{\mathbf{w}}'\beta' + \bar{\mathbf{u}}''\alpha'' + \bar{\mathbf{v}}''\beta'' + \bar{\mathbf{w}}''\gamma''] \frac{d\sigma}{r},$$

and upon comparing that with (1″), (2″), one will deduce that:

$$\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{V}}{\partial y} + \frac{\partial \mathbf{W}}{\partial z} = -\,\varepsilon\,\mu \int_S \frac{\partial \bar{\mathbf{E}}}{\partial t}\frac{dS}{r} - \varepsilon\,\mu \int_\sigma \frac{\partial \bar{\mathbf{e}}}{\partial t}\frac{d\sigma}{r}.$$

Since the right-hand side is equal to $-\varepsilon\,\mu \frac{\partial \mathbf{F}}{\partial t}$, by virtue of (5), one will get, by definition:

$$\frac{\partial \mathbf{U}}{\partial x} + \frac{\partial \mathbf{V}}{\partial y} + \frac{\partial \mathbf{W}}{\partial z} = -\varepsilon\,\mu \frac{\partial \mathbf{F}}{\partial t}, \tag{7}$$

which is the relation that was announced.

3. – In the preceding section, only the mathematical aspects of the potentials **F**, **U**, **V**, **W** were considered, namely, as functions that were defined by (5), (6). We shall now establish the physical attributes by showing how they relate to the *forces and electric and magnetic polarizations.*

Denote the components of the electrical force and polarization by *X*, *Y*, *Z*, **X**, **Y**, **Z**, and denote the components of magnetic force and polarization by *L*, *M*, *N*, **L**, **M**, **N**. One will then have:

$$\left\{ \begin{aligned} \mu\,L &= \mathbf{L} = A\left(\frac{\partial \mathbf{V}}{\partial z} - \frac{\partial \mathbf{W}}{\partial y} \right), \\ \mu\,M &= \mathbf{M} = A\left(\frac{\partial \mathbf{W}}{\partial x} - \frac{\partial \mathbf{U}}{\partial z} \right), \\ \mu\,N &= \mathbf{N} = A\left(\frac{\partial \mathbf{U}}{\partial y} - \frac{\partial \mathbf{V}}{\partial x} \right), \end{aligned} \right. \tag{8}$$

$$
(9) \qquad \begin{cases}
\dfrac{\mathbf{X}}{\varepsilon} = X = -\dfrac{\partial \mathbf{F}}{\partial x} - A^2 \dfrac{\partial \mathbf{U}}{\partial t}, \\[2mm]
\dfrac{\mathbf{Y}}{\varepsilon} = Y = -\dfrac{\partial \mathbf{F}}{\partial y} - A^2 \dfrac{\partial \mathbf{V}}{\partial t}, \\[2mm]
\dfrac{\mathbf{Z}}{\varepsilon} = Z = -\dfrac{\partial \mathbf{F}}{\partial z} - A^2 \dfrac{\partial \mathbf{W}}{\partial t}.
\end{cases}
$$

It would not be amiss to briefly recall how one justifies those equations, or if one prefers, what the (elementary) physical laws are that produce them. One can limit oneself to the case of a non-polarizable medium since the extension to the general case can be performed in an obvious way ([1]).

One has $\varepsilon = \mu = 1$ and $\mathbf{E} = E$, $\mathbf{e} = e$, $\mathbf{u} = u$, $\mathbf{v} = v$, $\mathbf{w} = w$, $\mathbf{L} = L$, $\mathbf{M} = M$, $\mathbf{N} = N$, $\mathbf{X} = X$, $\mathbf{Y} = Y$, $\mathbf{Z} = Z$. According to *Coulomb's law*, the electrostatic action the electrostatic action that is exerted at the instant t by the elementary electric mass $E(\xi, \eta, \zeta, t)\, dS$ on the mass + 1 that located at (x, y, z) will have the components:

$$
-E\frac{\partial \frac{1}{r}}{\partial x}\,dS, \quad -E\frac{\partial \frac{1}{r}}{\partial y}\,dS, \quad -E\frac{\partial \frac{1}{r}}{\partial z}\,dS.
$$

However, conforming to the hypothesis (I), we will assume that:

$$
-\frac{\partial \frac{\bar{E}}{r}}{\partial x}\,dS, \quad -\frac{\partial \frac{\bar{E}}{r}}{\partial y}\,dS, \quad -\frac{\partial \frac{\bar{E}}{r}}{\partial z}\,dS,
$$

which will involve a correction to the individual components that merely *of second order* ([2]) *in the inverse of the speed of light*. Analogously, in order to fix the action of a current element ($u\, dS$, $v\, dS$, $w\, dS$) on a magnetic pole, replace the components:

([1]) See: **W. Voigt**, *Kompendium der theoretischen Physik*, [Leipzig **2** (1896), pp. 48-58, 206-218].

([2]) Indeed, if one considers, e.g., the component along the x-axis:

$$
\frac{\partial \frac{\bar{E}}{r}}{\partial x} = E(\xi, \eta, \zeta, t - A\,r)\,\frac{\partial \frac{1}{r}}{\partial x} + \frac{1}{r}\,\frac{\partial E(\xi, \eta, \zeta, t - Ar)}{\partial x},
$$

and since (assuming that the function E is twice-differentiable with respect to t, one can write:

$$
E(\xi, \eta, \zeta, t - A\,r) = E(\xi, \eta, \zeta, t) - Ar\,\frac{\partial E(\xi, \eta, \zeta, t)}{\partial t} + \tfrac{1}{2}A^2 r^2\,\frac{\partial^2 E(\xi, \eta, \zeta, t - A\theta r)}{\partial t^2} \qquad (0 < \theta < 1),
$$

it will follow that:

$$A\left(v\frac{\partial\frac{1}{r}}{\partial z}-w\frac{\partial\frac{1}{r}}{\partial y}\right)dS\,,\qquad A\left(w\frac{\partial\frac{1}{r}}{\partial x}-u\frac{\partial\frac{1}{r}}{\partial z}\right)dS\,,\qquad A\left(u\frac{\partial\frac{1}{r}}{\partial y}-v\frac{\partial\frac{1}{r}}{\partial x}\right)dS\,,$$

which are inferred from the *Biot-Savart law* with the components:

$$A\left(\frac{\partial\frac{\bar v}{r}}{\partial z}-\frac{\partial\frac{\bar w}{r}}{\partial y}\right)dS\,,\qquad A\left(\frac{\partial\frac{\bar w}{r}}{\partial x}-\frac{\partial\frac{\bar u}{r}}{\partial z}\right)dS\,,\qquad A\left(\frac{\partial\frac{\bar u}{r}}{\partial y}-\frac{\partial\frac{\bar v}{r}}{\partial x}\right)dS\,,$$

so the difference in the present case will be *of order at least four* ([1]) in A.

Finally, if one recalls *F. Neumann's potential law*, which assigns the expressions:

$$-\frac{A^2}{r}\frac{\partial u}{\partial t},\qquad -\frac{A^2}{r}\frac{\partial v}{\partial t},\qquad -\frac{A^2}{r}\frac{\partial w}{\partial t}$$

to the components of the electric induction that is due to the current element ($u\,dS$, $v\,dS$, $w\,dS$) at a point (x, y, z) then the hypothesis (I) will lead to assume that:

$$-\frac{A^2}{r}\frac{\partial\bar u}{\partial t},\qquad -\frac{A^2}{r}\frac{\partial\bar v}{\partial t},\qquad -\frac{A^2}{r}\frac{\partial\bar w}{\partial t},$$

which differs from the first one and will also be *of order at least four* ([2]) in A.

Based upon all of the laws thus-modified, namely, of Coulomb, Biot-Savart, and F. Neumann, a simple integration over the entire region S will give essentially equations (8) and (9) when one sets $\varepsilon=\mu=1$ ([3]).

$$\begin{aligned}\frac{\partial\frac{\bar E}{r}}{\partial x}&=E\frac{\partial\frac{1}{r}}{\partial x}-Ar\frac{\partial E(\xi,\eta,\zeta,t)}{\partial t}\frac{\partial\frac{1}{r}}{\partial x}+\tfrac{1}{2}A^2r^2\frac{\partial^2E(\xi,\eta,\zeta,t-A\theta r)}{\partial t^2}\frac{\partial\frac{1}{r}}{\partial x}\\&\quad-\frac{1}{r}A\frac{\partial E(\xi,\eta,\zeta,t)}{\partial t}\frac{\partial r}{\partial x}+\frac{1}{2}\frac{1}{r}A^2\frac{\partial^2\{r^2E(\xi,\eta,\zeta,t-A\theta r)\}}{\partial t^2\,\partial x}\\&=E\frac{\partial\frac{1}{r}}{\partial x}+\frac{1}{2}A^2\left[r^2\frac{\partial^2E(\xi,\eta,\zeta,t-A\theta r)}{\partial t^2}\frac{\partial\frac{1}{r}}{\partial x}+\frac{1}{r}\frac{\partial^2\{r^2E(\xi,\eta,\zeta,t-A\theta r)\}}{\partial t^2\,\partial x}\right],\end{aligned}$$

which proves the assertion.

([1]) The proof is identical to the one that was indicated above for the components of the electrostatic action.

([2]) One can prove this even more simply than in the other two cases by truncating the Taylor development at the first term.

([3]) If one neglects the corrective hypothesis (I) then the equations will be equations (19.*b*) and (3.*b*) of the cited paper by Helmholtz: "Ueber die Bewegungsgleichungen der Elektricität für ruhende Körper." It should be noted that

4. – The time has come to achieve the desired goal of showing that the general equations (8) and (9) will descend to Hertz's equations.

In the first place, when one eliminates the function **F** from (9), one will have:

$$A^2\left\{\frac{\partial^2\mathbf{V}}{\partial z\,\partial t}-\frac{\partial^2\mathbf{V}}{\partial y\,\partial t}\right\}=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},$$

$$A^2\left\{\frac{\partial^2\mathbf{W}}{\partial x\,\partial t}-\frac{\partial^2\mathbf{U}}{\partial z\,\partial t}\right\}=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x},$$

$$A^2\left\{\frac{\partial^2\mathbf{U}}{\partial y\,\partial t}-\frac{\partial^2\mathbf{V}}{\partial x\,\partial t}\right\}=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y},$$

and a comparison of that with the derivatives of equations (8) with respect to time will give:

$$(8')\qquad \left\{\begin{aligned} A\frac{\partial\mathbf{L}}{\partial t}&=\frac{\partial Z}{\partial y}-\frac{\partial Y}{\partial z},\\ A\frac{\partial\mathbf{M}}{\partial t}&=\frac{\partial X}{\partial z}-\frac{\partial Z}{\partial x},\\ A\frac{\partial\mathbf{N}}{\partial t}&=\frac{\partial Y}{\partial x}-\frac{\partial X}{\partial y}.\end{aligned}\right.$$

However, if one differentiates (9) with respect to time, multiplies by $A\ \varepsilon\,\mu$, and replaces $A^2\varepsilon\,\mu\frac{\partial^2\mathbf{U}}{\partial t^2}$, $A^2\varepsilon\,\mu\frac{\partial^2\mathbf{V}}{\partial t^2}$, $A^2\varepsilon\,\mu\frac{\partial^2\mathbf{W}}{\partial t^2}$ with their values that one infers from (6′) then one will get:

$$A\mu\frac{\partial\mathbf{X}}{\partial t}=-A\left\{\frac{\partial}{\partial x}\left(\frac{\partial\mathbf{F}}{\partial t}\right)+\underset{2}{\Delta}\mathbf{U}+4\pi\,\mathbf{u}\right\},$$

$$A\mu\frac{\partial\mathbf{Y}}{\partial t}=-A\left\{\frac{\partial}{\partial y}\left(\frac{\partial\mathbf{F}}{\partial t}\right)+\underset{2}{\Delta}\mathbf{V}+4\pi\,\mathbf{v}\right\},$$

$$A\mu\frac{\partial\mathbf{Z}}{\partial t}=-A\left\{\frac{\partial}{\partial z}\left(\frac{\partial\mathbf{F}}{\partial t}\right)+\underset{2}{\Delta}\mathbf{W}+4\pi\,\mathbf{w}\right\}.$$

On the hand, when one recalls (7), one will get from (8) that:

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in the last section Helmholtz also discusses the influence of the electric and magnetic polarization by appealing to Poisson's theory. I will hastily add that our equations (8), (9) reflect the modern way of looking at polarizations.

$$\mu\left(\frac{\partial M}{\partial z}-\frac{\partial N}{\partial y}\right)=-A\left\{\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial t}\right)+\Delta_2\mathbf{U}\right\},$$

$$\mu\left(\frac{\partial N}{\partial x}-\frac{\partial L}{\partial z}\right)=-A\left\{\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{F}}{\partial t}\right)+\Delta_2\mathbf{V}\right\},$$

$$\mu\left(\frac{\partial L}{\partial y}-\frac{\partial M}{\partial x}\right)=-A\left\{\frac{\partial}{\partial z}\left(\frac{\partial \mathbf{F}}{\partial t}\right)+\Delta_2\mathbf{W}\right\}.$$

Ultimately, when one subtracts corresponding sides of those relations from the preceding ones and recalls (4):

$$(9') \qquad \begin{cases} A\dfrac{\partial \mathbf{X}}{\partial t}=\dfrac{\partial M}{\partial z}-\dfrac{\partial N}{\partial y}-4\pi A u, \\ A\dfrac{\partial \mathbf{Y}}{\partial t}=\dfrac{\partial N}{\partial x}-\dfrac{\partial L}{\partial z}-4\pi A v, \\ A\dfrac{\partial \mathbf{Z}}{\partial t}=\dfrac{\partial L}{\partial y}-\dfrac{\partial M}{\partial x}-4\pi A w. \end{cases}$$

The systems (8′) and (9′) also coincide in form with the Hertzian equations (9.a) and (9.b) ([1]). All that remains is to examine the boundary conditions.

Hertz assumed ([2]) that the tangential components of the electric and magnetic forces will remain continuous along any separation surface σ, but one will have the following discontinuities in the normal components (if one is dealing with isotropic media):

$$(10) \qquad \mu'\left(\frac{\partial L'}{\partial t}\alpha'+\frac{\partial M'}{\partial t}\beta'+\frac{\partial N'}{\partial t}\gamma'\right)+\mu''\left(\frac{\partial L''}{\partial t}\alpha''+\frac{\partial M''}{\partial t}\beta''+\frac{\partial N''}{\partial t}\gamma''\right)=0,$$

$$(11) \qquad \varepsilon'\left(\frac{\partial X'}{\partial t}\alpha'+\frac{\partial Y'}{\partial t}\beta'+\frac{\partial Z'}{\partial t}\gamma'\right)+\varepsilon''\left(\frac{\partial X''}{\partial t}\alpha''+\frac{\partial Y''}{\partial t}\beta''+\frac{\partial Z''}{\partial t}\gamma''\right)$$

$$=-4\pi(u'\alpha'+v'\beta'+w'\gamma'+u''\alpha''+v''\beta''+w''\gamma'').$$

In our case, we are dealing with a single medium, or better yet, if we keep in mind that there is a surface σ, several media that are endowed with the same dielectric and magnetization constants. Having taken that into account, we would like to see whether the functions X, Y, Z, L, M, N that are defined by (8) and (9) effectively satisfy Hertz's conditions.

([1]) *Ges. Werke*, Bd. II, pp. 225, and also volume 28 of this journal, pp. 207.
([2]) *Ibidem*, **§ 8**.

In the first place, it will result from **§ 2** that when one crosses a surface σ, the derivatives of the functions **U**, **V**, and **W** will remain continuous, so the same property will be true of the components of the magnetic force, and consequently (10) will be verified, and due to the fact that $\mu' = \mu'' = \mu$, it will express the continuity of the derivatives with respect to time of the normal components of the magnetic force.

If we move on to the components of the electric force then we will observe that from (9), the discontinuity can be of only electrostatic origin and that since (always in the spirit of **§ 2**) the tangential derivatives of the function **F** will stay continuous when one crosses the surfaces σ, the same thing will be true for the tangential components. As for the normal components, we will establish formula (11) in the following way:

If we differentiate (9) with respect to time and sum, after having multiplied by $\varepsilon\,\alpha$, $\varepsilon\,\beta$, $\varepsilon\,\gamma$, resp., in the usual way, then we will see that:

$$\varepsilon\left(\frac{\partial X}{\partial t}\alpha+\frac{\partial Y}{\partial t}\beta+\frac{\partial Z}{\partial t}\gamma\right)=-\,\varepsilon\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{F}}{\partial x}\alpha+\frac{\partial \mathbf{F}}{\partial y}\beta+\frac{\partial \mathbf{F}}{\partial z}\gamma\right)-A^2\varepsilon\left(\frac{\partial^2 \mathbf{U}}{\partial t^2}\alpha+\frac{\partial^2 \mathbf{V}}{\partial t^2}\beta+\frac{\partial^2 \mathbf{W}}{\partial t^2}\gamma\right).$$

Observe that the derivatives with respect to time of **U**, **V**, **W** are continuous functions (and even when we cross σ), so we will immediately get:

$$\varepsilon'\left(\frac{\partial X'}{\partial t}\alpha'+\frac{\partial Y'}{\partial t}\beta'+\frac{\partial Z'}{\partial t}\gamma'\right)+\varepsilon''\left(\frac{\partial X''}{\partial t}\alpha''+\frac{\partial Y''}{\partial t}\beta''+\frac{\partial Z''}{\partial t}\gamma''\right)=-\,\varepsilon\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{F}'}{\partial p}+\frac{\partial \mathbf{F}''}{\partial p''}\right),$$

and by virtue of (5.a), (2′), and (4′):

$$\varepsilon\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{F}'}{\partial p}+\frac{\partial \mathbf{F}''}{\partial p''}\right)=-\,4\pi\,\varepsilon\frac{\partial \mathbf{e}}{\partial t}=\frac{4\pi}{\mu}(\mathbf{u}'\alpha'+\mathbf{v}'\beta'+\mathbf{w}'\gamma'+\mathbf{u}''\alpha''+\mathbf{v}''\beta''+\mathbf{w}''\gamma'')$$
$$=\frac{4\pi}{\mu}(u'\alpha'+v'\beta'+w'\gamma'+u''\alpha''+v''\beta''+w''\gamma''),$$

so we will get (11), which was to be proved.