"ds^2 einsteiniani in campi newtoniani. I: Generalità e prima approssimazione," Rend. reale Accad. Lincei, classe di scienze fische, matematiche e naturali (5) **26** (2nd sem. 1917), 307-317.

Einsteinian ds^2 in Newtonian fields. I. Generalities and first approximation.

By T. Levi-Civita

Translated by D. H. Delphenich

The question that is the topic of the present note and some others that will follow is posed physically thus:

A region in space is the seat of a force field (in ordinary mechanics, it will certainly be Newtonian) that is due to the action of masses external to the field. Suppose that the mass is not displacing and that no other perturbing circumstance intervenes, so (while Einstein's new mechanics are valid) a static regime will be established that differs little from the one that the classical tradition answers to, so the conditions of equilibrium will be inferred from Einstein's gravitational equations. Our scope is precisely that of discussing the main consequences of those equations in the simple case that was just specified.

Since the quantitative deviation from the usual scheme is only observable in the more refined experiments, we are compelled intuitively to presume that any ordinary Newtonian potential is associated with a solution of the aforementioned differential system, since the degree of arbitrariness in its general integral will be that of harmonic functions. That will be exhibited, one might say, almost automatically, when one limits oneself to treating the differential equations in the first approximation. The Newtonian potential $-c^2\,\gamma$ (where c is the well-known universal constant) then keeps its ordinary significance in regard to statics (where one requires the work done to be the positional energy, with a change of sign, of a hypothetical material point that moves in the field), and the metric of the ambient space submits to only a formal alteration (with respect to the Euclidian metric that exists in the absence of the field) of modulus $1-\gamma$ that is very close to unity and has the line element:

$$dl = (1-\gamma)\, dl_0 ,$$

with dl_0 Euclidian.

In this note, we shall first take this occasion to recall some preliminaries (nos. **1** and **2**) for a mechanical observation of a general character. It is that in Einstein statics, the customary elementary notions of force function (for a conservative field) and positional energy (of a material point that moves in the field) still persist, but they are generally distinct. It is only in the first approximation that one is the opposite of the other, at least up to an inessential additive constant, as in ordinary mechanics.

I shall then (no. **3**) write the fundamental equations and occupy myself exclusively with their approximate integration (nos. **4-8**), with the result that was indicated already.

I refer to a subsequent note for the rigorous study of the differential system. In my next communication, I will obtain the integrability conditions, as illustrated by their geometric aspect.

1. – REVIEW OF THE MOTION OF A MATERIAL POINT IN A STATIC FIELD.

Let S be an arbitrary region in physical space. Let:

$$dl^2 = \sum_{i,k=1}^{3} a_{ik}\, dx_i\, dx_k \tag{1}$$

be the expression for the square of the line element, and let V be the speed of light at a generic point P of S.

The hypotheses that the fundamental quaternary form in Einstein's theory is free from cross terms in dt or has the type:

$$ds^2 = V^2\, dt^2 - dl^2, \tag{2}$$

and that the coefficients V^2, a_{ik} are functions of position P (i.e., of the coordinates x_1, x_2, x_3) that are *independent of t*, translate mathematically into limiting oneself to phenomena of a static character.

The motion of a material point (as usual, suppose that one can ignore its action on the field) is governed by the variational equation:

$$\delta \int ds = 0 .$$

For brevity, set:

$$x_i = \frac{dx_i}{dt} \qquad (i = 1, 2, 3),$$

$$v^2 = \frac{dl^2}{dt^2} = \sum_{i,k=1}^{3} a_{ik}\, dx_i\, dx_k , \qquad L = c\left|\sqrt{V^2 - v^2}\right| ,$$

in which c is a constant that arbitrary *a priori*, and one infers the equivalent Lagrange equations ([1]):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \qquad (i = 1, 2, 3) . \tag{3}$$

([1]) Cf., the note "Statica einsteiniana," in these Rendiconti **26** (1st sem. 1917), pp. 465.

Since L does not contain t explicitly, it will admit the well-known integral:

$$L - \sum_{i=1}^{3} \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i = E \qquad (E \text{ constant}), \tag{4}$$

which expresses the principle of conservation of energy. Indeed, the function of position and velocity of the moving point that appears on the left-hand side and that keeps its value during the motion can be interpreted as the energy (per unit mass) of that moving body: It is enough to attribute the value c to the constant (which is called canonical), which is the speed of light in the absence of any perturbing action, and to take the elementary case ($V = c$ and dl Euclidian) to be normal ([1]).

Observation. – In my cited note on Einsteinian statics, I adopted the definition $\left| \sqrt{V^2 - v^2} \right|$ for L (without the inessential factor c), so the dimension of L will that of a velocity. The same thing will happen in the left-hand side of the corresponding integral, which proves to be only proportional to the unitary energy of the moving body. With the present L, the left-hand side of (4) will really represent the aforementioned energy.

2. – MECHANICAL SIGNIFICANCE OF THE FUNCTION $-\frac{1}{2}V^2$.

If one annuls the velocity of the moving body – i.e., any of the $\dot{x}_i$ (the case of incipient motion starting from rest) – at a given instant then one will get from (3), in particular, that:

$$\sum_{k=1}^{3} a_{ik} \ddot{x}_k = -\frac{1}{2} \frac{\partial V^2}{\partial x_i} \qquad (i = 1, 2, 3), \tag{5}$$

which defines the $\ddot{x}_i$ (of course, the two dots above signify the second derivative with respect to t) as functions of position. The right-hand side:

$$X_i = -\frac{1}{2} \frac{\partial V^2}{\partial x_i} \tag{6}$$

(as derivatives of the same function $-\frac{1}{2}V^2$) obviously constitute a covariant system (under arbitrary transformations of the spatial coordinates). The reciprocal contravariant system:

([1]) Cf., A. Palatini, "La spostamento del perielio di Mercurio e la deviazione dei raggi luminosi secondo la theoria di Einstein," Nuovo Cim. (6) **14** (1917), page 40.

$$X^{(i)} = \sum_{k=1}^{3} a^{(ik)} X_k ,$$

in which the $a^{(ik)}$ denote the coefficients of the reciprocal quadric to dl^2, as usual.

The solution to (5) implies precisely that:

$$\ddot{x}_i = X^{(i)},$$

which exhibits the contravariant character of the incipient accelerations: I want to say, the $\ddot{x}_i$ that are associated with a material point in a static field when one assumes that all of the $\dot{x}_i$ are equal to zero.

It is known ([1]) that one can associate a single vector **F** to the two simple reciprocal systems X_i, $X^{(i)} = \ddot{x}_i$ (in the tangent Euclidian space that identifies any manifold with a first-order neighborhood of one of its generic points).

That vector **F** *obviously implies the static measure of* (unitary) *force of the field* (i.e., the incipient acceleration of a free material point, or if one prefers, the acceleration that it must gain in order to keep the point at rest).

Consider a point P' the is near to the point P and has the coordinates $x_i + dx_i$, along with the (invariant) trinomial:

$$\sum_{i=1}^{3} X_i \, dx_i = -\tfrac{1}{2} dV^2 .$$

Let dl denote the line element PP', so $-\dfrac{1}{2}\dfrac{dV^2}{dl}$ will present itself as the derivatives (at P) of the function $-\tfrac{1}{2}V^2$ along an arc (of any line) that extends from P to P'. On the other hand, the ratios dx_i / dt are the parameters that are associated with the direction PP', and the orthogonal projection (with the appropriate sign) of the vector **F** along that direction is expressed by the invariant:

$$\sum_{i=1}^{3} X_i \frac{dx_i}{dl} = -\frac{1}{2}\frac{dV^2}{dl} .$$

That is because when one defines the elementary work done by **F** relative to the displacement PP', as in ordinary Euclidian space, to be the product of the displacement with the orthogonal projection of the force, the identity:

$$-\frac{1}{2}\frac{dV^2}{dl} dl = -\tfrac{1}{2} dV^2$$

([1]) Ricci and Levi-Civita, "Méthodes de calcul différentiel absolu, etc.," Math. Ann. **14** (1900), page 137.

will show that $-\frac{1}{2}V^2$ *constitutes the potential function for the force that is exerted in the field under static conditions.*

It is worth noting that whereas in ordinary mechanics that potential function, with the opposite sign, can also be interpreted as an energy of position that is associated with a moving body is not generally true in Einstein's theory. Indeed, from (4), when the velocity is annulled, one will have the intrinsic (i.e., constant) and positional part of the energy of the moving body completely expressed by $c\ V$, which does not coincide with $-\frac{1}{2}V^2$, even up to an additive constant (which would be inessential with respect to the positional function $-\frac{1}{2}V^2$). The difference between the two expressions $c\ V$ and $\frac{1}{2}V^2$ is constant only in the first approximation – i.e., when the difference between V and the value of c is sufficiently small – because if one were to take $V = c\ (1 + \gamma)$ then it would be legitimate to regard γ as a first-order quantity (i.e., a pure number). One would then have:

$$c\ V = c^2\ (1 + \gamma), \qquad \tfrac{1}{2}V^2 = \tfrac{1}{2}c^2 (1 + 2\gamma),$$

which differ by $\frac{1}{2}c^2$.

3. – VACUUM FIELDS. INDEFINITE EQUATIONS.

Suppose that the region S in space to which we refer our considerations is completely vacuous, so it has an energy density (and therefore a matter density) that is everywhere zero. In addition, suppose that the specific forces are everywhere zero inside of S. Under those conditions, all of the elements of the energetic tensor (forces, density, and energy flux) will obviously be annulled.

The Einsteinian ds^2 (and with it, the spatial dl^2) will be rigorously Euclidian as long as the aforementioned tensor is zero in *all* of space ([1]). We now propose to indicate, more generally, the limitations that are derived from simply annulling it *locally* (i.e., in a finite region S of space).

The indefinite equations that were just discussed are obviously those of Einsteinian statics, with its right-hand side equal to zero (to be sure, the energetic tensor is zero); i.e., they are the seven following ones ([2]):

(I) $$\mathcal{M} = 0,$$

(II) $$\alpha_{ik} + \frac{V_{ik}}{V} = 0 \qquad (i, k = 1, 2, 3)\ .$$

([1]) The physical aspects of the statement are intuitive, and one can say that they reflect the starting point for Einstein's speculative construction. From the mathematical standpoint, however, it can be proved rigorously on the basis of the equations that now include the entire theory. I do not know if that proof exists but permit me to point out the observation that the theorem in question reduces to the constancy of any regular harmonic function in all of space in the limiting case of ordinary mechanics.

([2]) Page 464 of the note cited above "Statica einsteiniana."

One considers the spatial dl^2 (rather than Einstein's quadri-dimensional ds^2) to be fundamental in them: The α_{ik} are the Ricci symbols (which advantageously replace those of Riemann for the ternary form):

$$\mathcal{M} = \sum_{i,k=1}^{3} a^{(ik)}\, \alpha_{ik}$$

is the mean curvature of space. Since it is known that:

$$\Delta_2 V = \sum_{i,k=1}^{3} a^{(ik)}\, V_{ik}\ ,$$

(I) will be equivalent to the harmonicity condition:

(I′) $$\Delta_2 V = 0,$$

by virtue of (II).

4. – FIRST APPROXIMATION. RESULTING LINEARITY OF THE DIFFERENTIAL SYSTEM.

If one supposes that the expression (I) for ds^2 is very close to the Euclidian type, when referred to Cartesian spatial coordinates:

$$c^2\, dt^2 - \sum_{i=1}^{3} dx_i^2\ ,$$

then one should also set:

(5) $$V = c\,(1 + \gamma)\ ,$$

(6) $$a_{ik} = \varepsilon_{ik} + e_{ik} \qquad (i,\, k = 1,\, 2,\, 3),$$

with the usual meaning for the symbols ε_{ik} (viz., 0 for $i \neq k$ and 1 for $i = k$). One then has:

(6′) $$dl^2 = \sum_{i,k=1}^{3} a_{ik}\, dx_i\, dx_k \ = \ dl_0^2 + \sum_{i,k=1}^{3} e_{ik}\, dx_i\, dx_k\ ,$$

in which dl_0^2 is the elementary line element of Euclidian space when referred to Cartesian coordinates.

The e_{ik} are pure numbers, along with γ, and the assumed qualitative behavior of ds^2 is equivalent, in the first approximation, to treating all of those seven quantities as infinitesimal.

The Riemann symbols $a_{ij,\,hk}$ relative to the form (6′) [and therefore, to the coefficients (6)] reduce to ([1]):

$$a_{ij,\,hk} = \tfrac{1}{2}\left(\frac{\partial^2 e_{jh}}{\partial x_i\,\partial x_k} + \frac{\partial^2 e_{ik}}{\partial x_j\,\partial x_h} - \frac{\partial^2 e_{ih}}{\partial x_j\,\partial x_k} - \frac{\partial^2 e_{jk}}{\partial x_i\,\partial x_h}\right) \qquad (i, j, h, k = 1, 2, 3)\,,$$

accordingly.

Since the $a^{(jh)}$ keep their Euclidian values ε_{jk}, at least up to first-order terms, it will follow that:

$$G_{ik} = \sum_{j,h=1}^{3} a^{(jh)}\, a_{ij,kh} = \sum_{j=1}^{3} a_{ij,\,jk} = \tfrac{1}{2}\left(\frac{\partial^2 e_{jj}}{\partial x_i\,\partial x_k} + \frac{\partial^2 e_{ik}}{\partial x_j\,\partial x_j} - \frac{\partial^2 e_{ij}}{\partial x_j\,\partial x_k} - \frac{\partial^2 e_{jk}}{\partial x_i\,\partial x_j}\right).$$

In general, the G_{ik} are coupled with Ricci's α_{ik} by the relation [(14) in the note "Statica einsteiniana," that was cited twice above]:

$$\alpha_{ik} = G_{ik} + \mathcal{M}\, a_{ik}\,.$$

With the expression for G_{ik} that was just obtained, given that $\mathcal{M} = 0$ in the present case according to (I), it will result that:

(7) $$\alpha_{ik} = \tfrac{1}{2}\left(\frac{\partial^2 e_{jj}}{\partial x_i\,\partial x_k} + \frac{\partial^2 e_{ik}}{\partial x_j\,\partial x_j} - \frac{\partial^2 e_{ij}}{\partial x_j\,\partial x_k} - \frac{\partial^2 e_{jk}}{\partial x_i\,\partial x_j}\right).$$

The determination of the unknowns γ, e_{ik} must be deduced from (I), (II), or if one prefers, from their equivalents (I′), (II).

It is important to recall that, at least up to terms of order higher than one (in the γ, e_{ik}), the covariant derivatives V_{ik} of $V = c\,(1 + \gamma)$ with respect to the form (6′) will coincide with the corresponding ordinary derivatives of $c\,\gamma$. With that:

$$\Delta_2 V = c\,\Delta_2^0\,\gamma\,,$$

in which Δ_2^0 represents the ordinary second-order differential parameter with respect to dl_0^2, i.e., the operator $\sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$. (I′), (II) can then be written:

(8) $$\Delta_2^0\,\gamma = 0,$$

(9) $$\alpha_{ik} = -\frac{\partial^2 \gamma}{\partial x_i\,\partial x_k} \qquad (i, k = 1, 2, 3)\,,$$

([1]) Bianchi, *Lezioni di geometria differentiale*, v. I, Pisa, Spoerri, 1902, page 73, or Ricci and Levi-Civita, *loc. cit.*, page 142.

in which the α_{ik} are the linear combinations (second-order differentials) (7) of the unknowns e_{ik} .

5. – ISOLATIING THE STATIC PROBLEM.

Recall from no. **2** that $-\frac{1}{2}V^2$, or what amounts to the same thing, $-\frac{1}{2}(V^2 - c^2) = -c^2\gamma$, constitutes the (static) *potential of the field*, while (8) shows that (as in the classical theory of Newtonian attraction outside of the gravitating body) *it is subject to the restriction that it must be a harmonic function, as well as, of course, being regular in the field.* That field [when given the form (8)] will behave, in regard to the law of variation of the potential, as if it were Euclidian and referred to Cartesian coordinates.

(9) (if we accept it for the moment) implies no ultimate constraint on the function γ. Therefore, *conversely, any γ that is harmonic and regular in S will give rise to a possible field.* That is in perfect agreement with the ordinary picture, according to which the gradient of any function that is harmonic and regular in a field can be realized (in an infinitude of ways) by means of the attraction of a mass that is external to the field.

6. – THE GEOMETRIC PROBLEM. PARTICULAR SOLUTION.

Now let us turn to (9). We note that in the first place, we can take a particular solution to be:

$$e_{ik} = -2\,\varepsilon_{ik}\,\gamma \qquad (i, k = 1, 2, 3)\,. \tag{10}$$

The proof is immediate, from the expression (7) for the α_{ik} and the harmonicity of γ.

Since (3) then constitutes a linear, but not homogeneous, system in the e, the general integral is certainly composed (by way of a sum) of the more general solutions to the equations when they have a vanishing right-hand side:

$$\alpha_{ik} = 0\,.$$

Since γ no longer intervenes, that proves the statement in the preceding no. regarding the isolation of the static problem.

The general integral of the system $\alpha_{ik} = 0$ is well-known. However, as will be clarified below, it has not importance for us, since it corresponds to only a change of the reference coordinates.

7. – INESSENTIAL CHARACTER OF THE ARBITRARINESS FORMALLY REFLECTED IN THE GENERAL INTEGRAL.

Annulling the α_{ik} (rigorously, not just in our order of approximation) expresses the necessary and sufficient condition for the corresponding (ternary) dl^2 to be Euclidian, or reducible to the form

$\sum_{i=1}^{3} dy_i^2$ with a suitable choice of parameters. That is because when one is given the reference coordinates x_1, x_2, x_3 generically, the most general manner of defining a dl^2 that is Euclidian with respect to those x coordinates is obviously to introduce an arbitrary transformation between the y and x :

$$y_i = y_i\,(x_1, x_2, x_3) \qquad\qquad (i = 1, 2, 3),$$

and to take the coefficients a_{ik} to be the ones that result from expressing $\sum_{i=1}^{3} dy_i^2$ in terms of the differentials of the x. Assuming, as is always permissible, that the functions $y_i\,(x_1, x_2, x_3)$ have the form:

$$x_i + \xi_i\,(x_1, x_2, x_3)\ ,$$

(after actually introducing the corresponding differentials into the trinomial $\sum_{i=1}^{3} dy_i^2$) one will have:

$$dl^2 = \sum_{i,k=1}^{3} a_{ik}\, dx_i\, dx_k\ ,$$

with

$$a_{ik} = \varepsilon_{ik} + \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i}\right) + \sum_{j=1}^{3} \frac{\partial \xi_j}{\partial x_i}\frac{\partial \xi_j}{\partial x_k}\ .$$

In order to reflect the limitation of the difference $a_{ik} - \varepsilon_{ik}$ to first order, with the ultimate specification that the difference between the Cartesian coordinates of the y and the (curvilinear) ones of x has the same order ([1]), it is sufficient (and necessary) that one can treat the functions ξ as infinitesimal (along with their derivatives). It will result that:

$$e_{ik} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i}\right), \tag{11}$$

which constitute the formal expression for the general integral of the homogeneous system $\alpha_{ik} = 0$ [the α_{ik} depend upon the e linearly according to (7)].

([1]) In the absence of that specification, one demands only that the (numerical) quantity:

$$e_{ik} = \frac{1}{2}\left(\frac{\partial \xi_i}{\partial x_k} + \frac{\partial \xi_k}{\partial x_i}\right) + \sum_{j=1}^{3} \frac{\partial \xi_j}{\partial x_i}\frac{\partial \xi_j}{\partial x_k}$$

proves to be infinitesimal, and can therefore be obtained, as was shown by prof. Almansi ["L'ordinaria teoria dell'elasticità e la teoria delle deformazione finite," in these Rendiconti, **26** (2nd sem. 1917), 3-8.], even without the ξ being infinitesimal.

However, it is not that formal expression that is important to retain, but rather the fact that the terms (11) [which get added to (10) in order to have the general integral of the system (9) with a non-zero right-hand side] can always be made equal to zero by means of an opportune change of coordinates: That is, when one replaces the x with the combinations:

$$y_i = x_i + \xi_i\,(x_1, x_2, x_3)\,, \tag{12}$$

by which the expression for dl^2 will reduce to $\sum_{i=1}^{3} dy_i^2$, by construction, all of the differences $a_{ik} - \varepsilon_{ik}$ will be annulled.

If one chooses the variables to be the y then one must naturally subject the particular solution (10) to the transformation (12), as well. However, when one considers (10), (12) (when one regards the ξ as infinitesimal to the same order as γ) will reduce to the material substitution of the y for the x. The expression (11) for the particular solution, when also referred to y, will then remain unaltered, which is all that is interesting.

In addition, one should note that the elementary form (viz., the sum of second derivatives) of the parameter $\Delta_0^2\,\gamma$ will likewise remain unaltered.

8. – CANONICAL FORM FOR THE ds^2.

One sees from the preceding that *inside a vacuum field, the static potential* (in the first Newtonian approximation) $-c^2\,\gamma$ *is associated with a metric alteration of the ambient space. When the reference coordinates* (the y in the preceding no., which are denoted by x there) *are chosen opportunely,* γ *can be kept as a solution of the Laplace equation:*

$$\frac{\partial^2\gamma}{\partial x_1^2} + \frac{\partial^2\gamma}{\partial x_2^2} + \frac{\partial^2\gamma}{\partial x_3^2} = 0\,.$$

The coefficients a_{ik} in the square of the line element (to the same order of approximation) take on the expressions $a_{ik} - 2\varepsilon_{ik}\,\gamma$, which makes:

$$dl^2 = (1 - 2\gamma)\;(dx_1^2 + dx_2^2 + dx_3^2)\,.$$

As one sees, space does not generally remain Euclidian, not even in the first approximation, but is merely (in that approximation) representable as conformal to a Euclidian space.

By definition, Einstein's global ds^2 *that is due to an assigned Newtonian force field with the potential* $-c^2\,\gamma$ *is given by:*

$$ds^2 = c^2\,(1 + 2\gamma)\,dt^2 - (1 - 2\gamma)\;dl_0^2\,. \tag{13}$$

(dl_0 is the line element of a Euclidian space.)

The expression (13) for ds^2 for the field of a single mass (so $-c^2\gamma$ is inversely proportional to the distance from the mass) was already indicated explicitly by de Sitter ([1]). The case of as many masses as one desires (which corresponds substantially to an arbitrary harmonic function γ) was then implicit in a noteworthy formula in the second approximation that was established by J. Droste ([2]). With all of that, it would seem opportune to me to propose the systematic study of vacuum spaces and to also pose the results in the first approximation, and all the more so as they relate to an illuminated medium, and to obtain them in a more spontaneous manner without having to develop the calculations substantially.

([1]) Cf., Einstein, "Näherungweise Integration der Feldgleichungen der Gravitation," Sitz. Kgl. Preuss. Akad. Wiss. (1916), page 692.

([2]) "The field of *n* moving centres in Einstein's theory of gravitation," Kgl. Akad. van Wet. te Amsterdam, Proceedings **19** (1916), 447-455.