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THE RELATIVISTIC THEORY OF

GRAVITATION AND ELECTROMAGNETISM

GENERAL RELATIVITY AND UNITARY THEORIES

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PREFACE

To my mind, after more than thirty years have passed, the editing of this preface amounts to a nostalgic memory of the time in which the theories of relativity commenced their grand career. For me, it also evokes all of the work that I had the pleasure of seeing the birth of, thanks to A. Lichnerowicz and his students, with a richness in the results that fulfills my hopes and sometimes surpasses them. Without a doubt, that set is destined to be enlarged further, but it is time to make its present magnitude known. This is what A. Lichnerowicz did in his course at the Collège de France during the two years 1952-1953 and 1953-1954. It is the content of these two courses that constitute the present book.

Einstein's train of thought, starting with special relativity, may now be outlined in its grandiose simplicity, and hopefully remain faithful to him.

The invariant quadratic form that defines the interval between two neighboring events introduces a new geometry that is not properly Euclidian since it involves real elements of null length, a geometry of a spacetime that is admirably adapted to the propagation of electromagnetic waves.

However, this spacetime, although an excellent setting for physics, still has no physical reality. It does not define the enveloping field that unites the effects and causes of the energy distribution and its motion. The search for and study of the properties of this grand field for gravitation in a hyper-field that unites gravitation and electromagnetism is the passionate quest that spawned this book by A. Lichnerowicz. The construction is based on the genius of Riemann, who created general spaces in which the metric is given by a quadratic differential form, and Einstein, who extended the invariant interval of special relativity to that of a Riemannian space and divined what sort of fundamental equations must restrict the generality of these Riemannian spaces in order to define a true gravitational field.

The physical fecundity of the theory, which integrates and profoundly generalizes the preceding theories, also has an esthetic superiority due to the fact that, as we shall see, it seems to be the crowning glory of the theory of the propagation of waves.

Moreover, one sees in it a sort of mathematical fecundity, as well as new problems that are naturally provoked by the idea that the spacetime has physical reality, such as the following one: Any regular spacetime without matter seems to be necessarily locally Euclidian.

The deeper mathematical treatment of these hyper-geometries necessarily employs not only the tensorial calculus but also the style of intrinsic differential geometry that affords the necessary clarity and rigor that these difficult questions demand.

The first book is dedicated to general relativity, and Book II is dedicated to unitary theories. For the latter, A. Lichnerowicz has made a study of two examples that are both the most recent and the most suggestive from a physical standpoint: the Jordan-Thiry theory and the Einstein-Schrödinger theory.

One knows that in order to begin a theory of gravitation Einstein proposed a Riemannian geometry for spacetime. The fundamental quadratic form of four differentials:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta,$$

must be of the hyperbolic normal type, in the sense of J. Hadamard, i.e.:

$$ds^2 = (\omega^0)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2,$$

in which the ω are linearly independent Pfaff forms. One finds a real hypercone defined at each point that verifies the equation $ds^2 = 0$, just as in special relativity.

The gravitational tensor $g_{\alpha\beta}$ is subject to a system of equations that generalizes the Laplace equations for the exterior case and the Poisson equations for the interior case, which are the basis for potential theory. There are ten equations:

$$S_{\alpha\beta} = \chi\, T_{\alpha\beta}.$$

The two symmetric tensors $S_{\alpha\beta}$ and $T_{\alpha\beta}$ have a geometric nature and represent a generalization of the density that is described by the energy distribution and its motion, respectively. One verifies how the intuition of Einstein provided a solution to this dilemma that is quite unique by choosing $S_{\alpha\beta}$ to be the Ricci tensor.

The ten second-order equations for the derivatives of $g_{\alpha\beta}$, which are linear in these derivatives and verify the four conditions, are called conservation conditions of the tensor $S_{\alpha\beta}$. These conservation conditions are thus imposed on $T_{\alpha\beta}$. In order to use the fundamental equations, it is necessary to:

a) State in a precise fashion just what the spacetime manifold is, along with its various coordinate systems, and the hypotheses on the derivatives of various orders, and then state in a precise fashion what the $g_{\alpha\beta}$ are from that same standpoint.

b) Study the tensor T of matter and the electromagnetic field.

c) Study the Cauchy problem – or relativistic determinism – in order to exhibit the exceptional multiplicities.

d) Recall the relativistic viewpoint of the physical theories that are attached necessary properties of the tensor T, i.e., hydrodynamics.

e) Study the new problems, which are global in nature, that are posed by the physical reality that is attributed to spacetime.

It is in that order that the author proceeds, which is undoubtedly natural and is the reason that this work is profoundly satisfying. The work that is necessary for the reader to do will be, I think, facilitated if he has the elements of tensorial calculus (A. Colin, 1950) well within his reach, especially in the beginning.

I have already spoken of the importance of *a*). I would like to point out the elegant and profound treatment of *b*), and I must further insist upon *c*).

From the conservation conditions, the system of Einstein equations is a system in involution, i.e., the Cauchy data are assumed to verify four necessary conditions on the hypersurface upon which they are defined, and the six remaining equations determine a solution that verifies four supplementary conditions everywhere at that point in time.

The solution is locally unique in the case for which the data are analytic if these data are defined upon a hypersurface that is not tangent to the null-length hypercone at any point.

Of course, it is reasonable to assume that analyticity must play no role, and there is a place for the beautiful and difficult work that has been done in that regard. Definitive results have been obtained by Mme. Fourés under just the hypothesis of differentiability.

We have already seen the role of these exceptional manifolds – viz., the characteristic hypersurfaces that are everywhere tangent to the characteristic cone – that the theory of wave propagation has introduced, with the classical bicharacteristics playing the role of rays. The characteristic conoid that is generated by the bicharacteristics at a point of spacetime is nothing but the elementary wave that originates at this point and goes away.

These characteristics are moreover null-length geodesics. One therefore sees gravitational wave surfaces and rays appear in the exterior Cauchy problem that are the same for the electromagnetic field.

It is, moreover, remarkable that when one studies the solution of Mme. Fourés one sees the work on the propagation of waves in striking harmony with the previous work and with greater generality than before.

The Cauchy problem in the interior case enlarges the scope of these conclusions by introducing new characteristic manifolds that are generated by the flow lines, and finally (in the context of the perfect fluid), a third type of manifold that is the relativistic extension of the wave fronts of classical hydrodynamics. The speed of propagation, which is determined by starting with the equation of state, generalizes the classical value of Hugoniot.

The Cauchy problem for the case in which there also exists an electromagnetic field produces analogous results, but the set of givens must verify six conditions. Uniqueness may be further established by the method of Mme. Fourés for a non-characteristic hypersurface that is oriented in space. The identity of the propagation laws for the two fields is therefore established in full rigor.

It is appropriate to point out a result of great elegance that relates to a singular case for which the flow lines are null-length geodesics and led the author to say that the pure singular electromagnetic field describes a photon fluid.

When one makes a prolongation of the exterior to the interior, or conversely, the Cauchy problem leads us to specifically exhibit the matching conditions for the potentials that were actually included in *a*), but which merit a special study in light of the fact that they influence the interrelation between the field and material masses.

These conditions, which are familiar from potential theory, are the ones that demand the continuity of the potentials and their first derivatives, but due to the arbitrariness in the coordinates, that statement implies that special precautions must be taken here. In fact, there must exist a system that they verify.

A. Lichnerowicz shows that one has the right to use Gaussian coordinates that are deduced from the geodesics that are normal to a hypersurface.

The prolongation of the interior to the exterior then implies some conditions. For a perfect fluid, the separating hypersurface must be generated by the flow lines, and the pressure must be annulled on them.

The conditions are locally sufficient.

If one would like to go from the exterior to the interior then one must have that the separating hypersurface is generated by (timelike) geodesics of the exterior field, but then that hypersurface would be characteristic and the solution would not be guaranteed. The latter point should not be surprising because one concedes that there is either no distribution that produces the field or that several must be doing so.

If one applies the condition that relates to geodesics to a massive tube of very small cross-section then that will obviously entail that a material point must describe a geodesic. In other words, the geodesic principle is a consequence of Einstein's theory and some matching conditions that were often cited by Schwarzschild.

In *c*), we recover these problems of massive tubes in the field. We now go on to *d*). The relativistic hydrodynamics that is presented by A. Lichnerowicz is a context in which the relativistic generalizations that one makes of flow lines, vortex lines, Bernoulli's theorem, viscous fluids, relativistic Navier equations, posses not only rare elegance, but also indicate a very profound agreement with physical reality. We point out only that the speed of propagation of the hydrodynamical wave fronts in bounded, incompressible liquids is defined to be the one for which this propagation takes place at the speed of light.

As we have seen, problems of type *c*) are inspired by the physical reality that one attributes to the spacetime. Their solutions are, one may say, some internal coherence tests of Einstein's theory. This path has shown to be quite productive. The study of everywhere regular, stationary spacetimes (which appear to be necessarily locally Euclidian), and the fact that the necessity of singularities foreshadows the existence of massive tubes, as in a universe with gravitation, seem to offer some well-posed problems, although one senses that they are not consequences (and especially not simple consequences) of general axioms. The properties of the Laplace equation on a Riemannian manifold, and the existence of vectors whose flux gives rise to extensions of Gauss's theorem constitute a powerful means by which to make this study productive.

In that context, we cite only one theorem (which was researched by Einstein and Pauli in 1941, following the results of Serini, Racine, and the thesis of A. Lichnerowicz) and proved by A. Lichnerowicz in 1943:

Any complete stationary exterior spacetime with asymptotically Euclidian behavior is necessarily locally Euclidian.

One may also cite a very obscure result that was obtained by these methods (pp. ???), and which permits one to reduce the Schwarzschild postulates for the construction of ds^2. We recover analogous coherence considerations in the global study of unitary fields.

We now go on to Book II on the unitary theories. A. Lichnerowicz has dedicated Book II with an introduction to which I have nothing to add, although I would likewise say that I have already derived not only the word "hyperfield" for a truly unified field, but also a clear idea of what such a field must be from him.

The first unitary theory that is studied is that of Jordan-Thiry. It is a penta-dimensional in which the greater generality in the trajectories of charged particles corresponds to the introduction of a supplementary dimension. In truth, these trajectories are the geodesics of a Finsler manifold that minimizes the sum of the square root of a quadratic form and a linear form. A. Lichnerowicz and Y. Thiry have solved a very general problem of analytical mechanics that synthesizes the true central idea of a number of results, some of which are known and some of which are entirely new.

The result here is that one may introduce a five-dimensional Riemannian manifold whose metric is of hyperbolic normal type, and whose geodesics project onto the ordinary spacetime along the set of trajectories of charged particles. One thus has a hyper-field of fifteen potentials whose generality must be restricted by equations that were inspired by Einstein's equations and based on the conservative Ricci tensor; one thus has fifteen equations. The expression for the energy-momentum tensor and the physical interpretation for the various terms pose difficult questions that are inevitable in any unitary theory. The field equations have some imposed mathematical properties, but the physical interpretation itself is not imposed. One must arrive at it through persistence and prudence by starting with earlier results. The solution that is finally proposed by A. Lichnerowicz differs from that of Jordan and Thiry, for example.

The results that are deduced from the study of the Cauchy problem in the form of matching conditions are the same ones, i.e., they just as simple and suggestive as in general relativity. The problems of type *c*) give rise to a complete generalization for the Jordan-Thiry theory; the same study for the Kaluza-Klein theory is a little less satisfying. The latter theory therefore possesses a little less internal coherence.

The Einstein-Schrödinger theory is of a completely different type. It is a theory with an affine connection on a four-dimensional manifold. The geometric elements are a tensor $g_{\alpha\beta}$ with no symmetry and an affine connection $\Gamma^{\alpha}_{\beta\gamma}$. The field equations are obtained from the extremum of an integral, in the context of some variations of the $g_{\alpha\beta}$ and the $\Gamma^{\alpha}_{\beta\gamma}$. This mode of formation insures the existence of conservation identities.

The fundamental problem of the local integration of the equations presents the same general character that it does in general relativity.

The Cauchy data must verify a system of conditions, and the integration of the remaining system will provide some solutions that verify the conditions everywhere.

It seems that the methods of Mme. Fourés may be extended to that case.

What I would especially like to point is that the exceptional multiplicities introduce a characteristic cone that is not obviously null.

It is this tensor, to which correspond the null-length geodesics of a certain Riemannian metric, that must define the gravitational part of the unitary field.

One sees that as far as the very difficult problem of the physical interpretation of these fields is concerned, the Cauchy problem at least gives the answer to a fundamental question.

In concluding this overextended preface, in which I have given only a sketch of the relativistic revolution, one may see what a precious book A. Lichnerowicz has given us. By his profound novelty, his elegance, and his solidity, he cannot fail to attract new researchers to this work, along with the numerous students of the author. They are assured of finding problems in the scientific realm thus opened up that will unite modern geometry and physics with an uncontestable beauty.

G. DARMOIS.

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I. – SPACETIME AND THE EINSTEIN EQUATIONS

0. – Functions of class piecewise-C^2. – Here, $\mathbb{R}^n$ will denote the numerical space of n real variables ($x^1, x^2, \ldots, x^n$). A function φ with numerical values (i.e., a function of n real variables) is said to be *of class* C^ν (with ν a positive integer) if it admits continuous partial derivatives up to order ν. In all of what follows, we will set:

$$\frac{\partial \varphi}{\partial x^\alpha} = \partial_\alpha \varphi \qquad \frac{\partial^2 \varphi}{\partial x^\alpha \partial x^\beta} = \partial_{\alpha\beta} \varphi, \qquad \text{etc.}$$

We may denote the class of infinitely differentiable functions by C^∞, and the analytical functions of real variables by C^ω.

Having said this, we shall use the notion of a *function of class piecewise-C^2*. Consider a continuous function φ with numerical values that is defined in a neighborhood of the surface S ($x^n = 0$) in $\mathbb{R}^n$, and suppose that:

1) This function if of class C^2 in each of the neighborhoods $x^n > 0$ and $x^n < 0$.
2) The first and second derivatives of φ tend uniformly converge to definite limits when x^n goes to 0 for positive values (+ 0), and to limits that might be different when x^n goes to zero through negative values (– 0).

We may represent the discontinuity in a function that crosses S by placing the function insider brackets, []. From the classical argument that is due to Hadamard ([1]), the discontinuities of φ upon crossing S satisfy the relations:

$$[\partial_i \varphi] = 0 \qquad [\partial_n \varphi] = A \qquad \text{(here, } i, j, \text{ etc., } = 1, 2, \ldots, n-1),$$

and:

$$[\partial_{ij} \varphi] = 0 \qquad [\partial_{in} \varphi] = A \qquad [\partial_{in} \varphi] = B,$$

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(1) J. HADAMARD. *Leçons sur la propagation des ondes,* pp. 83-87. Hermann (1903).

in which A and B are functions that are defined on S.

After a coordinate change of class C^2 in $\mathbb{R}^n$, we will obtain a function φ and a "surface of discontinuity" S ($f = 0$), for which:

1) φ is a function of class C^2 in each of the neighborhoods $f > 0$ and $f < 0$.
2) The first and second derivatives of φ uniformly converge to definite limits when $f = + 0$ or $f = - 0$.

We say that a function φ that is defined in a domain of $\mathbb{R}^n$ is a *function of class picewise-C^2* if it is of class C^2 except in a neighborhood of a finite number of surfaces of discontinuities at which the preceding conditions are satisfied. These conditions are identical to the ones that appear in the theory of hydrodynamical waves.

1. – The spacetime manifold V_4. **–** In the relativistic theory of gravitation and electromagnetism, the primitive element is defined by a four-dimensional *spacetime manifold* V_4 that is endowed with the structure of a differentiable manifold, which we shall now specify.

It might be of value to briefly recall how one defines the notion of a *differentiable manifold.* An n-dimensional manifold V_n is a connected topological space in which each point possesses a neighborhood that is homeomorphic to the Euclidian n-ball. How does one define the structure of a differentiable manifold on such a manifold?

We call a topological representation of an open domain D of V_n in $\mathbb{R}^n$ a *local chart* of V_n, or, equivalently, a *local coordinate system; D* is called the *domain* of the chart. Such a chart associates a point ξ of $\mathbb{R}^n$ to each point x of D, and, as a result, n real numbers (x^α), which are the coordinates of ξ. The n real numbers (x^α) are called the *coordinates* of x in the chart considered. A differentiable manifold of class C^ν (ν is either a positive number, ∞ or ω) is an n-dimensional manifold V_n to which one may associate a set of charts – or *atlas* – A that satisfies the following two conditions:

A_1. – *The domains of the charts of A completely cover the manifold V_n ; in other words, the union of the domains of the charts of A is identical with V_n .*

A_2. – *If D_1 and D_2 are two domains of charts of A, and if $x \in D_1 \cap D_2$ then the coordinates of the point x in one of the charts are functions of class C^ν with non-null Jacobian of the coordinates of x in the other chart.*

Axioms A_1 and A_2 are the axioms of an atlas of class C^ν. We may define an equivalence relation between atlases of class C^ν on the same manifold: two atlases A and B are *equivalent* if their union is again an atlas of class C^ν. In order for this to be true, it is necessary and sufficient that this union should also satisfy axiom A_2 . It is clear that the relation so defined is indeed an equivalence relation: An atlas is equivalent to itself. The order in which one takes the atlases makes no difference. Finally, two atlases that are equivalent to a third one are equivalent to themselves. The *differential structure* of a manifold of class C^ν is defined by the equivalence relation between atlases; in other words, two equivalent atlases define the same differential structure of class C^{ν} on the manifold V_n .

A local coordinate system on V_n is called *compatible* with the differential structure on the manifold, or *admissible*, if its union with an atlas that defines V_n as a differentiable manifold is also an atlas of the same class.

We may then take the "*spacetime*" *manifold* V_4 to be a four-dimensional manifold that is endowed with a *differential structure of class* C^2 *and satisfies the following hypothesis,* which complements A_2, in addition: *We demand that the second derivatives of the coordinate functions be functions of class piecewise-*C^2 *in the intersection of the domains of two admissible coordinate systems.*

Therefore, in the intersection of the domains of two admissible coordinate systems, the coordinates of a point x in one of the systems will be four-times-differentiable functions of the coordinates of x in the other system with non-null Jacobians; the first and second derivatives must be continuous, but the third and fourth ones may present discontinuities of Hadamard type. These are the hypotheses on V_4 that will always be made in the sequel.

We may suppose that a Riemannian metric ds^2 of *everywhere hyperbolic normal* type is defined on the manifold V_4 . The local expression for this metric in an admissible coordinate system is:

(1-1) $$ds^2 = g_{\alpha\beta}(x^\lambda)\, dx^\alpha dx^\beta \qquad (\alpha, \beta, \text{ and every Greek index} = 0, 1, 2, 3).$$

The coefficients $g_{\alpha\beta}(x^\lambda)$ in (1-1) are called the *gravitational potentials* for the coordinate system envisioned. We essentially assume that the "*fundamental gravitation tensor*" *that is defined by the* $g_{\alpha\beta}$ *has components of class* C^1 *on* V_4 *and that the* $\partial_\gamma g_{\alpha\beta}$ *are functions of class piecewise-*C^2.

This axiom, which relates to the differentiability of the metric, is obviously compatible with the differential structure that is imposed on V_4. The hypothesis that was made on the type of the metric amounts to saying that at each point x of V_4 the ds^2 may be put into the form:

(1-2) $$ds^2 = (\omega_0)^2 - (\omega_1)^2 - (\omega_2)^2 - (\omega_3)^2,$$

in which the ω^α are a locally linearly-independent system of Pfaff forms. Therefore, the equation $ds^2 = 0$ defines a real cone C_x of directions that are tangent to V_4 at each point x of V_4, a cone that will be called the *elementary cone* at x.

In what follows, V_4 will designate the "spacetime" manifold endowed with a Riemannian metric that satisfies the specified hypotheses.

2. – Orientations in space and time. – *a*) A direction $d\mathbf{x}$ that is tangent to V_4 at x is oriented in time or space according to whether it is interior or exterior to the elementary cone. If we assume that it is defined by the components (dx^α) then one has:

$g_{\alpha\beta}\, dx^\alpha dx^\beta > 0$: $d\mathbf{x}$ is oriented in time

$g_{\alpha\beta}\, dx^\alpha dx^\beta < 0$: $d\mathbf{x}$ is oriented in space.

An elementary k-plane (k = 2 or 3) that is tangent to V_4 at x is oriented in space if all of its directions are oriented in space. It is oriented in time if it admits directions that are oriented in time.

Consider an elementary 3-plane Π_x that is defined by the equation $v_\alpha dx^\alpha = 0$. The direction that is defined by the vector v_α is the normal to the plane, i.e., the conjugate direction with respect to the elementary cone. As a result, Π_x is oriented in space or time according to whether v^α is oriented in time or space, respectively, i.e., according to the sign of $g_{\alpha\beta} v^\alpha v^\beta = g^{\alpha\beta} v_\alpha v_\beta$:

$$g^{\alpha\beta} v_\alpha v_\beta > 0: \qquad d\mathbf{x} \text{ is oriented in space}$$
$$g^{\alpha\beta} v_\alpha v_\beta < 0: \qquad d\mathbf{x} \text{ is oriented in time.}$$

Let η be unitary vector ($\eta^2 = + 1$) – hence, it is oriented in time – and let Π_x be the elementary 3-plane $\eta_\alpha dx^\alpha = 0$ that is orthogonal to it; hence, it is oriented in space. We say that η and Π_x define a *time* and an *associated space* at x. If $\mathbf{X}$ is an arbitrary vector then it will be the sum of a vector that is collinear with η and a vector that is orthogonal to η, hence, it is in Π_x. One immediately has:

$$\mathbf{X} = (\mathbf{X}\eta)\eta + \mathbf{X}^*, \qquad \text{with } \mathbf{X}^* \in \Pi_x. \tag{2-1}$$

The first vector of (2-1) is called the *temporal* component of $\mathbf{X}$, and $\mathbf{X}^*$ is called its *spatial* component relative to the time an associated space that is defined by η; the quantity $-(\mathbf{X}^*)^2$ is called the *spatial magnitude of the vector* $\mathbf{X}$ *relative to the direction of time* η.

A curve Γ in V_4 is oriented in time if the tangents to its various points are oriented in time. Along Γ, one will then have:

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta > 0.$$

For $ds^2 < 0$, Γ will be oriented in space.

A three-dimensional hypersurface S is oriented in space if its plane elements that are tangent to the various points are oriented in space. If S is locally defined by $f(x^\alpha) = 0$ then the plane element will admit an equation that has the coefficient form:

$$v_\alpha = \partial_\alpha f .$$

As a result $\Delta_1 f = g^{\alpha\beta} \partial_\alpha f \partial_\beta f > 0$ on S; for $\Delta_1 f < 0$, S is oriented in time.

b) In all of follows in this book, we agree that any Latin index i, j, etc., can take only the values 1, 2, 3.

The metric ds^2 that was introduced defines a *Minkowski space structure* on the vector space T_x that is tangent to V_4 at x, i.e., a structure of the spacetime of special relativity.

To any decomposition of the metric into a sum of squares:

$$ds^2 = (\omega_0)^2 - \sum_i (\omega^i)^2$$

there corresponds a basis (ω_0, ω_i) for the linear forms at x and a frame $(x, \mathbf{e}_0, \mathbf{e}_i)$ in T_x with:

$$d\mathbf{x} = \omega^\alpha \, \mathbf{e}_\alpha .$$

From the relation:

$$ds^2 = (d\mathbf{x})^2 = (\omega^\alpha \mathbf{e}_\alpha)(\omega^\beta \mathbf{e}_\beta),$$

one deduces:

(2-2) $\qquad \mathbf{e}_\alpha \mathbf{e}_\beta = 0 \qquad$ for $\alpha \neq \beta, \qquad (\mathbf{e}_0)^2 = +1, \qquad (\mathbf{e}_i)^2 = -1.$

A frame that satisfies the relations (2-2) will be called an *orthonormal frame* at x; the vector $\mathbf{e}_0$ and the 3-plane spanned by the $\mathbf{e}_i$ define *time* and an *associated space.*

Just as in Riemannian geometry, properly speaking, the geometric interpretation of the magnitudes that are defined at x is deduced by considering the Euclidian vector space that is tangent to x; similarly, the interpretation of a tensor that is defined at x is deduced by considering the Minkowski space that is tangent to x. *When referred to an orthonormal frame, this space must be identified with the spacetime of special relativity when referred to a Galilean frame. This immediately gives the physical interpretation in terms of the time and associated space for this frame.*

c) We shall have to evaluate the spatial magnitude of a vector $\mathbf{X}$ relative to a time direction that is defined by a unitary vector η on several occasions in the sequel. We adopt an *arbitrary* frame $(x, \mathbf{e}_l)$ at x such that the vector $\mathbf{e}_0$ is collinear with η. In this frame the fundamental quadratic form is given by:

$$(\mathbf{X})^2 = g_{\alpha\beta} X^\alpha X^b.$$

Since the vector $\mathbf{e}_0$ is oriented in time, one will have the equality:

(2-3) $$(\mathbf{e}_0)^2 = g_{00} > 0.$$

Since the vector η is unitary and collinear with $\mathbf{e}_0$, one will have:

(2-4) $$\eta = \frac{1}{\sqrt{g_{00}}} \mathbf{e}_0 .$$

Having said this, we seek to evaluate $(\mathbf{X}^*)^2$. Upon taking the square of (2-1) and taking (2-4) into account, it will follow that:

$$(\mathbf{X}^*)^2 = (X)^2 - (\mathbf{X} \cdot \eta)^2 = (\mathbf{X})^2 - \frac{1}{g_{00}} (\mathbf{X} \cdot \mathbf{e}_0)^2.$$

One deduces from this that:

(2-5) $$(\mathbf{X}^*)^2 = (\mathbf{X})^2 - \frac{(X^0)^2}{g_{00}}.$$

Namely, upon making this explicit with the aid of the contravariant components of **X**:

$$(\mathbf{X}^*)^2 = g_{\alpha\beta}\, X^\alpha X^\beta - \frac{(X^0)^2}{g_{00}} = g_{\alpha\beta}\, X^\alpha X^\beta - \frac{1}{g_{00}}\,(g_{0\alpha} X^\alpha)^2.$$

Now, if one decomposes the fundamental quadratic form with respect to the direction variable X^0 then it will follow that:

$$g_{\alpha\beta}\, X^\alpha X^\beta = \frac{1}{g_{00}}\,(g_{0\alpha}\, X^\alpha)^2 + \left(g_{ij} - \frac{g_{0i}\, g_{0j}}{g_{00}} \right) X^i X^j,$$

in which the last term on the right-hand side is a negative-definite quadratic form in the three variables X^i since g_{00} is positive. It results from this that:

(2-6) $$(\mathbf{X}^*)^2 = \left(g_{ij} - \frac{g_{0i}\, g_{0j}}{g_{00}} \right) X^i X^j,$$

which gives the desired spatial magnitude.

Consider the negative-definite quadratic form that we just created, and which admits the coefficients:

$$g^*_{ij} = g_{ij} - \frac{g_{0i}\, g_{0j}}{g_{00}}.$$

The quadratic form with the coefficients g^{*ij} that satisfy the relations:

(2-7) $$g^*_{ij}\, g^{*ij} = \delta_i^k,$$

is associated with this form.

Now, one may immediately verify that:

$$g^{*jk} = g^{jk}.$$

Because:

$$g^*_{ij}\, g^{ik} = g_{ij}\, g^{ik} - \frac{g_{0j}}{g_{00}}\, g_{0i}\, g^{ik} = g_{i\lambda}\, g^{\lambda k} - g_{i0}\, g^{0k} - g_{i0}\, g^{0k}\,(\, g_{0\lambda}\, g^{\lambda k} - g_{00}\, g^{0k}),$$

and taking into account that $g_{\lambda\alpha} g^{\lambda\beta} = \delta_\alpha^\beta$, we will get the relations:

$$g^*_{ij}\, g^{jk} = \delta_i^k.$$

In due course, we have established the following theorem:

THEOREM. – *For any frame* $(x, \mathbf{e}_\lambda)$ *for which* g_{00} *is positive the two associated quadratic forms have the coefficients:*

$$g^*_{ij} = g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}} \qquad\qquad g^{*ij} = g^{ij}$$

are negative definite. If $\mathbf{X}$ *is an arbitrary vector with components* (X^α) *then the spatial magnitude of* $\mathbf{X}$ *relative to the time direction that is defined by* $\mathbf{e}_0$ *is given by***:**

$$-(\mathbf{X})^2 = -g^*_{ij}\; X^i X^j.$$

Similarly, one establishes that the inequality $g_{00} > 0$ entails the negative-definite character of the two associated quadratic forms that admit the coefficients:

$$g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}, \qquad\qquad g_{ij},$$

respectively.

3. – The system of Einstein equations. – Now that we have discussed the geometric framework, the first step towards the relativistic theory of gravitation consists of choosing a tensorial system of partial differential equations that limit the generality of the fundamental tensor of gravitation $g_{\alpha\beta}$ and relate this tensor to the energetic distribution of spacetime V_4 . Einstein was led to these equations by two types of considerations: On the one hand, these equations must generalize the Laplace-Poisson equation that locally determines the Newtonian potential, and, on the other hand, they must give rise to what we call "conservation conditions."

We rewrite these equations in the form:

(3-1) $$S_{\alpha\beta} = \chi\, T_{\alpha\beta},$$

in which $S_{\alpha\beta}$ and $T_{\alpha\beta}$ are symmetric tensors, and c denotes a constant factor. The tensor $T_{\alpha\beta}$, which is of purely mechanical significance, must describe, at best, the character of the energy distribution (the interior case) at the point of V_4 considered, or it must be identically null in the regions of V_4 that are not swept out by energy (the exterior case). This tensor, which is called the energy-momentum tensor, or, more briefly, the energy tensor, thus generalizes the right-hand side of the Laplace-Poisson equation. We return to the choice of this tensor in the second part of this chapter.

The tensor $S_{\alpha\beta}$, which is of purely geometric significance (i.e., it depends upon only the Riemannian structure of the manifold V_4) is restricted by the following two conditions:

1) *The components $S_{\alpha\beta}$ depend upon only the potentials and their derivatives of the first two orders; they are linear with respect to the derivatives of the second order.*

2) *The tensor $S_{\alpha\beta}$ is conservative, i.e., it satisfies the relations:*

$$\nabla_\alpha S^\alpha_\beta = 0 \tag{3-2}$$

identically, in which ∇_α denotes the operation of covariant derivation.

One easily understands why it is necessary that there must exist a system of four identical equations between the $S_{\alpha\beta}$. By a convenient choice of local coordinate system, one may restrict four of the potentials to take the given values locally; ultimately, we will verify examples of these. If there does not exist a system of four relations between $S_{\alpha\beta}$ then the six remaining potentials must verify, for example, ten independent equations in the interior case. We verify that the four chosen conditions (3-2) are heavily laden with physical content.

The determination of the tensor $S_{\alpha\beta}$, which was found intuitively by Einstein and Weyl, was accomplished by a regular method by Élie Cartan ([1]). It proves that the only tensor that satisfies the preceding conditions is given by the formula:

$$S_{\alpha\beta} = h\,[R_{\alpha\beta} - \tfrac{1}{2}(R + k)\,g_{\alpha\beta}],$$

in which $R_{\alpha\beta}$ is the Ricci tensor of the Riemannian manifold and h, k are two arbitrary constants. Upon suppressing the extraneous factor h the corresponding partial differential equations may be written:

$$S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}(R + k)\,g_{\alpha\beta} = \chi\,T_{\alpha\beta}. \tag{3-3}$$

The constant k, which is called the *cosmological constant,* is only weakly involved with the study of cosmological problems, and plays no role in astronomical problems, properly speaking. Unless stated to the contrary, we will henceforth adopt the system:

$$S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}R\,g_{\alpha\beta} = \chi\,T_{\alpha\beta}$$

for the system of Einstein equations, and it is this system that we shall ponder. However, many of the results that we shall obtain may be trivially extended to the equations with non-null k.

([1]) É. CARTAN, J. Math. Pures et appliquées, **1** (1922), 141-203.

II. – THE ENERGY-MOMENTUM TENSOR

4. – The principal directions of spacetime. – One refers the *principal directions* of a Riemannian manifold at a point x when one wishes to describe the conjugate directions that are common to the elementary cone:

$$g(\mathbf{X}) = g_{\alpha\beta}X^{\alpha}X^{\beta} = 0 \tag{4-1}$$

at that point and the cone of the Ricci equation:

$$R_{\alpha\beta}X^{\alpha}X^{\beta} = 0,$$

in which $\mathbf{X}$ and (X^{α}) denote a tangent vector at x. It results from (3-4) or (3-3) that the principal directions for an Einstein spacetime are common conjugates to the cone (4-1) and the cone of the equation:

$$T(\mathbf{X}) = T_{\alpha\beta}X^{\alpha}X^{\beta} = 0 \tag{4-2}$$

that is associated with the energy tensor. The determination of the principal directions of spacetime is therefore found to reduce to the determination of the proper vectors of the matrix $(T_{\alpha\beta})$ relative to the matrix $(g_{\alpha\beta})$. The proper values of this matrix relative $(g_{\alpha\beta})$ are the roots of the equation in s:

$$| T_{\alpha\beta} - s\, g_{\alpha\beta} | = 0,$$

in which the || symbol denotes the determinant that is associated with the matrix that they enclose.

5. Orthonormal frames. The normal energy-momentum tensor. – In the rest of this chapter, we shall place ourselves at a given point x of V_4 and devote ourselves to the purely algebraic considerations that relate to the vectors and tensors that are associated with the tangent space V_4 at x. Recall that a frame $(x, \mathbf{e}_{\lambda})$ with origin x is called *orthonormal* if it composed of pair-wise orthogonal vectors, one of which $\mathbf{e}_0$ is oriented in time, while the others $\mathbf{e}_i$ are oriented in space and normed by $(\mathbf{e}_i)^2 = -1$.

We take the energy tensor to be a symmetric tensor $T_{\alpha\beta}$ that satisfies the following hypothesis: *$T_{\alpha\beta}$ admits a real, proper vector that is oriented in time.* Such a tensor will be called *normal*. If the associated quadratic form $T(\mathbf{X})$ is defined then one will see from the classical results on quadratic forms that such a tensor is always normal.

A proper vector $\mathbf{V}$ satisfies the relation:

$$(T_{\alpha\beta} - s\, S_{\alpha\beta})\, V^{\beta} = 0. \tag{5-1}$$

Let $\mathbf{V}_0$ be the real proper vector of $T_{\alpha\beta}$, which is oriented in time and assumed to be normal; we normalize $\mathbf{V}_0$ by the relation $[\mathbf{V}_0]^2 = 1$. The associated proper value is necessarily real. We adopt an orthonormal frame at the point x considered at an instant when $\mathbf{V}_0$ is the frame member that is oriented in time. In this frame, $g_{0i} = T_{0i} = 0$. The three missing proper values will then be the roots of:

$$| T_{ij} - s\, g_{ij} | = 0,$$

in which the g_{ij} determine a definite quadratic form. As a result, these proper values s_i will be real, and one may find three proper vectors $\mathbf{V}^i$ that are oriented in space such that the frame that is defined by the $\mathbf{V}^\lambda$ is orthonormal. This frame will be called a *principal frame* of the spacetime at the point x. Naturally, in the case for which one of the proper values is multiple there will exist an infinitude of principal frames at x.

Refer spacetime at a given instant and the point x to a principal frame and denote the components of a vector $\mathbf{X}$ by $X^{\alpha'}$. One has:

(5-2) $$g(\mathbf{X}) = (X^{0'})^2 - \sum_i (X^{i'})^2 .$$

By virtue of equations (5-1), one will then have:

(5-3) $$T(\mathbf{X}) = s_0 (X^{0'})^2 - \sum_i s_i (X^{i'})^2 .$$

One easily deduces an expression for the normal energy tensor from this relationby starting with its proper values and corresponding proper vectors. We perform a change of frame that consists of returning to an arbitrary frame, so if X_α and X_β denote the two systems of covariant components of a vector $\mathbf{X}$ then:

$$X_\alpha = A_\alpha^{\beta'} X_{\beta'} ,$$

in which $(A_{\beta'}^{\alpha})$ denotes the matrix of the frame change. Upon applying this formula to the covariant components of the vector $V^{(\lambda)}$, it will follow that:

(5-4) $$V_\alpha^{(0)} = A_\alpha^{0'} \qquad V_\alpha^{(i)} = -A_\alpha^{i'} .$$

As a result, from the classical tensorial formulae:

$$g_{\alpha\beta} = A_\alpha^{\lambda'} A_\beta^{\mu'} g_{\lambda'\mu'} \qquad T_{\alpha\beta} = A_\alpha^{\lambda'} A_\beta^{\mu'} T_{\lambda'\mu'} ,$$

and, starting with (5-2), (5-3), and (5-4), one will deduce the expressions:

(5-5) $$g_{\alpha\beta} = V_\alpha^{(0)} V_\beta^{(0)} - \sum_i V_\alpha^{(i)} V_\beta^{(i)}$$

and

(5-6) $$T_{\alpha\beta} = s_0 V_\alpha^{(0)} V_\beta^{(0)} - \sum_i s_i V_\alpha^{(i)} V_\beta^{(i)} ,$$

which are valid in an arbitrary frame.

6. Definite schema. The arbitrary fluid schema. - If we are given a field of symmetric tensors $T_{\alpha\beta}$ in an arbitrary domain of spacetime then we will say that this field defines an *energy schema* in this domain. A schema will be called *normal* if $T_{\alpha\beta}$ is normal. In particular, it will be definite if the quadratic form $T(\mathbf{X})$ that is associated with the tensor is everywhere positive-definite.

We shall study a definite schema. That hypothesis entails the existence of a real, proper vector that is oriented in time. We first interpret this vector as the *unitary velocity vector* of the schema considered and set:

$$u_\alpha = V_\alpha^{(0)}$$

to abbreviate the notations. This vector is defined up to a sign; we shall return to this indeterminacy later. From (5-3), the s_i are negative and s_0 is positive. We set:

$$\rho = s_0 \qquad\qquad p_{(i)} = -\, s_i$$

in which ρ and $p_{(i)}$ are therefore positive scalars. With these notations, formulas (5-5) and (5-6) take the form:

(6-1) $$g_{\alpha\beta} = u_\alpha u_\beta - \sum_i V_\alpha^{(i)} V_\beta^{(i)}$$

and

(6-2) $$T_{\alpha\beta} = \rho\, u_\alpha u_\beta + \sum_i p_{(i)} V_\alpha^{(i)} V_\beta^{(i)} \,.$$

Therefore, the energy tensor may be put into the form (6-2) in the case of the most general definite schema. We translate this result by saying that u_α is the *unitary velocity* of the fluid, and ρ is its *proper density;* the $p_{(i)}$ are called *proper partial pressures.* The three principal directions that are oriented in space and the corresponding pressures $p_{(i)}$ define the quadric of pressures at the point of the fluid considered.

7. – The "pure matter" schema and the "perfect fluid" schema. – A schema is called a *pure matter schema* if the corresponding energy tensor admits the proper values:

$$s_0 = \rho > 0 \qquad\qquad\qquad s_i = 0.$$

This schema is not definite, and the corresponding spacetime admits only one privileged principal direction that is oriented in time. One then has:

(7-1) $$T_{\alpha\beta} = \rho\, u_\alpha u_\beta .$$

A schema is called a "*perfect fluid schema*" if it is definite, and the proper partial pressures are equal. One will then have:

$$p_{(1)} = p_{(2)} = p_{(3)} = p,$$

and formula (6-2) will become:

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta + p \sum_i V_\alpha^{(i)} V_\beta^{(i)} .$$

Upon expressing the last term of the right-hand side with the aid of (6-1), one will obtain:

(7-2) $$T_{\alpha\beta} = (r + p)\; u_\alpha u_\beta - p\; g_{\alpha\beta}.$$

As in the pure matter schema, the corresponding spacetime will admit only one privileged principal direction that is oriented in time.

If we temporarily abandon the exclusively geometric viewpoint that we have adopted up till now and take the physical viewpoint then we might say that the energy-momentum tensor is capable of providing a representation for a definite energy distribution that is as complete as possible. This tensor therefore contains various terms that correspond to the various types of energy in the distribution. The most experimentally important one – when it exists – is always the one that corresponds to the ponderable energy. In particular, it prevails over the energy that originates in the pressures. If one takes these two types of energy into account exclusively (…missing line…) the corresponding material schema. However, the proper density ρ must be considered to be large compared to the numbers $p_{(i)}$ in this schema. One will note that the study of continuous media in special relativity, when carried out with the usual physical units, shows that it is the quotients (pressure / c^2) that are homogeneous in r and must appear in equations (6-2) and (7-2) instead of $p_{(i)}$ and p. The units that are used here are the ones for which $c = 1$.

The "pure matter" schema corresponds to the case for which one totally neglects the energy that originates in the pressures.

8. – The electromagnetic field. – Up till now, we have not assumed that an electromagnetic field was involved. The classical study of electromagnetism in special relativity leads us to represent such a field by an antisymmetric tensor $F_{\alpha\beta}$ – the electromagnetic field tensor ([1]). In a definite orthonormal frame the F_{i0} give the components of the electric field relative to the frame, and the F_{ij} give the components of the corresponding magnetic field. More precisely, the space vectors of the electric and magnetic field are the vectors that admit the space components:

(8-1) $$\begin{array}{ll} X = F_{10} = - F^{10} & L = F_{23} = F^{23} \\ Y = F_{20} = - F^{20} & M = F_{31} = F^{31} \\ Z = F_{30} = - F^{30} & N = F_{12} = F^{12}. \end{array}$$

One will note that since $g_{00} = 1$, $g_{ii} = -1$, in such a frame, the raising or lowering of a space index i is accomplished with a change of sign, whereas the same is not true for the temporal index 0.

([1]) See A. LICHNEROWICZ. *Éléments de calcul tensoriel,* A. Colin, Paris (1950), pp. 191-202.

One then finds that an antisymmetric tensor of order 2 is defined on the manifold V_4, which we will refer to admissible coordinates on V_4. When this is true, we shall always assume that the *components $F_{\alpha\beta}$ of the electromagnetic field in admissible coordinates are functions of class piecewise-C^2*; this hypothesis is analogous to the one that was made for the gravitational field. It may be convenient to introduce the exterior quadratic form:

$$F = \tfrac{1}{2} F_{\alpha\beta}\, dx^\alpha \wedge dx^\beta,$$

which will be called the *electromagnetic field form*. Later, we shall verify that it gives the differential equations that are presumed to be satisfied by the electromagnetic field.

We shall now introduce the indicators and tensor that are in common usage. In an n-dimensional Riemannian space V_n we define the *Kronecker tensor indicator* to be the tensor $\varepsilon^{\alpha_1\cdots\alpha_p}_{\beta_1\cdots\beta_p}$ ($p \leq n$), which is defined in the following manner: Its components equal +1 when the sequence of upper indices is an even permutation of the sequence of lower indices, which are assumed to be distinct, -1 in the case of an odd permutation, and 0 in all of the other cases. It is easy to verify that one obtains a tensor from this that is p-times contravariant and p-times covariant, and that *this tensor has a null covariant derivative* for any affine connection, and in particular, the Riemannian connection on V_4. In order to abbreviate the notation, we set:

$$\varepsilon^{\alpha_1\cdots\alpha_n}_{1\cdots n} = \varepsilon^{\alpha_1\cdots\alpha_n}, \qquad \varepsilon^{1\cdots n}_{\alpha_1\cdots\alpha_n} = \varepsilon_{\alpha_1\cdots\alpha_n}.$$

If V_4 is oriented then from this indicator and the existence of the metric, one immediately deduces the antisymmetric volume element tensor η, which also has a null covariant derivative for this metric. In the case of the spacetime manifold V_4 this tensor admits components that are defined by the formulas:

$$\eta_{\alpha\beta\gamma\delta} = \sqrt{|\,g\,|}\,\varepsilon_{\alpha\beta\gamma\delta}, \qquad \eta^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{|\,g\,|}}\,\varepsilon^{\alpha\beta\gamma\delta},$$

in which the presence of the – sign takes into account the negative character of g.

Having said this, the volume element tensor in an n-dimensional Riemannian space V_n permits us to deduce an antisymmetric tensor of order p ($n - p$) from any antisymmetric tensor p-tensor, which will be called its *adjoint*. We may deduce the adjoint tensor $(*F)_{\alpha\beta}$, which is also an antisymmetric tensor of order 2, from the tensor $F_{\alpha\beta}$. It is defined by the formula:

(8-2) $$(*F)_{\gamma\delta} = \tfrac{1}{2}\,\eta_{\alpha\beta\gamma\delta}\,F^{\alpha\beta}, \qquad (*F)^{\gamma\delta} = \tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,F_{\alpha\beta}.$$

One easily verifies that $(**F)_{\alpha\beta} = -F_{\alpha\beta}$. Conversely, one therefore has:

(8-3) $$F^{\alpha\beta} = -\tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}(*F)_{\gamma\delta}, \qquad F_{\alpha\beta} = -\tfrac{1}{2}\,\eta_{\alpha\beta\gamma\delta}(*F)^{\alpha\beta}.$$

It may be convenient to associate the exterior quadratic form:

$$*F = \tfrac{1}{2}(*F)_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta},$$

to the tensor $(*F)_{\alpha\beta}$. The interpretation of the components of $(*F)_{ab}$ in an orthonormal frame is immediately deduced from that of the components of the electromagnetic field. One thus has:

$$\begin{array}{ll} X = (*F)_{23} = (*F)^{23} & L = -(*F)_{10} = (*F)^{10} \\ Y = (*F)_{31} = (*F)^{31} & M = -(*F)_{20} = (*F)^{20} \\ Z = (*F)_{12} = (*F)^{12} & N = -(*F)_{30} = (*F)^{30}. \end{array} \tag{8-4}$$

One may attach two interesting scalars to the electromagnetic field: the scalar products of the form F with itself and with the form $*F$. We set:

$$\psi = \langle F, F\rangle = \tfrac{1}{2} F_{\alpha\beta}F^{\alpha\beta}, \qquad \Phi = \langle F, *F\rangle = \tfrac{1}{2} F_{\alpha\beta}(F)^{\alpha\beta}.$$

It is easy to obtain another representation for Φ. From the definition of the adjoint tensor, one has:

$$\Phi = \tfrac{1}{4}\,\eta^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} = \tfrac{1}{4}\,\tfrac{1}{4!}\,\eta^{\alpha\beta\gamma\delta}\varepsilon^{\lambda\mu\nu\rho}_{\alpha\beta\gamma\delta}F_{\lambda\mu}F_{\nu\rho}.$$

Now, the form $F \wedge F$ has precisely the components:

$$\tfrac{1}{4}\,\varepsilon^{\lambda\mu\nu\rho}_{\alpha\beta\gamma\delta}F_{\lambda\mu}F_{\nu\rho}.$$

It results from this that Φ is nothing but the scalar adjoint of that expression: $\Phi = *(F \wedge F)$. Now, from a well-known result on left-symmetric determinants:

$$F \wedge F = \sqrt{|F_{\alpha\beta}|}\; dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

From this, one deduces that:

$$\sqrt{|g|}\,\Phi = -\sqrt{|\mathrm{F}_{\alpha\beta}|},$$

and one sees that the vanishing of F expresses the vanishing of the determinant of the matrix $(F_{\alpha\beta})$, i.e., that the form $F \wedge F$ has rank less than four.

Finally, with the aid of formulas (8-1) and (8-4), one will observe that in an orthonormal frame the scalars Ψ and Φ have the following values in terms of the components of the electric and magnetic fields:

$$\Psi = L^2 + N^2 + M^2 - X^2 - Y^2 - Z^2, \tag{8-5}$$

and

$$\Phi = 2(LX + MY + NZ). \tag{8-6}$$

9. – Schemas with electromagnetic fields. – In special relativity ([1]) one establishes that it is possible to associate an energy-momentum tensor with an electromagnetic field by the formula:

(9-1) $$\tau_{\alpha\beta} = g_{\alpha\beta}(F_{\lambda\mu} F^{\lambda\mu}) - F_{\alpha\sigma} F_{\beta}{}^{\rho}.$$

One will note that:

(9-2) $$\tau \equiv g^{\alpha\beta} \tau_{\alpha\beta} = 0.$$

One takes $T_{\alpha\beta} = \tau_{\alpha\beta}$ in the case for which the energy distribution envisioned corresponds to just the electromagnetic field, and this tensor will be said to represent a *pure electromagnetic field schema.*

In order to represent a distribution that simultaneously involves matter and an electromagnetic field, one may add an electromagnetic energy tensor of the type (9-1) to the previously introduced matter tensors. For example, one thus envisions a "*pure matter-electromagnetic field*" schema with:

(9-3) $$T_{\alpha\beta} = \rho\, u_\alpha u_\beta + \tau_{\alpha\beta}$$

or a "*perfect fluid-electromagnetic field,*" for which:

(9-4) $$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta - p\, g_{ab} + \tau_{\alpha\beta}.$$

One must realize that such a pure and simple addition to the energy-momentum tensors that correspond to these schemas will give only a first approximation for the more refined energy-momentum tensors.

10. – The study of the energy-momentum tensor for the electromagnetic field. – We now propose to study the energy-momentum tensor (9-1) for the electromagnetic field, and, in particular, to indicate certain reduced forms in terms of which the components of the tensor $\tau_{\alpha\beta}$ may be expressed for convenient orthonormal frames. As we have said before, all of the analysis that follows is purely algebraic and valid at a given point x of the manifold V_4 .

If we adopt an arbitrary orthonormal frame at x then, with the aid of (8-1) and (9-1), it is easy to see that the components of the tensor $\tau_{\alpha\beta}$ are given as functions of the components (X, Y, Z) and (L, M, N) of the electric and magnetic fields relative to that frame by the elements of the matrix:

(10-9) $$\begin{bmatrix} \tau_{00} & 0 & 0 & \tau_{13} \\ & \tau_{1} & 0 & 0 \\ & & \tau_{22} = -\tau_{11} & 0 \\ & & & \tau_{33} \end{bmatrix};$$

this is a simple frame for $\tau_{\alpha\beta}$. Indeed, in such a frame one has:

([1]) See A. LICHNEROWICZ. *Éléments de calcul tensoriel*, A. Colin, Paris (1950), pp. 202-204.

$$XZ + LN = 0 \qquad LZ - XN = 0 \quad \text{and} \quad -MZ + YN = 0 \qquad YZ + MN = 0,$$

which entails that either $Z = N = 0$ or $X = L = Y = M = 0$; however, in the case for which $\tau_{22} = -\tau_{11}$, it further entails that $Z = N = 0$. Moreover, $\tau_{12} = XY + LM$ is null; the frame is therefore simple.

Having said this, it is easy to study the proper values of $\tau_{\alpha\beta}$ with respect to $g_{\alpha\beta}$ in the reduced form (10-8); they are the roots of the equations in s:

$$\left(\frac{\eta^2 - \xi^2}{2} + s\right)\left(\frac{\xi^2 - \eta^2}{2} + s\right)\left[\left(\frac{\xi^2 + \eta^2}{2}\right)^2 - s^2 - \xi^2\eta^2\right] = 0$$

so that:

$$\left[s^2 - \left(\frac{\xi^2 - \eta^2}{2}\right)^2\right]^2 = 0.$$

We may state (1):

THEOREM. – *The energy-momentum tensor for an electromagnetic field admits four pair-wise equal and opposite proper values: k, k, – k, – k.*

We have taken $k = (\xi^2 - \eta^2) / 2$. It is easy to obtain an intrinsic expression for k^2 with the aid of Ψ and Φ. Indeed, one has:

$$4k^2 = [(L^2 - Y^2) - (M^2 - X^2)]^2 = [L^2 - Y^2 + M^2 - X^2]^2 - 4(L^2 - Y^2)(M^2 - X^2);$$

now:

$$L^2 + M^2 - X^2 - Y^2 = \Psi,$$

and, from the trivial identity:

$$(L^2 - Y^2)(M^2 - X^2) = (XY + LM)^2 - (LX + MY)^2,$$

one has:

$$-4(L^2 - Y^2)(M^2 - X^2) = \Phi^2.$$

From this, one deduces that:

(10-10) $$k^2 = Y^2 + F^2.$$

From the theorem just stated, one sees that in order to pursue the reduction of the matrix ($\tau_{\alpha\beta}$) it is convenient to distinguish between two cases:

a) THE $k \neq 0$ CASE. There will then be two distinct proper values k and $-k$, and in order to pursue the reduction, it will suffice to observe that there exists an orthonormal frame ($\mathbf{W}^{(\lambda)}$) that is composed of proper vectors, and for which one has:

$$\tau_{\alpha\beta} = k\,[W_\alpha^{(0)} W_\beta^{(0)} - W_\alpha^{(1)} W_\beta^{(1)} + W_\alpha^{(2)} W_\beta^{(2)} + W_\alpha^{(3)} W_\beta^{(3)}];$$

(1) See SYNGE. Univ. of Toronto Studies. Appl. Math. Ser. 1 (1935) and RUSE. Proc. London Math. Soc. **41** (1936), 302-322.

the vectors $\mathbf{W}^{(2)}$ and $\mathbf{W}^{(3)}$ are two arbitrarily-chosen normed orthogonal vectors that are chosen in a 2-plane that is oriented in space, and the vectors $\mathbf{W}^{(0)}$ and $\mathbf{W}^{(1)}$ are two normed orthogonal vectors that are restricted only by the requirement that one of them must be oriented in time and the other must be oriented in space in a 2-plane that is oriented in time and totally orthogonal to the preceding one. One will note that $(\mathbf{W}^{(\lambda)})$ is a simple frame for which, for example, $\eta = 0$.

b) THE CASE $k = 0$; i.e., $\Psi = \Phi = 0$. – In this case, for any orthonormal frame the electric and magnetic fields will be orthogonal and have the same length. Here, the form (10-8) for the matrix $(\tau_{\alpha\beta})$ will reduce to:

$$(10\text{-}11) \qquad (\tau_{\alpha\beta}) = \begin{bmatrix} \xi^2 & 0 & 0 & -\varepsilon\xi^2 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & \xi^2 \end{bmatrix} \qquad (\varepsilon = \pm 1)$$

since $\xi^2 = \eta^2$. Introduce the vector:

$$\boldsymbol{l} = \mathbf{e}_0 + \varepsilon\, \mathbf{e}_3 .$$

One immediately sees that this vector will have null length, and that one may translate (10-11) into the tensorial equation:

$$(10\text{-}11) \qquad \tau_{\alpha\beta} = \xi^2\, l_\alpha\, l_\beta .$$

Therefore, in this case there will exist a vector of null length and a scalar ξ^2 such that $\tau_{\alpha\beta}$ may be put into the form (10-12). One will note that the form (10-12) will be preserved if one multiplies the vector $\boldsymbol{l}$ by an arbitrary factor upon modifying the scalar factor accordingly. This case will be called *singular case* for the electromagnetic field.

Finally, we study a lemma that will be useful to us soon.

LEMMA. – *Given an arbitrary* $\mathbf{t} \neq 0$ *that is oriented in time, it is always possible to find a simple frame for the tensor* $\tau_{\alpha\beta}$ *such that in that frame* $t^1 = t^2 = 0$.

Indeed, in the *a*) case the frame$(\mathbf{W}^{(\lambda)})$ is a simple frame. If we refer $\boldsymbol{l}$ to it then we have:

$$\mathbf{t} = \sum_\lambda \mathbf{t}^\lambda \mathbf{W}^{(\lambda)} \qquad [\text{with } (t^0)^2 - \sum_i (t^i)^2 > 0].$$

From the indeterminacy of $\mathbf{W}^{(2)}$ and $\mathbf{W}^{(3)}$, one may choose $\mathbf{W}^{(3)}$ to be collinear with the projection of $\mathbf{t}$ onto the 2-plane $(\mathbf{W}^{(2)}, \mathbf{W}^{(3)})$; i.e., such that $t^2 = 0$. Similarly, since $\mathbf{t}$ is oriented in time, one may choose $\mathbf{W}^{(0)}$ to be collinear with the projection of $\mathbf{t}$ onto the 2-plane $(\mathbf{W}^{(0)}, \mathbf{W}^{(1)})$; i.e., such that $t^1 = 0$.

In the *b*) case, the vector $\mathbf{t}$ will certainly not be orthogonal to $\boldsymbol{l}$ since the fact that $\boldsymbol{l}$ is of null length implies that the vectors that are orthogonal to $\boldsymbol{l}$ will be the ones in the 3-plane that is tangent to the cone C_x along $\boldsymbol{l}$; thus, they will be oriented in space. For example, one may choose $\mathbf{e}_0$ to be collinear with $\mathbf{t}$, and consider the vector $\mathbf{l} - \mathbf{e}_0$, whose scalar product $\mathbf{e}_0$ is:

$$(\mathbf{l} - \mathbf{e}_0) \cdot \mathbf{e}_0 = \boldsymbol{l} \cdot \mathbf{e}_0 - 1,$$

and since $\boldsymbol{l} \cdot \mathbf{e}_0$ is non- zero, one may multiply $\mathbf{t}$ by a factor such that $\boldsymbol{l} - \mathbf{e}_0$ will be orthogonal to $\mathbf{e}_0$, and, a result, it will have a square of –1. If $\mathbf{e}_3$ denotes the vector $\boldsymbol{l} - \mathbf{e}_0$ thus obtained then one will see that the frame $(\mathbf{e}_\lambda)$ is therefore a simple frame in which $t^1 = t^2 = 0$. In this case, we have created only one solution to the problem, whereas, it is obvious that others may exist. Be that as it may, our lemma is still proved.

11. – A theorem on perfect fluid-electromagnetic field schema. – Consider a perfect fluid-electromagnetic field schema. Such a schema is described by an energy-momentum tensor of the form:

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta - p\, g_{\alpha\beta} + \tau_{\alpha\beta},$$

in which $\tau_{\alpha\beta}$ is the energy tensor of the electromagnetic field. We then introduce the vector $\mathbf{U}$ that is oriented in time and defined by:

$$U_\alpha = \sqrt{\rho + p u_\alpha}\ . \tag{11-1}$$

If we know $\mathbf{U}$ then we will know the unitary vector $\mathbf{u}$, which is collinear with it in the same sense, and, by passing to the square, we will also know $(\rho + p)$. We may therefore substitute the vector $\mathbf{U}$ for these elements and take $T_{\alpha\beta}$ to have the form:

$$T_{\alpha\beta} = U_\alpha U_\beta - p\, g_{\alpha\beta} + \tau_{\alpha\beta}. \tag{11-2}$$

If we apply the preceding lemma to the vector $\mathbf{U}$ and the tensor $\tau_{\alpha\beta}$ then it results that there is a simple orthonormal frame for $\tau_{\alpha\beta}$ such $U_1 = U_2 = 0$. For such a frame the tensor $T_{\alpha\beta}$ is represented by:

$$(T_{\alpha\beta}) = \begin{bmatrix} U_0^2 - p + \dfrac{\xi^2 + \eta^2}{2} & 0 & 0 & \xi\eta + U_0 U_2 \\ & p + \dfrac{\eta^2 - \xi^2}{2} & 0 & 0 \\ & & p + \dfrac{\xi^2 - \eta^2}{2} & 0 \\ & & & U_3^2 + p + \dfrac{\xi^2 + \eta^2}{2} \end{bmatrix}. \tag{11-3}$$

If one performs a pseudo-rotation of the vectors $(\mathbf{e}_0, \mathbf{e}_3)$ in the 2-plane that is totally orthogonal to the fixed vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ then one notes that since $\mathbf{e}_1$ and $\mathbf{e}_2$ stay fixed the matrix ($\tau_{\alpha\beta}$) preserves the form (10-9) and, as a result, the frame remains simple for $\tau_{\alpha\beta}$. On the other hand, one obviously has $U_1 = U_2 = 0$ in this frame, too. The orthogonal frames for which ($T_{\alpha\beta}$) has the form (11-3) will therefore be defined only up to a pseudo-rotation of the pair of vectors ($\mathbf{e}_0$, $\mathbf{e}_3$), which is a pseudo-rotation that modifies only the elements of the partial matrix:

$$M = \begin{bmatrix} \tau_{00} & \tau_{03} \\ \tau_{30} & \tau_{33} \end{bmatrix}.$$

If **X** and **Y** denote two vectors in the 2-plane spanned by ($\mathbf{e}_0$, $\mathbf{e}_3$) then we set:

$$\varphi(\mathbf{X}, \mathbf{Y}) = \tau_{\mathrm{AB}} X^{\mathrm{A}} Y^{\mathrm{B}},$$

in which the indices A, B take only the values 0 and 3, and φ has scalar values. If we start with φ then the elements of the matrix M will be given by:

$$M = \begin{bmatrix} \varphi(\mathbf{e}_0, \mathbf{e}_0) & \varphi(\mathbf{e}_0, \mathbf{e}_3) \\ \varphi(\mathbf{e}_3, \mathbf{e}_0) & \varphi(\mathbf{e}_3, \mathbf{e}_3) \end{bmatrix}.$$

Having said that, we perform the pseudo-rotation that is defined by the formulas:

(11-4)
$$\mathbf{e}'_0 = \mathbf{e}_0 \cosh\theta + \mathbf{e}_3 \sinh\theta, \qquad \mathbf{e}'_3 = \mathbf{e}_0 \sinh\theta + \mathbf{e}_3 \cosh\theta$$

on ($\mathbf{e}_0$, $\mathbf{e}_3$).

The matrix M is replaced M', whose elements that are not situated on the principal diagonal have the values:

$$\varphi(\mathbf{e}'_0, \mathbf{e}'_3) = \tau_{00} \cosh\theta \sinh\theta + \tau_{03} (\cosh^2\theta + \sinh^2\theta) + \tau_{33} \cosh\theta \sinh\theta,$$

namely:

$$\varphi(\mathbf{e}'_0, \mathbf{e}'_3) = \tau_{03} \cosh 2\theta + \frac{\tau_{00} + \tau_{33}}{2} \sinh 2\theta.$$

If:

$$\Delta = (\tau_{00} + \tau_{33})^2 - 4\, (\tau_{03})^2 > 0$$

then there will exist a pseudo-rotation θ that annihilates this element.

Now, from (11-3) one has:

$$\Delta = (U_0^2 + U_3^2 + \xi^2 + \eta^2 + 2\eta\xi + 2U_0 U_3)\ (U_0^2 + U_3^2 + \xi^2 + \eta^2 - 2\eta\xi - 2U_0 U_3),$$

namely:

$$\Delta = [(\xi + \eta)^2 + (U_0 + U_3)^2]\ [(\xi - \eta)^2 + (U_0 - U_3)^2],$$

from which Δ is strictly positive, since the fact that **U** is oriented in time implies that:

(11-5) $$U_0^2 - U_3^2 > 0.$$

Therefore, if we are given the energy-momentum tensor $T_{\alpha\beta}$ for a perfect fluid-electromagnetic schema then *there will always exist an orthonormal frame for which the matrix of components of $T_{\alpha\beta}$ may be written:*

(11-6) $$(T_{\alpha\beta}) = \begin{bmatrix} U_0^2 - p + \dfrac{\xi^2+\eta^2}{2} & 0 & 0 & \xi\eta + U_0 U_2 \\ & p + \dfrac{\eta^2-\xi^2}{2} & 0 & 0 \\ & & p + \dfrac{\xi^2-\eta^2}{2} & 0 \\ & & & U_3^2 + p + \dfrac{\xi^2+\eta^2}{2} \end{bmatrix}$$

and $\xi\eta + U_0 U_3 = 0$.

We have therefore obtained an orthonormal frame that is composed of proper vectors of $T_{\alpha\beta}$ with respect to $g_{\alpha\beta}$. It then results from this that the energy tensor of a perfect fluid-electromagnetic field schema will always be a *normal* tensor.

Conversely, we now consider a *normal tensor $T_{\alpha\beta}$* and look for conditions that will make it possible to interpret it as the energy-momentum tensor of a perfect fluid-electromagnetic field schema. Let (s_0, s_i) be its proper values. In order to proceed with the identification with the expressions in (11-6) it is necessary to find five real quantities U_0, U_3, ξ, η, p, that satisfy the five equations:

(11–7) $$\xi\eta + U_0 U_3 = 0\,,$$

and:

(11-8) $$p + \frac{\eta^2-\xi^2}{2} = -s_1\,, \qquad p + \frac{\xi^2-\eta^2}{2} = -s_2\,,$$

(11-9) $$U_0^2 - p + \frac{\xi^2+\eta^2}{2} = s_0\,, \qquad U_3^2 + p + \frac{\xi^2+\eta^2}{2} = -s_3\,.$$

From (11-8), one immediately deduces that:

$$p = -\frac{s_1+s_2}{2}, \qquad k = \frac{\xi^2-\eta^2}{2} = \frac{s_1-s_2}{2}\,.$$

The electromagnetic field that one obtains will then be nonsingular if $s_1 \neq s_2$. From (11-9), one deduces the two relations:

(11-10) $$\begin{aligned} U_0^2 + U_3^2 + \xi^2 + \eta^2 &= s_0 - s_3\,, \\ U_0^2 - U_3^2 &= (s_0 + s_3) - (s_1 + s_2), \end{aligned}$$

and one must have $s_0 > s_3$. One deduces the values of:

$$(11\text{-}11) \qquad U_0^2 = \frac{(s_0 - s_1)(s_0 - s_2)}{s_0 - s_3}, \qquad U_3^2 = \frac{(s_1 - s_2)(s_2 - s_3)}{s_0 - s_3},$$
$$\xi^2 = \frac{(s_1 - s_3)(s_0 - s_2)}{s_0 - s_3}, \qquad \eta^2 = \frac{(s_2 - s_3)(s_0 - s_1)}{s_0 - s_3}$$

from (11-7) and (11-10) by an easy calculation.

As far as reality considerations are concerned, one sees that if one calculates the four quantities $U_0^2 + \xi^2$, $U_3^2 + \eta^2$, $U_0^2 + \eta^2$, $U_3^2 + \xi^2$ then one will necessarily have:

$$(11\text{-}12) \qquad s_0 > s_1, \qquad s_0 > s_2, \qquad s_1 > s_3, \qquad s_2 > s_3.$$

In addition, if $p \geq 0$ and $\rho > 0$ then one will deduce the necessary inequalities:

$$(11\text{-}13) \qquad s_1 + s_2 \leq 0, \qquad s_0 + s_3 > \frac{s_1 + s_2}{2}.$$

Therefore, assume that the inequalities (11-12) and (11-13) are satisfied. If, in addition, $s_1 \neq s_2$ then all of the proper values of the given tensor will be distinct, and the tensor will admit a unique orthonormal frame that is composed of proper vectors. Equations (11-11) will then give us the values of U_0 and U_3, up to sign. The hydrodynamical part of the energy tensor will be determined uniquely (in particular, this will be true for ρ and p), but **u** admits several determinations. Once **u** is chosen x, the η will be defined by (11-11) and (11-17), with a new sign indeterminacy. The electromagnetic part of the energy tensor will be determined uniquely, but once ξ and η are fixed the electromagnetic tensor is defined at a point x only up to a parameter α. Indeed, with respect to the frame considered, one will have:

$$X^2 + L^2 = \xi^2, \qquad Y^2 + M^2 = \eta^2, \qquad YL - XM = \xi\eta.$$

Thus, there exists an angle α such that:

$$X = \xi \cos\alpha, \qquad Y = \eta \sin\alpha,$$
$$L = \xi \sin\alpha, \qquad M = -\eta \cos\alpha.$$

If $s_1 = s_2$ then the electromagnetic part will be singular, and the frame envisioned will no longer be unique, but the results will obviously persist. If $s_1 + s_2 = 0$ then the schema that one obtains will be a matter-electromagnetic field schema. We state ([1]):

([1]) This theorem was established in 1947 by E. WILLIAMS, who used a different method. The method of proof that is given here is derived from that of G. Y. RAINICH. *Mathematics of Relativity*. John Wiley, New York (1950).

THEOREM. – *Any normal tensor that satisfies the inequalities* (11-12) *and* (11-13), *may be interpreted as the energy-momentum tensor of a perfect fluid-electromagnetic field schema.*

CHAPTER II

THE CAUCHY PROBLEM FOR THE EQUATIONS OF GRAVITATION AND ELECTROMAGNETISM

I. THE PURELY GRAVITATIONAL CASE

12. – The expression for the Ricci tensor. – As we shall see, since the system of partial differential equations that is due to Einstein presents a hyperbolic normal character, the first problem that we must pose for it is the Cauchy problem: Given the gravitational field on a hypersurface *S*, determine the gravitational field that satisfies the Einstein equations outside of *S*. The problem obviously translates into what one may call "relativistic determinism," and the fact that gravitation satisfies the schema of wave propagation will result immediately from this analysis.

In order to begin this problem, we must study the analytical expression for the Ricci tensor as a function of the potentials and their derivatives of the first two orders. In fact, except for the existence of conservation conditions, we use only the manner by which the second derivatives appear in the components of the Ricci tensor here.

From the expression for the curvature tensor in terms of the coefficients Γ of the Riemannian connection, viz.:

$$R_{\lambda\alpha,\ \ \beta}^{\ \ \ \ \mu} = \partial_\lambda \Gamma^\mu_{\alpha\beta} - \partial_\alpha \Gamma^\mu_{\lambda\beta} + \Gamma^\mu_{\lambda\rho}\Gamma^\rho_{\alpha\rho} - \Gamma^\mu_{\alpha\rho}\Gamma^\rho_{\lambda\beta},$$

one deduces the components of the Ricci tensor by contracting over λ and μ:

$$R_{\alpha\beta} = \partial_\rho \Gamma^\rho_{\alpha\beta} - \partial_\alpha \Gamma^\rho_{\rho\beta} + \Gamma^\sigma_{\sigma\rho}\Gamma^\rho_{\alpha\beta} - \Gamma^\sigma_{\alpha\rho}\Gamma^\rho_{\beta\sigma}. \tag{12-1}$$

If we want to draw attention to only the second-order derivatives of the potentials on the right-hand side of (12-1) then we may write:

$$R_{\alpha\beta} = g^{\rho\sigma}\{\partial_\rho[\alpha\beta,\sigma] - \partial_\alpha[\rho\beta,\sigma]\} + K_{\alpha\beta},$$

in which the [] expressions denote the Christoffel symbols of the first kind, and *K* depends only upon the potentials and their first derivatives. Upon developing the Christoffel symbols, it follows that:

$$R_{\alpha\beta} = \tfrac{1}{2} g^{\rho\sigma}\{\partial_{\beta\rho} g_{\alpha\sigma} + \partial_{\alpha\sigma} g_{\beta\rho} - \partial_{\beta\rho} g_{\alpha\sigma} - \partial_{\alpha\beta} g_{\rho\sigma}\} + K_{\alpha\beta}. \tag{12-2}$$

13. – The proof of the conservation conditions (3-2). – We commence by recalling that proof. Those conservation conditions are a simple consequence of the Bianchi identities that relate to the curvature tensor, and they result from differentiating the curvature forms.

The Bianchi identities may be written:

$$\nabla_\lambda R_{\alpha\beta,\mu\nu} + \nabla_\mu R_{\alpha\beta,\nu\lambda} + \nabla_\nu R_{\alpha\beta,\lambda\mu} = 0.$$

By contracting over β and ν, it will follow that:

$$\nabla_\lambda R_{\alpha\mu} - \nabla_\mu R_{\alpha\lambda} + \nabla_\nu R_{\alpha,}{}^{\nu}{}_{\lambda\mu} = 0.$$

Finally, we contract α and μ; we obtain:

$$\nabla_\lambda R - \nabla_\alpha R_\lambda{}^\alpha - \nabla_\nu R_\lambda{}^\nu = 0 \ ;$$

i.e., upon grouping the last two terms, which are identical, we will obtain:

(13-1) $$\nabla_\alpha R_\lambda{}^\alpha - \tfrac{1}{2}\nabla_\lambda R = 0.$$

If we set:

(13-2) $$S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}(R + k)g_{ab}$$

then (13-1) will be equivalent to:

$$\nabla_\alpha S_\lambda^\alpha = 0 \ ;$$

i.e., the desired conservation conditions.

One will note that from (13-2) one may deduce, by contraction:

$$S_\alpha^\alpha + k = -(R + k).$$

As a result:

(13-3) $$R_{\alpha\beta} = S_{\alpha\beta} - \tfrac{1}{2}(S_\rho^\rho + k)\, g_{\alpha\beta}.$$

14. – The exterior problem. Analysis of the equations. – We confine ourselves to the equations for a null cosmological constant; the modifications that must be made to the analysis when there is a cosmological constant are easy to obtain. Our purely local study will begin with the right-hand side of the Einstein equations. From (10-2) and (10-3), these equations may be written as either $S_{\alpha\beta} = 0$ or:

(14.1) $$R_{\alpha\beta} = 0.$$

The results that we obtain will be, moreover, useful for the analysis of the equations with a non-zero right-hand side. By definition, the gravitational field that we study is found to be determined by the gravitational derivatives and their first derivatives. Our problem is therefore the following one:

THE EXTERIOR PROBLEM. – *Given the potentials and their first derivatives on a hypersurface S, determine the potentials outside of S that are assumed to satisfy the Einstein equations* (14-1) *for the exterior case.*

First of all, we assume that S is *everywhere oriented in space* and that it has been locally represented by the equation $x^0 = 0$. One will then have $g^{00} > 0$. The data on S that is comprised of the potentials $g_{\mu\nu}$ determine the values of the first derivatives $\partial_i g_{\mu\nu}$ on S. We call the number of times that the index 0 appears in the ∂ symbol the *index* of a derivative of $g_{\mu\nu}$. The "Cauchy data" on S are therefore composed of the values on S of the potentials $g_{\mu\nu}$ (which are assumed to be at least three times continuously differentiable) and their first derivatives of index 1, namely, $\partial_0 g_{\mu\nu}$ (which is assumed to be at least twice continuously differentiable). We propose to determine the values of the derivatives of order higher than the first on S. This search will permit us to specify whether these derivatives are subject to discontinuities upon traversing S, and to embark upon a study of the structure of the system of Einstein equations.

A simple derivation on S gives the values of the second derivatives whose index is 0 or 1. As a result, the only second derivatives that may be discontinuous upon traversing S will be the derivatives of index 2, namely, $\partial_{00} g_{\lambda\mu}$. We are thus led to express these derivatives explicitly in equations (14-1), whose left-hand side may be expressed by (12-2). One therefore obtains:

(14-1) $$R_{ij} = -\tfrac{1}{2} g^{00}\partial_{00} g_{ij} + F_{ij}(\text{C.d.}) = 0,$$

(14-3a) $$R_{ij} \equiv \tfrac{1}{2} g^{00}\partial_{00} g_{ij} + \Phi_{ij}(\text{C.d.}) = 0,$$

(14-3*b*) $$R_{00} \equiv -\tfrac{1}{2} g^{00}\partial_{00} g_{ij} + \Psi(\text{C.d.}) = 0,$$

in which F, Φ, Ψ, denote quantities that are calculable on S using algebraic operations on the derivatives of S when one starts with the Cauchy data (C.d.).

Under the hypothesis that g_{00} is *non- zero*, one will see that the 6 equations (14-2) give the values of the derivatives $\partial_{00} g_{\lambda 0}$ on S. We must analyze this fact.

Our purely local study has been accomplished within the domain of a certain coordinate system. However, in the domain of the Cauchy data that is envisioned, the data on S will leave open the possibility of coordinate changes that preserve the numerical values of the coordinates at any point of S, as well as the Cauchy data. In order for us to account for this, consider the coordinate change that is defined by the formula:

$$x^{\lambda'} = x^\lambda + \frac{(x^0)^3}{6} + [\varphi(\lambda)(x^i) + \varepsilon^\lambda] \qquad (\lambda' = \lambda, \text{ numerically}),$$

in which ε^λ goes to zero when x^0 goes to 0. Under a change of coordinates, each point of S will preserve the same coordinates numerically, and, moreover, we will have:

(14-4) $$(A_\mu^{\lambda'})_S = \delta_\mu^\lambda, \qquad (\partial_\mu A_0^{\lambda'})_S = 0, \qquad (\partial_{i0} A_0^{\lambda'})_S = (\partial_{00} A_i^{\lambda'})_S = 0,$$

on S, in such a way that among the second derivatives of A, only the derivatives $(\partial_{00} A_0^{\lambda'})_S$ will be non-null on S:

$$(\partial_{00} A_0^{\lambda'})_S = \varphi^{(\lambda)}.$$

Under a coordinate change, the potentials and their first derivatives will transform according to the formulae:

$$g_{\lambda\mu} = A_\lambda^{\alpha'} A_\mu^{\beta'} g_{\alpha'\beta'} ,$$

(14-5) $$\partial_0 g_{\lambda\mu} = A_\lambda^{\alpha'} A_\mu^{\beta'} A_0^{\rho'} \partial_{\rho'} g_{\alpha'\beta'} + \partial_0 A_\lambda^{\alpha'} \bullet A_\mu^{\beta'} g_{\alpha'\beta'} + \partial_0 A_\mu^{\beta'} \bullet A_\lambda^{\alpha'} g_{\alpha'\beta'} \ .$$

From (14-4), one deduces that under the coordinate change considered, the Cauchy data on S will satisfy the relations:

$$g_{\lambda\mu} = g_{\lambda'\mu'} , \qquad \partial_0 g_{\lambda\mu} = \partial_{0'} g_{\lambda'\mu'} \ .$$

How does such a coordinate change influence the second derivatives (of index 2)? One has:

$$\partial_{00} g_{\lambda\mu} = A_\lambda^{\alpha'} A_\mu^{\beta'} A_0^{\rho'} A_0^{\sigma'} \partial_{\rho'\sigma'} g_{\alpha'\beta'} + \partial_{00} A_\lambda^{\alpha'} \bullet A_\mu^{\beta'} g_{\alpha'\beta'} + \partial_{00} A_\mu^{\beta'} \bullet A_\lambda^{\alpha'} g_{\alpha'\beta'} + \Theta,$$

in which Θ denotes terms that contain the first derivatives of A and are null on S. From this, one deduces that:

$$\partial_{00} g_{ij} = \partial_{0'0'} g_{i'j'} ,$$
$$\partial_{00} g_{\lambda 0} = \partial_{0'0'} g_{\lambda'0'} + \varphi^{(\rho)} g_{\lambda\rho} + \delta_\lambda^0 \varphi^{(\rho)} g_{0\rho} = \partial_{0'0'} g_{\lambda'0'} + \varphi_{(\lambda)} + \delta_\lambda^0 \varphi_{(0)} \ .$$

Therefore, the derivatives $\partial_{00} g_{ij}$ are not modified under such a coordinate change, whereas the $\partial_{00} g_{\lambda 0}$ may take on arbitrary values. Upon using a coordinate system in which the $\varphi^{(\lambda)}$ are different on either side of S – which is permissible in terms of the structure of V_4, since the $\varphi^{(\lambda)}$ appear only in the third coordinate on S – one sees that one may make the possible discontinuities in these second derivatives appear or disappear, which are discontinuities that are thus devoid of any intrinsic physical significance. In particular, one may restrict the $\partial_{00} g_{\lambda 0}$ to be continuous upon traversing S in a convenient coordinate system. One encounters such circumstances in the study of the Schwarzschild matching conditions.

Up to the preceding reservations, the second derivatives of the potentials will be *continuous* upon crossing the hypersurface S. In the case where the Cauchy data are differentiable up to a higher order, one sees that the same conclusions will extend to the successive derivatives of the potentials; in order to do that, it will suffice to explicitly differentiate equations (14-2) with respect to x^0. In all of our analysis up till now, only equations (14-2) were involved, to the exclusion of equations (14-3).

15. – The integration of the Einstein equations. – From the form of the left-hand sides of equations (14-2) and (14-3), it results that the quantities:

$$g^{0j} R_{ij} + g^{00} R_{i0}, \qquad\qquad g^{ij} R_{ij} + g^{00} R_{00}$$

do not contain any derivative of index 2 of the potentials; as a result, their values on S may be calculated by starting with the Cauchy data. These quantities are very simply expressed with the aid of the components of the tensor $S_{\alpha\beta}$. Indeed, one has:

$$S_i^0 \equiv R_i^0 \equiv g^{0j} R_{ij} + g^{00} R_{i0}.$$

On the other hand:

$$S_0^0 \equiv R_0^0 - \tfrac{1}{2} R \equiv g^{0\rho} R_{0\rho} + \tfrac{1}{2} (g^{ij} R_{ij} + 2\, g^{0i} R_{0i} + g^{00} R_{00}),$$

namely:

$$S_0^0 \equiv \tfrac{1}{2} (g^{00} R_{00} - g^{ij} R_{ij}).$$

Conversely, the data on R_{ij} and the 4 quantities S_α^0 permit us to evaluate the 4 quantities $R_{\alpha 0}$ for $g_{00} \neq 0$. From this, it result that under the hypothesis made the system composed of equations (14-2) and (14-3) is equivalent (in a neighborhood of S) to the system composed of equations (14-2) and the 4 equations:

(15-1) $$S_\alpha^0 = 0,$$

in which the left-hand sides contain no derivative of index 2. *Therefore, the four equations* (15-1) *give four conditions that must be necessarily verified on S by the Cauchy data.*

Therefore, consider Cauchy data that satisfy the four conditions $(S_\alpha^0)_S = 0$, and assume that we know a ds^2 that satisfies these Cauchy data, as well as equations (11-2). The left-hand sides of (15-1) are coupled with the right-hand sides of (14-2) by the conservation conditions:

$$\nabla_\alpha S_\lambda^\alpha = 0,$$

namely:

(15-2) $$\nabla_0 S_\lambda^0 + \nabla_i S_\lambda^i = 0.$$

Now, for a solution to equations (11-2) one has:

$$S_j^i = g^{i0} R_{0j} - \tfrac{1}{2} \delta_j^i (g^{00} R_{00} + 2 g^{0i} R_{0i}), \qquad\qquad S_0^i = g^{i\rho} R_{\rho 0},$$

and also:

$$S_i^0 = \mathrm{g}^{00} R_{i0}, \qquad\qquad S_0^0 = \tfrac{1}{2} \mathrm{g}^{00} R_{00}.$$

From this, one deduces that the four equations (15-2) may be put into the form:

(15-3) $$g^{00}\partial_0 S^0_\lambda = A^{i\rho}_\lambda \partial_0 S^0_\rho + B^\rho_\lambda S^0_\rho ,$$

in which A and B are of class C^0. Therefore, for any solution of (14-2) they satisfy a system of four linear homogeneous partial differential equations that are solvable with respect to the $\partial_0 S^0_\lambda$. Such a system admits no other solution but the null solution for null data $(S^0_\lambda)_S$ on S. One therefore sees that it results from the conservation conditions that if ds^2 satisfies equations (14-2) and – at least on S – equations (15-1) then it also satisfies equations (15-1) outside of S. One may translate this fact by saying that the system of Einstein equations is *in involution*, in the sense of E. Cartan.

We therefore see that our initial problem must be divided into two distinct problems.

PROBLEM I, or "the initial conditions problem." – This consists of looking for Cauchy data that satisfy the system $S^0_\alpha = 0$ – or the system of initial conditions – on S.

PROBLEM II, or "the evolution problem." – This consists of integrating the system $R_{ij} = 0$ for Cauchy data that satisfy the conditions of the first problem.

Suppose, for an instant, that all of the data in this second problem are real-analytic, even though this is, moreover, contrary to the axioms that we gave in the first chapter. With the aid of the Cauchy-Kowalewska existence theorem for partial differential equations, one may then establish that problem II admits one and only one real solution locally, up to a coordinate change that preserves the coordinates of any point of S and the Cauchy data. The coordinate changes permit us to give arbitrary values of $g_{0\lambda}$ outside of S that are compatible with the Cauchy data. However, the analyticity plays no role here, in fact, and this result must show us only that we are on the right track. The evolution problem has been recently solved by Fourés under the simple hypothesis of differentiability. We shall return to these results later.

The results of the preceding analysis are not appreciably modified if the hypersurface S is everywhere oriented in time ($g^{00} < 0$). The decomposition into two problems is accomplished in the same fashion, but these problems then present very different approaches to their solutions.

16. – Characteristic manifolds and bicharacteristics. – If the hypersurface S that is locally defined by $x^0 = 0$ is such that g^{00} is identically null on S – i.e., if it is tangent to the elementary cone at each of its points – then the results are found to be profoundly modified. The second derivatives of the potentials (i.e., the $\partial_{00} g_{ij}$) might be discontinuous upon crossing S; there can exist an infinitude of distinct solutions to the Einstein equations that correspond to the same Cauchy data on S. In this, one recognizes the classical results from the theory of partial differential equations that are concerned with characteristic manifolds, or, in the language of wave propagation, that characterize the wave surfaces. Therefore, we may state:

THEOREM. – *The elementary cones C_x are the characteristic cones for the system of Einstein equations. They admit the manifolds that are tangent to these cones for their characteristic manifolds.*

If S is locally represented by the equation $f(x^a) = 0$ then the characteristic manifolds will therefore be solutions of the first order partial differential equations:

(16–1) $$\Delta_1 f = g^{\alpha\beta}\partial_\alpha f\,\partial_\beta f = 0.$$

They will be denoted by V_3^C in the sequel. A solution to (16-1), i.e., a characteristic manifold V_3^C, may be generated by means of the characteristic strips of (16-1). We shall see that such a solution may be generated by means of the characteristic strips of V since such a strip consists of the set composed of a curve L_0 and a one-parameter family of elementary 3-planes that are tangent to this curve. We give the name of *bicharacteristics for the Einstein equations* to the characteristic curves of (16-1) L_0 . We shall now determine them.

Set:

$$H(x^\lambda, y_\mu) = g^{\alpha\beta} y_\alpha y_\beta,$$

and consider the partial differential equation:

(16–2) $$\Delta_1 f \equiv 2H\,(x^\lambda, \partial_\mu f) = C,$$

in which C is an arbitrary constant. In terms of the variables x^α, f, y_β, the characteristic strips of (13-2) will be given by the solutions to the differential system:

$$\frac{dx^0}{\dfrac{\partial H}{\partial y_0}} = \ldots = \frac{dx^3}{\dfrac{\partial H}{\partial y_3}} = \frac{df}{2H} = -\frac{dy_0}{\dfrac{\partial H}{\partial x^0}} = \ldots = -\frac{dy_3}{\dfrac{\partial H}{\partial x^3}} \;(= du),$$

which will have the first integral:

$$2H(x^\lambda, y_\mu) = C$$

that gives the value for the constant C. If one introduces the auxiliary variable u then the functions $x^\alpha(u)$, $y_\beta(u)$ will be given by the canonical system:

(16-3) $$\frac{dx^\alpha}{du} = \frac{\partial H}{\partial y_\alpha}, \qquad \frac{dy_\alpha}{du} = -\frac{\partial H}{\partial x_\alpha},$$

in the Hamiltonian function $H(x^\lambda, y_\mu)$. The first group of equations (16-3) may be explicitly written:

(16-4) $$\dot{x}^\alpha = g^{\alpha\beta} y_\beta \qquad \left(\dot{x}^\alpha = \frac{dx^\alpha}{du}\right).$$

Conversely:

$$y_\beta = g_{\alpha\beta}\,\dot{x}^\alpha .$$

Having said that, the solutions $x^\alpha(u)$ to (16-3) will be extremals for the Lagrangian function L that is defined by:

$$2L = g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta ,$$

since, by passing from the variables $(x^\alpha, \dot{x}^\beta)$ to the canonical variables (x^α, y_β), which are related by (3-4) and (3-5), one will have the classical relation between H and L:

$$H = \dot{x}^\alpha \frac{\partial L}{\partial \dot{x}^\alpha} - L \qquad (= L).$$

These solutions are the extremals that give the first integral:

$$2L = C \tag{16-6}$$

for the value of the constant C. Now, from the existence of this first integral the extremals thus defined are also extremals of:

$$\sqrt{2L} = \sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta} ,$$

which satisfy (16-6). It results from this that the $x^\alpha(u)$ define the geodesics of V_4. If C = 0 then the differential system of the characteristics of (16-1) admits the first integral f = constant, and the manifolds V_3^C may be generated by the strips of V_4 that are defined by *geodesics of null length* L_0, with the associated 3-plane being the plane that is tangent to the elementary cone along the tangent to L_0. From the theory of partial differential equations, one knows, moreover, that when one is given a manifold V_3^C that touches the elementary cone at x along a generator G the tangent to the curve L_0 that is associated to V_3^C at x is G, which shows, *a priori*, why the bicharacteristics have null length.

We may summarize the results of this analysis by stating:

THEOREM. – *The bicharacteristics of the Einstein equations are the null-length geodesics of the Riemannian manifold* V_4.

Upon referring to the theory of wave propagation that comes from the works of Hadamard, one sees that the gravitational field presents the *character of a propagation phenomenon* in relativity. The characteristic manifolds along which the discontinuities of the field may be produced play the role of gravitational *wave surfaces*. We shall later verify that the propagation of an electromagnetic field in V_4 involves exactly the same elements. In a neighborhood of a point x, the bicharacteristics that issue from x generate a hypersurface that admits x for a conic point and is called the *characteristic conoid* at x.

17. – The conservation conditions in material schemas. – We shall now return from the study of the Cauchy problem in the exterior case to an analogous study in the interior case; first, we shall consider the absence of an electromagnetic field. To that effect, we must study the physical consequences of the conservation conditions.

Let $T_{\alpha\beta}$ be the energy tensor for the material schema considered and write down the corresponding Einstein equations:

$$S_{\alpha\beta} = \chi\, T_{\alpha\beta}.$$

Since the tensor $S_{\alpha\beta}$ is not conservative, the same thing will also be true for the tensor $T_{\alpha\beta}$, which is constrained to differ from it by only a constant:

$$\nabla_\alpha T^\alpha_\beta = 0.$$

If we take a particular expression for the energy, with the aid of a certain number of physical elements, then we will determine a certain number of "conservation" properties for these physical elements by way of (3-2). By first reasoning in as general a manner as possible, we take the energy tensor $T_{\alpha\beta}$ in the form:

(17-1) $$T_{\alpha\beta} = r\, u_\alpha u_\beta - \theta_{\alpha\beta},$$

in which u_α is a unitary velocity vector (hence, it is oriented in time), r is a positive scalar (the pseudo-density), and $\theta_{\alpha\beta}$ is a symmetric tensor (the pressure pseudo-tensor). Since the vector u_α is unitary, one will have:

(17-2) $$g_{\alpha\beta} u^\alpha u^\beta = g^{\alpha\beta} u_\alpha u_\beta = 1,$$

and, upon differentiating:

(17-3) $$u^\beta \nabla_\beta\, u_\beta = 0.$$

If we define a vector **K** by the relation:

(17-4) $$\nabla_\alpha \theta^\alpha_\beta = r\, K_\beta$$

then the conservation conditions for the energy tensor may be written:

$$\nabla_\alpha (r\, u^\alpha u_\beta) = r\, K_\beta.$$

Therefore, the scalar r and the vector u_α will satisfy (17-2), (17-3), along with the relations:

(17-5) $$\nabla_\alpha (r\, u^\alpha)\, u_\beta + r\, u^\alpha\, \nabla_\alpha u^\beta = r\, K_\beta.$$

In order to simplify these equations, we may take (17-2) and (17-3) into account. By contracted multiplications with u^β, it follows that:

(17-6) $$\nabla_\alpha (r\, u^\alpha) = r\, K_\alpha u^\alpha.$$

Upon accounting for this relation in (17-5), one will obtain, after dividing by r:

(17-7) $$u^\alpha \nabla_\alpha u^b = (g_{\alpha\beta} - u_\alpha u_\beta)\, K^\alpha.$$

The left-hand side of equation (17-6) presents the aspect of an equation of continuity. On the other hand, we call the lines that are everywhere tangent to the unitary velocity vector – i.e., the trajectories of that vector – the *streamlines* of the schema considered. When the vector **K** is known, equations (17-7), in which $u^\alpha = \dfrac{dx^\alpha}{ds}$, will constitute a differential system that the streamlines must satisfy.

In order to find a simple interpretation for the foregoing, we study two particular cases. The first case will be that of the *pure matter schema.* In this case, one has:

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta,$$

and one may take, in turn:

$$r = \rho, \qquad \theta_{\alpha\beta} = 0, \qquad K_\beta = 0.$$

Equations (17-6) and (17-7) may therefore be written:

(17-8) $$\nabla_\alpha(\rho\, u^\alpha) = 0,$$

(17-9) $$u^\alpha \nabla_\alpha u_\alpha = 0.$$

The first equation expresses that the divergence of the product of the density of the medium with the unitary velocity vector is null. This is obviously the equation of continuity for the "atomized" medium considered. The system (17-9) expresses the idea that the streamlines of the current are auto-parallel, i.e., they are geodesics of the metric ds^2.

The second case that we envision is the one that I call the case of the *holonomic medium* (*or schema*). By definition, it is the case for which K_β is a gradient. If log F denotes the corresponding function of V_4 then F will be called the *index* of the holonomic medium considered and equations (17-6) and (17-7) will then take the form:

$$\nabla_\alpha(r\, u^\alpha) = r\, u^\alpha \frac{\partial_\alpha F}{F}, \qquad u^\alpha \nabla_\alpha u^\beta = \frac{\partial_\alpha F}{F}(g^\alpha_\beta - u^\alpha u_\beta).$$

We may verify that one may interpret the differential system of the streamlines in this case by saying that they are geodesics of the Riemannian metric that is defined in V_4:

$$d\bar{s}^2 = F^2\, ds^2,$$

which is conformal to the Einstein metric ds^2.

In particular, consider a perfect fluid in a state of motion such that one may deduce (from thermodynamical considerations, for example) the existence of an equation of state that relates the proper density to the pressure, namely $\rho = \varphi(p)$. In this case, the energy tensor may be written:

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta - p\, u_{\alpha\beta},$$

and one may take:

(17-10) $$r = \rho + p, \qquad \theta_{\alpha\beta} = p\, g_{\alpha\beta}, \qquad K_\beta = \frac{1}{\rho + p}\nabla_\alpha (p g^\alpha_\beta) = \frac{\partial_\beta p}{\rho + p},$$

in which $\rho = \varphi(p)$. It results from this that K_β is the gradient of:

$$\int_{p_0}^{p} \frac{dp}{\varphi(p) + p}.$$

Such a perfect fluid will therefore be a holonomic medium of index:

(17-11) $$F = \exp \int_{p_0}^{p} \frac{dp}{\varphi(p) + p}.$$

18. – The interior problem in the pure matter case. – We shall now pose the same local Cauchy problem in the interior case that corresponds to a pure matter schema. If we are given the gravitational field on a hypersurface S in terms of the Cauchy data that we have discussed then we will propose to determine that field outside of S when one assumes that the Cauchy data satisfy the equations of the pure matter schema:

(18-1) $$S_{\alpha\beta} = \rho\, u_\alpha u_\beta,$$

in which $\rho > 0$, and the vector u_α is unitary. We further assume that S is not tangent to the elementary cone; if it is locally represented by $x^0 = 0$ then one will have $g_{00} \neq 0$. It will results from the considerations of sec. **12** that under these conditions we may substitute an equivalent system for system (18-1) that will decompose into two subsystems; from (10-3), the first six equations can be written:

(18-2) $$R_{ij} = \chi\rho\, (u_i u_j - \tfrac{1}{2} g_{ij}),$$

whereas the other four are:

(18-3) $$S^0_\lambda = \xi\rho\, u_\lambda u^0.$$

It is convenient to add the equation:

(18-4) $$g^{\lambda\mu} u_\lambda u_\mu = 1$$

to them, as well as the inequality $\rho > 0$.

Any solution $(g_{\alpha\beta}, u_\lambda)$ to this system will therefore satisfy equations (17-8), (17-9), which will entail the conservative character of the tensor $T_{\alpha\beta}$. These equations may be put into the form:

(18-5) $$u_\alpha \nabla_\alpha u_\beta = u^0 \partial_0 u_\beta + \Phi_\beta(\text{C.d.}, u_\lambda, \partial_i u_\lambda) = 0$$

(18-6) $$\nabla_\alpha(\rho\, u^\alpha) \equiv u^0\partial_0\rho + \rho\,\partial_0 u^0 + F(\text{C.d.}, u_\lambda, \partial_i u_\lambda, \rho, \partial_i\rho) = 0.$$

Having said that, we assume that the Cauchy data $g_{\alpha\beta}$ and $\partial_0 g_{\alpha\beta}$ are three and two times continuously differentiable on S, respectively. This data the values of the S_λ^0 on S; it is then possible to determine the values of ρ and u_λ on S. One first has:

$$(\chi\rho\, u^0)^2 = g^{\lambda\mu}\, S_\lambda^0\, S_\mu^0 .$$

The right-hand side must be strictly positive; in other words, the vector that is locally defined by S_λ^0 must be oriented in time. We set:

$$g^{\lambda\mu} S_\lambda^0\, S_\mu^0 = (\Omega^0)^2 \ (> 0),$$

in such a way that:

$$\chi\rho\, u^0 = \Omega^0.$$

From this, and with the aid of (18-3), one deduces that:

$$u_\lambda = \frac{S_\lambda^0}{\Omega^0}, \qquad u^0 = \frac{S^{00}}{\Omega^0}, \qquad \chi\rho = \frac{(\Omega^0)^2}{S^{00}} .$$

Since ρ is positive, one must also have $S^{00} > 0$. On the other hand, from the indeterminacy in the sign of Ω^0, the Cauchy data must determine u_λ up to a sign. In what follows, we will assume that this sign has been chosen, once and for all.

Conforming to our prior analysis, equations (18-2) will then provide the values of the derivatives $\partial_0 g_{ij}$ on S. Since S^{00} is positive – in particular, it is non-zero – we will have $u^0 \neq 0$. As a result, equations (18-5) and (18-6) will provide the values of the derivatives $\partial_0 u_\beta$ and $\partial_0\rho$ on S, respectively. It will then result that the quantities u_λ, ρ, $\partial_0 g_{ij}, \partial_0 u_\lambda, \partial_0\rho$ will have definite values on a hypersurface S that satisfies the hypotheses that we made and *cannot be discontinuous upon crossing S*. If the Cauchy data are locally differentiable to a higher order then the same conclusions will extend to the higher derivatives of a solution ($g_{\alpha\beta}$, u_λ, ρ) on S; it suffices to differentiate either (18-2) or (18-5) and (18-6) with respect to x^0.

Consider a set ($g_{\alpha\beta}$, u_λ, ρ) that satisfies equations (18-2), (18-5), (18-6), in a neighborhood of S, and equations (18-3) and (18-4) on S. It results from (18-5) and (18-6) that $\nabla_\alpha(S_\beta^\alpha - \chi T_\beta^\alpha) = 0$. If we write this system of four equations for a solution of (18-2) then it will follow that:

$$g^{00}(S_\lambda^0 - \chi T_\lambda^0) = A_\lambda^{i\rho}\partial_i(S_\rho^0 - \chi T_\rho^0) + B_\lambda^\rho(S_\rho^0 - \chi T_\rho^0) ,$$

and, as a result, (18-3) is verified outside of S if it is verified on S. Similarly, from (18-5), it results that:

$$u^{\alpha}(u^{\beta}\nabla_{a}u_{\beta}) = \tfrac{1}{2}u^{\alpha}\,\partial_{\alpha}(g^{\lambda\mu}u_{\lambda}u_{\mu}) = 0,$$

and, as a result, the vector u will be of constant length along the streamlines. If it is unitary on S then it will be unitary outside of S.

Therefore, since (18-3) and (18-4) are satisfied on S, it will suffice to preoccupy ourselves with the system (18-2), (18-5), (18-6), for the integration. Under the (abusive) hypothesis of analyticity there will be one and only one Cauchy problem, up to a coordinate change that preserves S point-by-point, along with the Cauchy data if they satisfy the conditions:

$$g^{00} \neq 0, \qquad (\Omega^0)^2 = g^{\lambda\mu}\,S^0_{\lambda}S^0_{\mu} > 0, \qquad S^{00} > 0.$$

We examine what sort of hypersurfaces may produce discontinuities in a given interior field upon being traversed. This is the case for the following phenomena:

a) When S is tangent to the elementary cone or is the characteristic manifold V_3^C (g^{00} = 0);
b) When $\Omega^0 = 0$, which entails that $u^0 = 0$ and, as a result, $S^0_{\lambda} = 0$; from the fact that $u^0 = 0$, S will then be tangent to a streamline or generated by the streamlines.

The case for which $S^{00} = 0$ will reduce to the preceding one if, as we have assumed, the density ρ is finite. We therefore see a new sort of hypersurface appear as an exceptional hypersurface, along with the characteristic manifolds of the exterior problem: namely, the V_3^C, which are generated by the streamlines.

19. – The interior problem in the case of the perfect fluid. – The previous argument will be analogous in the case of a perfect fluid that is endowed with an equation of state. However, it is convenient to explicitly mention the calculations here because they present a new physically interesting circumstance. To the energy-momentum tensor:

$$T_{\alpha\beta} = (\rho + p)\,u_a u_{\beta} - p\,g_{\alpha\beta} \qquad [\rho = \varphi(x)],$$

there will correspond the system of field equations:

(19-1) $$R_{ij} = \chi\,[(\rho + p)\,u_i u_j - \tfrac{1}{2}(\rho - p)\,g_{ij}],$$

(19-2) $$S^0_{\lambda} = \chi\,[(\rho + p)\,u_{\lambda}u^0 - p\,g^0_{\lambda}],$$

to which it is necessary to add:

(19-3) $$g^{\lambda\mu}u_{\lambda}u_{\mu} = 1.$$

As for the conservation conditions, they may be written:

(19-4) $$u^\alpha \nabla_\alpha u_\beta - \frac{\partial_\alpha p}{\rho + p}(g^\alpha_\beta - u^\alpha u_\beta) = 0,$$

(19-5) $$\nabla_\alpha[(\rho + p)\, u^a] - u^\alpha \partial_\alpha p = 0.$$

For the moment, assume that the value of p is known on S. One deduces from (16-2) that:

$$\chi(\rho + p)\, u_\lambda u^0 = S^0_\lambda + \chi\, p\, g^0_\lambda,$$

and, upon taking (19-3) into account:

$$[\chi(\rho + p)\, u^0]^2 = g_{\lambda\mu}\, S^0_\lambda + \chi\, p\, g^0_\lambda (S^0_\lambda + \chi\, p\, g^0_\lambda)(S^0_\mu + \chi\, p\, g^0_\mu) = [\Omega^0(p)]^2,$$

in which we have denoted the function of p that appears in the right-hand side by $[\Omega^0(p)]^2$. One deduces from this that:

(19-6) $$u_\lambda = \frac{S^0_\lambda + \chi p g^0_\lambda}{\Omega^0(p)}, \qquad u^0 = \frac{S^{00} + \chi p g^{00}}{\Omega^0(p)}, \qquad \chi(\rho + p) = \frac{[\Omega^0(p)]^2}{S^{00} + \chi p g^{00}}.$$

If $\rho = \varphi(p)$ then the last equation above will give a finite equation in p that determines either p or the possible values of p. One then deduces the values of u_λ, ρ, and then, with the aid of (19-1), of the $\partial_0 g_{ij}$.

We now seek to calculate $\partial_0 \rho$, or $\partial_0 p$, which, taking into account the equation of state, is equivalent to it. Equation (19-4) is contravariant when $\beta = 0$, and equation (19-5) may be written:

(19-7) $$(\rho + p)\, u^0\, \partial_0 u^0 - [g^{00} - (u^0)^2]\, \partial_0 p = A,$$

(19-8) $$(\rho + p)\, \partial_0 u^0 + u^0 \varphi'(p) \partial_0 p = B,$$

in which the values of A and B are known on S. The simultaneous determination of $\partial_0 p$ and $\partial_0 u^0$ may be accomplished only if:

$$(u^0)^2 \varphi' + g^{00} - (u^0)^2 \neq 0,$$

namely:

$$g^{00} - (u^0)^2 (1 - \varphi') \neq 0.$$

If this is true for $\partial_0 p$ and $\partial_0 u^0$ then $\partial_0 u^1$ will be determined on S, as a result of the unused equations (19-4). The argument then proceeds as before.

One therefore obtains three cases of exceptional manifolds that may produce discontinuities in the field when crossed:

a) Manifolds that are tangent to the elementary cone, or characteristic manifolds V_3^C ($g^{00} = 0$);

b) Manifolds that are tangent to a streamline or are generated by streamlines V_3^1;

c) Manifolds V_3^H such that $g^{00} - (u^0)^2 (1 - \varphi') = 0$. When they are locally defined by $f(x^\lambda) = 0$ these will be the manifolds that satisfy the equation:

$$[\, g^{\lambda\mu} - u^\lambda u^\mu (1 - \varphi')] \partial_\lambda f \partial_\mu f = 0. \tag{19-9}$$

Discontinuities of the pressure gradient may be produced upon crossing these manifolds that constitute the relativistic extension of the wave fronts of classical hydrodynamics. We assume that these wave fronts are *oriented in time,* or, more rigorously, they are *tangent to the cone C_x*; we verify that this hypothesis is in accord with the demands of relativistic physics precisely. If it is true then:

$$\Delta_1 f \equiv g^{\lambda\mu} \partial_\lambda f \partial_\mu f < 0.$$

Now, from (19-6):

$$\Delta_1 f \equiv (u^\lambda \partial_\lambda f)^2 (1 - \varphi').$$

From this, one deduces that:

$$\varphi' \geq 1. \tag{19-10}$$

Having said that, it is easy to evaluate the quantity that constitutes the "velocity of propagation" of the hydrodynamical waves considered here. Take two neighborhing wave surfaces $(V_3^H)_0$ and $(V_3^H)_\varepsilon$, which are defined by the equations:

$$f(x^\lambda) = 0, \qquad\qquad f(x^\lambda) = \varepsilon,$$

and take ε to be infinitesimal, in principle. The streamline that issues from the point x of $(V_3^H)_0$ intersects $(V_3^H)_\varepsilon$ at a point that is easy to determine up to higher-order infinitesimals. If we denote this point by $x + \eta\mathbf{u}$ then η will be determined by the relation:

$$\eta\, u^\lambda \partial_\lambda f = \varepsilon. \tag{19-11}$$

Let $\mathbf{n}$ be the normalized vector ($\mathbf{n}^2 = -1$) that is normal to the wave surface $(V_3^H)_0$. This vector has the covariant coordinates at x:

$$n_\lambda = \frac{\partial_\lambda f}{\sqrt{-g^{\lambda\mu} \partial_\lambda f \partial_\mu f}}.$$

The orthogonal trajectory of V_3^H that issues from x intersects $(V_3^H)_\varepsilon$ at a point that may be written $x + \eta_1 \mathbf{n}$, up to higher-order infinitesimals, in which η_1 is determined by the relation:

(19-12) $$\eta_1 n^\lambda \partial_\lambda f = \varepsilon.$$

From this, one deduces that:

(19-13) $$\eta_1 = \frac{\varepsilon}{n^\lambda \partial_\lambda f} = \frac{\varepsilon\sqrt{-g^{\lambda\mu}\partial_\lambda f \partial_\mu f}}{g^{\lambda\mu}\partial_\lambda f \partial_\mu f} = -\frac{\varepsilon}{\sqrt{-g^{\lambda\mu}\partial_\lambda f \partial_\mu f}}.$$

The vector $\mathbf{t} = \eta\mathbf{u} - \eta_1\mathbf{n}$ is obviously tangent to the wave surface. Indeed, one has:

$$\eta\,(\mathbf{u}\cdot\mathbf{n}) = \eta\frac{u^\lambda \partial_\lambda f}{\sqrt{-g^{\lambda\mu}\partial_\lambda f \partial_\mu f}} = \frac{\varepsilon}{\sqrt{-g^{\lambda\mu}\partial_\lambda f \partial_\mu f}} = -\eta_1,$$

and, as a result:

$$\mathbf{t}\cdot\mathbf{n} = (\eta\mathbf{u} - \eta_1\mathbf{n})\cdot\mathbf{n} = 0.$$

The vector $\mathbf{t}$ is oriented in time because its square:

$$\eta_0^2 = (\mathbf{t})^2 = \eta^2 - \eta_1^2 - 2\eta\eta_1(\mathbf{u}\cdot\mathbf{n}) = \eta^2 + \eta_1^2$$

is positive. The vector $\eta\mathbf{u}$ thus appears to be the sum of two vectors: One of them is orthogonal to the wave surface and oriented in space, and the other one is tangent to the surface and oriented in time. The "velocity of propagation" v of the wave will therefore be found to be defined by the limit of the ratio of the norms of the vectors, namely:

$$v = \lim_{\varepsilon\to 0}\left|\frac{\eta_1}{\eta_0}\right|.$$

One thus has:

$$v^{-2} = \lim_{\varepsilon\to 0}\frac{\eta_0^2}{\eta_1^2} = \lim_{\varepsilon\to 0}\left(1+\frac{\eta^2}{\eta_1^2}\right),$$

so that upon replacing η and η_1 with their values, one will have:

$$v^{-2} = 1 - \frac{g^{\lambda\mu}\partial_\lambda f \partial_\mu f}{(u^\lambda \partial_\lambda f)^2} = 1 - (1-\varphi') = \varphi'.$$

The velocity of propagation of the waves is thus $\dfrac{1}{\sqrt{\varphi'}}$. This value calls for two remarks: First, it generalizes the value that is obtained by Hugoniot's theorem in classical hydrodynamics. Second, under our hypotheses ($\varphi' \geq 1$) the velocity of propagation will be less than or equal to the velocity of light, which is taken to be unity; this is an essential necessity from the relativistic viewpoint.

II. – THE CASE IN WHICH THERE EXISTS AN ELECTROMAGNETIC FIELD

20. – The relativistic equations of electromagnetism. – Here, we adopt a classical (i.e., non-quantum) viewpoint and first indicate the equations that may appear as the rigorous equations of electromagnetism in general relativity. The essential element that enters into these equations – viz., the electromagnetic field tensor $F_{\alpha\beta}$ – varies considerably in spacetime in the neighborhood elementary particles. The corresponding macroscopic magnitudes, which vary relatively little in spacetime domains that are filled with a sizable portion of charged matter and are deduced from microscopic magnitudes by taking means will be dispensed with here. We shall therefore outline what appears to be a *provisional* relativistic theory of electromagnetism.

One deduces from the classical results of special relativity that the electromagnetic tensor $F_{\alpha\beta}$ must satisfy two systems of four partial differential equations in V_4 that are nothing but the translation of Maxwell's equations into the language of general relativity. We take these equations in the following form: First of all, the $F_{\alpha\beta}$ must satisfy the four inhomogeneous equations:

(20-1) $$D^\alpha \equiv \nabla_\beta F^{\alpha\beta} = \chi' J^\alpha,$$

in which χ' is a constant that we may reduce to 1 by a judicious choice of units, and the vector J^α describes the electrical current in spacetime domains that are filled with charged matter. The vector that appears in the right-hand side of equations (20-1) plays the same role in these equations that the energy tensor does in the right-hand side of the Einstein equations. We shall return later to the possible expressions for vector J^α, to which one gives the name of *electric current vector*.

As for the second system of Maxwell equations, it may, for an orientable manifold, be put into the form:

(20-2) $$E^\alpha \equiv \tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \nabla_\beta F_{\gamma\delta} = 0,$$

in which the left-hand side is a vector of V_4.

One may thus express the left-hand side of the Maxwell equations with the aid of the adjoint tensor $(*F)_{\alpha\beta}$, which may be substituted for $F_{\alpha\beta}$ in order to represent the electromagnetic field. From (8-3), one has:

$$F^{\beta\alpha} = -\tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} (*F)_{\gamma\delta} = 0.$$

Since η has a zero covariant derivative, one deduces from this that:

$$D^\alpha \equiv \nabla_\alpha F^{\beta\alpha} = -\tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \nabla_\beta (*F)_{\gamma\delta},$$

namely:

(20-3) $$D^\alpha \equiv -\tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \nabla_\beta (*F)_{\gamma\delta}.$$

Similarly, one has:

$$E^\alpha \equiv \tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \nabla_\beta F_{\gamma\delta} = \nabla_\beta [\tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} F_{\gamma\delta}],$$

namely:

(20-4) $$E^\alpha \equiv \nabla_\beta (*F)^{\beta\alpha}.$$

One confirms that by passing to the adjoint, the expressions for D^α and E^α that are given by (20-1) and (20-2) will correspond to the left-hand sides of equations (20-3) and (20-4), respectively.

Upon specifying $\nabla_\beta F_{\gamma\delta}$ in equations (20-2), one may put these equations into an equivalent form that has an interesting interpretation. Indeed, one has:

$$\nabla_\beta F_{\gamma\delta} = \partial_\beta F_{\rho\delta} - \Gamma^\rho_{\beta\gamma} F_{\rho\delta} - \Gamma^\rho_{\beta\delta} F_{\gamma\rho} .$$

Since the Γ are symmetric with respect to the lower two indices, and η is antisymmetric:

$$E^\alpha = \tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \nabla_\beta F_{\gamma\delta} = \tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \partial_\beta F_{\gamma\delta}.$$

One may interpret this result by saying that E^α is the adjoint vector to the exterior differential dF of the form F that is associated with the electromagnetic field. Similarly, one will have:

$$D^\alpha = -\tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \nabla_\beta (*F)_{\gamma\delta} = -\tfrac{1}{2}\, \eta^{\beta\gamma\delta\alpha} \partial_\beta (*F)_{\gamma\delta},$$

with an analogous interpretation. Equations (20-2) thus say that $dF = 0$. Therefore F will be *locally* the exterior differential of a linear form; in other words, $F_{\alpha\beta}$ will be *locally* the rotation of a vector field φ_α:

$$F_{\alpha\beta} = \partial_\alpha \varphi_\beta - \partial_\beta \varphi_\alpha .$$

φ_α is called a *vector-potential* for the electromagnetic field. Naturally, φ_α is defined only locally and up to an additive gradient. The transformations $\varphi_\alpha \to \varphi_\alpha + \partial_\alpha S$ (S is a scalar-valued function) have classically received the name of *gauge transformations* in theoretical physics. Note that in general there is no reason for there to exist a vector field φ_α on every manifold V_4 such that $F_{\alpha\beta}$ is its rotation. If one wants this to be true then it will be necessary to explicitly specify this or make a convenient hypothesis on the topology of V_4 or the domain that is occupied by the field.

One is often forced to raise the indeterminacy of the vector-potential by restricting it with supplementary conditions. The one that is classically employed is:

(20-5) $$\nabla_\alpha \varphi^\alpha = 0.$$

If φ_α is an arbitrary known vector-potential then the vector-potential that satisfies (20-5) will be deduced by the addition of a gradient of a function S, such that:

$$\Delta_2 S \equiv g^{\alpha\beta} \nabla_\beta\, \partial_\beta S = -\nabla_\alpha \varphi^\alpha.$$

S is therefore a solution to a second-order equation of hyperbolic type. In the case of an elliptic metric and a compact orientable manifold, the Hodge-de Rham theory of harmonic forms shows that if $F = d\varphi$ then there will exist one and only one global vector-potential φ_α that satisfies (20-5). No analogous theorem exists in the case of a hyperbolic signature.

Recently Dirac has introduced the supplementary condition:

$$g^{\alpha\beta}\varphi_\alpha\varphi_\beta = \frac{1}{k^2}, \tag{20-6}$$

instead of (20-5), in which k is a given constant. Under the same conditions, the function S will be defined by the first-order partial differential equation:

$$g^{\alpha\beta}(\partial_\alpha S + \varphi_\alpha)(\partial_\beta S + \varphi_\beta) = \frac{1}{k^2},$$

in which the form is completely related to that of Hamilton-Jacobi. The fact that this equation is one of the first order is encouraging in the eyes of global analysis.

Finally, let $T_{\alpha\beta}$ be an energy tensor that defines a schema that involves an electromagnetic field. From the postulates of general relativity, the electromagnetic field and the metric tensor must be related by not only the Maxwell equations, but also the Einstein equations for the interior case:

$$S_{\alpha\beta} = \chi\, T_{\alpha\beta}, \tag{20-7}$$

in which one term of $T_{\alpha\beta}$ is composed of the Maxwell tensor $\tau_{\alpha\beta}$, which is defined by (9-1).

In summation, the relativistic equations of electromagnetism are composed of the Maxwell equations (20-1) and (20-2) and the Einstein equations (20-7).

21. – Conservation conditions for the electromagnetic case. – The vectors D^α and E^α that appear in the left-hand sides of the Maxwell equations enjoy a fundamental property: *Their respective divergences are zero.* This originates from the fact that they are, up to sign, the adjoint vectors of the exterior differentials of the forms F and $*F$. For example, consider the vector:

$$E^\alpha = \tfrac{1}{2}\eta^{\beta\gamma\delta\alpha}\partial_\beta F_{\gamma\delta}.$$

Its divergence is given by the formula:

$$\nabla_\alpha E^\alpha = \frac{1}{\sqrt{|g|}}\partial_\alpha(\sqrt{|g|}E^\alpha) = \frac{1}{2\sqrt{|g|}}\varepsilon^{\beta\gamma\delta\alpha}\partial_{\alpha\beta}F_{\gamma\delta}.$$

From the symmetry of $\partial_{\alpha\beta}F_{\gamma\delta}$ with respect to the indices α and β and the antisymmetry of ε with respect to the same indices, it results that:

(21-1) $$\nabla_\alpha E^\alpha = 0.$$

When one substitutes the adjoint tensor ($*F_{\gamma\delta}$) for $F_{\alpha\beta}$ one similarly establishes:

(21-2) $$\nabla_\alpha D^\alpha = 0.$$

Since the vector J^α is restricted by equation (20-1) to differ from D^α only by a constant factor, one must necessarily have:

$$\nabla_\alpha J^\alpha = 0,$$

which we translate by saying that *the electric vector-current J^α is conservative*.

In the conservative tensor $T_{\alpha\beta}$ for a schema that involves an electromagnetic field, there appears the term:

(21-3) $$\tau^\alpha_\beta = \tfrac{1}{4} g^\alpha_\beta F_{\lambda\mu} F^{\lambda\mu} - F^{\alpha\rho} F_{\beta\rho},$$

which is associated with the electromagnetic field $F_{\alpha\beta}$. We seek to evaluate $\nabla_\alpha \tau^\alpha_\beta$, which appears in the conservation conditions. One immediately has:

$$\nabla_\alpha \tau^\alpha_\beta = \tfrac{1}{2} F^{\lambda\mu} \nabla_\beta F_{\lambda\mu} - F^{\alpha\beta} \nabla_\alpha F_{\beta\rho} - F_{\beta\rho} \nabla_\alpha F^{\alpha\rho}.$$

We evaluate the second term on the right-hand side. One has, by an obvious transformation:

$$F^{\alpha\rho} \nabla_\alpha F_{\beta\rho} = \tfrac{1}{2} F^{\alpha\beta} [\nabla_\alpha F_{\beta\rho} + \nabla_\rho F_{\alpha\beta}].$$

Upon taking into account the Maxwell equations (20-2), which may be written:

$$\nabla_\alpha F_{\beta\rho} + \nabla_\beta F_{\rho\alpha} + \nabla_\rho F_{\alpha\beta} = 0,$$

this takes the form:

$$F^{\alpha\rho} \nabla_\alpha F_{\beta\rho} = \tfrac{1}{2} F^{\alpha\rho} \nabla_\beta F_{\alpha\rho}.$$

This term thus annihilates the first term of the right-hand side, and what will remain is:

$$\nabla_\alpha \tau^\alpha_\beta = F_{\rho\beta} \nabla_\alpha F^{\alpha\rho}.$$

If one replaces $\nabla_\alpha F^{\alpha\rho}$ by the value it gets from the Maxwell equations (20-1) then it will follow that:

(21-4) $$\nabla_\alpha \tau^\alpha_\beta = F_{\rho\beta} J^\rho.$$

Consider a pure electromagnetic field schema. In such a schema, the vector-current is zero by hypothesis. On the other hand, from the conservative character of the energy tensor, one will have $\nabla_\alpha \tau^\alpha_\beta = 0$; equations (21-4) show that these conditions are quite compatible. Similarly, one may note that they are equivalent, in general. In order for this to be true it is necessary and sufficient that the determinant of the matrix $F_{\alpha\beta}$ be different from zero. From the considerations of sec. 8, the hypothesis made is therefore that $\Phi \neq 0$; i.e., the electric field and the magnetic field cannot be orthogonal in a local physical interpretation.

22. – The Cauchy problem for a pure electromagnetic field schema. – From the considerations of sec. **20**, the Maxwell equations admit conservation conditions that are analogous to the ones that are verified by the Einstein equations. In order to study the interaction of the gravitational field and the electromagnetic field, we are therefore led to study the Cauchy problem that relates to the two fields locally, and, as a consequence, the set of Maxwell and Einstein equations together. We put ourselves in the "exterior in the unitary sense" case, i.e., under the hypothesis of an electromagnetic schema in the absence of matter. The system of equations to be studied may be written:

(22-1) $$D^{\alpha} = 0, \qquad E^{\alpha} = 0,$$

and:

(22-2) $$Q_{\alpha\beta} \equiv S_{\alpha\beta} - \chi\, \tau_{\alpha\beta} = 0,$$

in which $\tau_{\alpha\beta}$ is defined by (21-1). Relative to these equations, our problem is then the following one:

PROBLEM. – *Given electromagnetic and gravitational fields on a hypersurface S***,** *determine these fields outside of S under the assumption that they satisfy equations* (22-1) *and* (22-2).

If S is locally represented by $x^0 = 0$ then the Cauchy data will be the values of the $g_{\alpha\beta}$ (which is at least three-times continuously differentiable) on S, and the values of the $\partial_0 g_{\alpha\beta}$ and the components $F_{\alpha\beta}$ of the electromagnetic field (which is at least twice-continuously differentiable) on S. First of all, we assume that S is not tangent to the elementary cone in the local domain considered ($g^{00} \neq 0$).

The only field derivatives that may be discontinuous upon crossing S are the derivatives $\partial_{00} g_{\alpha\beta}$ and $\partial_0 F_{\alpha\beta}$, since a simple differentiation on S gives the values of the other derivatives of the same order for these fields. We shall thus make these derivatives explicitly evident in the Maxwell and Einstein equations.

The Maxwell equations (17-1) may be written:

(22-3) $$D_{\alpha} \equiv g^{\beta\gamma}\, \nabla_{\beta} F_{\gamma\alpha} = 0.$$

If we make the derivatives $\partial_0 F_{\alpha\beta}$ appear in them explicitly then it will follow that:

(22-4) $$D_i \equiv g^{00}\, \partial_0 F_{0i} + g^{0k}\, \partial_0 F_{ki} + \delta_i\,(\text{C.d.}) = 0,$$

and:

(22-5) $$D_0 \equiv g^{0i}\, \partial_0 F_{i0} + \delta_0\,(\text{C.d.}) = 0,$$

in which the δ_λ are quantities on S that are deduced from the Cauchy data by algebraic operations and derivations on S. From this, one immediately deduces the quantity:

$$g^{0i} D_i + g^{00} D_0 = D^0,$$

which does not contain any derivative of index 1 of the electromagnetic field. We are thus led to replace the system (22-3) with the equivalent system (since $g^{00} \neq 0$) that is composed of the three equations (22-4) and the equation:

$$D^0 = 0. \tag{22-6}$$

Similarly, the Maxwell equations (17-2) may be decomposed into one system of three equations:

$$E^i \equiv \tfrac{1}{2}\eta^{\beta\gamma\delta i}\partial_\beta F_{\gamma\delta} = \tfrac{1}{2}\eta^{\beta\gamma\delta i}\partial_\beta F_{\gamma\delta} + \varepsilon^i \text{(C.d.)} = 0, \tag{22-7}$$

and the equation:

$$E^0 \equiv \tfrac{1}{2}\eta^{jkl0}\partial_j F_{kl} = 0, \tag{22-8}$$

in which the ε^i are quantities on S that are calculated by starting with the Cauchy data, and E^0 enjoys this same property. One will note that (22-8) expresses the fact that the form $\frac{1}{2} F_{kl}\, dx^k \wedge dx^l$ that is induced on S has a zero exterior derivative.

Finally, from the considerations of sec. **12**, the system of Einstein equations is equivalent to a system that decomposes into two. The first six equations will be the following ones ([1]):

$$R_{ij} - \chi(\tau_{ij} - \tfrac{1}{2} g_{ij}\tau) = 0 \qquad (\tau = \tau_\alpha^\alpha). \tag{22-9}$$

As for the other four, they are written:

$$Q_\lambda^0 \equiv S_\lambda^0 - \chi\tau_\lambda^0 = 0, \tag{22-10}$$

in which the left-hand sides are quantities that are calculated on S by starting with the Cauchy data.

Having said this, equations (22-7) give the values of the three derivatives $\partial_0 F_{jk}$ on S. Since the quantity g is different from zero, equations (22-4) give the values of the three derivatives $\partial_0 F_{0i}$ on S, and equations (22-9) give those of the derivatives $\partial_{00} g_{ij}$. All tolled, with the same reservations as before, the second derivatives of the potentials and the first derivatives of the electromagnetic field will be continuous upon crossing the hypersurface S. This conclusion may be obviously extended to the higher-order derivatives by differentiating equations (22-4), (22-7), and (22-9) with respect to x^0, since the present analysis has not used equations (22-4), (22-7), and (22-9), up till now.

([1]) Naturally, $\tau = 0$ for the energy-momentum tensor of an electromagnetic field; however, we do not profit from this circumstance here, in such a way that the present calculations will be valid under more general conditions (see sec. **25**).

23. – The integration of the Maxwell-Einstein equations. – The left-hand sides of equations (22-6), (22-8), and (22-10) – namely, D^0, E^0, and Q_λ^0 – are quantities that are directly calculable on S from starting with the Cauchy data. It will then result that the chosen Cauchy data must be such that the six preceding quantities are zero on S.

Hence, consider Cauchy data that satisfies these six conditions on S, and assume that we know a pair of tensors ($g_{\alpha\beta}$, $F_{\alpha\beta}$) that correspond to these Cauchy data and satisfy equations (22-4), (22-7), and (22-9). First of all, the left-hand sides of (17-1) and (17-2) are related by the conservation conditions $\nabla_\alpha D^\alpha = 0$ and $\nabla_\alpha E^\alpha = 0$. For such a solution of (22-4) and (22-7), these conditions will take the form:

$$\partial_0 D^0 = A^i \partial_i D^0 + BD^0 \qquad (A^i \text{ and } B \text{ are of class } C^0),$$

and:

$$\partial_0 E^0 = -\Gamma^\lambda_{\lambda 0} E^0 .$$

Therefore, D and E each satisfy a linear homogenous first order partial differential equation which is obtained by solving for the derivative with respect to x^0; for null data on s there will correspond only zero solutions. Therefore, our pair ($g_{\alpha\beta}$, $F_{\alpha\beta}$) satisfies the Maxwell equations (17-1) and (17-2) outside of S.

From (21-2), since the system (17-1) is satisfied, the tensor $\tau_{\alpha\beta}$ will be conservative. The same will therefore be true for the tensor $Q_{\alpha\beta} \equiv S_{\alpha\beta} - \chi\tau_{\alpha\beta}$. Therefore, the left-hand sides of (22-9) and (22-10) will be coupled by relations that stem from the fact that $\nabla_\alpha Q^\alpha_\beta = 0$. In order to find the form of the conditions when (22-9) is satisfied, it will suffice to calculate the values of Q_i^0 and Q_0^0 for the pair envisioned ($g_{\alpha\beta}$, $F_{\alpha\beta}$). If we take (22-9) into account then an easy calculation will give:

$$Q_i^0 = g^{00}[R_{i0} - \chi(\tau_{i0} - \tfrac{1}{2} g_{i0}\tau)],$$
$$Q_0^0 = \tfrac{1}{2} g^{00}[R_{00} - \chi(\tau_{00} - \tfrac{1}{2} g_{00}\tau)].$$

For such a pair, it will then result that the expression:

$$R + \chi\tau = g^{\alpha\beta}[R_{\alpha\beta} - \chi(\tau_{\alpha\beta} - g_{\alpha\beta}\tau)]$$

can be expressed by a homogeneous linear combination with continuous coefficients ($g^{00} \neq 0$) of the Q_λ^0. In turn, the same thing will true for the components:

$$Q_{\alpha\beta} = S_{\alpha\beta} - \chi\tau_{\alpha\beta} = [R_{\alpha\beta} - \chi(t_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta}\tau)] - \tfrac{1}{2} g_{\alpha\beta}(R + \chi\tau)$$

that is true for the components Q^i_β. It will then result that for a solution of (22-9) the conservation conditions for the tensor Q^α_β will reduce to the form:

$$g^{00}\partial_0 Q_\lambda^0 = A_\lambda^{i\rho}\partial_i Q_\rho^0 + B_\lambda^0 Q_\rho^0 ,$$

in which the A and B are continuous, and the same reasoning applies. Our pair ($g_{\alpha\beta}$, $F_{\alpha\beta}$) then satisfies the Maxwell-Einstein equations outside of S.

Our initial problem is thus found to be subdivided further into two more distinct problems.

PROBLEM I – Find Cauchy data that satisfy:

(23-1) $$D^0 = 0 \qquad E^0 = 0 \qquad Q^0_\lambda = 0,$$

on S.

PROBLEM II – Integrate the system of equations (22-4), (22-7), and (22-9), for Cauchy data that satisfy the system (23-1) on S.

Under the (abusive) hypothesis that the data of the second problem are all real-analytic, the corresponding system will again admit one and only one real-analytic solution, up to a change of coordinates that preserves the coordinates of the points of S and the Cauchy data. The method of Fourés is then applied to the present system, which permits us to obtain the theorems of existence and "physical uniqueness" under the hypothesis of simple differentiability, under the condition that g^{00} is positive (S is oriented in space).

If the manifold S is such that g^{00} is identically zero then the derivatives $\partial_{00} g_{ij}$ and $\partial_0 F_{0i}$ of the gravitational field and the electromagnetic field might be discontinuous upon crossing S. In particular, one may construct solutions to the Maxwell-Einstein equations such that the derivatives of the gravitational field are continuous upon crossing S but the derivatives $\partial_0 F_{0i}$ are discontinuous. The manifold S appears as the characteristic manifold of the Maxwell equations or as the electromagnetic wave front. We may therefore state:

THEOREM. – *There is an identity between the characteristic manifolds of the Einstein equations and those of the Maxwell equations. These manifolds are the solutions to the equation* $\Delta_1 f = 0$.

We have thus established the identity of the gravitational waves and the electromagnetic waves in a completely rigorous manner, and, as a result, the identity of the laws of propagation of the two fields. In particular, the null-length geodesics that constitute the gravitational rays also constitute the electromagnetic rays, or light rays.

23 (cont.). – The singular electromagnetic field and null-length geodesics. –

a) Consider a *pure electromagnetic field* $F_{\alpha\beta}$ in a spacetime domain that is *singular* in the sense of sec. **10**. The space vectors of the electric field and the magnetic field are orthogonal and of equal norm relative to any orthonormal frame. From the considerations of sec. **10**, there exist simple orthonormal frames ($\mathbf{e}_\alpha$) at each point x of this domain such that the components (X, Y, Z) and (L, M, N) of the electric and magnetic fields relative to these frames are:

$$\begin{array}{lll} X = \xi, & Y = 0, & Z = 0, \\ L = 0, & M = \varepsilon\xi, & N = 0, \end{array}$$

in which $\varepsilon = \pm 1$. From (8-1) and (8-4), one deduces that the only non-zero components of the tensors $F^{\alpha\beta}$ and $(*F)^{\alpha\beta}$ are:

$$\begin{array}{rlrl} F^{01} & = \xi, & F^{31} & = \varepsilon\xi, \\ (*F)^{23} & = \xi, & (*F)^{20} & = \varepsilon\xi, \end{array}$$

respectively.

As for the tensor $\tau_{\alpha\beta}$, its components with respect to the frame envisioned are given by the matrix (10-11). The vector:

$$\boldsymbol{l} = \mathbf{e}_0 + \varepsilon\, \mathbf{e}_3$$

defines an isotropic proper direction of $\tau_{\alpha\beta}$, and one has the tensorial equation:

(23*a*-1) $$\tau_{\alpha\beta} = \xi^2\, l_\alpha\, l_\beta .$$

b) Refer to the spacetime domain envisioned to the simple orthonormal frames $(\mathbf{e}_\alpha)$. One has:

$$ds^2 = (\omega^0)^2 - \sum_i (\omega^i)^2 .$$

Let ω_α^β be the Pfaff forms that define the Riemannian connection that is attached to the metric ([1]) when expressed with respect to these frames. We set:

$$\omega_\alpha^\beta = \gamma_{\alpha\rho}^\beta \omega^\rho \qquad\qquad \gamma_{\alpha\beta\rho} = g_{\alpha\sigma}\gamma_{\beta\rho}^\sigma$$

in which the $\gamma_{\alpha\beta\rho}$ generalize the Ricci rotation coefficients. The covariant derivative of the metric tensor is given by:

$$\nabla_\rho g_{\alpha\beta} = \partial_\rho g_{\alpha\beta} - \gamma_{\alpha\rho}^\sigma g_{\alpha\beta} - \gamma_{\alpha\rho}^\sigma g_{\alpha\beta} = -(\gamma_{\alpha\beta\rho} + \gamma_{\beta\alpha\rho}),$$

in which ∂_ρ represents a Pfaffian derivative. From this, one deduces that:

(23*a*-2) $$\gamma_{\alpha\beta\rho} + \gamma_{\beta\alpha\rho} = 0,$$

which are relations that will be useful to us in what follows.

c) The pure electromagnetic field envisioned satisfies the Maxwell equations:

([1]) The reader is referred to Book II, Chapter IV.

(23*a*-3) $$\nabla_\beta F^{\beta\alpha} = 0,$$
and
(23*a*-4) $$\nabla_\beta ({}^*F^{\beta\alpha}) = 0.$$

In the frame envisioned, equations (23*a*-2), for $\alpha = 0$ or $\alpha = 3$, may be written explicitly in the form:

$$\nabla_\beta F^{\beta 0} = -\partial_1 \xi + \xi(\gamma_{122} + \gamma_{133} - \varepsilon\gamma_{301} + \varepsilon\gamma_{103}) = 0,$$
$$\varepsilon \nabla_\beta F^{\alpha 3} = -\partial_1 \xi + \xi(\gamma_{123} - \gamma_{100} - \varepsilon\gamma_{301} + \varepsilon\gamma_{130}) = 0,$$

(23*a*-5) $$\gamma_{100} + \gamma_{133} + \varepsilon(\gamma_{103} + \gamma_{130}) = 0 .$$

Similarly, upon specifying (23*a*-4) for $\alpha = 0$ and $\alpha = 3$, it follows that:

(23*a*-6) $$\gamma_{200} + \gamma_{233} + \varepsilon(\gamma_{203} + \gamma_{230}) = 0.$$

We then evaluate the components of the vector:

$$t_\beta \nabla_\beta l^\alpha = l_\beta\, \partial_\beta l^\alpha + \gamma^\alpha_{\rho\beta} l^\rho l^\beta ,$$

in which ***l*** admits the components $(1, 0, 0, \varepsilon)$ in the frame envisioned. First, it follows that:

$$l^\beta \nabla_\beta l^0 = \varepsilon\,\gamma^0_{30} + \gamma^0_{33} = \varepsilon\,\gamma_{300} + \gamma_{303} ,$$
$$l^\beta \nabla_\beta l^0 = \varepsilon\,\gamma^0_{30} + \gamma^0_{33} = \varepsilon\,(\varepsilon\,\gamma_{300} + \gamma_{303}).$$

From the other part of (23*a*-5) and (23*a*-6), one deduces that:

$$l^\beta \nabla_\beta l^1 = \gamma^1_{00} + \gamma^1_{33} + \varepsilon(\gamma^1_{03} + \gamma^1_{30}) = \gamma_{100} + \gamma_{133} + \varepsilon\,(\gamma_{103} + \gamma_{130}) = 0,$$
$$l^\beta \nabla_\beta l^2 = \gamma^2_{00} + \gamma^2_{33} + \varepsilon(\gamma^2_{03} + \gamma^2_{30}) = \gamma_{200} + \gamma_{233} + \varepsilon\,(\gamma_{203} + \gamma_{230}) = 0.$$

If one sets:

$$a = \varepsilon\,\gamma_{300} + \gamma_{303},$$

in the frame envisioned then there will exist a scalar *a* such that the tensorial equation:

(23*a*-7) $$l^\beta \nabla_\beta l^\alpha = a\, l^\alpha$$

will be satisfied. This equation says that the direction of the vector ***l*** is parallel-transported along its proper direction. Therefore, *the trajectories of the vector **l** will be null-length geodesics in the spacetime envisioned* ([1]).

One sees from (23*a*-1) and (23*a*-7) that a singular, pure electromagnetic field schema may be interpreted as a singular field medium whose "streamlines" are null-length

([1]) This result is due to LOUIS MARIOT. Comptes rendus (1954).

geodesics. We interpret these results by saying that a singular, pure electromagnetic field describes a fluid of photons.

24. The Lorentz transfer equation. – Except for the case of a purely electromagnetic distribution, the fundamental equations for electromagnetism that we gave before will remain incomplete as long as one does not know an expression for the electric vector-current J^λ. One may complete these equations with a supplementary equation that depends upon the vector-current and gives us a way of implicitly interpreting the electronic hypothesis. We assume that, in reality, any elementary electronic current is a convection current; in other words, the electric current vector J^λ must be collinear with the unitary velocity vector u^λ of the material. We are therefore led to add the equation.

(24-1) $$J^\lambda = \mu\, u^\lambda,$$

in which μ is a scalar variable that we call the *proper electric charge density*. We give equation (24-1) the name of the *Lorentz transfer equation*. A great number of consequences result from the hypotheses that this equation suggests.

Consider the case of a matter-electromagnetic field schema. It corresponds to the energy-momentum tensor:

(24-2) $$T_{\alpha\beta} = \rho\, u_\alpha u_\beta + \tau_{\alpha\beta}.$$

One may associate the vector:

$$\rho\, K_\beta = -\nabla_\alpha \tau^\alpha_\beta = -F_{\rho\beta} J^\rho$$

with this form of the tensor, and [from (17-6) and (17-7)] the conservation conditions (24-4) for the tensor will take the form:

(24-3) $$\nabla_\alpha(\rho\, u^\alpha) = -F_{\rho\beta} J^\rho u^\sigma,$$

and

(24-4) $$u^\alpha \nabla_\alpha u_\beta = (g_{\alpha\beta} - u_\alpha u_\beta)\frac{1}{\rho} F^{\alpha\rho} J_\rho.$$

These conservation conditions, as well as the ones that are associated with the Maxwell equation (17-1), take a very simple form when one accepts the validity of the Lorentz transfer equation. From the conservation of the current vector, one first deduces that:

(24-5) $$\nabla_\alpha(\mu\, u^\alpha) = 0.$$

(24-6)

Moreover, one will have the term:

$$F_{\rho\beta} J^\rho u^\sigma = \mu\, F_{\rho\beta} u^\rho u^\sigma = 0$$

in the right-hand side of (24-3) as a result of the antisymmetry of $F_{\rho\sigma}$. Therefore, (24-3) will take the simple form:

(24-7) $$\nabla_\alpha(\rho\, u^\alpha) = 0.$$

Similarly, the last term of the right-hand side of (24-4) is null for the same reason if $J_\rho = \mu\, u_\rho$, and (24-4) then takes the form:

(24-7) $$u^\alpha \nabla_\alpha u_\beta = \frac{\mu}{\rho}\, F_{\alpha\beta} u^\alpha,$$

which is the differential system of the streamlines.

From equations (24-5) and (24-6), which translate into the conservation of electricity and matter, respectively, one derives a physically interesting consequence. Indeed, these two equations may be written:

$$\nabla_\alpha u^\alpha + u^\alpha \frac{\partial_\alpha \mu}{\mu} = 0,$$

and:

$$\nabla_\alpha u^\alpha + u^\alpha \frac{\partial_\alpha \rho}{\rho} = 0.$$

It follows by subtraction that:

$$u^\alpha \left(\frac{\partial_\alpha \mu}{\mu} - \frac{\partial_\alpha \rho}{\rho} \right) = 0$$

or:

$$u^\alpha \partial_\alpha \left(\log \frac{\mu}{\rho} \right) = 0.$$

Therefore, the ratio $k = \dfrac{\mu}{\rho}$ *will remain constant along the streamline.* In general, the ratio $k = \dfrac{\mu}{\rho}$ may vary from one streamline to another. We are especially interested in the "homogeneous" schemas, for which k is constant over the entire spacetime domain considered.

25. – The Cauchy problem for the matter-electromagnetic field schema. – We conclude this chapter by studying the local Cauchy problem for a set of two fields in the case of a matter-electromagnetic field schema that satisfies the Lorentz transfer equation. Given gravitational and electromagnetic fields on a hypersurface S, we propose to determine the fields outside of S when one assumes that they satisfy the equations:

(25-1) $$D^\alpha = \mu\, u^\alpha, \qquad E^\alpha = 0,$$

(25-2) $$Q_{\alpha\beta} \equiv S_{\alpha\beta} - \chi\, T_{\alpha\beta} = 0,$$

in which:

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta + \tau_{\alpha\beta};$$

$\tau_{\alpha\beta}$ denotes the energy-momentum tensor for the electromagnetic field. We set:

$$P_{\alpha\beta} = S_{\alpha\beta} - \chi\tau_{\alpha\beta},$$

and equations (25-2) may be written:

$$Q_{\alpha\beta} \equiv P_{\alpha\beta} - \chi\,\rho\, u_\alpha u_\beta = 0.$$

It is convenient to add the equation:

(25-3) $$g^{\alpha\beta} u_\alpha u_\beta = 1,$$

as well as the inequality $\rho > 0$.

We further assume that S is not tangent to the elementary cone; if it is locally represented by the equation $x^0 = 0$ then one will have $g^{00} \neq 0$. We know that under these conditions we may replace the system (25-1), (25-2), and (25-3) with an equivalent system that decomposes into two systems. The first system will be composed of the following equations:

(25-4) $$D_i = \mu\, u_i\,, \qquad E^i = 0,$$

and:

(25-4) $$R_{ij} - \chi\,(T_{ij} - \tfrac{1}{2}\, g_{ij} T) = 0.$$

The second system will be composed of the equations:

(25-6) $$D^0 = \mu\, u^0, \qquad E^0 = 0,$$

and:

(25-7) $$Q_\lambda^0 \equiv P_\lambda^0 - \chi\, u_\lambda u^0 = 0,$$

to which we add equation (25-3).

Any solution ($g_{\alpha\beta}$, $\mathrm{F}_{\alpha\beta}$, u_α, μ, ρ) of the system considered satisfies equations (24-5), (24-6), and (24-7), which originate in the conservation conditions, namely:

(25-8) $$\nabla_\alpha(\mu\, u^\alpha) \equiv u^0 \partial_0 \mu + \mu \partial_0 u^0 + G(\text{C.d.}, u_\lambda, \partial_i u_\lambda, \mu, \partial_i \mu) = 0,$$

(25-9) $$\nabla_\alpha(\rho\, u^\alpha) \equiv u^0 \partial_0 \rho + \rho \partial_0 u^0 + F(\text{C.d.}, u_\lambda, \partial_i u_\lambda, \rho, \partial_i \mu) = 0,$$

(25-10) $$u^\alpha \nabla_\alpha u_\beta - \frac{\mu}{\rho} F_{\alpha\beta} u^\alpha \equiv u^0 \partial_0 u_\beta + \Phi_\beta(\text{C.d.}, u_\lambda, \partial_i u_\lambda, \frac{\mu}{\rho}) = 0.$$

Having said this, assume that the Cauchy data $g_{\alpha\beta}$ and $\partial_0 g_{\alpha\beta}$, F_{ab} are three and two times continuously differentiable on S, respectively, and naturally, that they satisfy $E^0 = 0$ (i.e., the existence of a vector-potential for F_{ij} on S). This data will determine the values of the P_λ^0 on S. A calculation that is identical to an earlier one will then permit us to determine the values of the u_λ and ρ on S. First, one has:

$$(\chi\rho\, u^0)^2 = g^{\lambda\mu}\, P_\lambda^0 P_\mu^0 .$$

One assumes that the right-hand side is strictly positive, and sets:

$$g^{\lambda\mu} P_\lambda^0 P_\mu^0 = (\Omega^0)^2.$$

One then deduces:

$$u_\lambda = \frac{P_\lambda^0}{\Omega^0}, \qquad u^0 = \frac{P^{00}}{\Omega^0}, \qquad \chi\rho = \frac{(\Omega^0)^2}{P^{00}}, \qquad\qquad (P^{00} > 0)$$

and (25-6) will give the value of μ, since D^0 is known on S.

Under these conditions, equations (25-5) will give the values of the derivatives $\partial_{00} g_{ij}$ on S, and equations (25-4) will give the values of the derivatives $\partial_0 F_{ij}$ and $\partial_0 F_{0i}$. Then, since $u^0 \neq 0$, equations (25-10), (25-9), and (25-8) will give the equations of $\partial_0 u_\beta, \partial_0 \rho$, and $\partial_0 \mu$ on S. Therefore, these quantities will have well-defined values on a hypersurface S that satisfies the hypotheses that were made, and might not be discontinuous upon traversing it.

Consider a set ($g_{\alpha\beta}$, $F_{\alpha\beta}$, u_λ, ρ, μ) that satisfies equations (25-4), (25-5), (25-8), (25-9), and (25-10) in a neighborhood of S, and satisfies equations (25-3), (25-6), (25-7) *on* S. From (25-10), it results that:

$$u^\alpha (u^\beta \nabla_\alpha u_\beta) = 0;$$

therefore u_λ has a constant length along the streamlines. Since it is unitary on S, it is unitary outside of S, and (25-3) is satisfied outside of S. Similarly, for a solution of (25-4), the conservation conditions $\nabla_\alpha E^\alpha = 0$, $\nabla_\alpha (D^\alpha - \mu u^\alpha) = 0$, one of which is satisfied identically and the other of which follows from (25-8), may be written:

$$\partial_0 E^0 = -\Gamma_{\lambda 0}^0 E^0,$$
$$\partial_0 (D^0 - \mu u^0) = A^i \partial_0 (D^0 - \mu u^0) + B (D^0 - \mu u^0).$$

As a result, (25-6) are satisfied outside of S.

Since this is true, equations (25-9) and (25-10) will entail the conservative character of T_β^α, and, as a result, that of $Q_\beta^\alpha = S_\beta^\alpha - \chi T_\beta^\alpha$. The calculations that were made in sec. **24**, in which it sufficed to substitute the tensor T_β^α for the tensor $\tau_{\alpha\beta}$, show that for any solution of (25-5), the conservation conditions $\nabla_\alpha Q_\beta^\alpha = 0$ may be written:

$$\partial_0 Q_\lambda^0 = A_\lambda^{i\rho} \partial_i Q_\rho^0 + B_\lambda^0 Q_\rho^0,$$

in which the A's and B's are continuous. Therefore, equations (25-7) are satisfied outside of S.

Since (25-3), (25-6), and (25-7) are satisfied on S, it results that it suffices to preoccupy ourselves with the system (25-4), (25-5), (25-8), (25-9), and (25-10) from the standpoint of integration.

Here, one may further establish that, under the hypotheses made, there exists one and only one solution to the Cauchy problem, up to a change of coordinates that preserves the coordinates of any point of S. As in the case of the pure matter schema, the exceptional manifolds are the characteristic manifolds V_C^3, and the manifolds that are generated by the streamlines $V_3^{(1)}$.

An analogous theory may be developed in the case for which T corresponds to a *perfect fluid-electromagnetic field* schema ([1]).

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[1] See chapter VI.

CHAPTER III

MATCHING CONDITIONS

I. – THE GRAVITATIONAL CASE

26. – GAUSSIAN coordinates. – Consider a hypersurface S in V – for example, one oriented in space – and suppose that the coordinates $(x^{\lambda'})$ are locally defined by the equation $x^{0'} = 0$. Construct a geodesic through each point ξ of S that is normal to the S, a geodesic that is therefore oriented in time. We shall establish that these geodesics are orthogonal to an infinitude of hypersurfaces (Fig. 1).

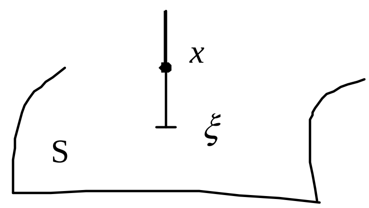

Figure 1.

Indeed, we let **V** be a normal vector ($\mathbf{V}^2 = +1$) that is tangent to a geodesic and study its rotation, which is written:

$$\omega_{\alpha\beta} = \nabla_\alpha V_\beta - \nabla_\beta V_\alpha = \partial_\alpha V_\beta - \partial_\beta V_\alpha$$

in a system of arbitrary coordinates.

Since the trajectories of **V** are geodesics one has:

$$V^\alpha \nabla_\alpha V_\beta = 0.$$

On the other hand, since **V** is normal:

$$V^\alpha \nabla_\beta V_\alpha = 0.$$

One deduces from this by subtraction that:

(26-1) $$V^\alpha \omega_{\alpha\beta} = 0.$$

For the (x^λ) coordinates, take coordinates with the property that the lines along which the (x^i) are constant are the geodesics considered. If a geodesic encounters S at x then one may, for example, take $x = x^{i'}(\xi)$. Note that one then has $V^i = 0$. Suppose, moreover, that $x^0 = 0$ defines S. Since the geodesics are orthogonal to S, one then has $V_i = 0$ on S.

Having made these assumptions, (26-1) is written:

(26-2) $$\omega_{0i} = 0$$

in these coordinates.

One the other hand, since $\omega_{\alpha\beta}$ is a rotation:

$$\varepsilon^{\alpha\beta\gamma\delta}\partial_\alpha\omega_{\beta\gamma} = 0.$$

For $\delta = i$, upon taking (26-2) into account, it will follow that:

$$\partial_0\omega_{jk} = 0. \tag{26-3}$$

Now, one has $V_i = 0$ on S, and, as a result:

$$\omega_{jk} = \partial_j V_k - \partial_k V_j = 0.$$

From (26-2) and (26-3) it will then result that $\omega_{\alpha\beta} = 0$ in a neighborhood of S. As a result, there will locally exist a function φ such that **V** is its gradient:

$$V_\alpha = \partial_\alpha\varphi,$$

and the geodesics are orthogonal trajectories to the surfaces φ = constant. Since $\partial_i\varphi = 0$ on S, φ will reduce to a constant, which one may assume to be zero. If one adopts x^0 to be the normal coordinate then ds^2 will take the form:

$$ds^2 = g_{00}\,(dx^0)^2 + g_{ij}(x^\lambda)\,dx^i\,dx^j.$$

However, one has:

$$g^{00} = \Delta_1\varphi = g^{\alpha\beta}V_\alpha V_\beta = 1.$$

Thus:

$$g_{00} = 1,$$

and the metric may be written:

$$ds^2 = (dx^0)^2 + g_{ij}(x^\lambda)\,dx^i\,dx^j. \tag{26-4}$$

The hypersurface $x_0 = c$ is therefore obtained by measuring off an arc of length c along each normal geodesic that starts on S.

The coordinates (x^i, x^0), whose existence we just proved, are called the *Gaussian coordinates* that are associated with S and the coordinates $x^{i'}$ $(= x^i)$ on S. Conversely, suppose that one has a coordinate system (x^λ) in a neighborhood of a hypersurface S $(x^0 = 0)$, for which the metric takes the form (26-4). It is easy to see that the lines along which x^0 alone varies will be geodesics, because:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\lambda}{ds}\frac{dx^\mu}{ds} = \Gamma^\lambda_{00} = 0,$$

along these geodesics, and the coordinates (x^λ) will be the Gaussian coordinates that are associated with S.

The results that we just established for a hypersurface S that is oriented in space persist, with a slight modification, when S is oriented in time. The vector **V** must then be normalized by $\mathbf{V}^2 = -1$ in such a way that $g^{00} = g_{00} = -1$. In this case the Gaussian coordinates lead to a metric of the form:

$$ds^2 = -(dx^0)^2 + g_{ij}(x^\lambda)\, dx^i\, dx^j. \tag{26-5}$$

27. – Matching conditions. **–** We propose to represent a model of several gravitating masses on a spacetime manifold V_4. Each mass generates a "world tube" that is bounded by a hypersurface S. On one side of S, there exists a metric that satisfies the Einstein equations in the interior case. Exterior to all of the masses we have a metric that satisfies the equations for the exterior case. In each of these domains the potentials, as well as their first derivatives, are continuous relative to an admissible coordinate system.

What happens when one crosses the hypersurface S? Conforming to our general axioms (see sec. **1**), we must impose the following conditions, whose first usage dates back to Schwarzschild.

MATCHING CONDITIONS. – *For any point x of S there exists an admissible coordinate system whose domain contains x such that the potentials and their first derivatives are continuous upon crossing S relative to this system.*

Since the manifold V_4 is twice-differentiable, the potentials and their first derivatives will therefore be precisely continuous for any admissible coordinate system and at any point of V_4. The second derivatives of the potentials will themselves be continuous upon crossing S. The matching conditions will obviously be contained implicitly in our general axioms, but, because of their importance, we have chosen to state them explicitly here.

We verify that it results from the matching conditions that the manifolds S are necessarily oriented in time.

28. – GAUSSIAN coordinates and matching conditions. – Consider one of the hypersurfaces S (which is oriented in time), and assume that it is locally defined by the equation $x^{0'} = 0$ in an admissible coordinate system $(x^{\lambda'})$ of V_4. Conforming to the matching conditions, the potentials $g_{\lambda'\mu'}$ and their first derivatives $\partial_0 g_{\lambda'\mu'}$ are assumed to be *continuous* upon crossing S.

Now refer both parts of a neighborhood of S to the Gaussian coordinates that are associated with S and the coordinates $(x^{\lambda'})$ on S. We propose to rigorously establish that *relative to the Gaussian coordinates, the potentials are continuous upon crossing S, and that under these conditions the Gaussian coordinates collectively form an admissible coordinate system for* V_4.

First of all, for $x^0 = 0$ the coordinates x^i and $x^{i'}$ are identical and $x^{0'} = 0$. From this one deduces that:

$$x^{i'} = x^i + x^0\varphi(x^\lambda), \qquad x^{0'} = x^0\varphi(x^\lambda),$$

in a neighborhood of S.

As a result:

$$(A_j^{i'})_S = \delta_j^i, \qquad (A_j^{0'})_S = 0,$$

and the $g_{ij} = g_{i'j'}$ will have well-defined values on S.

Now, let us study the values of $A_0^{\lambda'}$. These quantities are the components of a vector that is tangent to the lines x^i = constant, i.e., to the geodesics that are normal to S. Moreover:

$$g_{\lambda'\mu'} A_0^{\lambda'} A_0^{\mu'} = -1.$$

The $A_0^{\lambda'}$ are therefore the components $V^{\lambda'}$ of the normal vector **V** that is tangent to the geodesics:

$$V^{\lambda'} = A_0^{\lambda'}.$$

The vector **V** admits the covariant components $V_{i'} = 0$ and $V_{0'}$ on S in the $(x^{\lambda'})$ coordinates, such that $g^{00}(V_{0'})^2 = -1$. It will then result that the quantities:

$$(A_0^{\lambda'})^2 = (V^{\lambda'})_S = g^{\lambda'0'} V_{0'}$$

have well-defined values on S and are subject to discontinuities upon crossing S.

One immediately deduces from this that the first derivatives:

$$\partial_0 g_{ij} = \partial_0 (A_i^{\lambda'} A_j^{\mu'} g_{\lambda'\mu'}) = \partial_i A_0^{\lambda'} A_j^{\mu'} g_{\lambda'\mu'} + \partial_j A_0^{\mu'} A_i^{\lambda'} g_{\lambda'\mu'} + A_0^{\rho'} A_i^{\lambda'} A_j^{\mu'} \partial_{\rho'} g_{\lambda'\mu'}$$

are themselves continuous upon crossing S.

Finally, we show that the derivatives $\partial_\nu A_\mu^{\lambda'}$ are continuous upon crossing S. The problem is posed only for the derivatives $\partial_0 A_0^{\lambda'}$, since the other ones are deduced from the $A_\mu^{\lambda'}$ by differentiation on S. Now, since the $V^{\lambda'}$ satisfy the geodesic equation:

$$\frac{dV^{\lambda'}}{ds} + \Gamma^{\lambda'}_{\mu'\nu'} V^{\mu'} V^{\nu'} = 0,$$

the $\dfrac{dV^{\lambda'}}{ds}$ will be continuous upon crossing S because the derivative is taken along a geodesic and, on the other hand:

$$\frac{dV^{\lambda'}}{ds} = \frac{d}{ds}(A_0^{\lambda'}) = \partial_\mu A_0^{\lambda'} \frac{dx^\mu}{ds} = \partial_0 A_0^{\lambda'} \frac{dx^0}{ds}.$$

One sees that the $\partial_0 A_0^{\lambda'}$ are then continuous upon crossing *S*, as well. Since the passage from Gaussian coordinates to an admissible system is twice-differentiable, the Gaussian coordinates themselves will also be twice-differentiable.

In the actual determination of the metrics, it is generally not very convenient to express the matching conditions in an arbitrary admissible coordinate system. It results from the preceding study that the stated matching conditions are equivalent to the following ones:

1) The potentials, as well as their first derivatives, match from one side of *S* to the other relative to the Gaussian coordinates that are associated with *S*.

2) The set of Gaussian coordinates on both sides of *S* form admissible coordinates for V_4; i.e., they are compatible with the differentiable structure of V_4.

From a practical standpoint, it might not be convenient to use this new form of the conditions. One will note that 1) does not imply 2), and that, as a result, it will be convenient to either verify the second condition or to define the structure of V_4 *a posteriori* in such a way that it *is* true.

29. – Local prolongation from the interior of matter to the exterior. – For the moment, we place ourselves in the absence of an electromagnetic field. In a region that is bounded by a hypersurface *S* we give ourselves an interior ds^2 that corresponds to a pure matter or perfect fluid schema. We propose to look for the condition under which there exists an exterior ds^2 on the other side of *S* that matches with the given interior ds^2 on *S*. We say that we are addressing the problem of prolonging the interior to the exterior.

Therefore, assume that there exists such an exterior ds^2. Since the hypersurface *S* is locally defined by $x^0 = 0$ in an admissible coordinate system, the quantities $S_{\lambda'}^0$ that are associated with the exterior ds^2 will be zero identically. Now, on *S* they depend upon only the potentials, their first derivatives, and their second derivatives of index 1. From the matching of the interior and exterior ds^2, the quantities S_λ^0 that are associated with the interior field must therefore also be zero on *S*, and one will have:

$$T_\lambda^0 \equiv (\rho + p)u_\lambda u^0 - pg_\lambda^0 = 0$$

on *S*. One necessarily deduces from this that:

(29-1) $$u^0 = 0, \qquad p = 0$$

on *S*.

In other words: *S is generated by the streamlines of the interior schema*, and the pressure goes to zero on *S*.

Conversely, suppose that this is true. Since the hypersurface S is generated by the streamlines, it will be oriented in time everywhere. On the other hand, the quantities S_λ^0 that are associated with the interior ds^2 are zero on S ($x^0 = 0$). An exterior ds^2 that matches with the interior ds^2 on S will therefore be a solution to the exterior Cauchy problem relative to S and also to the Cauchy problem that is defined by the interior ds^2 on S whose data satisfy the condition:

$$S_\lambda^0 = 0$$

on S.

One knows that this problem locally admits a physically unique solution under these conditions. The existence of such a regular solution may be assured only in a certain neighborhood of S.

We may state the following:

THEOREM. – *Under the hypothesis of a pure matter or perfect fluid schema, in order for the problem of prolonging from the interior to the exterior to admit a solution upon crossing a hypersurface S, it is necessary that:*

1) *The hypersurface S must be generated by the streamlines of the interior schema.*
2) *The pressure on S must be zero for the perfect fluid schema hypothesis.*

These conditions are, moreover, sufficient for the local existence of a solution.

Under the hypothesis of an arbitrary fluid schema:

$$T_{\alpha\beta} = \rho\, u_\alpha u_\beta + \sum_i p_{(i)} V_\alpha^{(i)} V_\beta^{(i)} .$$

Reasoning that is analogous to the foregoing shows that the possibility condition for the problem translates into the nullity of the four quantities:

$$T_\lambda^0 = \rho\, u_\lambda u^0 + \sum_i p_{(i)} V_\lambda^{(i)} V^{(i)0}$$

on S. One must therefore have:

$$u^\lambda T_\lambda^0 = \rho\, u^0 = 0$$

on S. From this, one deduces the following form for the possibility conditions:

$$u^0 = 0, \qquad p_i V^{(i)0} = 0 \qquad (i = 1, 2, 3).$$

The first equation once more says that S must be generated by the streamlines. The other equations relate to the composition of the pressures on the surface.

30. – Prolonging from the exterior to the interior. The geodesic principle. – We now study the inverse problem, which presents itself in a completely different light. We give ourselves an exterior ds^2 that is bounded by a hypersurface S, and propose to examine whether there exists an interior ds^2 that matches up with it on S when we restrict ourselves to the pure matter schema.

Therefore, assume that there exists such an interior ds^2. The study of the direct problem imposes an extremely interesting condition on the hypersurface S: S must be generated by the streamlines of the interior schema, lines that are time-oriented geodesics of the interior ds^2, and thus, from the matching conditions, they will also be time-oriented geodesics of the exterior ds^2. Therefore, *the hypersurface S must be generated by the time-oriented geodesics of the exterior* ds^2.

Suppose that this condition is satisfied. The problem thus posed is an interior Cauchy problem relative to S and the Cauchy domains that are deduced from the exterior ds^2 on S. However, one has $S_\lambda^0 = 0$ for these data on S ; i.e., $u^0 = 0$. We are therefore concerned with one of the exceptional cases for which the Cauchy problem is not correctly posed – viz., one for which the hypersurface S is not generated by the streamlines – and we conclude nothing on the subject of our prolongation problem. One may interpret this circumstance *grosso modo* by considering that an exterior field might not be compatible with an effective distribution of masses, or else that it may be compatible with several such distributions; this is what one confirms in the study of the Schwarzschild ds^2.

Meanwhile, from the fact that S, by virtue of the matching conditions, must be generated by the geodesics of the exterior ds^2, one may infer a fundamental result.

Consider a very small test mass in a given exterior gravitational field. Internal interactions are negligible for such a mass so the interior field of the mass may be represented by a pure matter schema. This mass will describe a world-tube S in V_4 that has a very small cross-section and is generated by the time-oriented exterior geodesics of ds^2. If one passes to the limit and neglects the cross-section of the tube then one sees that the trajectory of a material particle is necessarily a time-oriented exterior geodesic of ds^2. We state the following:

GEODESIC PRINCIPLE. – *The spatio-temporal trajectory of any material point in a given exterior gravitational field is a time-oriented geodesic of the exterior* ds^2.

One sees how this principle may be regarded, on the one hand, as a consequence of the conservation conditions – i.e., the Einstein conditions – and, on the other hand, the matching conditions.

31. – Global problems. – We are now in position to reveal, in full clarity, what might be the fundamental instrument for the representation of gravitation in general relativity.

A ds^2 is called *regular* in a domain of V_4 when it satisfies the conditions regarding its type and differentiability that we stated in the first chapter.

I propose to call a manifold V_4 a *world model* when it is endowed with an everywhere-regular metric that satisfies the following conditions – and it is good to once more discuss them in detail:

a) In any domain of V_4 that is filled with an energy distribution and bounded by a hypersurface S, the metric is regular and satisfies the Einstein equations for the interior case.

b) In the domains of V_4 that do not contain any energy distribution, the metric is regular and satisfies the Einstein equations for the exterior case.

c) Conforming to the matching conditions, the potentials and their first derivatives are continuous upon crossing the hypersurface S.

When it is possible to construct such a world model the exterior field *may be regarded as the gravitational field that is effectively produced* by the various masses or energy distributions that were introduced.

Only one such world model is susceptible to physical interpretation. A metric is not susceptible to any such interpretation in any domain D_0 in which it is not regular. In order to ultimately reach a world model, one must see if it is possible to find such a domain, i.e., to choose a hypersurface S that bounds a domain D that contains D_0, and to construct an energy distribution and a metric that are related by the Einstein conditions, such that the metric is everywhere regular in S and matches with the previously-given metric on S. Note that such problems are of an essentially global nature, whereas our prior analysis was local. One knows only a few things about the general solution to such problems very well.

In a world model (in the sense that we just defined), it must be impossible to introduce new energy distributions whose associated metrics match with the exterior field; otherwise, the preceding definition would lose all of its interest. One is therefore led to think that, conversely, it must be impossible to introduce an energy distribution that is compatible with an exterior field in a domain where this field is regular. Therefore, one must study the validity of the following proposition in relativity:

PROPOSITION A. – *The introduction of an energy distribution into a given exterior field may be accomplished only in the case of domains for which this field is not regular.*

This proposition is not satisfied for the general axioms that we have adopted up till now, but we verify that one may establish it under very broad conditions that have a simple physical interpretation.

If one admits the necessity of Proposition A then one will see that it must be impossible to introduce an energy distribution into a world that is composed of an everywhere-regular exterior ds^2 on V_4. Such a world model would have to be the *vacuum* world model, and, as a result, with the hypothesis of a zero cosmological constant, such a world model would have to be one of a universe without gravitation, i.e., a locally-Euclidian universe.

One may further say that a very small test mass that is introduced into such a universe can be regarded as everywhere isolated. Therefore, there must locally exist coordinates (x^λ) that are interpretable in terms of space and time such that the motion of the test mass is uniform rectilinear motion with respect to these coordinates. Therefore, the geodesics of such a Riemannian manifold may be locally represented by equations of the form:

(31-1) $$x^i = a^i x^0 + b^i \qquad (i = 1, 2, 3).$$

One immediately sees that the Riemannian spaces in which the geodesics may be defined, equations (31-1) will give those of the locally-Euclidian spaces. We are therefore led to study the validity of the following proposition for the exterior ds^2:

PROPOSITION B. – *An exterior ds^2 that satisfies the axioms of general relativity and is everywhere-regular must be locally-Euclidian.*

We verify that this proposition is not satisfied under the general axioms, but it may be established under physically interesting conditions.

II. THE CASE IN WHICH THERE EXISTS AN ELECTROMAGNETIC FIELD

32. – Prolongation in the case for which there exists an electromagnetic field. – All of what was said above about matching conditions for the metric may be applied to the case in which the energy distributions in V_4 are composed of charged matter of charge density μ if we introduce an electromagnetic field $F_{\mu\nu}$. However, we must add certain considerations that relate to this electromagnetic field and the Maxwell equations. Conforming to our axioms, *$F_{\alpha\beta}$ must be continuous in any admissible system of coordinates on V_4*. In particular, this must be true upon crossing a hypersurface S on which the charge density μ is discontinuous.

From the previous analysis in sec. 25, it results from the equation $D^0 = \mu u^0$ that one will necessarily have $D^0 = u^0 =$ on S ($x^0 = 0$), and, as a result, S must be generated by the streamlines of a charged matter schema.

In particular, this will be true upon crossing hypersurfaces that bound the charged matter and along which μ passes from a value that is different from 0 to the value 0. More precisely, we may thus establish the following:

THEOREM. – *Given a distribution of charged matter or a charged, perfect fluid that is bounded by a hypersurface S, as well as gravitational and electromagnetic fields ($g_{\alpha\beta}$, $F_{\alpha\beta}$) that are related to the distribution by the Maxwell-Einstein equations, in order for there to exist gravitational and electromagnetic fields that correspond to the pure electromagnetic field schema and match the preceding ones on S, it is necessary that:*

1) *The hypersurface S be generated by the streamlines of the distribution;*

2) *The pressure on S is zero under the perfect fluid hypothesis.*

These conditions are sufficient for the local existence of a solution.

Indeed, assume that there exist gravitational and electromagnetic fields that correspond to the pure electromagnetic schema and match with the given fields on S.

Since the quantities $Q_\lambda^0 = S_\lambda^0 - \chi\tau_\lambda^0$, D^0, E^0 are continuous upon crossing S, the same quantities for the given fields must be necessarily annulled on S. Now, for the distribution envisioned, one will have:

$$Q_\lambda^0 = \chi[(\rho + p)u_\lambda u^0 - p\,g_\lambda^0], \qquad D^0 = \mu\,u^0, \qquad E^0 = 0.$$

In order for these quantities to be zero on S, it is necessary and sufficient that $u^0 = p = 0$ on S.

Conversely, if this is true then since S is generated by the streamlines, it will be everywhere time-oriented. The desired field ($g_{\alpha\beta}$, $F_{\alpha\beta}$) will be a solution of the Cauchy problem that relates to S, and the Cauchy data will be provided by the field of the distribution on S, which are data that satisfy the conditions:

$$Q_\lambda^0 = D^0 = E^0 = 0.$$

One knows that under these conditions the Cauchy problem for the pure electromagnetic schema will locally admit one solution, which will be physically unique.

33. – The trajectories of a charged, material particle. – Consider a very small *charged* test mass in a given field ($g_{\alpha\beta}$, $F_{\alpha\beta}$) that corresponds to a pure electromagnetic field. The interior field for such a mass may be represented by a pure matter-electromagnetic field schema. This mass will describe a world-tube S in V_4 that has a very small cross-section and will be generated by the streamlines of its interior field, which will satisfy the differential system (34-7), namely:

$$u^\alpha \nabla_\alpha u^\beta = k\,(F_{\beta\alpha})_i\,u_\alpha \qquad (k = \frac{\mu}{\rho})$$

in which $(F_{\beta\alpha})_i$ denotes the interior field of the mass. However, from the continuity of the electromagnetic field, one will has $(F_{\beta\alpha})_i = F_{\beta\alpha}$ on S, and, as a result, S will be generated by the solution curves to the differential system:

$$u^\alpha \nabla_\alpha u^\beta = k\,F_{\beta\alpha}\,u_\alpha\,;$$

namely:

(33-1)
$$\frac{d^2x^\beta}{ds^2} + \Gamma^\beta_{\lambda\mu}\frac{dx^\lambda}{ds}\frac{dx^\mu}{ds} = k\,F^\beta{}_\alpha\frac{dx^\alpha}{ds}\,.$$

If one passes to the limit and neglects the cross-section of the tube then one will see that the trajectory of a charged material particle of negligible cross-section in a given field ($g_{\alpha\beta}$, $F_{\alpha\beta}$) will satisfy the differential system (33-1), in which k is the constant ratio of the charge to the mass of the particle.

One will note that the union of the gravitational and electromagnetic fields leads us to pose global problems that are analogous to the ones that we discussed for the gravitational field alone. In particular, if one wants to attribute the electromagnetic field to the presence of charged matter uniquely then one will be led to study the validity of the following proposition:

PROPOSITION C. – *If a gravitational field and an electromagnetic field that is everywhere-regular on* V_4 *satisfy the equations of the pure electromagnetic schema (the unitary exterior case) and the axioms of general relativity then the electromagnetic field must be zero and the* ds^2 *must be locally-Euclidian.*

The validity of this proposition was established by Y. Thiry ([1]) in the stationary case.

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II. – ROTATIONAL AND IRROTATIONAL MOTIONS

CHAPTER IV

THE RELATIVISTIC HYDRODYNAMICS OF HOLONOMIC MEDIA

I. THE PRINCIPLE OF THE EXTREMUM AND INTEGRAL INVARIANTS

34. – Holonomic media. The differential system of the streamlines. – Suppose that a certain domain of V_4 is occupied by a material distribution whose energy-momentum tensor may be put into the form:

$$T_{\alpha\beta} = r\, u_\alpha u_\beta - \theta_{\alpha\beta},$$

in which r is a positive scalar and u_α is a unitary vector. In sec. 17, we agreed that a medium that is described by $T_{\alpha\beta}$ should called *holonomic* if the vector field K_β, which is defined by:

$$r\, K_\beta = \nabla_\alpha \theta^\alpha_\beta,$$

is a gradient vector field. If this is true then we will set:

$$K_\beta = \partial_\beta \log F .$$

r is the pseudo-density of the medium, **u** is its unitary velocity vector, and F is its *index*. We recall that a perfect fluid with the state equation $\rho = \varphi(p)$ is a holonomic medium; in this case:

$$r = \rho + p, \qquad\qquad F = \exp \int_{p_0}^{p} \frac{dp}{\rho + p}.$$

The streamlines of the holonomic medium considered satisfy the differential system:

$$\text{(34-1)} \qquad \frac{dx^\alpha}{ds} = u^\alpha, \qquad\qquad u^\alpha \nabla_\alpha u_\beta = \frac{\partial_\alpha F}{F}(g_{\alpha\beta} - u^\alpha u_\beta).$$

In all of what follows in this chapter we will place ourselves in a domain of V_4 that is swept out by a holonomic medium. First of all, we propose to geometrically interpret the streamlines of such a medium.

35. – The variation of an integral. – To that effect, we commence by briefly recalling a formula from the calculus of variations that is slightly classical. Consider a differentiable manifold V_n, the "fiber" space W_{2n} of vectors that are tangent to the various points of V_n, and the "fiber" space D_{2n-1} of directions that are tangent to the various points of V_n . If one adopts local coordinates on V_n then an element of W_{2n} will be defined by the coordinates (x^α) of the corresponding point x of V_n and the n quantities $\dot{x}^\alpha$, which are the contravariant components of the vector in the natural frame that is associated with the (x^α) at x. For an element of D_{2n-1}, the $\dot{x}^\alpha$ will be defined only proportionally in the form of direction parameters of the direction.

Let C be a curve in V_n, which we represent by giving it a parametric representation as a function of an arbitrary scalar parameter u. In local coordinates, one will therefore have:

$$x^\alpha = x^\alpha(u).$$

The curve C is the projection of the curve Γ in D_{2n-1} onto V_n that is defined by the directions that are tangent to C at its various points. A curve $L_{(u)}$ in W_{2n} corresponds to the *parametric representation* of C, which is defined by the derivatives of the x^α with respect to u at the various points x of C. In local coordinates, one will therefore have:

$$\dot{x}^\alpha = \frac{dx^\alpha}{du}$$

for $L_{(u)}$.

Let r be a function with scalar values that is defined on the space W_{2n} of vectors (x,**V**) that are tangent to the various points of V_n and are such that for fixed x: $f(\lambda\mathbf{V}) = \lambda f(\mathbf{V})$. In local coordinates, such a function will be represented by $f(x^\alpha, \dot{x}^\alpha)$, and it will be *homogenous of the first degree* with respect to the $\dot{x}^\alpha$.

To f and the arc of the curve C that joins the points x_0, x_1, one may associate the following integral, which is calculated along $L_{(u)}$:

$$\Phi = \int_{u_0}^{u_1} f(x^\alpha, \dot{x}^\alpha)du = \int_{x_0}^{x_1} f(x^\alpha, dx^\alpha) .$$

This integral is attached to f and C intrinsically since it will not be modified if one changes the parametric representation of C.

We now calculate the variation of the integral Φ for an arbitrary variation of the arc C with variable extremities. First of all, we suppose that C belongs to the domain of a system of local coordinates. It will then follow that:

$$\delta F = f_{u_1}\delta u_1 - f_{u_0}\delta u_0 + \int_{u_0}^{u_1} \delta f\, du .$$

A classical argument from the calculus of variations gives:

$$\int_{u_0}^{u_1} \delta f\, du = \left[\frac{\partial f}{\partial \dot{x}^\alpha}\delta x^\alpha\right]_{u=u_0}^{u=u_1} - \int_{u_0}^{u_1} P_\alpha \delta x^\alpha du$$

in which P_α denotes the right-hand side of the Euler equations that are associated with *f*. The quantities $(\delta x^\alpha)_{u=u_0}$ and $(\delta x^\alpha)_{u=u_1}$ that appear in the brackets are obviously expressed in terms of the vectors $\delta\mathbf{x}_0$ and $\delta\mathbf{x}_1$ by the formulae:

$$(\delta x^\alpha)_{u=u_0} = \delta x_0^\alpha - \dot{x}_0^\alpha \delta u_0\,, \qquad (\delta x^\alpha)_{u=u_1} = \delta x_1^\alpha - \dot{x}_1^\alpha \delta u_1\,.$$

One deduces from this that:

(35-1) $$\delta\Phi = [\omega(\delta)]_{x_1} - [\omega(\delta)]_{x_0} - \int_{u_0}^{u_1} P_\alpha \delta x^\alpha du$$

in which $\omega(\delta)$ denotes the form:

$$\omega(\delta) = \frac{\partial f}{\partial \dot{x}^\alpha}\delta x^\alpha - \left[\dot{x}^\alpha \frac{\partial f}{\partial \dot{x}^\alpha} - f\right]\delta u\,,$$

which will reduce to:

(35-2) $$\omega(\delta) = \frac{\partial f}{\partial \dot{x}^\alpha}\delta x^\alpha\,,$$

as a result of the homogeneity of *f*, and the indices x_0 and x_1 in (35-1) signify that the form ω is evaluated at x_0 and x_1 for the vectors $\mathbf{x}_0$ and $\mathbf{x}_1$. We remark that ω – or, if one prefers, the *n* quantities $\frac{\partial f}{\partial \dot{x}^\alpha}$ – define a covariant vector field on D_{2n-1}. Moreover, it is well known that the *P* will be the components of a covariant vector *P* for a certain parametric representation; therefore, the scalar product $<P\, d\mathbf{x}>$ will appear under the $\int$ sign. Since ω is defined in local coordinates by (35-2), in which the $\dot{x}^\alpha$ are the ones that are defined to be proportional to *C*, one may put (35-1) into the form:

(35-3) $$\delta\Phi = [\omega(\delta)]_{x_1} - [\omega(\delta)]_{x_0} - \int_{u_0}^{u_1} < P\,\delta\mathbf{x} > du$$

Now, if the arc of the curve *C* is arbitrary then formula (35-3) extends without modification by the addition of the variation of the integrals that relate to the arcs of the curve that are interior to the same domain as the local coordinates.

36. – An extremal principle for the streamlines. – We now return to the manifold V_4 and apply the results that we just discussed to the function:

$$f = F\,\frac{ds}{du} = F\sqrt{g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta}\ ,$$

in which F is a function that is defined on V_4. Let x_0 and x_1 be two points of V_4 that may be joined by a time-oriented curve, and consider the integral:

$$\overline{s} = \int_{x_0}^{x_1} F\ ds = \int_{x_0}^{x_1} F\sqrt{\mathrm{g}_{\alpha\beta}\,dx^\alpha dx^\beta}\ ,$$

which is an integral that is necessarily evaluated along a time-oriented curve C. From:

$$f^2 = F^2 g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta\ ,$$

one immediately deduces by derivation that:

$$\text{(36-1)}\qquad f\,\frac{\partial f}{\partial \dot{x}^\alpha} = F^2 g_{\alpha\beta}\,\dot{x}^\beta\ ,\qquad f\,\frac{\partial f}{\partial x^\alpha} = F\,[\partial_\alpha F\ g_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma + \tfrac{1}{2}F\,\partial_\alpha g_{\beta\gamma}\dot{x}^\beta\dot{x}^\gamma],$$

which will then give the values of the partial derivatives.

Consider a three-parameter family of time-oriented curves and study the variation of $\overline{s}$, when it is taken along a curve of this family, when one varies this arc within that family. We take the real curvilinear abscissa s to be the parameter u of the curve C. The vector:

$$\dot{x}^\alpha = \frac{dx^\alpha}{ds} = u^\alpha$$

will then be the unitary vector that is tangent to C, which will locally define a vector field. The formulae (36-1) then reduce to:

$$\text{(36-2)}\qquad \frac{\partial f}{\partial \dot{x}^\beta} = F\,u_\beta\ ,\qquad \frac{\partial f}{\partial x^\beta} = \tfrac{1}{2}F\,\partial_\beta g_{\alpha\beta}u^\alpha u^\beta + \partial_\beta F = F\,[\alpha\beta, \rho]\,u^\alpha u^\beta + \partial_\beta F\ ,$$

in which [ ] denotes Christoffel symbol of the first type. The components P_β of P that correspond to this parametric representation are written:

$$P_\beta = \frac{d}{ds}\frac{\partial f}{\partial \dot{x}^\beta} - \frac{\partial f}{\partial x^\beta} = \frac{d}{ds}(Fu_\beta) - F[\alpha\beta,\gamma]u^\alpha u^\beta - \partial_\beta F\ .$$

Therefore, upon specifying the total derivative:

$$P_\beta = F(u^\alpha\partial_\alpha u_\beta - [\alpha\beta,\gamma]u^\alpha u^\beta) - \partial_\alpha F(g_\beta^\alpha - u^\alpha u_\beta)\,,$$

one thus obtains:

$$\text{(36-3)}\qquad P_\beta = F\,[u^\alpha\,\nabla_\alpha u_\beta - \frac{\partial_\alpha F}{F}(g_\beta^\alpha - u^\alpha u_\beta)]\,.$$

One thus has the following formula for $\bar{s}$:

(36-4) $$\delta\bar{s} = [\omega(\delta)]_{x_1} - [\omega(\delta)]_{x_0} - \int_{s_0}^{s_1} < P\ \delta\mathbf{x} > ds\ ,$$

in which $\omega(\delta)$ is locally defined by:

(36-4) $$\omega(\delta) = F\ u_\alpha dx^\alpha,$$

and P admits the covariant components that are locally defined by (36-3).

Now, apply formula (36-4) to the case in which C varies with fixed extremities x_0 and x_1, so $\delta\mathbf{x}_0 = \delta\mathbf{x}_1 = 0$, and, as a result of the variation of $\bar{s}$ it reduces to:

$$\delta\bar{s} = -\int_{s_0}^{s_1} < P\ \delta\mathbf{x} > ds\ ,$$

and in order for $\bar{s}$ to be an extremum under these conditions it is necessary and sufficient that the unitary vector of C be such that $P = 0$, i.e., that:

(36-5) $$u^\alpha \nabla_\alpha u_\beta - \frac{\partial_\alpha F}{F}(g^\alpha_\beta - u^\alpha u_\beta) = 0,$$

equations that are formally identical with (34-1). We may state the following theorem ([1]):

THEOREM. – *During any motion of a holonomic medium, the streamlines are the time-oriented lines that realize the extremum of the integral:*

$$\bar{s} = \int_{x_0}^{x_1} F\ ds$$

for variations with fixed extremals.

In other words, the streamlines are the (time-oriented) geodesics of the Riemannian metric that is conformal to the spacetime metric ds^2 and defined by:

(36-7) $$d\bar{s}^2 = F^2\ ds^2 = F^2\ g_{\alpha\beta}\ dx^\alpha dx^\beta.$$

We are therefore led to endow the domain of V_4 under consideration with either the spacetime metric or the new metric that is defined by the index of the medium. For this latter metric, the fundamental tensor will have the components:

([1]) This theorem has been proved by a different method by Eisenhart in the case of perfect fluids, Trans. Amer. Math. Soc., **26** (1924), 205-220. See also J. L. SYNGE, Proc. London Math. Soc., **43** (1937), 376-416.

(36-8) $\bar{g}_{\alpha\beta} = F^2\, g_{\alpha\beta}\,, \qquad \bar{g}^{\alpha\beta} = F^2\, g^{\alpha\beta}.$

37. – The relative integral invariant of hydrodynamics. – In all of what follows, we will further consider a definite motion of a holonomic medium. The streamlines locally form a congruence of curves in V_4, the unitary vector field u^α satisfies equations (36-6), and the form $\omega = F\, u_\alpha\, \delta x^\alpha$ is defined V_4. Such a motion may be defined, for example, by the solution to a correctly posed interior Cauchy problem, such as the one that was studied in sec. **19**. Trace a cycle Γ of dimension 1 on a hypersurface S that is not tangent to the streamlines, and consider the streamlines that issue from the various points x_0 of Γ_0. These streamlines will generate a two-dimensional differentiable manifold $\mathfrak{T}$ (at least in a neighborhood of S) that we call a *flow tube*. If we bound each streamline by x_0 and a point x_1 that generates a cycle Γ_1 that is traced in $\mathfrak{T}$ and is homotopic to Γ_0 on $\mathfrak{T}$ then we may apply formula (36-4) to each of the arcs of the streamlines, which are arcs for which $P = 0$. Since the total variation of the integral $\bar{s}$ is zero when x_0 describes Γ_0, it will follow that:

(37-1) $$\int_{\Gamma_1} \omega(\delta) - \int_{\Gamma_0} \omega(\delta) = 0\,.$$

We translate this result into the following statement:

THEOREM. – *The differential system of the streamlines:*

$$\frac{dx^\alpha}{ds} = u^\alpha$$

of a motion of a holonomic medium admits the relative integral invariant (in the sense of Poincaré):

(37-2) $$\int_\Gamma \omega = \int_\Gamma F u_\alpha \delta x^\alpha$$

in which Γ is a one-dimensional cycle.

In the expression (37-2) for the integral invariant there appears a covariant vector that has the components:

(37-3) $$C_\alpha = F\, u_\alpha\,.$$

We give the name of *current vector* for the medium considered to the vector **C** that admits these components on the Riemannian manifold that is defined by ds^2. Since F is scalar, the contravariant components of this vector will be:

(37-4) $$C^\alpha = F\, u^\alpha.$$

Consider the vector $\bar{\mathbf{C}}$ with the same covariant components $\bar{C}_\alpha = C_\alpha$ on the Riemannian manifold that is defined by $d\bar{s}^2$. This vector will then be unitary under this metric since, from (36-8):

$$\bar{g}^{\alpha\beta}\,\bar{C}_\alpha\,\bar{C}_\beta = F^2\,g_{\alpha\beta}(F\,u_\alpha)(F\,u_\beta) = 1.$$

The contravariant components of the vector $\bar{\mathbf{C}}$ will then be:

(37-5) $$\bar{C}^\alpha = \bar{g}^{\alpha\beta} C_\beta = F^{-1}\,u_\alpha .$$

One notes that if $\bar{\nabla}_\alpha$ is the covariant derivative operator that is associated with the metric $d\bar{s}^2$ then the fact that the streamlines are geodesics of this metric will translate into the equations:

(37-6) $$\bar{C}^\alpha\,\bar{\nabla}_\alpha \bar{C}^\beta = 0.$$

One may directly verify that equations (37-6) are equivalent to equations (36-6).

Upon introducing the current vector, one may say that ω defines its elementary circulation, and one may give the following equivalent statement to the preceding theorem:

THEOREM – *Given a one-dimensional cycle Γ that is not tangent to the streamlines, the circulation of the current vector along Γ will remains invariant when one deforms Γ on a tube $\mathfrak{T}$ of the streamlines that are defined by Γ.*

One recognizes that this is the relativistic generalization of the classical theorem of the conservation of circulation.

38. – The invariant form $d\omega$ and the vorticity tensor. – As is well known, one immediately deduces an absolute integral invariant (in the sense of Poincaré) from the integral invariant that is defined by ω by an application of Stokes's formula. If D_2 is a two-dimensional differentiable chain that is transversal [1] to the streamlines, and ∂D_2 is its boundary then one will have, in fact:

$$\int_{\partial D_2} \omega = \iint_{D_2} d\omega ,$$

in which $d\omega$ denotes the exterior derivative of the form ω. In local coordinates:

(38-1) $$d\omega = dC_\beta \wedge dx^\beta = \tfrac{1}{2}(\partial_\alpha C_\beta - \partial_\beta C_\alpha)\,dx^\alpha \wedge dx^\beta .$$

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[1] We use the word "transversal" in the sense of not being tangent, and not in its proper sense from the calculus of variations.

In the language of Élie Cartan, we translate this result by saying that *the form* $d\omega$ *is an invariant of the differential system:*

(38-2)
$$\frac{dx^0}{u^0} = \frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \frac{dx^3}{u^3}$$

for the streamlines. This signifies that the form $d\omega$ admits a local expression as a function of the first integrals of this differential system.

The antisymmetric rotation tensor of the current vector is associated with the form $d\omega$, which has the components:

(38-3)
$$\Omega_{\alpha\beta} = (\partial_\alpha C_\beta - \partial_\beta C_\alpha) .$$

We give this tensor the name of the *vorticity tensor* for the motion that is associated with the current vector; it constitutes a true extension of the rotation of the velocities that is introduced in classical hydrodynamics. One notes that for a perfect fluid it is the rotation of a vector that depends upon the dynamical elements ρ, p, of the fluid. In fact, the influence of these elements is very weak, and the correction that is introduced is a purely relativistic correction of the order c^{-2}. Indeed, for such a fluid, F differs from 1 only by terms of that order since upon re-establishing the usual physical units, i.e., upon replacing p by pc^{-2}, one will get the following expression for F:

$$F = \exp \int_{p_0}^{p_1} \frac{dp}{c^{-2}\rho + p} = 1 + \int_{p_0}^{p_1} \frac{dp}{c^{-2}\rho + p} ,$$

up to terms in c^{-4}.

II. – ROTATIONAL AND IRROTATIONAL MOTIONS. VORTEX LINES.

39. – The characteristic system of the form $d\omega$. – We have seen that the differential system (38-2) for streamlines admits the integral $\int \omega$ as a relative integral invariant, or, equivalently, it admits the form $d\omega$ as an invariant form. We now propose to look for all vector fields V^α such that the differential system of their trajectories enjoys the same property with respect to ω and $d\omega$. To that effect, we must form the characteristic system of the form $d\omega$, and since $d\omega$ is closed that system reduces to the associated system ([1]):

$$\Omega_{\alpha\beta} dx^\beta = 0.$$

Any vector V^α that satisfies:

(39-1)
$$\Omega_{\alpha\beta} V^\beta = 0$$

([1]) On the subject of the notions of associated system and characteristic system, see ÉLIE CARTAN, *Les systèmes differentials exterieurs,* chap. II, Hermann (1945).

corresponds to a differential system of trajectories that leave the form $d\omega$ invariant. We are therefore led to study the rank of the system of four linear equations in four unknowns (39-1).

From a classical theorem ([1]) on the rank of an exterior quadratic form, this rank is necessarily even. Now, it is certainly less than four since the vector u^β that defines $\Omega_{\alpha\beta}$ satisfies the system (39-1). It results from this that the rank is certainly two or zero.

1) If the characteristic system (39-1) is of rank two then *the motion of the medium considered will be rotational.* The form $d\omega$ then admits characteristic manifolds that are generated by the streamlines, which we will study in a moment.

2) If the characteristic system (39-1) is of rank zero – i.e., if the vorticity tensor $\Omega_{\alpha\beta}$ is identically zero – then *the motion of the medium considered will be irrotational.*

We confine ourselves to a simply-connected domain of V_4. In order for the motion to be irrotational in this domain, it is necessary and sufficient that C_α be the gradient of a function φ such that $C_\alpha = \partial_\alpha \varphi$; i.e., the streamlines must be the trajectories that are orthogonal to a family of hypersurfaces φ = constant.

40. – The study of an irrotational motion. – It results from the study of Gaussian coordinates that we made in sec. **26** that the irrotational motions of a holonomic medium possess a property that generalizes a notion that translates into Lagrange's theorem in classical hydrodynamics, namely, the property of "permanence," to a relativistic context.

We adopt the metric $d\bar{s}^2$ for the metric in the domain considered. The circumstances for this metric are identical with the ones in sec. **26**; as a result, if the streamlines that are geodesics for $d\bar{s}^2$ are orthogonal to a hypersurface S then it will result that they are orthogonal trajectories for the family of hypersurfaces that are "parallel to S" for the metric $d\bar{s}^2$, i.e., they are obtained by moving continuously along the time-oriented streamlines through a current arc $\bar{s}$, starting with S. The motion considered is therefore irrotational. We state the following:

THEOREM. – *In order for the motion of a holonomic medium to be irrotational it is necessary and sufficient that the streamlines be orthogonal to the same (local) hypersurface.*

This result may be extended slightly. In the study of sec. **26**, one first established that $\Omega_{\alpha\beta} = 0$ on S, and then that this result would be valid outside of S.

Suppose that *there exists a space-oriented hypersurface S such that* $\Omega_{\alpha\beta} = 0$ *on S.* Locally adopt coordinates such that the streamlines are represented by x^i = constant, and the hypersurface S is represented by $x^0 = 0$. With the notations of the present chapter, relation (26-1) for the metric $d\bar{s}^2$ may then be written:

([1]) For example, see ÉLIE CARTAN, *Les systèmes differentials exterieurs,* chap. I, Hermann (1945).

(40-1) $$\Omega_{\alpha\beta}\bar{C}^{\beta} = 0,$$

and it will also say that the form $d\omega$ is an invariant for the differential system (38-2) of the streamlines. In the local coordinates that were adopted, (40-1) is written:

(40-2) $$\Omega_{0i} = 0.$$

$\Omega_{ij} = 0$ on S, and, since $d(d\omega) = 0$, reasoning that is analogous to that of sec. 26 will show that:

$$\partial_0\Omega_{ij} = 0$$

on a neighborhood of S.

It results that $\Omega_{\alpha\beta} = 0$ on a neighborhood of S and that, as a result, the motion that corresponds to it is irrotational.

41. – Vorticity vector. Vortex lines. – In what follows, unless stated to the contrary, we always put ourselves in the case of irrotational motion. At the point x of V_4 under consideration, we study the two-dimensional vector space Π_x of vectors **V** such that:

(41-1) $$\Omega_{\alpha\beta}V^{\beta} = 0.$$

We are already acquainted with the vector **u** in Π_x, which is unitary with respect to ds^2 and tangent to the streamline at x. In order to succeed in determining Π_x , it will therefore suffice to look for a second non-zero vector θ, which we choose to be orthogonal to the first. We are therefore led to determine a vector θ_α that satisfies the equations:

(41-2) $$\Omega_{\alpha\beta}\,\theta^{\beta} = 0 \qquad u_{\beta}\,\theta^{\beta} = 0.$$

We provisionally adopt local coordinates for which the streamlines are the lines x^i = constant. One then has $u_0\,u^0 = 1$, and, conforming to (40-2):

(41–3) $$\Omega_{0i} = 0.$$

Equations (41-2) reduce to the equations:

(41-4) $$\Omega_{12}\,\theta^2 + \Omega_{13}\theta^3 = 0,\quad \Omega_{21}\,\theta^1 + \Omega_{23}\theta^3 = 0,\quad \Omega_{21}\theta^1 + \Omega_{32}\theta^2 = 0,$$

and:

(41-5) $$u_{\beta}\,\theta^{\beta} = 0.$$

Solving equations (41-4) gives:

(41-6) $$\theta^1 = -\lambda\,\Omega_{23},\quad \theta^2 = -\lambda\,\Omega_{31},\quad \theta^3 = -\lambda\,\Omega_{12},$$

in which λ denotes an arbitrary factor. One deduces from (41-5) that:

(41-7) $$\theta^0 = \lambda\, u^0 (u_1\,\Omega_{31} + u_2\,\Omega_{31} + u_3\,\Omega_{21}).$$

If we choose:

$$\lambda = \frac{u_0}{\sqrt{|g|}}$$

then we may put equations (41-6) and (41-7) into the form:

$$\theta^1 = \eta^{1023}\,u_0\,\Omega_{23}, \qquad \theta^2 = \eta^{2031}\,u_0\,\Omega_{31}, \qquad \theta^3 = \eta^{3012}\,u_0\,\Omega_{12},$$

and:

$$\theta^0 = \tfrac{1}{2}\,\eta^{0ijk}\,u_i\,\Omega_{jk}.$$

Let:

(41-8) $$\theta^\alpha = \tfrac{1}{2}\,\eta^{\alpha\beta\gamma\delta}\,u_\beta\,\Omega_{\gamma\delta},$$

in which η is the volume element tensor for ds^2. One will note that since λ is non-zero, from equations (41-3) and (41-6), the fact that $\theta^\alpha = 0$ will imply that $\Omega_{\alpha\beta} = 0$. It results from this that the vector θ that is defined in arbitrary coordinates by (41-8) will solve the problem. We give this vector the name of *vorticity vector.* The trajectory lines of the vector field θ will be called *vortex lines.* These lines, which are orthogonal to the streamlines, are everywhere space-oriented.

42. – Vortex tubes. – Since the vorticity vector satisfies equations (41-1), it will result that the form $d\omega$ is invariant for the differential system of the vortex lines:

$$\frac{dx^0}{\theta^0} = \frac{dx^1}{\theta^1} = \frac{dx^2}{\theta^2} = \frac{dx^3}{\theta^3}.$$

Let Γ be a one-dimensional cycle that is not tangent to the vortex lines and does not pass through a point at which θ goes to zero. The vortex lines T that issue from the points x of Γ will generate a two-dimensional differentiable manifold (at least in a certain neighborhood of Γ) that will be the *vortex tube* that is defined by Γ. We may state a theorem on the subject of vortex tubes that is identical to the one that was stated for the tubes of streamlines:

THEOREM. – *Given a one-dimensional cycle* Γ *that is not tangent to the vortex lines and on which* θ *does not go to zero, the circulation of the current vector along* Γ *remains invariant when one deforms* Γ *on the vortex tube* Θ *that is defined* Γ**.**

Consider a fundamental system of cycles on Θ. Conforming to the convention that is used in hydrodynamics, the periods of the form ω for these cycles will be called the *moments* of the vortex tube considered, Θ.

Let $\mathfrak{T}$ be a flow tube, and let Γ and Γ' be two cycles of $\mathfrak{T}$ that are homotopic on $\mathfrak{T}$ and satisfy the hypotheses of the preceding theorem. Each of these cycles defines a vortex tube, which we will denote by Θ and Θ', respectively. If Γ_1 is a cycle of Θ that is homotopic to Γ on Θ and Γ_1' is a cycle of that is homotopic to Γ' on Θ' then it will follow that:

$$\int_{\Gamma_1'} \omega = \int_{\Gamma'} \omega = \int_{\Gamma} \omega = \int_{\Gamma_1} \omega ,$$

and therefore:

(42-1) $$\int_{\Gamma_1'} \omega = \int_{\Gamma_1} \omega .$$

Equation (42-1) constitutes the relativistic generalization of a classical theorem of Helmholtz. If we change the roles that are played by the streamlines and the vortex lines then this will naturally lead to an analogous result.

43. – The characteristic manifold. – The field of elementary 2-planes Π_x that is defined by the characteristic system of the form $d\omega$:

$$\Omega_{\alpha\beta} dx^{\beta} = 0$$

is a completely-integrable field. One gives the name of *characteristic manifolds* of the form $d\omega$ to the two-dimensional manifolds W_2 that are its integrals. These manifolds may be generated by the streamlines and by the vortex lines, which makes their construction immediate.

In order to construct the characteristic manifold $W_2^{(0)}$ that passes through a point x_0 of V_4, one deals with the streamline C_0 and the vortex line T_0 that issues from that point. The streamlines C that pass through the points of T_0 generate the characteristic manifold $W_2^{(0)}$ that passes through x_0. The same will be true for the vortex lines T that issue from the points of C_0, and these lines will be trajectories that are orthogonal to the streamlines on $W_2^{(0)}$. Conversely, it is clear that any trajectory that is orthogonal to the streamlines on $W_2^{(0)}$ will be a vortex line. Therefore:

THEOREM. – *If one deals with the streamlines that pass through the points of a vortex line then the trajectories that are orthogonal to these streamlines on the surface that they generate will be vortex lines.*

This result must be considered to be the relativistic analog of the following theorem of classical hydrodynamics: If a line in a fluid is a vortex line at one instant then it will be a vortex line at any other instant.

The property of streamlines and vortex lines that their differential systems both admit the same integral invariant $\int \omega$ thus permits us to construct a theory of vorticity in relativistic hydrodynamics that best generalizes the theory of vorticity in classical hydrodynamics. Meanwhile, note an interesting peculiarity of relativistic hydrodynamics:

Consider a vector field **V** such that the vector $\mathbf{V}_x$ that corresponds to x belongs to Π_x ; i.e., it is tangent to the characteristic manifold that passes through x. One will then have:

$$\mathbf{V} = \lambda\,\mathbf{u} + \mu\,\boldsymbol{\theta},$$

in which λ and μ denote two scalars. From the results that were discussed in sec. **38**, the form $d\omega$ will be an invariant for the differential system for the trajectories of the vector field **V**, and these lines will enjoy all of the properties that relate to the circulation of the current vector that were stated for the streamlines and vortex lines. We call them *quasi-vortex lines.* Since the 2-plane Π_x is time-oriented, one might find that such lines are either time-oriented (like the streamlines) or space-oriented (like the vortex lines), or, similarly, of *null length.* The latter correspond to a vector **V**, which is such that:

$$\mathbf{V}^2 = (\lambda\,\mathbf{u} + \mu\,\boldsymbol{\theta})^2 = \lambda^2 - \mu^2\theta^2 = 0$$

in which θ denotes the real number:

$$\theta = \sqrt{-(\boldsymbol{\theta})^2}\,.$$

Therefore, there exist null-length quasi-vortex lines in relativistic hydrodynamics, which are the trajectories of the two vector fields:

$$\mathbf{V} = \boldsymbol{\theta} \pm \theta\,\mathbf{u}\,.$$

III. – PERMANENT MOTIONS.

44. – Spacetimes that are stationary in a domain. – We first place ourselves in the domain of a local coordinate system. Suppose that the local coordinates (x^0, x^i) may be chosen in such a manner that the corresponding potentials $g_{\alpha\beta}$ are independent of the variable x^0, with the lines along which the variable x^0 varies being only time-oriented (g_{00} > 0). If that is true then we will say that the metric is *locally stationary* in the domain envisioned. We note that we make no hypothesis on the orientation of the local hypersurfaces x^0 = constant. We recall that the inequality $g_{00} > 0$ entails that $g_{\alpha\beta}$ has the opposite character to the quadratic form in three variables:

$$\left(g_{ij} - \frac{g_{0i}\,g_{0j}}{g_{00}}\right)X^i X^j\,. \tag{44-1}$$

Now consider a definite four-dimensional domain D_4 of V_4, and suppose that the Riemannian manifold that is defined by D_4 and is endowed with the spacetime metric ds^2 admits a *connected, one-parameter group of global isometries that leave no point of* D_4 *invariant and whose trajectories z are time-oriented.*

Since the isometries are global, D_4 must be generated by the z. We assume, moreover, that:

a) The z are homeomorphic to the real line $\mathbb{R}$.

b) One may find a three-dimensional differentiable manifold D_3 that satisfies the same differentiability hypothesis as V_4, and is such that there exists a differentiable homeomorphism of class C^2 of D_4 with the product manifold $D_3 \times \mathbb{R}$, in which the z project onto the right-hand factor. This homeomorphism is, moreover, assumed to be piecewise continuous up to order 4.

With these conditions, we say that the Riemannian spacetime V_4 is *stationary* in D_4. We call the z trajectories the *timelines*.

Let ξ be the infinitesimal generator of the group of isometries. Since no point of D_4 is invariant, $x \neq 0$ at every point of D_4. One knows that this vector satisfies the Killing equations:

(44-2)
$$X\, g_{\alpha\beta} = \nabla_\beta\, \xi_\alpha + \nabla_\alpha\, \xi_\beta = 0,$$

in which X denotes the Lie derivative operator with respect to the vector ξ.

Consider a system of local coordinates $(x^{i'})$ in D_3. We may define local coordinates $(x^{\lambda'})$ in D_4 in the following manner: The data $(x^{i'})$ determine a timeline. In order to determine a point on this line, we look at the manifold $(x^{0'})$ = constant to which it belongs since these manifolds will be homeomorphic to D_3 by the homeomorphism in b). In the local coordinates $(x^{i'})$, the timelines, which are trajectories of the vector field ξ, will be the lines $(x^{i'})$ = constant; as a result, the contravariant components of x will be:

$$\xi_i = 0, \qquad \xi^{0'} \neq 0.$$

We perform the change of local coordinates that is defined by:

$$x^i = x^{i'}, \qquad x^0 = f(x^{0'}, x^{j'}).$$

One may substitute new manifolds in D_4, x^0 = constant, for the manifolds $x^{0'}$ = constant, such that the new component:

$$\xi^0 = A^0_{0'}\, \xi^{0'} = \frac{\partial f}{\partial x^{0'}}\, \xi^{0'}$$

is *equal to* 1. It suffices to take:

$$\frac{\partial f}{\partial x^{0'}} = \frac{1}{\xi^{0'}} \qquad (\xi^{0'} \neq 0),$$

and the function f is found to be defined up to an additive function of the coordinates $x^{j'}$. Fix this function by taking a submanifold of D_4 that is intersected at one and only one point by each timeline in D_4 to represent x^0 = constant. Note that the timelines z establish a homeomorphism between the manifolds that are homeomorphic to D_3, x^0 = constant, and the manifolds $x^{0'}$ = constant, and that there exists a homeomorphism of D_4 with $D_3 \times \mathbb{R}$ that satisfies the hypotheses of b) and maps the manifolds x^0 = constant to the factor manifolds, which are homeomorphic to D_3 ([1]).

In the preceding system of coordinates (x^0, x^i) one has:

$$\xi_\alpha = g_{\alpha 0} .$$

Notice how the Killing equations (44-2). It follows that:

$$\nabla_\beta \xi_\alpha = \nabla_\beta g_{\alpha 0} + \Gamma^\alpha_{\beta 0}\, g_{\alpha 0} = [\beta\, 0,\ \alpha].$$

One deduces from this that:

$$\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta = [\beta\, 0,\ \alpha] + [\alpha\, 0,\ \beta] = \partial_0 g_{\alpha\beta} .$$

Therefore, $\partial_0 g_{\alpha\beta} = 0$ in the coordinates envisioned, and the $g_{\alpha\beta}$ will be independent of the variable x^0. The metric is locally stationary in a neighborhood of any point of D_4.

The coordinate systems (x^0, x^i) whose existence we just showed will be said to be *adapted to the stationary character*. It is clear that the coordinate changes that allow us to pass from one adapted system to the other are of the form:

$$x^{i'} = \psi^{i'}(x^j) \qquad\qquad x^{0'} = x^0 + \psi(x^j),$$

in which the $\psi^{i'}$ and ψ are arbitrary functions of the x^j. In a coordinate system that is adapted to the stationary character, the symbol of the infinitesimal transformation that the generates the group of isometries:

$$X\varphi = \xi^\alpha \partial_\alpha \varphi \qquad\qquad (\varphi \text{ is a scalar function})$$

will be given by:

$$X\varphi = \partial_0 \varphi .$$

The isometries that we consider are defined in adapted coordinates by:

$$x^i \to x^i, \qquad\qquad x^0 \to x^0 + h.$$

45. – The notion of a permanent motion. – If we are given a holonomic medium in motion that is associated with a spacetime metric ds^2 then we may say that the motion of

([1]) The preceding considerations will be useful to us in the global theory of stationary spacetimes.

the medium is *permanent* if the Riemannian manifold is stationary in a domain D and the group of isometries leaves the unitary velocity vector **u** and the index F of the medium invariant. One will therefore have:

$$XF = 0, \qquad X\mathbf{u} = 0,$$

in which XF and $X\mathbf{u}$ are defined by:

$$XF = \xi^\alpha \partial_\alpha F \qquad X\, u_\beta = \xi_\alpha \nabla_\alpha u_\beta + u_\alpha \nabla_\beta \xi^\alpha,$$

in an arbitrary coordinate system.

If the coordinates are adapted to the stationary character then one obviously has:

$$XF = \partial_0 \varphi\,,$$

and F is constant along the timelines. Moreover, from the expression for the operator X:

$$X\, u_\beta = \nabla_0\, u_\beta + \Gamma^\alpha_{\beta 0} u_\alpha = \partial_0 u_\beta\,,$$

and the u_β are also constant along the timelines; the same thing will be true for u^β.

For example, consider a motion of a perfect fluid that admits an equation of state $\rho = \varphi(p)$ such that the associated Riemannian spacetime is stationary in D_4. Let x be an arbitrary point of D_4 and choose an adapted coordinate system (x^α) such that x belongs to the manifold S ($x^0 = 0$) and S is not a *hydrodynamical wave front*, i.e., an exceptional manifold for the Cauchy problem stat was studied in sec. **19**. From the study of that problem, it results that on the manifold $x^0 = 0$ and the neighboring manifolds $x^0 =$ constant one will have, from (19-6):

(45-1) $$\chi\,(S^{00} + \chi\, p\, g^{00})(\rho + p) = g^{\lambda\mu}\,(S^0_\lambda + \chi\, p\, g^0_\lambda)(S^0_\mu + \chi\, p\, g^0_\mu)\,.$$

On the other hand, one obviously has $\partial_0 g_{\alpha\beta} = 0$ and $\partial_0 S^0_\lambda = 0$ in adapted coordinates. By differentiating (45-1) with respect to x^0, it will follow that:

$$[(S^{00} + \chi\, p\, g^{00})\,(\varphi' + 1) + \chi\, g^{00}\,(\rho + p) - 2(S^{00} + \chi\, p\, g^{00})]\,\partial_0\, p = 0,$$

namely:

$$[g^{00} - (1 - \varphi')\frac{S^{00} + \chi\, p\, g^{00}}{\chi(\rho + p)}]\,\partial_0\, p = 0\,;$$

i.e., from (19-6):

$$[g^{00} - (1 - \varphi')(u^0)^2\,]\,\partial_0\, p = 0.$$

One deduces from this that $\partial_0\, p = 0$ on S and, as a result, that $\partial_0 \rho = 0$ and $\partial_0 u_\beta = 0$.

Therefore one has $Xp = 0$ and $X\rho = 0$ at the point x; hence:

$$XF = X \exp \int_{p_0}^{p} \frac{dp}{\rho + p} = 0,$$

and, on the other hand:

$$X\mathbf{u} = 0.$$

The motion considered will thus be permanent. We state the following:

THEOREM. – *If one is given a perfect fluid that admits an equation of state in a domain D_4 then in order for the motion of this fluid in D_4 to be permanent it is necessary and sufficient that the associated Riemannian spacetime be stationary in D_4.*

46. – The first integral *H*. – Consider a permanent motion of a holonomic medium. The Riemannian manifold that is defined by D_4, when endowed by the metric $d\bar{s}^2$, will admit the same group of isometries. Moreover, the vector field $\bar{\mathbf{C}}$ on this manifold will satisfy $X\bar{\mathbf{C}} = 0$.

The streamlines of a permanent motion are the trajectories of a vector field that admit the infinitesimal transformations X. One says that the corresponding differential system admits this infinitesimal transformation. As in classical hydrodynamics ([1]), the study of permanent motions will therefore be reduced to the study of a differential system that simultaneously admits an integral invariant and an infinitesimal transformation.

Let:

$$d\omega = dC_\beta \wedge dx^\beta$$

be the invariant for the differential system (38-2) for the streamlines. If $d\mathbf{x}_1$ and $d\mathbf{x}_2$ are two vectors that are tangent to V_4 at x then the alternating bilinear form:

$$d\omega(d\mathbf{x}_1, d\mathbf{x}_2) = d_1 C_\beta \, d_2 x^\beta - d_2 C_\beta \, d_1 x^\beta$$

will be canonically associated with the form $d\omega$, in which d_1C_β and d_2C_β are the values of the differential dC_β for both vectors. The existence of the infinitesimal transformation X that is defined by x permits us to deduce ([2]) an invariant linear form from the quadratic form $d\omega$. By (38-2), it is the form $d\omega(\xi, d\mathbf{x})$, namely:

$$d\omega(\xi, d\mathbf{x}) = \xi^\alpha \partial_\alpha C_\beta \, dx^\beta - \xi^\beta dC_\beta .$$

In local adapted coordinates, $\xi^i = 0$ and $\xi^0 = 0$, and one has:

$$d\omega(\xi, d\mathbf{x}) = - dC_0 .$$

([1]) E. CARTAN. *Leçons sur les invariants intégraux,* Hermann, Paris (1922), pp. 86.
([2]) E. CARTAN. *Leçons sur les invariants intégraux,* Hermann, Paris (1922), pp. 82-84.

Now, say that dC_0 is invariant because of (38-2); i.e., say that C_0 is a first integral of this system. Therefore, *the component C_0 of the current vector in adapted coordinates preserves a constant value along each streamline.* Now, C_0 is the expression in adapted coordinate for the scalar function H that is defined by:

(46-1) $$H = \xi^\alpha C_\alpha$$

in arbitrary local coordinates.

We therefore state the following:

THEOREM. – *The scalar function H that is defined by* (46-1) *preserves a constant value along each streamline for any permanent motion of a holonomic medium.*

The vorticity tensor and the vorticity tensor θ also admit the infinitesimal transformation X, since $\partial_0 \Omega_{\alpha\beta} = 0$ and $\partial_0 \theta^\alpha = 0$ in adapted coordinates. It results from the fact that $X\theta = 0$ that the differential system of the vortex lines admits the same properties as the differential system of the streamlines as far as $d\omega$ and X are concerned. Therefore, H will also preserve a constant value along each vortex line, and as a result, H will be constant on each characteristic manifold W_2.

47. – The differential of the function *H*. – We seek to evaluate the differential of the function H. In a coordinate system that is adapted to the stationary character one has:

$$dH = dC_0 = \partial_i C_0 \, dx^i = \Omega_{0i} \, dx^i,$$

which may be written:

(47-1) $$dH = \Omega_{\alpha\beta} \, \xi^\beta \, dx^\alpha,$$

which is a formula that is then valid in arbitrary coordinates. The results of sec. **46** will be obvious from this formula.

We wish to find the case in which H is constant not only on each characteristic manifold but also on every D_4. One deduces from (47-1) that in order for this to be true it will be necessary and sufficient that:

$$\Omega_{\alpha\beta} \, \xi^\beta = 0.$$

This will be the case when the permanent motion considered is either precisely irrotational or precisely rotational since the characteristic manifolds are generated by the timelines. The form $d\omega$ will then be invariant for the differential system of the timelines.

Let us return to the adapted coordinates. It is easy to express dH with the aid of the Ω_{ij} and the components of the velocity vector **u**. Indeed, one has:

$$\Omega_{i\alpha} u^\alpha = \Omega_{i0} \, u^0 + \Omega_{ij} \, u^j = 0.$$

Upon substituting the expression for Ω_{i0} so obtained for the one in dH in adapted coordinates, it follows that:

(47-2) $$u^0 \, dH = \Omega_{ij} \, u^j \, dx^i.$$

This formula is the relativistic extension of a formula in classical hydrodynamics that is due to Beltrami.

48. – Bernoulli's theorem. – One may put the results of the preceding sections into a form that closely recalls Bernoulli's theorem of classical hydrodynamics. To that effect, we introduce the *spatial magnitude* v^2 *of a unitary velocity vector* $\mathbf{u}$ relative to the direction of time ξ ([1]).

We adopt and *adapted local coordinate system* and, in order to abbreviate the notions, we set:

$$U = g_{00} ,$$

in which U, which is strictly positive, is called the *principal potential.* The direction of the vector ξ will coincides with that of the vector $\mathbf{e}_0$ in the natural frame at x that is associated with the adapted coordinates. It will therefore follow from (2-5) that:

$$-v^2 = \mathbf{u}^2 - \frac{u_0^2}{g_{00}} = 1 - \frac{u_0^2}{\mathrm{U}} .$$

One deduces from this that:

$$(u_0)^2 = U\,(1 + v^2)$$

and, as a result, that:

(48-1) $$(C_0)^2 = F^2\, U\,(1 + v^2).$$

We are therefore led to state the theorem of sec. **46** in the following form ([2]):

THEOREM. – *Along each streamline, the permanent motion of a holonomic medium satisfies the condition:*

$$F^2\, U\,(1 + v^2) = \text{constant},$$

in which F is the index of the medium, v^2 *is the spatial magnitude of the velocity vector relative to the direction of the timelines, and U is the principal potential of gravitation.*

We suppose that our holonomic medium is a perfect fluid that admits an equation of state and re-establish the velocity of light c in the expression (38-1) for the first integral. It then follows that:

$$(C_0)^2 = F^2\, U\,(1 + c^{-2}\, v^2),$$

in which F^2 is given by the formula:

$$F^2 = \exp \int_{p_0}^{p} \frac{dp}{c^2 \rho + p}$$

([1]) See sec. **2** for this notion.

([2]) The results of secs. **46**, **47**, **48**, figure in A. LICHNEROWICZ, Ann. École Normale, **58** (1941), 285-304.

in the usual physical units.

Suppose that the quotients $c^{-2}p / \rho$ and $c^{-2} v^2$ are small with respect to unity, and then formulate the expression for $(C_0)^2$ up to terms in c^{-4}. One first has the approximate expression:

$$F^2 = 1 + 2\int_{p_0}^{p} \frac{dp}{c^2 \rho},$$

and from this one deduces that:

$$(C_0)^2 = U + U\left(\frac{v^2}{c^2} + 2\int_{v_0}^{v} \frac{dp}{c^2 \rho}\right).$$

One sees that, up to terms in c^{-2}, the statement that:

$$\tfrac{1}{2}c^2 U + U\left(\frac{v^2}{c^2} + 2\int_{v_0}^{v} \frac{dp}{c^2 \rho}\right) = \text{constant}$$

is valid along each streamline, which is a result that closely recalls the classical statement of Bernoulli's theorem.

CHAPTER V

ESSAY ON THE RELATIVISTIC HYDRODYNAMICS OF VISCOUS FLUIDS

49. – The incompressible fluid. – We saw in sec. **19** that if we are given a perfect fluid that admits an equation of state $\rho = \varphi(p)$ then the velocity of propagation of the hydrodynamical wave fronts will be given by the scalar:

$$\frac{1}{\sqrt{\varphi'(p)}},$$

and that the equation of state is admissible from a relativistic standpoint only if $\varphi' \geq 1$. The case in which the wave fronts propagate with unit velocity (i.e., the velocity of light) is the one for which $\varphi' = 1$, i.e., the one for which the equation of state is of the form:

$$\rho - p = \text{constant}. \tag{49-1}$$

Such a relation represents – if I may say so – the maximum incompressibility that a perfect fluid may admit in relativity. Therefore, a fluid that satisfies (49-1) must be considered to be an *incompressible* perfect fluid from a relativistic point of view.

From (49-1) one may deduce a simple relation that depends upon the current vector C^α. The equation of continuity of the perfect fluid:

$$\nabla_\alpha[(\rho + p)\, u^\alpha] = u^\alpha \partial_\alpha p, \tag{49-2}$$

may be into the form:

$$(\rho + p)\, \nabla_\alpha u^\alpha + u^\alpha \partial_\alpha p = 0. \tag{49-2'}$$

By virtue of (49-1) the derivative of $(\rho - p)$ along each streamline is zero, namely:

$$u^\alpha \partial_\alpha (\rho - p) = 0. \tag{49-3}$$

From (49-2) and (49-3) one deduces that:

$$\nabla_\alpha u^\alpha + \frac{u^\alpha \partial_\alpha p}{\rho + p} = 0.$$

If F denotes the index of the fluid then this is:

$$\nabla_\alpha u^\alpha + u^\alpha \frac{\partial_\alpha F}{F} = 0. \tag{49-4}$$

Now, relation (49-4) is none other than:

(49-5) $$\nabla_\alpha C^\alpha = \nabla_\alpha (F u^\alpha) = 0.$$

More generally, an arbitrary fluid that is *endowed with an index F* will be called *incompressible* if it satisfies the relation $\nabla_\alpha C^\alpha = 0$.

50. – The index of a fluid and the associated metric. – In the course of this chapter, we propose to study what the energy-momentum tensor and the equations of motion of a viscous fluid might be, and, in particular, to explore how the classical equations of Navier may be generalized.

Consider the energy-momentum tensor of an arbitrary fluid. In the course of sec. **6**, we saw that this tensor may be put into the form:

$$T_{\alpha\beta} = \rho\, u_\alpha u_b + \pi_{\alpha\beta},$$

in which ρ is the proper density of the fluid, **u** is its unitary velocity vector, and $\pi_{\alpha\beta}$ is its pressure tensor. From the study of sec. **6** this symmetric tensor $\pi_{\alpha\beta}$ must satisfy the relations:

(50-1) $$\pi_{\alpha\beta}\; u^\beta = 0.$$

We assume that the pressure tensor involves a pressure scalar p that appears in the equation of state for the fluid, and once more we call the scalar function:

$$F = \exp \int_{p_0}^{p} \frac{dp}{\rho + p}$$

the *index* of the fluid at its various points.

As in the case of a perfect fluid, we introduce the metric:

$$d\bar{s}^2 = F^2\, ds^2$$

that is conformal to the spacetime metric, as well as the vectors **C** and $\bar{\mathbf{C}}$ on the Riemannian spaces that are defined by $d\bar{s}^2$ and ds^2, respectively. These vectors admit the covariant and contravariant components:

$$C_\alpha = F\, u_\alpha, \qquad C^\alpha = F\, u^\alpha,$$

and:

$$\bar{C}_\alpha = C_\alpha = F\, u_\alpha, \qquad \bar{C}_\alpha = F^{-1}\, u^\alpha,$$

so the vector $\bar{\mathbf{C}}$ is collinear with **u** in the same sense and unitary for the metric $d\bar{s}^2$.

We denote the covariant derivative operator for the metric $d\bar{s}^2$ by $\bar{\nabla}_\alpha$. It is easy to evaluate this operator when applied to a vector as a function of the analogous operator ∇_α

relative to the metric ds^2. If $\Gamma^\rho_{\alpha\beta}$ and $\bar{\Gamma}^\rho_{\alpha\beta}$ define the Riemannian connections for ds^2 and $d\bar{s}^2$, respectively, and if:

$$K_\alpha = \partial_\alpha \log F ,$$

then it will follow that:

$$\bar{\Gamma}^\rho_{\alpha\beta} = \bar{g}_{\alpha\beta}[\bar{\alpha}\bar{\beta}, \bar{\sigma}] = F^{-2} g^{\rho\sigma}\{[F^2 [\alpha\beta, \sigma] + F(g_{\beta\sigma}\partial_\alpha F + g_{\alpha\sigma}\partial_\beta F - g_{\alpha\beta}\partial_\sigma F)\}.$$

From this, one deduces the formula:

$$\text{(50-2)} \qquad \bar{\Gamma}^\rho_{\alpha\beta} = \Gamma^\rho_{\alpha\beta} + K_\alpha g^\rho_\beta + K_\beta g^\rho_\alpha - K^\rho g_{\alpha\beta}.$$

In particular, if v_β is an arbitrary covariant vector then one will deduce from (50-2) that:

$$\text{(50-3)} \qquad \bar{\nabla}_\alpha v_\beta = \nabla_\alpha v_\beta - K_\alpha v_\beta - K_\beta v_\alpha + K^\rho v_\beta g_{\alpha\beta}.$$

51. – The energy-momentum tensor of a viscous fluid. – The study of perfect fluids that was made in the preceding chapter leads us to think that there is no point in separating the purely kinematical properties from the dynamical properties here – as one does in classical hydrodynamics – and that the metric $d\bar{s}^2$ and its corresponding elements must be introduced for the study of the properties of the fluid that generalize the purely kinematical properties from the classical viewpoint.

We therefore consider a viscous fluid, which we characterize – from the standpoint of internal deformations – by its metric $d\bar{s}^2$, and, as a result, by the associated unitary vector **C**. We refer this fluid to an orthonormal frame for the metric $d\bar{s}^2$ at a point x in the domain that it sweeps. Relative to this frame:

$$\bar{\mathbf{C}}^i = C_i = 0, \qquad \bar{\mathbf{C}}^0 = C_0 = 1.$$

In such a frame, we may take the following expressions for the space components of the pressure tensor of a viscous fluid:

$$\text{(51-1)} \qquad \pi_{ij} = -\pi\bar{g}_{ij} + \tfrac{1}{2}\mu(\bar{\nabla}_i C_j + \bar{\nabla}_j C_i),$$

in which π is a scalar and μ is a viscosity coefficient. One will observe that in the frame envisioned the space tensor π_{ij} differs from the classical pressure tensor for a viscous fluid by relativistic corrections of order c^{-2}. One the other hand, from equations (50-1), one has:

$$\text{(51-2)} \qquad \pi_{\alpha 0} = 0,$$

in this frame.

It is easy to put equations (51-1) and (51-2) into a completely invariant form. The spacetime tensor whose only non-zero components are $\bar{g}_{ij}$ in this particular frame is none other than the tensor:

$$\bar{g}_{\alpha\beta} - C_\alpha C_\beta = F^2 (g_{\alpha\beta} - u_\alpha u_\beta).$$

On the other hand, consider the tensor $\gamma_{\alpha\beta}$ that is defined by the formula:

$$2\gamma_{\alpha\beta} = \bar{\nabla}_\alpha C_\beta + \bar{\nabla}_\beta C_\alpha - \bar{C}^\lambda (\bar{\nabla}_\lambda C_\alpha C_\beta + \bar{\nabla}_\lambda C_\beta C_\alpha). \tag{51-3}$$

In the frame envisioned, it admits space components that are given by:

$$2\gamma_{ij} = \bar{\nabla}_i C_j + \bar{\nabla}_j C_i. \tag{51-4}$$

On the other hand, one has:

$$2\gamma_{i0} = \bar{\nabla}_i C_0 + \bar{\nabla}_0 C_i - \bar{\nabla}_0 C_i = \bar{\nabla}_i C_0$$

in this frame and:

$$2\gamma_{00} = 2\bar{\nabla}_0 C_0 - 2\bar{\nabla}_0 C_0 = 0.$$

Now, since $\bar{\mathbf{C}}$ is unitary for the metric $d\bar{s}^2$, one will have:

$$\bar{C}^\lambda \bar{\nabla}_\alpha C_\lambda = 0 \tag{51-5}$$

in any frame. In particular:

$$\bar{\nabla}_\alpha C_0 = 0$$

in the frame that we adopted. Therefore, in the frame envisioned, the tensor $\gamma_{\alpha\beta}$ that is defined by formula (51-3) will admit components that are defined by (51-4) and:

$$\gamma_{\alpha 0} = 0.$$

It results from this that the pressure tensor $\pi_{\alpha\beta}$ that we adopted may be expressed by the invariant formula:

$$\pi_{\alpha\beta} = -\pi F^2 (g_{\alpha\beta} - u_\alpha u_\beta) + \mu\, \gamma_{\alpha\beta}.$$

Finally, in order to make the incompressibility of the fluid, which must appear in π along with the pressure, more obvious, we take:

$$\pi = F^{-2} (p - \lambda\, \nabla_\rho C^\rho),$$

in which p is the pressure that we introduced, which serves to define the index for us, and λ denotes the second viscosity coefficient.

$$T_{\alpha\beta} = (\pi + p - \lambda\, \nabla_\rho C^\rho)\, g_{\alpha\beta} + \mu\, \gamma_{\alpha\beta}, \tag{51-6}$$

in which the tensor $\gamma_{\alpha\beta}$ is given by the formula:

(51-7) $$2\gamma_{\alpha\beta} = \bar{\bar{\nabla}}_\alpha C_\beta + \bar{\bar{\nabla}}_\beta C_\alpha - \bar{C}^\lambda (\bar{\nabla}_\lambda C_\alpha C_\beta + \bar{\nabla}_\lambda C_\beta C_\alpha).$$

In all of what follows, we will assume that the viscosity coefficients λ and μ are constant throughout the fluid domain envisioned.

52. – The streamlines of a viscous fluid. – In a domain that is occupied by the fluid the gravitational metric, ds^2 is related to the energy-momentum tensor $T_{\alpha\beta}$ by the Einstein equations:

(52-1) $$S_{\alpha\beta} = \chi\, T_{\alpha\beta}.$$

As a result, the energy-momentum tensor (51-6) must satisfy the conservation conditions:

(52-2) $$\nabla_\alpha T_\beta^\alpha = 0,$$

which give the continuity equation, and the differential system that is satisfied by the velocity vector **u** that is tangent to the streamlines. In order to abbreviate notions, we set:

(52-3) $$\bar{\rho} = \rho + p - \lambda\, \nabla_\rho C^\rho.$$

If $T_{\alpha\beta}$ is given by (51-6) then equations (52-2) may be written:

(52-4) $$\nabla_\alpha(\bar{\rho}\, u^\alpha) u_\beta + \bar{\rho}\, u^\alpha \nabla_\alpha u_\beta - \partial_\beta(p - \lambda \nabla_\rho C^\rho) + \mu \nabla_\rho \gamma_\beta^\rho = 0.$$

Upon scalar multiplying (52-4) by u^β, one will obtain the continuity equation:

(52-5) $$\nabla_\alpha(\bar{\rho}\, u^\alpha) = u^\alpha[\partial_\beta(p - \lambda \nabla_\rho C^\rho) + \mu \nabla_\rho \gamma_\beta^\rho].$$

Upon substituting the expression for $\nabla_\alpha(\bar{\rho} u^\alpha)$ that one derives from (52-5) into (52-4), it will follow that:

(52-6) $$\bar{\rho}\, u^\alpha \nabla_\alpha u_\beta = (g_\beta^\alpha - u^\alpha u_\beta)[\partial_\beta(p - \lambda \nabla_\rho C^\rho) - \mu \nabla_\rho \gamma_\beta^\rho].$$

The differential system (52-6) that regulates the streamlines may be considered to be the relativistic extension of the Navier equations.

53. – The calculation of the vector $\nabla_\rho \gamma_\alpha^\rho$. – The vector $\nabla_\rho \gamma_\alpha^\rho$ appears in the right-hand side of equations (52-6), and it becomes necessary for us to evaluate this expression by starting with (51-7) and (50-3). The vorticity tensor of the fluid, i.e., the rotation of C_α, is always denoted by $\Omega_{\alpha\beta}$, and we have:

(53-1) $$\Omega_{\alpha\beta} = (\partial_\alpha C_\beta - \partial_\beta C_\alpha) = (\nabla_\alpha C_\beta - \nabla_\beta C_\alpha) = (\nabla_\alpha C_\beta - \nabla_\beta C_\alpha).$$

From the relation $\bar{C}^\lambda \bar{\nabla}_\alpha C_\lambda = 0$ that we established in sec. **51**, we see that formula (51-7) may be put into the form:

$$2\gamma_{\alpha\beta} = \bar{\nabla}_\alpha C_\beta + \bar{\nabla}_\beta C_\alpha - \bar{C}^\lambda (\Omega_{\alpha\beta} C_\beta + \Omega_{\beta\alpha} C_\alpha),$$

or again, upon destroying the symmetry in the first two terms and introducing u^λ instead of $\bar{C}^\lambda$:

$$2\gamma_{\alpha\beta} = 2\bar{\nabla}_\beta C_\alpha + \Omega_{\alpha\beta} - u^\lambda (\Omega_{\alpha\beta} u_\beta + \Omega_{\beta\alpha} u_\alpha).$$

In the sequel, we will introduce the tensor:

(53-2) $$2\Theta_{\alpha\beta} = \Omega_{\alpha\beta} - u^\lambda (\Omega_{\lambda\beta}\, u_\beta + \Omega_{\lambda\beta}\, u_\alpha),$$

in which the components depend on the vorticity tensor linearly. One will therefore have:

$$\gamma_{\alpha\beta} = \bar{\nabla}_\beta C_\alpha + \Theta_{\alpha\beta}.$$

By virtue of (50-3), let:

$$\gamma_{\alpha\beta} = \nabla_\beta C_\alpha - K_\alpha C_\beta + K_\beta C_\alpha - K^\rho C_\rho g_{\alpha\beta} + \Theta_{\alpha\beta}.$$

Start with this expression for the tensor $\gamma_{\alpha\beta}$ and take the contracted covariant derivative for the metric ds^2. It will follow that:

$$\nabla_\alpha \gamma^\alpha_\beta = \nabla_\alpha \nabla_\beta C^\alpha - (\nabla_\alpha K^\alpha)\, C_\beta - (\nabla_\alpha C^\alpha)\, K_\beta - C^\alpha \nabla_\alpha K_\beta$$
$$- K^\alpha \nabla_\alpha C_\beta + \nabla_\beta (K^\alpha C_\alpha) + \nabla_\alpha \Theta^\alpha_\beta.$$

However, upon observing that K_β is a gradient and introducing the vorticity tensor again, one will get:

$$C^\alpha \nabla_\alpha K_\beta + K^\alpha \nabla_\alpha C_\beta = C^\alpha \nabla_\beta K_\alpha + K^\alpha \nabla_\beta C_\alpha + \Omega_{\alpha\beta} K^\alpha = \nabla_\beta (K^\alpha C_\alpha) + \Omega_{\alpha\beta} K^\alpha.$$

One deduces from this that:

(53-3) $$\nabla_\alpha \gamma^\alpha_\beta = \nabla_\alpha \nabla_\beta C^\alpha - (\nabla_\alpha K^\alpha)\, C_\beta - (\nabla_\alpha C^\alpha)\, K_\beta - \Omega_{\alpha\beta} K^\alpha + \nabla_\alpha \Theta^\alpha_\beta.$$

On the other hand, from a classical formula that relates to the curvature tensor, one will have:

$$\nabla_\alpha \nabla_\beta C^\alpha - \nabla_\beta \nabla_\alpha C^\alpha = R_{\alpha\beta,}{}^\alpha{}_\rho\, C^\rho = R_{\alpha\beta} C^\alpha.$$

Now, by virtue of the Einstein equations it will follow that:

$$S_{\alpha\beta} C^\alpha = R_{\alpha\beta} C^\alpha - \tfrac{1}{2} R C_\beta = \chi\, T_{\alpha\beta} C^\alpha = \chi\rho\, C^\beta,$$

since C is an eigenvector of the energy-momentum tensor with the eigenvalue ρ. From this one deduces that:

$$\nabla_\alpha \nabla_\beta C^\alpha = \partial_\beta (\nabla_\alpha C^\alpha) + (\chi\rho + \tfrac{1}{2} R)\, C_\beta,$$

and since K_α is the gradient of log F:

(53-4) $$\nabla_\alpha \gamma_\beta^\alpha = \partial_\beta (\nabla_\alpha C^\alpha) - \frac{\partial_\beta F}{F} \nabla_\alpha C^\alpha + (\chi\rho + \tfrac{1}{2} R - \Delta \log F)\, C_\beta - \Omega_{\alpha\beta} K^\alpha + \nabla_\alpha \Theta_\beta^\alpha,$$

a formula that may also be written:

(53-5) $$\nabla_\alpha \gamma_\beta^\alpha = (\nabla_\alpha C^\alpha)\, \partial_\beta \log \frac{\nabla_\alpha C^\alpha}{F} + (\chi\rho + \tfrac{1}{2} R - \Delta \log F)\, C_\beta - \Omega_{\alpha\beta} K^\alpha,$$

in which Δ denotes the Laplacian operator for the metric ds^2.

Substitute this value for the vector $\nabla_\alpha \gamma_\beta^\alpha$ into the system (52-6) that regulates the streamlines, and note that:

$$(g_\beta^\alpha - u^\alpha u_\beta)\, C_\alpha = F\, (g_\beta^\alpha - u^\alpha u_\beta)\, u_\alpha = 0.$$

will then therefore follow for this system that:

(53-6) $$\begin{aligned} \bar\rho\, u^\alpha \nabla_\alpha u_\beta &= (g_\beta^\alpha - u^\alpha u_\beta) \\ &\times \left[\partial_\alpha (p - \lambda \nabla_\rho C^\rho) - \mu (\nabla_\rho C^\rho \partial_\alpha \log \frac{\nabla_\rho C^\rho}{F} - \Omega_{\alpha\beta} K^\rho + \nabla_\rho \Theta_\alpha^\rho) \right]. \end{aligned}$$

This system therefore defines the motion of a viscous fluid. One will note that the proof of (53-6) involves the Einstein equations explicitly even though the curvature of ds^2 is eliminated in the final result.

54. – The irrotational motion of a viscous fluid. – We say that the viscous fluid considered admits an irrotational motion when the vorticity tensor $\Omega_{\alpha\beta}$ is zero for this motion; suppose that this is true. From the identity:

$$\bar C^\alpha \bar{\bar\nabla}_\beta C_\alpha = 0,$$

one will derives:

(54-1) $$\bar C^\alpha\, \bar{\bar\nabla}_\alpha C_\beta = 0$$

under these conditions, and the streamlines will necessarily be geodesics of the metric $d\bar{s}^2$. Therefore, the equations of motion (53-6) must coincide with the ones that relate to a perfect fluid with density ρ and pressure p ([1]).

We look for the case in which this is true precisely. For an irrotational motion, (53-6) will reduce to:

$$(54\text{-}2)\qquad (\rho + p - \lambda\,\nabla_\rho C^\rho)\, u^\alpha\,\nabla_\alpha u_\beta$$

$$= (g^\alpha_\beta - u^\alpha u_\beta)\left[\partial_\alpha(p - \lambda\nabla_\rho C^\rho) - \mu\nabla_\rho C^\rho \partial_\alpha \log\frac{\nabla_\rho C^\rho}{F}\right],$$

and equations (54-1) may be written:

$$(54\text{-}3)\qquad (\rho + p)\, u^\alpha\,\nabla_\alpha u^\beta = (g^\alpha_\beta - u^\alpha u_\beta)\partial_\alpha p \qquad \text{or} \qquad u^\alpha\,\nabla_\alpha u^\beta = (g^\alpha_\beta - u^\alpha u_\beta)\frac{\partial_\alpha F}{F}.$$

Upon taking (54-3) into account, it will then follow that:

$$u^\alpha\,\nabla_\alpha u^\beta = (g^\alpha_\beta - u^\alpha u_\beta)\left[\frac{1}{\nabla_\rho C^\rho}\partial_\alpha(\nabla_\rho C^\rho) + \frac{\mu}{\lambda}\partial_\alpha \log\frac{\nabla_\rho C^\rho}{F}\right],$$

which is a formula that may be put into the form:

$$(54\text{-}4)\qquad u^\alpha\,\nabla_\alpha u^\beta = (g^\alpha_\beta - u^\alpha u_\beta)\left[\frac{\partial_\alpha F}{F} + \left(1 + \frac{\mu}{\lambda}\right)\partial_\alpha \log\frac{\nabla_\rho C^\rho}{F}\right].$$

In order for (54-4) to be equivalent to (54-3), it is necessary and sufficient that either $\lambda + \mu = 0$ or that the vector:

([1]) The difference between this result and the analogous result in classical hydrodynamics points to a genuine difficulty in the theory: The definition of the rotation in a motion can be made only by means of an index F. It is true that this index is largely indeterminate, and one will have to try to remedy the difficulty that just presented itself by leaving the index F temporarily indeterminate. All of the formulae that follow will remain true, with the exception of the one that relates F to ρ and p. In an irrotatonal motion, the streamlines naturally remain geodesics of the ds^2, but this metric will no longer necessarily correspond to a fluid that is assumed to be perfect. If one wants to express the possibility of irrotational motions in any case then one will come to the following condition: There must exist a scalar λ such that:

$$\frac{\partial_\alpha F}{F} = \frac{\partial_\alpha[p - (\lambda + \mu)\theta]}{\rho + p - (\lambda + \mu)\theta} + \psi u_\alpha \qquad (\theta = \nabla_\rho C^\rho),$$

but there is no reason for this to be true, in general. We prefer the viewpoint that is adopted here. In fact, the indeterminacy that prevails in the definition of pressure makes the difference obvious without seriously inconveniencing the theory.

$$\text{grad}\log\frac{\nabla_\rho C^\rho}{F}$$

must be collinear with the vector **u**; i.e., that the quotient $\frac{\nabla_\rho C^\rho}{F}$ must be constant on the hypersurfaces that have the streamlines for their orthogonal trajectories. In particular, this will be the case if $\nabla_\rho C^\rho = 0$; i.e., if the fluid is incompressible. We then see that one has the following:

THEOREM._ – *An incompressible, viscous fluid may admit an irrotational motion for which the equations of the streamlines are identical with the ones for an irrotational motion of an incompressible, perfect fluid.*

CHAPTER VI

THE RELATIVISTIC HYDRODYNAMICS OF A CHARGED, PERFECT FLUID

55. – The conservation conditions for a charged, perfect fluid. – In this chapter, we propose to study whether it is possible to extend the relativistic hydrodynamics of a perfect fluid that was developed in Chapter IV to the case of a charged, perfect fluid. The case of the charged, pure matter schema will be treated as a particular case.

We will always assume that the energy distribution in the domain of V_4 envisioned is given by the energy-momentum tensor that was studied in detail in sec **11**:

$$T_{\alpha\beta} = (\rho + p)\, u_\alpha u_\beta - p\, g_{\alpha\beta} + \tau_{\alpha\beta}, \tag{55-1}$$

in which $\tau_{\alpha\beta}$ is the energy-momentum tensor for an electromagnetic field $F_{\alpha\beta}$. This tensor will satisfy the Maxwell equations (20-1), which we write as:

$$\nabla_\beta F^{\alpha\beta} = \mu\, J^\alpha. \tag{55-2}$$

The current vector **J** is given by the Lorentz transport equation, and μ is the proper charge density of the fluid. Here, we assume that there exists a *global vector potential* in the domain envisioned, i.e., a vector field φ_α such that:

$$F_{\alpha\beta} = \partial_\alpha \varphi_\beta - \partial_\beta \varphi_\alpha . \tag{55-3}$$

In particular, this is always the case if the domain envisioned is simply-connected. The linear differential form:

$$\varphi = \varphi_\alpha dx^\alpha,$$

which we call the *vector potential form*, is associated with this vector potential. Its exterior differential is obviously the electromagnetic field form.

Finally, we assume that there exists an equation of state $\rho = \varphi(p)$, and introduce the index of the fluid ([1]):

$$F = \exp \int_{p_0}^{p} \frac{dp}{\rho + p} . \tag{55-4}$$

First, it is necessary for us to form the continuity equations and the streamlines that are deduced from the conservation conditions (55-1) for the energy-momentum tensor. The energy-momentum tensor that is adopted will be of the form:

([1]) In this chapter, the notation F will be reserved for the index of a fluid, and we will not explicitly introduce the electromagnetic field form.

$$T_{\alpha\beta} = r\, u_\alpha u_\beta - \theta_{\alpha\beta},$$

with:

$$r = \rho + p \qquad\qquad \theta_{\alpha\beta} = pg_{\alpha\beta} - \tau_{\alpha\beta}.$$

We must therefore introduce the vector K_β, which is defined by:

$$(\rho + p)\, K_\beta = \nabla_\alpha\,(pg_\beta^\alpha - \tau_\beta^\alpha) = \partial_\beta\, p - \nabla_\alpha \tau_\beta^\alpha .$$

Now, from equations (55-2) one has:

$$\nabla_\alpha \tau_\beta^\alpha = F_{\rho\beta} J^\rho = \mu\, F_{\rho\beta} u^\beta.$$

One deduces from this that:

(55-5) $$(\rho + p)\, K_\beta = \partial_\beta\, p + \mu\, F_{\beta\rho} u^\rho.$$

Under these conditions, the continuity equation will be written, from (17-6):

$$\nabla_\alpha[(\rho + p)\, u^\alpha] = u^\alpha(\partial_\beta\, p + \mu F_{\alpha\rho} u^\rho) .$$

Hence, from the antisymmetry of $F_{\alpha\beta}$ we will get:

(55-6) $$\nabla_\alpha[(\rho + p)\, u^\alpha] = u^\alpha \partial_\beta\, p ,$$

which is formally identical with the equation for a perfect fluid in the absence of charge and an electromagnetic field.

From (11-7), the differential system that determines the streamlines is:

(55-7) $$u^\alpha\, \nabla_\alpha u^\beta = (g_\beta^\alpha - u^\alpha u_\beta)\left(\frac{\partial_\alpha p}{\rho + p} + \frac{\mu}{\rho + p} F_{\alpha\rho} u^\rho\right).$$

Finally, the conservation of charge translates into the relation:

(55-8) $$\nabla_\alpha(\mu\, u^\alpha) = 0.$$

Having said this, observe that equations (55-6) and (55-8) may be put into the form:

$$\nabla_\alpha u^\alpha + \frac{u^\alpha \partial_\alpha p}{\rho + p} = 0,$$

and:

$$\nabla_\alpha u^\alpha + \frac{u^\alpha \partial_\alpha \mu}{\mu} = 0,$$

respectively.

It follows by subtraction that:

$$u^{\alpha}\left(\frac{\partial_{\alpha}\mu}{\mu}-\frac{\partial_{\alpha}p}{\rho+p}\right)=0. \tag{55-9}$$

Now introduce the positive scalar ρ^*, which is defined by the relation:

$$\rho^*=\frac{\rho+p}{F}, \tag{55-10}$$

which is a scalar that differs very little from the density ρ. Indeed, up to terms in c^{-4}, one will have:

$$\rho^*=\rho\left(1+\frac{c^{-2}p}{\rho}-c^{-2}\int_{p_0}^{p}\frac{dp}{\rho}\right)$$

in the usual physical units. More rigorously, one has for the scalar ρ^*:

$$\frac{d\rho^*}{\rho^*}=\frac{d(\rho+p)}{\rho+p}-\frac{dp}{\rho+p}=\frac{d\rho}{\rho+p}.$$

One deduces from this that equations (55-9) may be put into the form:

$$u^{\alpha}\partial_{\alpha}\log\frac{\mu}{\rho^*}=0,$$

which expresses the fact that *the ratio $k=\mu/\rho^*$ stays constant all along the streamline.* In the sequel, we shall study *only fluids that are charged in a homogenous manner*, i.e., ones for which the ratio k stays constant throughout the domain of spacetime that is envisioned.

56. – The extremal principle for the streamlines. – In a given spacetime domain, we therefore give ourselves a perfect fluid schema that is charged in a homogeneous manner and admits the index F and the vector potential φ. Consider the integral:

$$\sigma=\int_{x_0}^{x_1}[F\,ds+k\varphi], \tag{56-1}$$

which is evaluated on an arc of a time-oriented curve C that joins two points x_0 and x_1. If we parametrically represent C with the aid of the arbitrary parameter u then:

$$\sigma=\int_{u_0}^{u_1}[F(g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta})^{1/2}+k\varphi_{\alpha}\dot{x}^{\alpha}]\,du \qquad \left(\dot{x}^{\alpha}=\frac{dx^{\alpha}}{du}\right).$$

Introduce the function $f(x^\lambda, \dot{x}^\mu)$, which is homogeneous of the first degree with respect to the $\dot{x}^\mu$ and is defined by:

$$f = F(g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{1/2} + k\,\varphi_\alpha \dot{x}^\alpha .$$

Its partial derivatives are defined by the relations:

(56-2)

$$\frac{\partial f}{\partial \dot{x}^\alpha} = F\frac{g_{\alpha\beta}\dot{x}^\beta}{(g_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma)^{1/2}} + k\varphi_\alpha ,$$

$$\frac{\partial f}{\partial x^\alpha} = F\frac{\partial_\alpha g_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma}{2(g_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma)^{1/2}} + \partial_\alpha F(g_{\beta\gamma}\dot{x}^\beta \dot{x}^\gamma)^{1/2} + k\partial_\alpha \varphi_\beta \dot{x}^\beta .$$

Consider a three-parameter family of time-oriented curves, and study the variation of σ when taken over an arc of a curve in the family when one varies this arc within the family. If we adopt the curvilinear abscissa s on the arc C then the vector:

$$\dot{x}^\alpha = \frac{dx^\alpha}{ds} = u^\alpha$$

will be the unitary tangent vector to C, and it will locally define a vector field. Under these conditions, formulas (56-2) can be written:

(56-3) $$\frac{\partial f}{\partial \dot{x}^\alpha} = F\,u_\beta + k\,\varphi_\beta ,$$

(56-4) $$\frac{\partial f}{\partial x^\alpha} = \tfrac{1}{2}F\,\partial_\beta\, g_{\alpha\rho}u^\alpha u^\rho + \partial_\beta F + k\partial_\beta \varphi_\alpha u^\alpha .$$

In this parametric representation, the components P_β of the Euler vector are:

$$P_\beta = \frac{d}{ds}(F\,u_\beta + k\,\varphi_\beta) - F[\alpha\beta, \rho]\,u^\alpha u^\rho - \partial_\beta F - k\partial_\beta \varphi_\alpha u^\alpha ,$$

so that upon specifying the total derivative with respect to s we will get:

$$P_\beta = F\left[u^\alpha \nabla_\alpha u_\beta - (g^\alpha_\beta - u^\alpha u_\beta)\frac{\partial_\alpha F}{F}\right] - k\,F_{\beta\rho}u^\rho .$$

Now obviously:

$$(g^\alpha_\beta - u^\alpha u_\beta)\,F_{\alpha\rho}u^\rho = F_{\beta\rho}u^\rho.$$

Therefore the components P_β of the covariant Euler vector may be written:

(56-5) $$P_\beta = F\left[u^\alpha \nabla_\alpha u_\beta - (g^\alpha_\beta - u^\alpha u_\beta)\left(\frac{\partial_\alpha F}{F} + \frac{k}{F} F_{\alpha\rho} u^\rho\right)\right].$$

From the argument in secs. **35** and **36**, one will thus have the formula:

(56-6) $$\delta\sigma = [\omega(\delta)]_{x_1} - [\omega(\delta)]_{x_0} - \int_{s_0}^{s_1} < P\delta\mathbf{x} > ds,$$

for the variation of σ, in which, from (56-3) $\omega(\delta)$ is defined locally by:

(56-7) $$\omega(\delta) = (F\, u_\alpha + k\, \varphi_\alpha)\, dx^\alpha.$$

Now apply formula (56-6) to the case in which C varies with fixed endpoints x_0 and x_1. In order that σ be an extremum under these conditions, it is necessary and sufficient that $P = 0$, i.e., that:

(56-8) $$u^\alpha \nabla_\alpha u_\beta - (g^\alpha_\beta - u^\alpha u_\beta)\left(\frac{\partial_\alpha F}{F} + \frac{k}{F} F_{\alpha\rho} u^\rho\right) = 0.$$

If we note that:

$$\frac{\partial_\alpha F}{F} = \frac{\partial_\alpha p}{\rho + p} \qquad\qquad \frac{k}{F} = \frac{\mu}{\rho^* F} = \frac{\mu}{\rho + p}$$

then we will see that equations (56-8) are formally identical to equations (55-7), which regulate the streamlines. We may then state the:

THEOREM. – *In any motion of a perfect fluid that is charged in a homogenous manner the streamlines are the time-oriented curves that realize the extremum for the integral:*

$$\sigma = \int_{x_0}^{x_1} (F\, ds + k\, \varphi),$$

for variations with fixed endpoints, in which φ is the vector potential form, and k is the constant ratio μ / ρ^*.

The streamlines that appear here are the geodesics, not of a Riemannian metric, but of a Finslerian metric that depends upon the parameter k. One knows ([1]) that the form (56-1) in the integrand is one from which one may obtain a Riemannian metric by the introduction of a supplementary dimension.

57. – The invariant form $d\omega$ and the vorticity tensor of a charged, perfect fluid. – Consider a definite motion of a perfect fluid that is charged in a homogenous manner. One deduces results from formula (56-6) and the preceding extremal principle that are

([1]) See LICHNEROWICZ and Y. THIRY, C.R. Soc. Acad. Sc., **224** (1947), 529.

analogous to the ones that were established in Chapter IV for the streamlines of such a motion. Since the reasoning is identical to that of the rest of that chapter, we will confine ourselves to stating the results and exhibiting the various differences that they present.

First of all, the differential system of the streamlines:

$$\frac{dx^0}{u^0} = \frac{dx^1}{u^1} = \frac{dx^2}{u^2} = \frac{dx^3}{u^3} \tag{57-1}$$

admits the relative integral invariant:

$$\int_\Gamma \omega = \int_\Gamma (F\, u_\alpha + k\varphi_\alpha)\, dx^\alpha \;. \tag{57-2}$$

If $C_\alpha = Fu_\alpha$ denotes the hydrodynamical current vector of the fluid then the form ω will define the covariant vector:

$$I_\alpha = Fu_\alpha + k\varphi_\alpha = C_\alpha + k\varphi_\alpha. \tag{57-3}$$

We give the name of *momentum vector* for the charged fluid considered to the vector **I** of the Riemannian manifold that is defined by ds^2 and admits these components. This vector will admit the contravariant components:

$$I^\alpha = Fu^\alpha + k\varphi^\alpha = C^\alpha + k\varphi^\alpha.$$

The trajectories of this vector field will be called *momentum lines.* We note that the orientation of these lines may be arbitrary, *a priori*. The form ω appears as the elementary circulation of the momentum vector:

$$\omega = I_\alpha dx^\alpha.$$

As a result, we will have the following:

THEOREM – *If* Γ *is a one-dimensional cycle that is not tangent to the streamlines then the circulation of the momentum vector along* Γ *is invariant when one deforms* Γ *on the flow tube that is defined by* Γ.

Consider the form:

$$d\omega = dI_\beta \wedge dx^\beta = \tfrac{1}{2}(\partial_\alpha I_\beta - \partial_\alpha I_\beta)\, dx^\alpha \wedge dx^\beta. \tag{57-4}$$

One may further translate the preceding result by saying that *the form* $d\omega$ *is an invariant of the differential system* (57-1) *for the streamlines.*

The antisymmetric tensor that is the rotation of the momentum vector is associated with the form $d\omega$:

$$\Pi_{\alpha\beta} = \partial_\alpha I_\beta - \partial_\alpha I_\beta = \Omega_{\alpha\beta} + kF_{\alpha\beta}, \tag{57-5}$$

in which $\Omega_{\alpha\beta}$ is the rotation of the hydrodynamical current vector. We give the name of *vorticity tensor for a charged fluid* to the tensor $\Pi_{\alpha\beta}$. The characteristic system of the form $d\omega$ is then the system:

(57-6) $$\Pi_{\alpha\beta}\, dx^{\beta} = 0.$$

If the vorticity tensor is non-zero then this system will have rank two, and the motion envisioned will be called *rotational*. If the vorticity tensor is identically zero then the motion will be called *irrotational*.

58. – The study of an irrotational motion of a charged, perfect fluid. – First, we suppose that the motion envisioned is irrotational and confine ourselves to a simply-connected domain of that fluid. In order that $\Pi_{\alpha\beta} = 0$ in such a domain, it is necessary and sufficient that the momentum vector I^{α} be the gradient of a scalar function, i.e., that *the momentum lines be the orthogonal trajectories to a one-parameter family of hypersurfaces.*

An irrotational motion possesses a property of "permanence" that translates into the following statement:

THEOREM – *If there exists a spatially-oriented (or, more generally, transverse to the streamlines) hypersurface S upon which the vorticity tensor is annulled then the motion of the charged fluid envisioned will be irrotational.*

This theorem is established by reasoning that is identical to that of sec. **40** upon adopting local coordinates in which the streamlines are the lines x^i = constant, and the hypersurface S is represented by $x^0 = 0$. In particular, *if the momentum lines are orthogonal to a hypersurface S* that is transverse to the streamlines then the motion will be irrotational.

56. – The study of a rotational motion. – In the sequel, we will assume that the motion envisioned is irrotational. The vorticity vector τ of the charged perfect fluid will then defined by the relation:

$$\tau^{\alpha} = \tfrac{1}{2}\, \eta^{\alpha\beta\gamma\delta} u_{\beta} \Pi_{\gamma\delta};$$

the vortex lines will be its trajectories. These lines are orthogonal to the streamlines, and their differential system leaves the form $d\omega$ invariant. One deduces from this that the circulation of the momentum vector along a one-dimensional cycle enjoys the same properties with respect to the vortex lines that the circulation of the hydrodynamical current vector does in the absence of charge.

The two-dimensional characteristic manifolds of the form $d\omega$ are once more generated by the streamlines and the vortex lines.

60. – The permanent motion of a charged, perfect fluid. – A motion of a charged, perfect fluid will be called "permanent" in a domain D_4 if the Riemannian space that is defined by the associated metric ds^2 is stationary in D_4 and the corresponding group of isometries leaves the potential vector φ_α invariant.

If X denotes the Lie derivative operator that is associated with the infinitesimal generator ξ of the group of isometries then one will have:

$$X\varphi_\alpha = 0.$$

One will therefore easily see that:

$$XF_{\alpha\beta} = 0 \qquad Xp = X\rho = XF = 0 \qquad Xu_\alpha = 0.$$

One deduces the existence of the invariant linear form $d\omega(\xi, d\mathbf{x})$ for the differential system of the streamlines from the existence of the invariant quadratic form $d\omega$ and the infinitesimal transformation X. In adapted coordinates this form is written:

$$d\omega(\xi, d\mathbf{x}) = -dI_0 .$$

The following theorem results from this:

THEOREM – *In any permanent motion of a charged, perfect fluid, the component:*

$$I_0 = Fu_0 - k\varphi_0$$

of the momentum vector relative to an adapted coordinate system will preserve a constant value along each characteristic manifold, and therefore along each streamline and vortex line, in particular.

The first integral of the system of streamlines that comes into existence may be written in arbitrary coordinates:

(60-1) $$H = I_\alpha \xi^\alpha.$$

Its differential may be expressed by:

(60-2) $$dH = \Pi_{\alpha\beta}\xi^\beta dx^\alpha.$$

As a result, H will be constant over all of D_4 if $\Pi_{\alpha\beta}\xi^\beta = 0$, i.e., if the characteristic manifolds are also generated by the timelines or if the motion is irrotational. Under the former hypothesis, the form $d\omega$ will be an invariant of the differential system of the streamlines.

61. – The case of a pure, charged matter schema. – All of the preceding results apply when one substitutes a pure matter schema that is charged in a homogenous manner for a perfect fluid that is charged in a homogenous manner. In the preceding equations, it suffices to set:

$$p = 0 \qquad\qquad F = 1 \qquad\qquad \rho^* = \rho.$$

The momentum vector is then given by:

$$I_\alpha = u_\alpha + k\varphi_\alpha \qquad\qquad (k = \mu / \rho = \text{constant}),$$

and this equation is precisely in accord with the one that is adopted in special relativity in order to define the momentum vector of a charged particle.

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III. – GLOBAL STUDY OF STATIONARY SPACETIMES

CHAPTER VII

RICCI TENSOR FOR A SPACE THAT ADMITS A CONNECTED, ONE-PARAMETER GROUP OF ISOMETRIES. APPLICATIONS

I. – STATIONARY SPACETIME

62. – Notion of a stationary, Riemannian spacetime. – We say that a Riemannian spacetime V_4 is *stationary* if it is stationary everywhere on its manifold; i.e., if there exists a connected one-parameter group of global isometries on V_4 with time-oriented trajectories z that leave no point of V_4 invariant, and the family of lines z – or *timelines* – satisfies the following hypotheses:

a) The timelines are homeomorphic to the real line $\mathbb{R}$.

b) One may find a three-dimensional, differentiable manifold V_3 that satisfies the same differentiability hypotheses as V_4 such that there exists a differentiable homeomorphism of class C^2 of the manifold V_4 with the product manifold $V_3 \times \mathbb{R}$ in which the z map to the right-hand factors. Moreover, this homeomorphism is assumed to be piecewise-continuous up to order four.

We may naturally identify V_3 with the space whose points z are the timelines; V_3 will be called the quotient space, or, more briefly, *space.* We have seen that there then exist local coordinates (x^λ) in V_4, which are said to be *adapted* to the stationary character, that enjoy the following properties:

1. The (x^i) define an arbitrary local coordinate system on V_3. The manifolds x^0 = const. are globally-defined manifolds on V_4 and are homeomorphic to V_3. The homeomorphisms of *b*) may be assumed to map the manifolds x^0 = const onto the manifolds that are homeomorphic to V_3 in the product manifold $V_3 \times \mathbb{R}$.

2. The potentials $g_{\alpha\beta}$ are independent of the variable x^0 relative to adapted coordinates. The vector $\boldsymbol{\xi}$, which is the infinitesimal generator of the isometry group, admits the contravariant components:

$$\xi^i = 0, \qquad \xi^0 = 1.$$

The square of this vector is:

(62-1) $$\xi^2 = g_{00} > 0 \qquad (\xi = \sqrt{\xi^2} > 0).$$

2. These coordinates are defined, up to a coordinate change, by:

(62-2) $$x^{i'} = \psi^{i'}(x^j), \qquad x^{0'} = x^0 + \psi(x^j),$$

in which ψ denotes the restriction of an arbitrary function $\psi(x)$ that is defined on V_3 to a local chart of V_3.

In all of what follows in this part, we will introduce only adapted local coordinate systems. In such a system, the manifolds x^0 = const will be called *spatial sections* of that system.

Let W_3 be a definite spatial section of V_3; this manifold will be homeomorphic to V_3, and the (x^i) will define local coordinates on it. The metric of V_4 and its group of isometries will define some tensors on each W_3. This is true because the g_{ij} define a symmetric tensor (the induced metric) on W_3, the g_{0i} define a covariant vector, and the g_{00} define a scalar, since these quantities transform according to the classical tensorial laws under a change of local coordinates:

(62-3) $$x^{i'} = \psi^{i'}(x^j), \qquad x^{0'} = x^0,$$

that preserves W_3. Naturally, one tensor must map to the other one under the map that is induced by the isometry that maps one spatial section of the same system to another

Among these tensors, certain ones, of which the scalar $\xi^2 = g_{00}$ is the simplest example, must map to each other under the map that is induced by the homeomorphism of two spatial sections with different systems. They may thus be considered to be *defined on the quotient space* V_3. In order for a tensor of W_3 to be defined on V_3, it is necessary and sufficient that it be invariant under the change:

(62-4) $$x^{i'} = x^i, \qquad x^{0'} = x^0 + \psi(x^j),$$

which we will call a *change of the system of spatial sections*.

63. – The Riemannian spaces V_3 and W_3. – We assign an orthonormal frame $(\mathbf{e}_\alpha)$ to each point x of a neighborhood of V_4, whose first vector $\mathbf{e}_0$ is the unitary vector that is tangent at x to the timeline $z(x)$ that passes through x. The vector $d\mathbf{x}$ is represented by:

$$d\mathbf{x} = \omega^\alpha \mathbf{e}_\alpha = \omega^0 \mathbf{e}_0 + \omega^i \mathbf{e}_i,$$

in which the ω^α are local Pfaff forms. The metric of V_4 may then be written:

(63-1) $$ds^2 = (\omega^0)^2 - \sum_i (\omega^i)^2,$$

and the ω^j will be annulled along the timelines. Such an orthonormal frame will be called *adapted.*

Having said this, let (x^l) be an adapted system of local coordinates. Since the ω^j are Pfaff forms with respect to the dx^i, one will see that (63-1) is nothing but a decomposition of the fundamental quadratic form into squares with the variable dx^0 playing the role of directrix variable. Upon performing this decomposition into squares one has:

$$ds^2 = (\omega^0)^2 + ds^{*2}, \tag{63-2}$$

with:

$$\omega^0 = \frac{1}{\xi}(g_{00}\,dx^0 + g_{0i}\,dx^i), \tag{63-3}$$

and

$$ds^{*2} = -\sum(\omega^i)^2 = \left(g_{ij} - \frac{g_{0i}\,g_{0j}}{g_{00}}\right)dx^i\,dx^j\,. \tag{63-4}$$

From formula (63-4), it results from the last expression that the quadratic form ds^{*2} is independent of the variable x^0, and, from the second expression that it is independent of any system of spatial sections. From this, one deduces that *it determines a negative-definite Riemannian metric on* V_3. The quantities:

$$\dot{g}_{ij} = g_{ij} - \frac{g_{0i}\,g_{0j}}{g_{00}}\,,$$

are the components of a tensor on V_3, or a "*spatial tensor.*" The corresponding contravariant tensor is $\dot{g}^{ij} = g^{ij}$.

In the sequel, we shall always assume that the manifolds V_3 and V_4 have been given the structure of a Riemannian manifold that is defined by (63-4). They will therefore be isometric ([1]).

64. – The spatial tensor H_{ij}. – Consider the covariant vector ξ_λ; the component $\xi_0 = \xi^2$ defines a scalar on W_3 and V_3. The $\xi_i = g_{0i}$ define a covariant vector field on W_3. We study the variation of the ξ_i under a change of system of spatial sections:

$$\xi_i = \xi_{i'} + A_i^{0'}\xi_{0'} = \xi_{i'} + A_i^{0'}\xi^2\,.$$

Therefore, consider the vector φ_λ that is defined on V_4 by:

$$\varphi_\lambda = \frac{\xi_\lambda}{\xi^2} = \frac{g_{0\lambda}}{\xi^2} \qquad (\varphi_0 = 1). \tag{64-1}$$

([1]) Note that in our study W_3 is *not* assumed to be endowed with the structure of a Riemannian manifold that is defined by its embedding in V_4, i.e., by the tensor g_{ij} on W_3 (the corresponding metric might then be of elliptic or hyperbolic signature).

The components φ_i define a vector field on W_3 that transforms according to the formula:

(64-2) $$\varphi_i = \varphi_{i'} + A_i^{0'} .$$

under a change of systems of spatial sections.

Therefore, if one introduces the rotation $H_{\lambda\mu}$ of φ_λ then one will see that the components $H_{0\lambda}$ of this tensor on V_4 are:

$$H_{0\lambda} = \partial_0 \varphi_\lambda - \partial_\lambda \varphi_0 = 0,$$

and that, from (64-2), *the $H_{\lambda\mu}$ define a tensor on V_3*.

It is easy to see the geometric significance of the vanishing of the tensor H_{ij}. For a neighborhood U of V_3, it is possible to find local spatial sections x^0 = const such that the timelines that correspond to U are the orthogonal trajectories in the adapted coordinate system relative to these spatial sections $\xi_i = g_{0i} = 0$, $\varphi_i = 0$, and, as a result, that $H_{ij} = 0$. Conversely, if $H_{ij} = 0$ then the tensor $H_{\lambda\mu}$ will be zero, and ($\varphi_0 = 1$, φ_i) will locally define a gradient field. Therefore, in a neighborhood U of V_3 there will exist a function $f(x^i)$ such that:

$$\varphi_\lambda = \partial_0 [x^0 + f(x^i)].$$

Therefore, $H_{ij} = 0$ says that the timelines are orthogonal trajectories to the local spatial sections in the neighborhoods U of V_3.

We have therefore defined a scalar x in the space V_3, a negative definite Riemannian metric:

(64-3) $$(ds^*)^2 = \dot{g}_{ij} dx^i dx^j ,$$

and an antisymmetric tensor H_{ij}. On each W_3, one finds, other than these elements, a vector field φ_i such that:

(64-4) $$\dot{g}_{ij} = g_{ij} - \xi^2 \varphi_i \varphi_j , \qquad H_{ij} = \partial_i \varphi_j - \partial_j \varphi_i .$$

65. – Passing from an orthonormal frame to a natural frame. – When one says that a tensor is referred to a system of local coordinates, what that really means is that it is referred to the natural frame of this local coordinate system. If ($\mathbf{e}_\alpha$) denotes the natural frame at x of an adapted local coordinate system (x^α), and if ($\mathbf{e}_\alpha$) is an adapted orthonormal frame, then one will have:

$$d\mathbf{x} = dx^\alpha \mathbf{e}_\alpha = \omega^\alpha \mathbf{e}_\alpha$$

and the ($\mathbf{e}_i$) define an orthonormal frame for $(ds^*)^2$. From (63-3), it follows that:

(65-1) $$\omega^0 = \xi (dx^0 + \varphi_i dx^i)$$

and since the ω^j are local Pfaff forms with respect to the dx^i:

(65-2) $$\omega^j = A^i_j \, dx^j .$$

Denote the inverse matrix to the matrix (A^i_j) by $(\bar{A}^i_j)$. By solving (65-2), it will then follow that:

(65-3) $$dx^i = \bar{A}^i_j \, \omega^j .$$

Formulas (65-2) and (65-3) define the passage from local coordinates (x^i) to the orthonormal frame $(\mathbf{e}_i)$ in V_3 or W_3 . Let $\bar{\varphi}_i$ be the components with respect to the $(\mathbf{e}_i)$ of the vector φ_i on W_3 :

(65-4) $$\bar{\varphi}_i = \bar{A}_i^j \varphi_j .$$

(65-1) may then be put into the form:

(65-5) $$\omega^0 = \xi \, (dx^0 + \bar{\varphi}_i \, \omega^i) ,$$

and conversely one will deduce that:

(65-6) $$dx^0 = \frac{\omega^0}{\xi} - \bar{\varphi}_i \, \omega^i .$$

The change of frame in V_4 is carried out with the help of the coefficient matrix (A^α_β) of dx^β in terms of the ω^α and the inverse matrix. In particular, one will have:

(65-7) $$A^0_0 = \xi, \quad A^0_i = \xi\varphi_i, \quad A^i_0 = 0; \qquad \bar{A}^0_0 = \frac{1}{\xi}, \quad \bar{A}^0_i = -\varphi_i, \quad \bar{A}^i_0 = 0.$$

II. - Ricci tensor of a space V_{n+1} that admits a one-parameter group of isometries

66. – The fundamental formulas of Riemannian geometry in orthonormal frames. – The preceding analysis obviously applies to a Riemannian space V_{n+1} that admits a connected, one-parameter group of isometries that satisfies hypotheses that are pointless to repeat. We denote the manifolds that are analogous to V_3 and W_3 by V_n and W_n , resp. Since the following calculations are purely local, we may assume that V_n reduces to one of its neighborhoods. Finally, in order for our calculations to be easily adapted to the various hypotheses on the signature of V_{n+1}, we will assume to begin with that V_{n+1} admits a positive-definite metric. In that subset, one will therefore have:

$$ds^2 = \sum_\alpha (\omega^\alpha)^2 = (\omega^0)^2 + \sum_i (\omega^i)^2 \qquad (\alpha = 0, 1, \ldots, n;\ i = 1, \ldots, n).$$

All of the other formulas of secs. **63**, **64**, **65** will remain unchanged or undergo an obvious change.

Having said this, we propose to evaluate the Ricci tensor on V_{n+1} by starting with the ones on V_n and the tensors that we just introduced. In order to perform the calculations that relate the elements that pertain to the metric ds^2 on the ones that pertain to the metric ds^{*2}, we will impose the use of orthonormal frames. The hypothesis on the signature of the metric permits us to not distinguish between the covariant and contravariant indices in such a frame. We will therefore put the indices in the lower position, while preserving the summation convention that relates to a twice-repeated index. Up till now, we have notated elements that are expressed with respect to an orthonormal frame with an overbar. Since only orthonormal frames will be at issue in the present section of this chapter, we will temporarily suppress the overbar, and later re-establish it.

We commence by recalling the fundamental formulas of Riemannian geometry in terms of orthonormal frames. The formulas of moving frames may be written:

(66-1) $$d\mathbf{x} = \omega_\alpha \mathbf{e}_\alpha ,$$

(66-2) $$d\mathbf{e}_\alpha = \omega_{\alpha\beta} \mathbf{e}_\beta .$$

Since the frames are orthonormal, the metric on V_{n+1} may be written:

(66-3) $$ds^2 = \sum_\alpha (\omega^\alpha)^2 = (\omega^0)^2 + \sum_i (\omega^i)^2 .$$

The local Pfaff forms $\omega_{\alpha\beta}$ define the Riemannian connection on V_{n+1} and are defined by two types of conditions. First, one has:

$$\omega_{\alpha\beta} = -\omega_{\beta\alpha} ,$$

which says that the moving frame stays orthonormal. One then must express the vanishing of the torsion of the Riemannian connection, which leads to the formulas:

(66-4) $$d\omega_\alpha = \omega_\beta \wedge \omega_{\beta\alpha} .$$

Finally, the exterior differentials of the $\omega_{\alpha\beta}$ are related to the curvature forms on V_{n+1} by the formulas:

(66-5) $$d\omega_{\alpha\beta} = \omega_{\alpha\beta} \wedge \omega_{\beta\alpha} - \Omega_{\alpha\beta}$$

in which the local quadratic differential forms $\Omega_{\alpha\beta}$ are given by:

(66-6) $$\Omega_{\alpha\beta} = \tfrac{1}{2} R_{\alpha\beta,\lambda\mu}\, \omega_\lambda \wedge \omega_\mu ,$$

in which the $R_{\alpha\beta,\lambda\mu}$ denote the components of the curvature tensor on V_{n+1} in the orthonormal frame.

The quantities that replace the classical Christoffel symbols Γ here are the Ricci rotation coefficients $\gamma_{\alpha\beta\lambda}$, which are found to be defined by the relations:

$$\omega_{\alpha\beta} = \gamma_{\alpha\beta\lambda}\ \omega_\lambda \qquad\qquad (\gamma_{\alpha\beta\lambda} = -\gamma_{\beta\alpha\lambda}).$$

The calculation of the $\gamma_{\alpha\beta\lambda}$ will be carried out as a function of the coefficients of the quadratic forms $d\omega_\alpha$ with the aid of formulas (66-4). We set:

(66-7) $$d\omega_\alpha = \tfrac{1}{2}\, c_{\lambda\mu\alpha}\ \omega_\lambda \wedge \omega_\mu\,, \qquad\qquad (c_{\lambda\mu\alpha} = -c_{\mu\lambda\alpha}).$$

Upon identifying the terms in $\omega_\lambda \wedge \omega_\mu$ in (66-4) and (66-7), it will follow that:

(66-8) $$c_{\lambda\mu\alpha} = \gamma_{\lambda\alpha\mu} - \gamma_{\mu\alpha\lambda} = \gamma_{\lambda\alpha\mu} + \gamma_{\mu\alpha\lambda}.$$

Conversely, one immediately solves equations (66-8) with respect to the γ by a calculation that is analogous to the one that led to the Christoffel symbols, and one will have:

(66-9) $$\gamma_{\alpha\beta\lambda} = \tfrac{1}{2}\, c_{\alpha\beta\lambda} + c_{\alpha\lambda\beta} - c_{\beta\lambda\alpha}.$$

If f denotes an arbitrary function then we will set:

(66-10) $$df = \partial_\alpha f\, \omega_\alpha$$

in all of what follows in this section.

67. – Calculation of the Ricci coefficients for V_{n+1} ([1]). – Consider the Riemannian manifold V_n, endowed with the metric $(ds^*)^2$. To any adapted orthonormal frame on V_{n+1}, i.e., to any decomposition of the metric ds^2 on V_{n+1} into squares of the type:

$$ds^2 = (\omega_0)^2 + \sum_i (\omega_i)^2$$

there corresponds a decomposition of ds^{*2} into squares:

$$ds^{*2} = \sum_i (\dot\omega_i)^2\,,$$

with $\dot\omega_i = \omega_i$; i.e., an arbitrary orthonormal frame of V_n. The formulas of Riemannian geometry in orthonormal frames may be applied to the manifold V_n ; we agree to denote the elements that relate to it with the symbol *. Hence, $\overset{*}{\nabla}$ will denote the covariant

([1]) The calculations that follow appear in Y. THIRY, Journ. math. pures et appl. (9) (1951), 275-396.

derivative for the metric ds^{*2}. We propose to calculate the Ricci rotation coefficients $\gamma_{\alpha\beta\lambda}$ of V_{n+1} as a function of the ones on V_n, $\dot{\gamma}_{ijk}$, and the tensors that were introduced.

We commence with the calculation of the $c_{\lambda\mu\alpha}$. First, one has:

$$d\omega_i = \tfrac{1}{2}\, c_{jki}\, \omega_j \wedge \omega_k + c_{j0i}\, \omega_j \wedge \omega_0 .$$

On the other hand, one has:

$$d\omega_i = d\overset{*}{\omega}_i = \tfrac{1}{2}\, \dot{c}_{jki}\, \omega_j \wedge \omega_k .$$

in V_n. One will deduce from this that:

(67-1) $$\dot{c}_{jki} = c_{jki} \qquad c_{j0i} = 0.$$

In order to calculate the $c_{\lambda\mu 0}$, consider the form ω_0. It is given by (65-5) in adapted local coordinates. If one introduces the components of φ in the orthonormal frame, which are components that one denotes by φ_i here, then one will have:

(67-2) $$\omega_0 = \xi\,(dx^0 + \varphi_i\, \omega_i).$$

Upon taking the exterior differential of (67-2), one will get:

$$d\omega_0 = d\xi \wedge \frac{\omega_0}{\xi} + \xi\, d(\varphi_i\, \omega_i).$$

Upon developing the differentials in the right-hand side, one will get, taking (67-1) into account:

$$d\omega_0 = \frac{\partial_i \xi}{\xi}\, \omega_i \wedge \omega_0 + x\,[\partial_i \varphi_j\, \omega_i \wedge \omega_i + \tfrac{1}{2}\, \varphi_h\, c_{ijh}\, \omega_i \wedge \omega_j].$$

On the other hand:

$$d\omega_0 = \tfrac{1}{2}\, c_{ij0}\, \omega_i \wedge \omega_j + c_{i00}\, \omega_i \wedge \omega_0 .$$

One deduces the formulas:

(67-3) $$c_{ij0} = \xi\,(\partial_i \varphi_j - \partial_j \varphi_i + c_{ij0}\varphi_h), \qquad c_{i00} = \frac{\partial_i \xi}{\xi}$$

from this.

Since the c's are known, one may then evaluate the γ's with the aid of formula (66-9). Obviously, one will first have:

$$\gamma_{ijk} = \dot{\gamma}_{ijk} ,$$

which will permit us put the first of formulas (67-3) into a more manageable form.

Indeed, one has:

$$c_{ijh} = \gamma_{ihj} - \gamma_{jhi} = \dot{\gamma}_{ihj} - \dot{\gamma}_{jhi} ,$$

and the first formula in (67-3) may be written:

$$c_{ij0} = \xi\,[(\partial_i \varphi_j - \dot{\gamma}_{jhi}\, \varphi_h) - (\partial_j \varphi_i - \dot{\gamma}_{ihj}\, \varphi_h)]$$

namely:

(67-4) $$c_{ij0} = \xi\,(\overset{*}{\nabla}_i\, \varphi_j - \overset{*}{\nabla}_j\, \varphi_i) = \xi\, H_{ij} ,$$

since H_{ij} is the rotation of φ_i on W_3 .

If one or the other of the indices of γ are zero then one will have:

$$\begin{cases} \gamma_{i0k} = \tfrac{1}{2}(c_{i0k} + c_{ik0} - c_{0ki}) = \tfrac{1}{2} c_{ik0} \\ \gamma_{ik0} = \tfrac{1}{2}(c_{ik0} + c_{i0k} - c_{k0i}) = \tfrac{1}{2} c_{ik0} \\ \gamma_{i00} = \tfrac{1}{2}(c_{i00} + c_{i00} - c_{00i}) = c_{i00} . \end{cases}$$

One then deduces the following expressions for the Ricci rotation coefficients:

(67-5) $$\gamma_{ijk} = \dot{\gamma}_{ijk} , \qquad \gamma_{i0k} = \gamma_{0ki} = \gamma_{ik0} = \tfrac{1}{2}\, \xi\, H_{ik}, \qquad \gamma_{i00} = \frac{\partial_i \xi}{\xi} .$$

68. – Calculation of the components of the curvature tensor in an orthonormal frame. – From the preceding expressions, it results that:

(68-1) $$\omega_{ij} = \dot{\omega}_{ij} + \tfrac{1}{2}\, \xi\, H_{ij}\, \omega_0 ,$$

(68-2) $$\omega_{i0} = \frac{\partial_i \xi}{\xi}\, \omega_0 + \tfrac{1}{2}\, \xi\, H_{ir}\, \omega_r .$$

We now propose to evaluate the components of the curvature tensor on V_{n+1} as a function of the components of the curvature tensor on V_n and the various tensors that were introduced. To that effect, we apply formulas (66-5) to the Riemannian manifolds V_{n+1} and V_n .

When (66-5) is applied to the form ω_{ij} , it will result that:

$$d\omega_{ij} = \omega_{i0} \wedge \omega_{0j} + \omega_{i0} \wedge \omega_{0j} - \Omega_{ji} ,$$

namely, from (68-1):

(68-3) $$d\omega_{ij} = \omega_{i0} \wedge \omega_{0j} + (\dot{\omega}_{ir} + \tfrac{1}{2}\, \xi\, H_{ir}\, \omega_0) \wedge (\dot{\omega}_{rj} + \tfrac{1}{2}\, \xi\, H_{rj}\, \omega_0) - \Omega_{ij} .$$

On the other hand, by differentiating (68-1), one will get:

(68-4) $$d\omega_{ij} = d\dot{\omega}_{ij} + \tfrac{1}{2}\, d(\xi\, H_{ij}) \wedge \omega_0 + \tfrac{1}{2}\, \xi\, H_{ij}\, \omega_k \wedge \omega_{k0} .$$

From formulas (68-3) and (68-4) and applying formula (66-5) on V_n, one deduces that:

(68-5) $\Omega_{ij} - \dot{\Omega}_{ij} = -\omega_{i0} \wedge \omega_{j0} + \frac{1}{2}\,[\dot{\omega}_{rj}\,\xi H_{rj} + \dot{\omega}_{jr}\,\xi H_{ir} - d(H_{ij}) \wedge \omega_0 - \frac{1}{2}\,\xi H_{ij}\omega_k \wedge \omega_{k0}.$

One recognizes the absolute derivative $\overset{*}{\nabla}(\xi H_{ij})$ in the term in brackets. It results from this and (68-2) that (68-5) may be put into the form:

$$(68\text{-}6)\qquad \begin{cases} \Omega_{ij} - \dot{\Omega}_{ij} = -\left(\dfrac{\partial_i \xi}{\xi}\omega_0 + \frac{1}{2}\xi H_{ir}\omega_r\right) \wedge \left(\dfrac{\partial_i \xi}{\xi}\omega_0 + \frac{1}{2}\xi H_{js}\omega_s\right) \\ \qquad -\frac{1}{2}\overset{*}{\nabla}_l(\xi H_{ij})\omega_l \wedge \omega_0 - \frac{1}{2}\xi H_{ij}\omega_k \wedge \left(\dfrac{\partial_i \xi}{\xi}\omega_0 + \frac{1}{2}\xi H_{kl}\omega_l\right). \end{cases}$$

Identifying the coefficients of the term $\omega_l \wedge \omega_0$ in both sides leads to the following relation:

$$(68\text{-}7)\qquad R_{ij,kl} = \dot{R}_{ij,kl} - \frac{\xi^2}{4}(H_{ik}H_{jl} - H_{il}H_{jk} + 2H_{ij}H_{kl}],$$

If we now identify the coefficients of the term $\omega_l \wedge \omega_0$, we get:

$$R_{ij,l0} = -\frac{1}{2}[\overset{*}{\nabla}_l(\xi H_{ij}) + 2\partial_l\xi H_{ij} - \partial_i\xi H_{jl} + \partial_l\xi H_{ij}],$$

Namely, upon developing the covariant derivative:

$$(68\text{-}8)\qquad R_{ij,l0} = -\frac{1}{2}[\xi\overset{*}{\nabla}_l H_{ij} + 2\partial_l\xi H_{ij} - \partial_i\xi H_{jl} + \partial_l\xi H_{ij}].$$

(68-9)

If we apply the same formula (66-5) to the form ω_{i0} then we will get:

$$d\omega_{i0} = \omega_{ir} \wedge \omega_{r0} - \Omega_{i0},$$

namely, from (68-1) and (68-2):

$$(68\text{-}10)\qquad d\omega_{i0} = (\dot{\omega}_{ir} + \frac{1}{2}\xi H_{ir}\omega_0) \wedge \left(\frac{\partial_r\xi}{\xi}\omega_0 + \frac{1}{2}\xi H_{rs}\omega_s\right) - \Omega_{i0}.$$

We are interested in only the terms of $d\omega_{i0}$ that contain ω_0 as a factor. By differentiating (68-2), one will get:

$$(68\text{-}10)\qquad d\omega_{i0} \equiv d\left(\frac{\partial_i\xi}{\xi}\right) \wedge \omega_0 + \frac{\partial_i\xi}{\xi}\omega_k \wedge \frac{\partial_k\xi}{\xi}\omega_0 \qquad \text{(mod terms in } \omega_k \wedge \omega_l).$$

Upon identifying the terms in (68-9) and (68-10) that contain ω_0 as a factor, one obtains:

$$\left[d\left(\frac{\partial_i\xi}{\xi}\right)-\dot{\omega}_{ir}\frac{\partial_i\xi}{\xi}\right]\wedge\frac{\partial_i\xi\,\partial_k\xi}{\xi^2}\omega_k\wedge\omega_0-\frac{\xi^2}{4}H_{ir}H_{kr}\omega_k+\Omega_{i0}\equiv 0.$$

From this, one deduces the relation:

$$R_{i0,l0}=-\overset{*}{\nabla}_l\left(\frac{\partial_i\xi}{\xi}\right)-\frac{\partial_i\xi\,\partial_l\xi}{\xi^2}+\frac{\xi^2}{4}H_{ir}H_{lr}\,,$$

Hence, upon developing the covariant derivative:

$$R_{i0,l0}=-\frac{1}{\xi}\overset{*}{\nabla}_l\left(\partial_i\xi\right)+\frac{\xi^2}{4}H_{ir}H_{lr}\,. \tag{68-11}$$

We have therefore obtained the following formulas for the components of the curvature tensor on V_{n+1}:

$$R_{ij,kl}=\dot{R}_{ij.kl}-\frac{\xi^2}{4}(H_{ik}H_{jl}-H_{il}H_{jk})-\frac{\xi^2}{2}H_{ij}H_{kl} \tag{68-12}$$

$$R_{ij,l0}=-\tfrac{1}{2}[\xi\overset{*}{\nabla}_l H_{ij}+2\partial_l H_{ij}-\partial_l\xi H_{ij}-\partial_i\xi H_{jl}+\partial_j\xi H_{il}] \tag{68-13}$$

$$R_{i0,l0}=-\frac{1}{\xi}\overset{*}{\nabla}_l\left(\partial_i\xi\right)+\frac{\xi^2}{4}H_{ir}H_{lr}\,. \tag{68-14}$$

69. – Calculation of the components of the Ricci tensor on V_{n+1} in an orthonormal frame. – Finally, we now propose to valuate the components of the Ricci tensor on V_{n+1} in an adapted orthonormal frame as a function of its components on V_n and the various tensors introduced. The formulas that we introduce are useful, not only in the present theory, but also in the unitary field theory. First, one has:

$$R_{ik}=\sum_j R_{ij,kj}+R_{i0,k0}\,.$$

Now, from (68-12):

$$\sum_j R_{ij,kj}=\dot{R}_{ik}-\tfrac{3}{4}\xi^2 H_{ij}H_{jk}\,.$$

From this and (68-14), one deduces that:

$$R_{ik}=\dot{R}_{ik}-\frac{1}{\xi}\overset{*}{\nabla}_k(\partial_i\xi)-\frac{\xi^2}{2}H_{ij}H_{kj} \tag{69-1}$$

Similarly, one has:

$$R_{i0} = \sum_j R_{ij,0j} = -\sum_j R_{ij,j0} .$$

One deduces from this and (68-13) that:

(69-2) $$R_{i0} = \tfrac{1}{2}[\xi \dot{\nabla}_j H_{ij} + 3\partial_j \xi\, H_{ij}] = \frac{1}{2\xi^2}\dot{\nabla}_j(\xi^2 H_{ij}) .$$

Finally, the last component of the Ricci tensor is:

$$R_{00} = \sum_i R_{0i,0i} = \sum_i R_{i0,i0} .$$

One deduces from this and (68-14) that:

(69-3) $$R_{00} = -\frac{1}{\xi}\dot{\Delta}\xi + \frac{\xi^2}{2}H^2 ,$$

in which $\dot{\Delta}$ denotes the Laplacian for the metric $(ds^*)^2$, and we have set:

(69-4) $$H^2 = \tfrac{1}{2}\sum_{i,j}(H_{ij})^2 .$$

If we abandon the current notations then we shall reunite with the formulas that were proved before by re-establishing the overbar and locating the lower and upper indices in such a way that they satisfy the classical Einstein convention. We will get:

(69-5) $$\begin{cases} \bar{R}_{ik} = \dot{R}_{ik} - \dfrac{1}{\xi}\dot{\nabla}_k(\partial_i \xi) - \dfrac{\xi^2}{2}\bar{H}_i{}^j \bar{H}_{kj} \\ \bar{R}_{i0} = \dfrac{1}{2\xi^2}\dot{\nabla}_j(\xi^2 \bar{H}_i{}^j) \\ \bar{R}_{00} = -\dfrac{1}{\xi}\dot{\Delta}\xi + \dfrac{\xi^2}{2}H^2 \end{cases}$$

in which we have set:

(69-5) $$H^2 = \frac{1}{2}\bar{H}_{ij}\,\bar{H}^{ij} .$$

III. – APPLICATON TO STATIONARY SPACETIMES

70. –The fundamental equations for a hyperbolic signature. – Return to the stationary spacetime V_4 that was envisioned in the first part of this chapter, which admits a metric of the hyperbolic normal type:

$$ds^2 = (\omega^{\underline{0}})^2 - \sum_j (\omega^{\underline{j}})^2 \; . \tag{70-1}$$

We will get ds^2 as a sum of four squares upon performing the following transformation on the local Pfaff forms:

$$\bar{\varpi}^{\underline{0}} = \omega^{\underline{0}} \qquad\qquad \bar{\varpi}^{\underline{j}} = i\omega^{\underline{j}} \, .$$

We may establish the formulas that were established in the second part of this chapter for the new form of ds^2:

$$ds^2 = (\bar{\varpi}^{\underline{0}})^2 + \sum_j (\bar{\varpi}^{\underline{j}})^2 \; .$$

However, the components in the hyperbolic metric may be derived from the ones in the elliptic form of the metric by the following rule: Any contravariant index $j = 1, 2, 3$ corresponds to multiplication by i, any covariant index j, to multiplication by $-i$, and the index 0, to multiplication by 1. Formulas (69-5) and (69-6) will obviously remain unchanged under this transformation, and one will therefore have:

$$(ds^*)^2 = -\sum_j (\bar{\varpi}^{\underline{j}})^2 \tag{70-2}$$

under our new hypotheses (69-5) and (69-6), in which the starred elements are now expressed with respect to the negative-definite metric.

71. – Calculation of the component R_0^0 relative to an adapted, natural frame on V_4. – Now assume that V_4 is referred to a definite, adapted, local coordinate system, and let W_3 be a corresponding spatial section. We propose to evaluate the R_0^0 component of the Ricci tensor in the natural frame that is associated with the local coordinates with the aid of the preceding formulas. In the difference of two tensors that appears in formulas (69-5), R_0^0 has precisely a scalar character on W_3, but not on V_3.

One deduces from classical formulas for the change of frame applied to the Ricci tensor:

$$R_0^0 = A^0_{\underline{\alpha}} \, A^{\underline{\beta}}_0 \, R^{\underline{\alpha}}_{\underline{\beta}} \, .$$

Formulas (65-7) give us the values of the preceding A. One will therefore have:

$$R_0^0 = \xi\left[\frac{1}{\xi}R_{\underline{0}}^{\underline{0}} - \varphi_{\underline{i}}R_{\underline{0}}^{\underline{i}}\right] = R_{\underline{0}}^{\underline{0}} - \xi\varphi_{\underline{i}}R_{\underline{0}}^{\underline{i}}\,.$$

Now, from the signature of the metric, one has:

$$R_{\underline{0}}^{\underline{0}} = R_{\underline{00}} \qquad\qquad R_{\underline{0}}^{\underline{i}} = -R_{\underline{i}\,\underline{0}}\,.$$

Upon turning all of the indices into lower ones, one deduces from this and (69-5) that:

(71-1) $$R_0^0 = -\frac{1}{\xi}\dot{\Delta}\xi - \frac{\varphi_{\underline{i}}}{2\xi}\dot{\nabla}_{\underline{j}}(\xi^3 H_{\underline{ij}}) + \frac{\xi^2}{2}H^2\,.$$

Now, upon integrating by parts, one will obtain:

$$\varphi_{\underline{i}}\dot{\nabla}_{\underline{j}}(\xi^3 H_{\underline{ij}}) = \dot{\nabla}_{\underline{j}}(\xi^3\varphi_{\underline{i}}H_{\underline{ij}}) - \xi^3 H_{\underline{ij}}\dot{\nabla}_{\underline{j}}\varphi_{\underline{i}}$$

namely, from the antisymmetry of H_{ij}:

$$\varphi_{\underline{i}}\dot{\nabla}_{\underline{j}}(\xi^3 H_{\underline{ij}}) = \dot{\nabla}_{\underline{j}}(\xi^3\varphi_{\underline{i}}H_{\underline{ij}}) - \xi^3 H^2.$$

When one substitutes this into (71-1), one will deduce that:

$$R_0^0 = -\frac{1}{\xi}[\dot{\Delta}\xi + \frac{1}{2}\dot{\nabla}_{\underline{j}}(\xi^3\varphi_{\underline{i}}H_{\underline{ij}})]$$

so, upon slightly modifying the notations and raising certain indices, one will get:

(71-2) $$R_0^0 = -\frac{1}{\xi}[\dot{\Delta}\xi + \frac{1}{2}\dot{\nabla}_{\underline{j}}(\xi^3\varphi_{\underline{k}}H^{\underline{kj}})]\,.$$

Upon expressing the Laplacian explicitly and converting the indices into lower or upper ones, according to the Einstein convention, we get:

$$R_0^0 = -\frac{\dot{g}^{\underline{ij}}}{\xi}\dot{\nabla}_{\underline{j}}[\partial_{\underline{i}}\xi + \frac{\xi^3}{2}\varphi_{\underline{k}}H^{\underline{k}}{}_{\underline{i}}]\,.$$

We are thus led to introduce the vector **h** on W_3 that is defined by:

(71-3) $$h_{\underline{i}} = \partial_{\underline{i}}\xi + \frac{\xi^3}{2}\varphi_{\underline{k}}H^{\underline{k}}{}_{\underline{l}}\,,$$

and we will have the following simple formula for R_0^0:

(71-4) $$R_0^0 = -\frac{1}{\xi}\nabla_{\underline{j}} h^{\underline{j}} = -\frac{1}{\xi} div\,\mathbf{h}\,.$$

From the tensorial character of the various terms, the preceding formulas may be written immediately, while making only components relative to the local coordinates appear. If $H^k{}_i = g^{kl}H_{li}$ then one will have the relations [1]:

(71-5) $$h_i = \partial_i \xi + \frac{\xi^2}{2}\varphi_k H^k{}_i \qquad\qquad h^i = g^{ij} h_j$$

and:

(71-6) $$R_0^0 = -\frac{1}{\xi}\dot{\nabla}_i h^i\,.$$

72. – A divergence formula in spacetime. – Formulas (71-4) or (71-6) involve the divergence, relative to the Riemannian manifold W_3, of a vector on this manifold in their right-hand sides. It is possible to transform this formula in such a fashion as to involve the divergence, relative to spacetime V_4, of a vector of a vector on V_4 in terms of the metric on V_4 and the system of spatial sections. One recalls that in an orthonormal frame the covariant derivative of a vector η_β is given by:

(72-1) $$\nabla_{\underline{\alpha}}\eta_{\underline{\beta}} = \partial_{\underline{\alpha}}\eta_{\underline{\beta}} - \gamma_{\beta\rho\alpha}\eta^{\underline{\rho}}\,.$$

Take a vector on V_4 that is orthogonal to $\boldsymbol{\xi}$ to be $\boldsymbol{\eta}$; its components in an adapted orthonormal frame will be:

$$\eta^{\underline{0}} = \eta_{\underline{0}} = 0 \qquad\qquad \eta^{\underline{i}} = -\eta_i\,.$$

Evaluate its divergence:

$$\nabla_{\underline{\alpha}}\eta^{\underline{\beta}} = \nabla_{\underline{0}}\eta^{\underline{0}} + \nabla_{\underline{i}}\eta^{\underline{i}} = \nabla_{\underline{0}}\eta_{\underline{0}} - \nabla_{\underline{i}}\eta_{\underline{i}}\,.$$

Now, from (72-1), one has:

$$\nabla_{\underline{0}}\eta_{\underline{0}} = -\gamma_{\underline{0}\underline{j}\underline{0}}\eta^{\underline{j}} = \gamma_{\underline{j}\underline{0}\underline{0}}\eta^{\underline{j}} = \frac{\partial_{\underline{j}}\xi}{\xi}\eta^{\underline{j}}$$

$$\nabla_{\underline{i}}\eta_{\underline{i}} = \partial_{\underline{i}}\eta_{\underline{i}} - \gamma_{\underline{i}\underline{j}\underline{i}}\eta^{\underline{j}} = \dot{\nabla}_{\underline{i}}\eta_{\underline{i}}\,.$$

One deduces from this that:

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[1] Note that these relations use the fact that $\dot{g}^{ij} = g^{ij}$ and that $H^k{}_i = g^{k\lambda}H_{\lambda i}$, since $H_{0i} = 0$. Therefore, our notations are quite coherent.

(72-2) $$\nabla_{\underline{\alpha}}\eta^{\underline{\beta}} = \frac{\partial_{\underline{j}}\xi}{\xi}\eta^{\underline{j}} + \dot{\nabla}_{\underline{j}}\eta^{\underline{j}} = \frac{1}{\xi}\dot{\nabla}_{\underline{j}}(\xi\eta^{\underline{j}}).$$

Therefore, consider the vector $\boldsymbol{\eta}$ on V_4, which admits the following covariant components in an orthonormal frame:

(72-3) $$\eta_{\underline{i}} = \frac{h_{\underline{i}}}{\xi} = \frac{\partial_{\underline{i}}\xi}{\xi} + \frac{\xi^2}{2}\varphi_{\underline{k}}H^{\underline{k}}{}_{i}, \qquad \eta_{\underline{0}} = 0.$$

One deduces from (71-4) that:

(72-4) $$R_0^0 = -\nabla_{\underline{\alpha}}\eta^{\underline{\alpha}} = -\operatorname{div}\boldsymbol{\eta}.$$

Formulas (72-3) and (72-4) may be written immediately in local coordinates. The vector $\boldsymbol{\eta}$ admits the covariant components:

(72-5) $$\eta_i = \frac{\partial_i\xi}{\xi} + \frac{\xi^2}{2}\varphi_k H^k{}_i, \qquad \eta_0 = 0$$

and:

(72-6) $$R_0^0 = -\nabla_\alpha\eta^\alpha.$$

One may then replace the vector $\boldsymbol{\eta}$ by a vector $\boldsymbol{\xi}$ that enjoys the same properties *but whose contravariant component ζ^0 is zero.* Indeed, upon developing (72-6), one will get:

$$-R_0^0 = \partial_i\eta^i + \Gamma^\alpha_{\alpha i}\eta^i + \partial_0\eta^0 + \Gamma^\alpha_{\alpha 0}\eta^0 = \partial_i\eta^i + \Gamma^\alpha_{\alpha i}\eta^i,$$

since the spacetime is stationary, so the η^0 and $\sqrt{|g|}$ will not depend upon x^0 in adapted coordinates. As a result, if $\boldsymbol{\zeta}$ denotes the vector on V_4 with contravariant components:

$$\zeta^i = \eta^i, \qquad \zeta^0 = 0,$$

then one will have, moreover, that:

(72-7) $$R_0^0 = -\nabla_\alpha\zeta^\alpha,$$

with:

(72-8) $$\zeta^i = g^{ij}\left(\frac{\partial_j\xi}{\xi} + \frac{\xi^2}{2}\varphi_k H^k{}_j\right), \qquad \zeta^0 = 0.$$

73. – Another expression for the components of ζ in local coordinates. – One may obtain a simple expression for the components of **ζ** in local coordinates with the aid of the Christoffel symbols, which will be an expression that somewhat masks its vectorial character.

First transform the expression for η_i. Taking into account the relation $g_{\alpha\beta}g^{\alpha\beta} = g_\alpha^\beta$, one will obtain:

$$\eta_i = \frac{\partial_i g_{00}}{2g_{00}} + \tfrac{1}{2} g_{0k}\,\mathrm{g}^{kl} H_{li} = \frac{\partial_i g_{00}}{2g_{00}} - \tfrac{1}{2} g^{0l}\,\mathrm{g}_{00} H_{li}\,.$$

Upon specifying H_{li} with the aid of the potentials, one will obtain:

$$\begin{aligned}\eta_i &= \frac{\partial_i g_{00}}{2g_{00}} - \tfrac{1}{2} g^{0l}\mathrm{g}_{00}\left[\partial_l\left(\frac{g_{0i}}{g_{00}}\right) - \partial_i\left(\frac{g_{0l}}{g_{00}}\right)\right] \\ &= \frac{\partial_i g_{00}}{2g_{00}} - \tfrac{1}{2} g^{0l}(\partial_l g_{0i} - \partial_i g_{0l}) + g^{0l}\,\frac{g_{0i}\partial_l g_{00} - g_{0l}\partial_i g_{00}}{2g_{00}}\,,\end{aligned}$$

so, by obvious transformations, we will obtain:

$$\eta_i = \tfrac{1}{2} g^{00}\partial_i g_{00} + g^{0i}[0i, l] + g_{0i}\,\frac{g^{0l}\partial_l g_{00}}{2g_{00}}\,.$$

Upon introducing the Christoffel symbols Γ, that will give:

$$\eta_i = \Gamma^0_{0i} - \Gamma^0_{00}\frac{g_{0i}}{g_{00}} \qquad\qquad \eta_0 = 0.$$

One deduces from this that:

$$\eta_\lambda = \Gamma^0_{0\lambda} - \Gamma^0_{00}\frac{g_{0\lambda}}{g_{00}}\,, \qquad\qquad \eta^\lambda = \Gamma^0_{\mu 0} g^{\lambda\mu} - \frac{\Gamma^0_{00}}{g_{00}} g_0^\lambda\,,$$

and, as a result:

(73-1) $$\zeta^\lambda = \Gamma^0_{0\mu} g^{\lambda\mu}\,.$$

One easily verifies by direct calculation that the divergence of **ζ** that is given by (73-1) is $-R_0^0$ ([1]).

([1]) Cf. A. LICHNEROWICZ, *Problèmes globaux en mécanique relativiste,* Hermann, (1939), pp. 64.

74. – Another formula for the divergence on a spatial section W_3 . – Always assume that the V_4 is referred to a specific adapted local coordinate system, and let W_3 be a corresponding spatial section. One can valuates the value of the components R_{00} of the Ricci tensor in the natural frame that is associated with the local coordinates immediately e with the aid of the preceding formulas. Indeed, from (65-7), one has:

$$R_{00} = A_0^{\underline{\alpha}} A_0^{\underline{\beta}} R_{\underline{\alpha}\underline{\beta}} = \xi^2 R_{\underline{0}\underline{0}};$$

namely, from (69-5):

$$\frac{R_{00}}{g_{00}} = -\frac{1}{\xi}\dot{\Delta}\xi + \frac{\xi^2}{2}H^2. \tag{74-1}$$

On the other hand, (71-4) may be written:

$$R_0^0 = -\frac{1}{\xi} div\,\mathbf{h},$$

in which **h** is the vector with covariant components:

$$h_i = \partial_i \xi + \frac{\xi^2}{2}\varphi_k H^k{}_i .$$

Introduce the vector **p** with the covariant components:

$$p_i = \partial_i \xi - h_i = \frac{\xi^2}{2}\varphi_k H^k{}_i . \tag{74-2}$$

Formula (71-4) may then be put into the form:

$$R_0^0 = -\frac{1}{\xi}\dot{\Delta}\xi + \frac{1}{\xi} div\,\mathbf{p}. \tag{74-3}$$

Upon subtracting both sides of (74-1) from (74-3), one will obtain:

$$\xi\left[\frac{R_{00}}{g_{00}} - R_0^0\right] = \frac{\xi^2}{2}H^2 - div\,\mathbf{p}.$$

Now one obviously has:

$$R_{00} = g_{00} R_0^0 + g_{0i} R_0^i,$$

so that:

$$\frac{R_{00}}{g_{00}} - R_0^0 = \frac{g_{0i}}{g_{00}} R_0^i = \varphi_i R_0^i .$$

One will therefore obtain:

(74-4) $$\xi\varphi_i R_0^i = \frac{\xi^2}{2} H^2 - div\,\mathbf{p}\,.$$

Let C be a three-dimensional differentiable chain in W_3 with boundary ∂C. By integrating (74-4) over C, one will obtain an integral formula that will be very useful to us ([1]):

(74-5) $$\iiint_C \xi\varphi_i R_0^i\, d\tau = \iiint_C \frac{\xi^2}{2} H^2 d\tau - flux_{\partial C}\,\mathbf{p}\,,$$

in which $d\tau$ denotes the volume element on W_3.

([1]) From the consideration of secs. **72** and **73**, each of the divergence formulas that we established relative to W_3 corresponds to a divergence formula in spacetime V_4. One will find the formula that is equivalent to (74-5) in A. LICHNEROWICZ and Y. FOURÉS, C.R. Acad. Sc., **226** (1948), 432.

CHAPTER VIII

EVERYWHERE-REGULAR, STATIONARY SPACETIMES

I. – STUDY OF THE PROPERTIES OF THE LAPLACE EQUATION ON A RIEMANNIAN MANIFOLD

75. – Complete, Riemannian manifolds. – In the first part of this chapter, we will commence by establishing some theorems that relate to the Laplacian on a Riemannian manifold *with a* (positive-) *definite metric,* which will be a manifold that we assume to be *complete*. To my knowledge, these theorems have not been stated and established under the hypotheses that are necessary for us. A purely local theorem was given by E. Hopf and Georges Giraud ([1]), and I have been led to use the result that relates to the very simple case in which V_n is compact and orientable. In the latter case, this result served as a basis for Bochner and myself ([2]) in our research on the real cohomology of Riemannian manifolds, thanks to the technique of harmonic forms.

Let us recall the definitions and principal results that concern complete, Riemannian manifolds ([3]).

Given two points x_1 and x_2 of V_n, consider the lengths of the piecewise-continuously differentiable paths that join the points. We refer to the *distance* between the two points – which we will denote by $d(x_1, x_2)$ – when we mean the *lower bound of the lengths of all such paths* that join x_1 to x_2. It is clear that $d(x_1, x_2)$ will satisfy the triangle inequality. This distance associates the structure of a metric space with the given Riemannian structure on V_n that is compatible with the differentiable topology on V_n .

Consider an infinite sequence of points x_p (p = 1, 2, …) on V_n . This sequence is called a *Cauchy sequence* for the metric space structure if one may associate any ε with an integer P such that the inequalities:

$$p, q > P$$

imply that:

$$d(x_p , x_q) < \varepsilon.$$

One knows that this definition is equivalent to the following one: Consider a set $\mathcal{E}$ that bounds the points of V_n . Such a set is *bounded* if there exists a point a and a positive number r such that for any x of $\mathcal{E}$:

$$d(a, x) \leq r.$$

([1]) E. HOPF, Preuss. Akad. Wiss. Sitz., **19** (1927), 147-152; GEORGES GIRAUD, Bull. des Sc. math., **56** (1932), 9.

([2]) S. BOCHNER, Bull. Amer. Math. Soc., **52** (1946), 776-797; Ann. of Math., **49** (1948), 349-390; A. LICHNEROWICZ, C.R. Acad. Sc., (1948), 1678, and Comptes rendus du Congrés de Harvard (1950).

([3]) H. HOPF and W. RINOW, Comm. Math. Helv., **3** (1931), 209-225; G. DE RHAM, Comm. Math. Helv., **26** (1952), 328-343.

In order for V_n to be *complete* it is necessary and sufficient that $\mathcal{E}$ be relatively compact, i.e., that its adherence $\bar{\mathcal{E}}$ be compact.

There exists another equivalent definition of a complete, Riemannian manifold that might be useful to state. Let a be an arbitrary point of V_n, and let T_a be the vector space that is tangent to V_n at a, which is a space that we assume to be endowed with a Euclidean space structure. If we are given a geodesic arc that originates at a then we will refer to the *initial vector* of the arc when we mean the vector y in T_a that is tangent to a in the same sense and has a length that is equal to the length of the geodesic arc. Having said this, in order for V_n to be complete, it is necessary and sufficient that *any vector* of T_a *must be the initial vector of a geodesic arc,* which will be an arc that is necessarily unique.

That arc may be naturally double-checked; one recovers it if it is part of a closed geodesic. One may translate the preceding property by saying that, as far as length is concerned, the geodesics that originate at a are closed or infinite in both senses. We let ψ denote the *map of* T_a *into* V_n that makes any vector y in T_a correspond to the extremity x of the geodesic arc with origin at a and initial vector y:

$$\psi: T_a \rightarrow V_n .$$

One proves that one has the following properties of a complete Riemannian manifold:

a) The set $C_r(a)$ of points x in V_n such that $d(a, x) \leq r$, where r is an arbitrary positive number, is *compact.*

b) Given two arbitrary points x_1, x_2, of V_n that are at a distance $d(x_1, x_2)$ apart, there exists at least one path that joins x_1 to x_2 whose length is $d(x_1, x_2)$, and that path will be a geodesic.

One deduces from this that the map ψ is a map from T_a onto V_n .

Complete, Riemannian manifolds may be divided into two classes:

1. If the set of distances $d(a, x)$ from the points x of V_n to a fixed point a is bounded ([1]) then the complete, Riemannian manifold will be a *compact space*.
2. If the set of distance $d(a, x)$ is not bounded then the complete manifold V_n will be non-compact. One further says that it admits a *domain at infinity:* i.e., given an arbitrarily large r, there exist x in V_n such that:

$$d(a, r) > r.$$

We shall consider each of these classes in turn.

([1]) It results naturally from the triangle inequality that the distances between all pairs of points of V_n will then be bounded.

76. – Green's formula. – In this section and the following one, we shall make, at the very least, the following differentiability hypotheses on the Riemannian manifold envisioned:

a) V_n is a differentiable manifold of class C^2 and the second derivatives of the admissible coordinate changes are piecewise twice-continuously differentiable.

b) The metric on V_n :

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1, 2, \ldots, n)$$

is positive-definite, and the components g_{ij} of the fundamental tensor will be functions of class C^1 in admissible coordinates; the first derivatives will be piecewise C^1.

A scalar function U is called *regular* in a domain if it is of class C^1 in this domain and its first derivatives are piecewise C^1. If U and V are two such functions then one will have:

$$\operatorname{div}(U \operatorname{grad} V) = g^{ij}\,\nabla_j\,(U\,\partial_j V) = U\, g^{ij}\,\nabla_i\,\partial_j V + g^{ij}\,\partial_i U\,\partial_j V\,,$$

in the domain envisioned; namely, if we denote the Laplacian on the Riemannian manifold V_n by Δ:

$$\operatorname{div}\,(U \operatorname{grad} V) = U\,\Delta V + \operatorname{grad} U \cdot \operatorname{grad} V.$$

If Γ is an n-dimensional differentiable chain in the regularity domain of U and V, and $\partial\Gamma$ its boundary then by an application of the generalized Stokes formula, one will get:

(76-1) $$\int_{\partial\Gamma} U \operatorname{grad} V\, \mathbf{n}\, d\sigma = \int_{\Gamma} (U\Delta V + \operatorname{grad} U \cdot \operatorname{grad} V)\, d\tau\,,$$

in which $d\tau$ denotes the volume element of V_n, $d\sigma$ denotes the area element on $\partial\Gamma$, and $\mathbf{n}$ denotes the unit normal vector to $\partial\Gamma$, with the corresponding orientation. If we take $V = U$ then we will immediately deduce the uniqueness theorem for solutions of the Dirichlet problem for $\Delta U = 0$ from (76-1).

Upon changing the role of U and V in (76-1) and subtracting both sides we will get the Green formula:

(76-2) $$\int_{\partial\Gamma} (U \operatorname{grad} V - V \operatorname{grad} U)\, n\, d\sigma = \int_{\Gamma} (U\Delta V - V\Delta U)\, d\tau\,.$$

77. – Case for which V_n is compact and orientable. – The desired global theorem is immediate in the case where the Riemannian manifold envisioned is a compact, orientable manifold. Let U be a regular function on V_n that is a solution of:

(77-1) $$\Delta U = f(x),$$

in which *f is a continuous function on V_n with values that are positive or zero.* Upon integrating (77-1) over the fundamental cycle of V_n, one will first have:

$$\int_{V_n} f(x)\,d\tau = \int_{V_n} \Delta U\ d\tau = 0$$

and, as a result, f may only be identically zero. If we apply formula (76-1) to the fundamental cycle of V_n, with $V = U$ then we get:

$$\int_{V_n} [U\Delta V + (\mathrm{grad}\,U)^2]\,d\tau = 0$$

namely:

$$\int_{V_n} (\mathrm{grad}\,U)^2\,d\tau = 0\,.$$

One deduces from this that grad $U \equiv 0$, and that U is constant on V_n. We then state:

THEOREM. – *Any function U that is regular on a compact, orientable, Riemannian manifold V_n and is a solution of:*

$$\Delta U = f(x)$$

in which $f(x)$ is a continuous function whose values are positive or zero, reduces to a constant on V_n identically.

Ultimately, we shall drop the orientability condition.

78. – A theorem of E. Hopf ([1]). – Let E_n be the real properly Euclidian space of dimension n, and let D be an n-dimensional open set of E_n that is arbitrarily large, but bounded. Suppose that we are given an operator of elliptic type on D:

(78-1) $$L(U) = g^{ij}\,\partial_{ij}U + h^i\,\partial_i U$$

in which the g^{ij} and the h^i are continuous functions on D, and the quadratic form $g^{ij}\lambda_i\,\lambda_j$ is positive definite. We may establish the following theorem:

THEOREM. – *If U is a function of class C^2 on D such that $L(U) \geq 0$, and there exists a point a in the interior of D such that $U(x) \leq U(a)$ for any x of D then the function U is constant in D.*

We reason by absurdity and assume that that there exists at least one point x_0 in D for which $U(x_0) < U(a)$. We set:

$$U(a) = M.$$

In order to reach a contradiction, we first propose to construct a closed ball B and a function V on this ball such that $L(V)$ is strictly positive on B and attains its maximum on B at an interior point of B.

([1]) For the proof of this theorem, see also K. YANO and S. BOCHNER, *Curvature and Betti Numbers*, Ann. of Math. Studies, pp. 26.

a) To that effect, observe that if one starts with the point x_0 as a center [$U(x_0) < M$] and traces a sphere S_r of radius r then it results from the continuity of U that for a sufficiently small r one has $U(x) < M$ on the closed ball that is defined by S_r. On the contrary, for a certain value r_0 of r there will exist at least one point x_1 on S_{r_0} for which $U(x_1) = M$, although for x interior to S_{r_0}, one will have $U(x) < M$.

Trace out a sphere S of center O and radius R that is tangent to S_{r_0} at x_1 and interior to $S_{r_0} \cap D$. One therefore has $U(x) < M$ for any x on the closed ball that is defined by S, except at the point x_1 for which $U(x_1) = M$.

Finally, consider a sphere Σ with center x_1 and a radius that is sufficiently small that O is exterior to it. The surface of S divides Σ into two "caps." We denote the cap – including its boundary – that corresponds to the interior of S by C_i, and the cap – not including its boundary – that corresponds to the exterior of S by C_e. There will then exist a number $\varepsilon > 0$ such that:

(78-1) $$U(x) \leq M - \varepsilon \quad \text{on } C_i$$

whereas one has only:

(78-2) $$U(x) \leq M \qquad \text{on } C_e\,.$$

Once the sphere S has been so chosen, we denote the closed ball that it defines by B.

Having said this, adopt the center O of S to be the origin of a system of ordinary orthogonal coordinates (x), and consider the function:

(78-3) $$W = e^{-\alpha r^2} - e^{-\alpha R^2}$$

in which α is a positive constant, and:

$$r^2 = \overline{Ox}^2 = \sum_i (x_i)^2\,.$$

One then has:

$$\partial_i W = -2\alpha e^{-\alpha r^2} x_i \qquad\qquad \partial_{ij} W = 4\alpha^2 e^{-\alpha r^2} x_i x_j - 2\alpha e^{-\alpha r^2} \delta_{ij}\,.$$

One deduces from this that:

(78-4) $$L(W) = e^{-\alpha r^2}\, 2\alpha [2\alpha\, g^{ij} x_i x_j - g^{ij} \delta_{ij} - h^i x_i]\,.$$

Since O is exterior to B, $g^{ij}\, x_i\, x_j$ will be strictly positive on B. If we choose a sufficiently large quantity for α then we may then suppose that:

$$L(W) > 0 \qquad \text{on } B.$$

On the other hand, one has $W(x) < 0$ on C_e, whereas $W(x_1) = 0$.

Finally, we take our function V to be the function:

(78-5) $$V = U + \delta W,$$

in which δ is a positive number that is sufficiently small to make $V(x) < M$ on C_i, which is possible from (78-1). One will thus have:

$$V(x) < M \qquad \text{on } C_i, \qquad\qquad V(x) < M \qquad \text{on } C_e,$$

therefore:

$$V(x) < M \qquad \text{on } \Sigma.$$

Now:

$$V(x_1) = M.$$

Therefore, in the closed ball B, the function V *will certainly attain its maximum at an interior point* ξ.

On the other hand, since:

$$L(U) \geq 0, \qquad L(W) > 0$$

on B, one will have:

(78-6) $$L(V) > 0 \qquad \text{on } B.$$

b) We shall now show that it is possible for a function V such that $L(V) > 0$ on B to attain its maximum on B at an interior point. If this is the case then the $\partial_i V$ will be zerol at this point ξ. We adopt this point to be the origin of a new coordinates system (x^i) and develop V with the aid of Taylor's formula. One obtains:

$$V(x) = V(\xi) + \tfrac{1}{2}\, x^i x^j \,[\,(\partial_{ij} V)_\xi + \varepsilon(x^k)]$$

in which $\varepsilon \to 0$ with the x^k. Denote the distance ξx by ρ and set $x^i = \rho\,\lambda^i$. One will thus have

$$\lambda^i \lambda^j \,[\,(\partial_{ij} V)_\xi + \varepsilon] \leq 0$$

at ξ. It results from this that for any system of quantities λ^i:

$$(\partial_{ij} V)_\xi\, \lambda^i \lambda^j \leq 0.$$

As a result, since the g^{ij} are the coefficients of a positive definite form:

$$[g^{ij}\, \partial_{ij} V]_\xi \leq 0.$$

Now, $L(V)$ reduces to:

$$L(V) = g^{ij}\, \partial_{ij} V$$

at ξ, and it will be strictly positive, by construction.

79. – Geodesically-normal coordinates. – In the rest of this section, we shall consider a differentiable manifold V_n *of class* C^4 that is given a positive-definite, Riemannian metric *of class* C^3. One knows that one may associate a positive number $\rho(a)$ to every point a of V_n such that for any x for which $d(x, a) < \rho(a)$ there will exist one and only one geodesic arc of length $d(a, x)$ that joins a to x. The number $\rho(a)$ may be chosen in such a way that it defines a continuous function of a.

Therefore, the restriction of ψ to the open Euclidian ball of radius $\rho(a)$ in T_a will be a bijective map of that ball onto the neighborhood $d(a, x) < \rho(a)$ of the point a in V_n. The geodesic normal coordinates of a point x of this neighborhood will then be defined by the components of the vector such that $\psi y = x$, relative to a given frame with origin a. In other words, if r denotes the distance $d(a, x)$, and (θ^i) denotes the n components of the unit vector that is tangent at a to the geodesic arc in question that joins a to x then the geodesic normal coordinates of x at the origin a will be the numbers:

$$y^i = \theta^i r.$$

The determination of ψ and its geodesic normal coordinates is related to the integration of the differential system for the geodesics. We write this system in the form:

$$\text{(79-1)} \qquad \frac{du^i}{dr} = -\Gamma^i_{jk}\, u^j u^k , \qquad \frac{dx^i}{dr} = u^i \qquad (g_{ij} u^i u^j = 1).$$

We must integrate (79-1) for the following initial conditions for $r = 0$:

$$\text{(79-2)} \qquad (x^i)_{r=0} = (x^i)_a , \qquad (u^i)_{r=0} = \theta^i,$$

and concern ourselves with the dependency of the solution upon the θ^i. For $d(a, x) < r(a)$, it results from classical theorems on differential systems that, under the hypotheses we made, the system (79-1) will admits a solution $x^i(r, \theta^j)$ that is twice-continuously differentiable with respect to r and θ^j. As a result, the (x^i) will be functions of the normal coordinates (y^i) of class C^2.

In other words, ψ defines a differentiable homeomorphism of class C^2 of the open Euclidian ball of radius $\rho(a)$ onto the neighborhood $d(a, x) < \rho(a)$ of V_n.

80. – Geometric considerations. – Suppose that we have adopted geodesic normal coordinates with origin a, and that for $d(a, x) < \rho(a)$ we refer the tensors to these coordinates. In particular, let $(g_{ij})_x$ be the components of the fundamental tensor at x, and let $\partial_i r$ be the derivatives of $r = d(a, x)$ with respect to the normal coordinates of x. From sec. **79**, the $(g_{ij})_x$ are functions of class C^1 of these normal coordinates. On the other hand, one knows ([1]) that:

$$\partial_i r = (g_{ij})_a\, \theta^j = (g_{ij})_x \theta^j$$

([1]) For example, see A. LICHNEROWICZ, Bull. Soc. Math. France, **72** (1944), 146-150.

in such a way that the $\theta_i = \partial_i r$ will be the covariant components of the unit vector **v** that is tangent at x to the geodesic arc that joins a to x and has the same sense as that arc. The geodesics that issue from a will be orthogonal trajectories of the geodesic spheres $S_r(a)$ with center a and radius $r < \rho(a)$, and **v** will therefore be the unit normal vector to $S_r(a)$ at x that is oriented towards the exterior. The system of spheres $S_r(a)$ and geodesic arcs that issue from a will be homeomorphic to the corresponding system of concentric Euclidian spheres in T_a with the same radii.

Finally, if $d\Sigma$ denotes the area element of $S_r(a)$ one has:

$$\lim_{r\to 0}\frac{d\Sigma}{r^{n-1}} = d\omega(0),$$

in which $d\omega(0)$ is the area element of the unit Euclidian sphere. We may set:

$$d\Sigma = r^{n-1} d\omega(r). \tag{80-1}$$

81. – Theorems on maxima. – *a*) Consider a function U of class C^2 on a local coordinate domain D in V_n that has the property that the quantity:

$$L(U) = \Delta U + l^i \partial_i U , \tag{81-1}$$

in which ΔU is the Laplacian of the function, and l^i is a continuous vector field, will be either positive or zero on D. If U attains its maximum in D at an interior point b then we may consider U to be a function of the local normal coordinates with origin a and apply the theorem of Hopf (sec. **78**) to that function. The function U is then necessarily constant in D.

In particular, we may use a neighborhood $d(a, x) < \rho(a)$, which is represented by means of the normal coordinates of origin a, to be the local coordinate domain D. Since the homeomorphism of this neighborhood onto the open Euclidian ball in T_a of radius $\rho(a)$ has class C^2, U will appear to be a function of class C^2 of the normal coordinates. Moreover, in normal coordinates:

$$\Delta U = g^{ij}\,\partial_{ij}U - g^{ij}\,\Gamma^k_{ij}\,\partial_k U$$

admits g^{ij} coefficients of class C^1 and continuous coefficients $h^k = -g^{ij}\Gamma^k_{ij}$. If U attains its maximum at an interior point of the domain $d(a, x) < \rho(a)$ then we may apply the theorem of Hopf, and the function U will necessarily be constant on $d(a, x) < \rho(a)$.

b) Now let U be a function of class C^2 on V_n such that one has $L(U) \geq 0$ on V_n, and $L(U)$ actually attains its maximum on V_n at a point a of that manifold. We propose to show that U is constant on the entire manifold, *which is assumed to be complete.*

Indeed, let x_0 be an arbitrary point of V_n, and let R be a number that is greater than $d(a, x_0)$. The set of points x such that $d(a, x) \leq R$ is a compactum $C_R(a)$. The continuous function $\rho(x)$ admits a minimum $\rho_0 \neq 0$ on this compactum. Let m be a fixed positive number that is less than ρ_0.

The point x_0 may be joined to a by a path l of finite length L (for example, a geodesic arc). Since the function U attains its maximum at a, it will be constant in $C_\mu(a)$. Let q be an integer such that $L < q\mu$, and divide the arc l into q intervals of equal length with subdivision points $a, x_1, x_2, \ldots, x_{q-1}, x_0$. One will then have $d(a, x_1) < \mu$. As a result, U attains its maximum at x_1 in $C_\mu(x_1)$; hence, U is constant in $C_\mu(x_1)$, and, as a result, in $C_\mu(a) \cup C_\mu(x_1)$, and so on. The function U is finally constant in:

$$C_\mu(a) \cup C_\mu(x_1) \cup \ldots \cup C_\mu(x_0),$$

and one has:

THEOREM – *If a bounded function U of class C^2 on a complete Riemannian manifold is such that $L(U)$ is positive or zero and attains its maximum at a point a of V_n then it will be constant on V_n.*

82. – Behavior of the function U at infinity. – *a*) Suppose that the complete, Riemannian manifold V_n admits a domain at infinity, and consider a function U of class C^2 on V_n such that $L(U) = 0$. We say that U uniformly tends to a constant k in the domain at infinity of V_n ; i.e., that if a denotes a given point then one may associate any positive ε with a number R that depends only upon e and has the property that:

$$d(a, x) \geq R$$

implies that:

$$| U(x) - k | < \varepsilon.$$

As a result, either $U(x) = k$ on V_n or $U(x)$ takes various values of k on V_n . For example, suppose that there exists an x_0 such that:

$$U(x_0) > k.$$

(One likewise treats the contrary hypothesis by reasoning on $-U$).

Take:

$$\varepsilon < U(x_0) - k$$

and let R be the associated positive number. For $d(a, x) \geq R$, one has:

$$U(x) < k + \varepsilon < U(x_0) .$$

Consider the domain $C_R(a)$ that is defined by $d(a, x) \leq R$. It contains x_0 in its interior. The maximum of U on $C_R(a)$ will therefore be attained at an interior point b, and one will have:

$$U(x) \leq U(b), \qquad \text{for } d(a, x) \leq R,$$
$$U(x) < U(x_0) \leq U(b), \qquad \text{for } d(a, x) \geq R.$$

U therefore attains its maximum on V_n at the point b. As a result, since the function U satisfies the hypotheses of the preceding theorem, it may only be constant on V_n. One has:

THEOREM – *If a function U of class C^2 on a complete Riemannian manifold is such that $L(U) = 0$ and U uniformly tends to a constant k in the domain at infinity of V_n then it will be constant on V_n.*

b) Consider a function U on V_n of class C^2 such that $L(U)$ is positive or zero. Suppose furthermore that U uniformly tends to a constant k in the domain at infinity, but with values that are greater than or equal to k. There exists an R such that for $d(a, x) \geq R$ one has:

$$U(x) \geq k.$$

Then, by the preceding argument, it results that U will attain its maximum at a point b of V_n. As a result:

THEOREM – *If a function U of class C^2 on a complete Riemannian manifold is such that $L(U) \geq 0$ and it uniformly tends to a constant k by values that are greater than or equal to k in the domain at infinity of V_n then it will be constant on V_n.*

83. – The stationary spacetimes that are envisioned. – We return to the stationary spacetimes V_4 in the sense that was defined in sec. **62**.

We say that *such a spacetime is spatially complete, or, more briefly, complete if the associated Riemannian manifold V_3 with the metric $(ds)^2$ is complete.*

In what follows, we shall envision stationary spacetimes that satisfy the postulates of relativity, and, in particular, the Einstein equations, in the various cases. We shall put ourselves in a domain D in which the metric satisfies the Einstein equations of a *given schema*. Of course, under these conditions, the manifolds V_3 or W_3 will satisfy the differentiability hypotheses that were made in sec. **62**. However, due to the Einstein equations themselves, there is more.

Consider the potentials $g_{\alpha\beta}$ in a neighborhood of a point x of V_4, and a hypersurface S, which we represent locally in adapted coordinates by $x^1 = 0$. From isometry, the derivatives of the potentials $g_{\alpha\beta}$ on S may present discontinuities, and, as a result, the values of these potentials on S will be of class C^3. On the other hand, since S is generated by timelines, it will be non-characteristic. As a result, all of the second derivatives of the potentials $g_{\alpha\beta}$ will b continuous upon traversing S, with the exception of the derivatives $\partial_1 g_{\lambda 1}$ and their possible discontinuities are obviously constant along a timeline. It is then possible to perform a local change of adapted coordinates on one side of S that annuls the discontinuities of these derivatives. Since the result extends immediately to third derivatives, one will obtain local adapted coordinates in a neighborhood of x such that the corresponding $g_{\alpha\beta}$ are of class C^3.

For example, suppose, as we shall in the following sections, that we are concerned with everywhere exterior stationary spacetimes. We will therefore see that it is possible to assume that spacetime is homeomorphic of class (C^2, piecewise C^4) to a manifold V_4 of class C^4 that is the topological product of a V_3 of class C^4 with the real line $\mathbb{R}$, with V_4 being endowed with a Riemannian metric of class C^3. This is what we shall assume in what follows.

II. – STATIC, LEVI-CIVITA SPACETIMES

84. – Notion of a static spacetime. – Consider a stationary spacetime such that the spatial tensor H_{ij} is identically zero. We say that such a spacetime is *static* in the Levi-Civita sense. We recall that a space for which $H_{ij} = 0$ enjoys the following property: If U is a neighborhood of V_3 then the corresponding timelines will be orthogonal trajectories to the local spatial sections over U. One may therefore find a local, adapted coordinate system such that in the domain of V_4 that projects onto U, ds^2 takes the form:

$$ds^2 = \xi^2 (dx^0)^2 + \dot{g}_{ij}\; dx^i dx^j \qquad (\dot{g}_{ij} = g_{ij}) \tag{84-1}$$

in which the $\dot{g}_{ij}$ and ξ depend only upon the variables (x^k). This is the property that Levi-Civita originally used in order to define static spacetimes.

In the case of a static spacetime equations (69-5) take the simple form:

$$\left\{\begin{aligned} R_{\underline{i}\underline{k}} &= \dot{R}_{\underline{i}\underline{k}} - \tfrac{1}{\xi}\dot{\nabla}_{\underline{k}}(\partial_{\underline{i}}\xi), \\ R_{\underline{i}\underline{0}} &= 0, \\ R_{\underline{0}\underline{0}} &= -\tfrac{1}{\xi}\dot{\Delta}\xi. \end{aligned}\right. \tag{84-2}$$

85. – Complete, static, exterior spacetimes. – Consider a complete, static spacetime that everywhere satisfies the Einstein equations for the exterior case. We assume, moreover, that if V_3 admits a domain at infinity then ξ will uniformly tends to a constant $k \neq 0$. It will then be easy to apply the theorems that were established in the first part of this chapter to V_3 since, from the last equation (84-2), the function ξ will satisfy:

$$\dot{\Delta}\xi = 0.$$

If V_3 is compact then ξ reduces to a constant identically. If V_3 is not compact then we are dealing with the conditions of the first theorem of sec. **82**, and ξ again reduces to the constant k identically. Therefore, ξ = const. in either case, and, from (84-2), the Einstein equations of the exterior case reduce to the equations:

$$\dot{R}_{ik} = 0.$$

Therefore, the three-dimensional Riemannian manifold V_3 admits a Ricci tensor that is zero. One knows that, for this dimension, since the curvature tensor is equivalent to the Ricci tensor, the vanishing of the Ricci tensor will imply a locally-Euclidian character. Thus, in any domain of V_4 that projects onto the neighborhood U of V_3 the metric of V_4 may be written:

$$ds^2 = k^2 (dx^0)^2 + (d\dot{s})^2$$

in which $(d\dot{s})^2$ is a locally-Euclidian metric. It results from this that V_4 is locally-Euclidian. We state the:

THEOREM – *Any everywhere regular static exterior spacetime with V_3 compact is necessarily locally-Euclidian. Any complete, everywhere-regular, static, exterior spacetime for which ξ uniformly tends to* 1 *in the domain at infinity is necessarily locally-Euclidian* ([1]).

If the manifold V_3 is complete and, in addition, simply-connected then one knows, moreover, that it will be Euclidian.

Note that in order to establish the preceding theorem we have used only the quotient space V_3, and not the existence of global spatial sections W_3. This theorem will then be true under hypotheses that are more general than the ones that were stated in Chapter VII. For example, it will suffice to assume that the timelines, which are trajectories of the isometry group, define a convenient fibration on V_4. If two points of V_4 are considered to be equivalent when they are located on the same timeline then the quotient of V_4 by that equivalence relation will be a space V_3 that enjoys the properties that were studied in Chapter VII. The preceding theorem is therefore valid under the more general hypotheses.

III. – EXTERIOR, STATIONARY SPACETIMES

86. – The case in which V_3 is compact. – In the sequel, we shall envision only complete, exterior, stationary spacetimes. First, suppose that V_3 is compact. From (69-5), the Einstein equation will take the form:

(86-1)
$$-\dot{\Delta}\xi = -\frac{\xi^3}{2} H^2$$

with:

$$H^2 = \tfrac{1}{2} g^{ik} g^{jl} H_{ij} H_{kl} .$$

Since the function $\xi^3 H^2$ is positive or zero on V_3, the function ξ will satisfy $-\dot{\Delta}\xi \leq 0$ on V_3. Since $\xi \neq 0$, one will deduce that the quantity $H^2 = 0$ on V_3, i.e.:

$$H_{ij} = 0.$$

([1]) See CH. RACINE, *Thèse*, *Paris* (1934).

Such an exterior spacetime is therefore necessarily static in the Levi-Civita sense, and, as a result, locally-Euclidian. We state:

THEOREM. – *Any everywhere regular stationary exterior spacetime with V_3 compact is necessarily locally Euclidian.*

87. – Case in which V_3 is non-compact. First theorem. – Now consider a complete, stationary, exterior spacetime with a space V_3 that admits a domain at infinity. Suppose that ξ uniformly tends to 1 in the domain at infinity of V_3 by values that are less than or equal to 1.

The Einstein equation (86-1) may be written:

$$-\Delta(-\xi) = \frac{\xi^3}{2} H^2 .$$

One sees that the function $-\xi$ satisfies the hypotheses of the second theorem of sec. **82**. As a result, one will have $\xi = 1$, $H^2 = 0$, on V_3, and we will again come back to a static spacetime. We may state:

THEOREM – *Any everywhere-regular, stationary, exterior spacetime with a complete V_3 for which ξ uniformly tends to* 1 *in the domain at infinity of V_3 is necessarily locally-Euclidian.*

In the two preceding theorems, the argument involves only the manifold V_3, and the same remark that was made about static spacetimes will be valid; the theorems are true under the more general hypothesis that V_4 is fibered by timelines.

88. – Asymptotically-Euclidian behavior. – Physicists frequently use another sort of hypothesis in the non-compact case that is not always presented clearly, but which we shall try to schematize mathematically in a correct manner. Here, $d(x_0, x_1)$ will denote the distance between two points of the Riemannian manifold W_3.

Consider a three-dimensional Euclidian space $\mathcal{E}_3$ that admits a negative-definite metric $\overline{ds}^2$. We refer $\mathcal{E}_3$ to a privileged coordinate system (y^i) for which:

$$\overline{ds}^2 = \delta_{ij}\, dy^i\, dy^j,$$

in which:

$$\delta_{ij} = 0 \qquad \text{for } i \neq j, \qquad \delta_{ii} = -1.$$

We say that V_4 admits an asymptotically-Euclidian behavior on W_3 when, for a point a of W_3 and a sufficiently large number R:

1. There exists a homeomorphism h of class C^2 in the domain $d(a, x) > R$ of W_3 onto a domain of $\mathcal{E}_3$ that is homeomorphic and complementary to a closed ball of $\mathcal{E}_3$. (This

homeomorphism h will thus define a Euclidian space structure on the domain envisioned in W_3 .)

2. If (y^α) is the privileged coordinate system on V_4 that is defined by the (y^i) and the variable x^0 in the domain of V_4 that corresponds to the domain $d(a, x) > R$ of W_3 then we will have:

(88-1) $$|g_{\alpha\beta} - \delta_{\alpha\beta}| < \frac{M}{r} \qquad |\partial_\gamma g_{\alpha\beta}| < \frac{M}{r^2} \qquad [r = d(a, x)]$$

in that domain, in which M is a fixed positive number and the potentials and their derivatives are expressed relative to the privileged coordinates (y^α) and $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$, $\delta_{00} = +1$.

It is clear that the point a plays only an auxiliary role here.

In the rest of this section, we shall envision only elements of W_3 . Let x_0 be a point of W_3 such that $r_0 = d(a, x)$ is greater than R, y_0 is its image in $\mathcal{E}_3$ by h, and ρ_0 is the ordinary distance from y_0 to the origin 0 in $\mathcal{E}_3$:

$$\rho_0^2 = \sum_i (y_i)^2 .$$

We propose to compare the numbers ρ_0 and r_0.

Let L be a geodesic arc that joins a to x_0 and has a length r_0. Let L' be the connected arc $\overset{\frown}{x, x_0}$ of L for all points x such that $d(a, x) \geq R$. If r_0' denotes its length then we will have $r_0 = R + r_0'$. In the domain $d(a, x) > R$ of W_3 , we will have, from (88-1):

$$\delta_{ij} = \dot{g}_{ij} + \frac{m_{ij}}{r} \qquad [r = d(a, x)]$$

and, as a result:

$$\overline{ds}^2 = (d\dot{s})^2 + \frac{1}{r} m_{ij}\, dy^i\, dy^j,$$

in which the m_{ij} are bounded in absolute value. By integration over $\overset{\frown}{x, x_0}$ we will obtain:

(88-2) $$\int_{\overset{\frown}{x,x_0}} |\overline{ds}| \leq \int_{\overset{\frown}{x,x_0}} \sqrt{1 + \frac{1}{r}\left|\frac{m_{ij}dy^i dy^j}{(d\dot{s})^2}\right|}\, |d\dot{s}|.$$

Now, under our hypotheses there exists a fixed positive number K_1 such that:

$$\left|\frac{m_{ij}\, dy^i dy^j}{(d\dot{s})^2}\right| < K_1 .$$

From this, it results that:

$$\int_{\widehat{x,x_0}} |\,\overline{ds}\,| \leq K_2 \int_{\widehat{x,x_0}} |\,ds^*\,| = K_2\, r_0',$$

in which K_2 denotes a fixed positive number. Now, we obviously have:

$$\rho_0 < \int_{\widehat{x,x_0}} |\,\overline{ds}\,| + K_3,$$

in which K_3 is a fixed positive number. Therefore, there exists a fixed positive number K such that for any x_0 for which $d(a, x_0) > R$ one will have:

$$\rho_0 < K\, r_0. \tag{88-3}$$

89. – The study of the flux vector p. – Let Σ_ρ be a sphere in $\mathcal{E}_3$ with center O and radius ρ, and let S_{r_0} be the set of points of W_3 such that $d(a, x) = r_0 > R$. We choose ρ to be sufficiently large that Σ_ρ contains the image of S_{r_0} by h, and denote the image of the sphere Σ_ρ in W_3 by $\mathcal{S}_\rho$.

Consider the compactum B_ρ of W_3 that is defined by:

a) The points x for which $d(a, x) \leq r_0$;
b) The points x for which $d(a, x) \geq r_0$ whose image in E_3 is interior to Σ_ρ or on that sphere.

The boundary of B_ρ is $\mathcal{S}_\rho$ with the outward orientation. If **p** denotes the vector that was introduced in sec. **74** whose covariant components are:

$$p_i = \frac{\xi^3}{2}\,\varphi_k\, H_i^k$$

then we shall propose to study the behavior of the flux **p** as it crosses $\mathcal{S}_\rho$ when $\rho \to \infty$.

For the moment, let us adopt a frame that is orthonormal for the metric $(ds^*)^2$ at a point x of W_3. One has:

$$p_{\underline{i}} = -\frac{\xi^3}{2}\sum_k \varphi_{\underline{k}} H_{\underline{i}\,\underline{k}}\,.$$

From the Schwarz inequality:

$$(p_i)^2 \leq \left(\frac{\xi^3}{2}\right)^2 \sum_k (\varphi_{\underline{k}})^2 \sum_j (H_{\underline{i}\,\underline{j}})^2\,.$$

One deduces from this that:

$$\sum_i (p_i)^2 \le \left(\frac{\xi^3}{2}\right)^2 \sum_k (\varphi_{\underline{k}})^2 \sum_{i,j} (H_{\underline{ij}})^2 .$$

We introduce the positive scalars that measure the magnitudes of the tensors:

$$p = \sqrt{|\mathbf{p}^2|}, \qquad \varphi = \sqrt{|\boldsymbol{\varphi}^2|}, \qquad H = \sqrt{H^2} .$$

One will thus have:

$$p \le \frac{\xi^2}{\sqrt{2}} \varphi H ,$$

and the flux of $\mathbf{p}$ through $\mathcal{S}_\rho$ may be majorized by:

$$|\operatorname{flux}_{\mathcal{S}_\rho} \mathbf{p}| \le \iint_{\mathcal{S}_\rho} p \, d\Sigma^* \le \frac{1}{\sqrt{2}} \iint_{\mathcal{S}_\rho} \xi^3 \varphi H \, d\Sigma^* ,$$

in which $d\Sigma^*$ is the area element of $\mathcal{S}_\rho$ for the metric $(ds^*)^2$. Under our hypotheses, for $x \in \mathcal{S}_\rho$ one will have:

$$\xi^3 \varphi H < \frac{C_1}{r^3}$$

in which C_1 is a fixed number and $r = d(a, x)$. It results from (88-3) that there exists a fixed number C_2 such that:

$$\xi^3 \varphi H < \frac{C_2}{\rho^3} .$$

On the other hand:

$$\int_{\mathcal{S}_\rho} d\Sigma^* = \int_{\mathcal{S}_\rho} \left[1 + o\left(\frac{1}{r}\right)\right] d\bar{\Sigma} < C_3 \int_{\mathcal{S}_\rho} d\bar{\Sigma} ,$$

in which $d\bar{\Sigma}$ is the Euclidean area. We will thus have:

$$|\operatorname{flux}_{\mathcal{S}_\rho} \mathbf{p}| < \frac{1}{\sqrt{2}} \frac{C_2}{\rho^2} C_3 4\pi\rho^2 < \frac{C}{\rho} ,$$

in which C is a fixed number. We have established that under our hypotheses:

(89-1) $$\lim_{\rho \to \infty} \operatorname{flux}_{\mathcal{S}_\rho} \mathbf{p} = 0.$$

90. – The case in which V_3 is non-compact. Second theorem. – Therefore, consider a complete, exterior, stationary spacetime that admits an asymptotically-Euclidian behavior in the domain at infinity of W_3. Apply the integral relation (74-5) to W_3, which is extended to a domain B_ρ for a sufficiently large ρ. One will get:

$$\iiint_{B_\rho} \xi \varphi_i R_0^i \, d\tau = \iiint_{B_\rho} \frac{\xi^3}{2} H^2 d\tau - \text{flux}_{\mathcal{S}_\rho} \mathbf{p} .$$

Since the space is exterior, we will get:

$$\text{flux}_{\mathcal{S}_\rho} \mathbf{p} = \iiint_{B_\rho} \frac{\xi^3}{2} H^2 d\tau .$$

Suppose that H^2 is positive at a point of W_3. It is then positive in a certain neighborhood D of this point, and one will have:

$$\text{flux}_{\mathcal{S}_\rho} \mathbf{p} \le \iiint_{D} \frac{\xi^3}{2} H^2 d\tau .$$

Now, when $\rho \to \infty$, the left-hand side will go to zero, which contradicts the preceding inequality. One will thus have $H \to 0$ on W_3 or V_3, and the spacetime envisioned will necessarily reduces to a static, Levi-Civita spacetime. We may thus state:

THEOREM – *Any complete, exterior, stationary spacetime that admits an asymptotic-Euclidian behavior is necessarily locally Euclidian.*

This theorem involves the existence of global spatial sections in an essential manner.

IV. – APPLICATION OF THE DIVERGENCE FORMULAS TO STATIONARY UNIVERSES

91. – The matching of stationary gravitational fields. – In this last part, we shall study certain stationary spacetimes that satisfy the Einstein equations in different cases.

Consider a hypersurface S ($x^1 = 0$) on a spacetime manifold V_4 that carries the Cauchy data for an exterior problem, and assume that these data are such that $\partial_0 g_{\alpha\beta} = 0$ on S, while the lines in S along which only x^0 varies are assumed to be time-oriented. It results from the theorems of Mme. Fourés that the corresponding solution of the exterior Cauchy problem can only be locally stationary. We are then led to think that an exterior gravitational field that extends a stationary field will itself be stationary. We shall assume this, from now on.

Consider a stationary field – interior or exterior – and an exterior, stationary field that agrees the preceding one on S. One easily establishes that if (x^λ) is an adapted, local coordinate system relative to the first field (with S being defined by $x^1 = 0$) then one may obtain an adapted, local coordinate system relative to the second field that satisfies:

$$x^{\lambda'} = x^\lambda + \frac{(x^1)^3}{6}[\varphi^{(\lambda)}(x^A) + \varepsilon^\lambda] \qquad (A = 0, 2, 3) \tag{91-1}$$

in which $\varepsilon^\lambda \to 0$ when $x^1 \to 0$. In particular, the spatial sections can be assumed to have a second-order contact on S. One may say that (91-1) translates into the matching of the isometries on S.

92. – The sign of R_0^0 for a spatially-oriented W_3 . – Some of the results that follow involve the sign of the R_0^0 component of the Ricci tensor for an interior stationary ds^2, and *a spatial section* W_3 *that is spatially oriented.* For such a W_3, one has $g_{00} > 0$ along with $g^{00} > 0$, and the two quadratic forms with the coefficients:

$$g_{ij} \qquad \text{and} \qquad g^{ij},$$

respectively, are both negative-definite.

Having said this, we adopt the perfect fluid-electromagnetic field schema for the energy-momentum tensor envisioned. From (13-3), one has:

$$R_0^0 = \chi\left(T_0^0 - \tfrac{1}{2}T\right)$$

in which the energy tensor $T_{\alpha\beta}$ is given by:

$$T_{\alpha\beta} = (r + p)\, u_\alpha u_\beta - p\, g_{\alpha\beta} + \tau_{\alpha\beta};$$

$\tau_{\alpha\beta}$ corresponds to the electromagnetic field $F_{\alpha\beta}$. One obviously has $T = \rho - 3p$. As a result:

$$R_0^0 = \chi[(\rho + p)u_0 u^0 - p - \tfrac{1}{2}(\rho - 3p) + \tau_0^0]$$

namely:

$$R_0^0 = \chi[\rho\left(u_0 u^0 - \tfrac{1}{2}\right) + p\left(u_0 u^0 + \tfrac{1}{2}\right) + \tau_0^0].$$

Let us study the sign of the various terms in brackets. First, since u_α is unitary we have:

$$u_0 u^0 - \tfrac{1}{2} = g^{00}(u_0)^2 + g^{0i} u_0 u_i - [g^{00}(u_0)^2 + 2g^{0i} u_0 u_i + g^{ij} u_i u_j] ,$$

namely:

$$u_0 u^0 - \tfrac{1}{2} = \tfrac{1}{2} [\, g^{00}(u_0)^2 - g^{ij} u_i u_j] > 0,$$

under the hypotheses that were made. The same will be true *a fortiori* for $u_0 u^0 + \tfrac{1}{2}$. We now study the sign of:

$$\tau_0^0 = \tfrac{1}{4} F_{\lambda\mu} F^{\lambda\mu} - F_{0\lambda} F^{0\lambda}.$$

If we single out the index 0 and the Latin indices then we will get:

$$\tau_0^0 = \tfrac{1}{4}(F_{ij}F^{ij} + 2F_{0i}F^{0i}) - F_{0i}F^{0i},$$

namely:

$$\tau_0^0 = \tfrac{1}{4}F_{ij}F^{ij} - \tfrac{1}{2}F_{0i}F^{0i}.$$

If we express the electromagnetic field in terms of its covariant components (for example) then we have:

$$\tau_0^0 = \tfrac{1}{4}(g^{ik}g^{jl}F_{ij}F_{kl} + 2g^{ik}g^{j0}F_{ij}F_{k0}) - \tfrac{1}{2}(g^{00}g^{ij}F_{0i}F_{0j} + g^{0j}g^{ik}F_{0j}F_{ik} + g^{0j}g^{0i}F_{0i}F_{j0}).$$

One deduces from this that:

(92-2)
$$\tau_0^0 = \tfrac{1}{4}g^{ik}g^{jl}F_{ij}F_{kl} - \tfrac{1}{2}(g^{00}g^{ij} - g^{0i}g^{0j})\,F_{0i}F_{0j},$$

which is, moreover, *positive* under the hypotheses that were made. One sees that R_0^0 *is strictly positive for a spatially oriented* W_3, and that this result will persist in the case of the pure matter, perfect fluid, and pure electromagnetic field schemas. In what follows, we shall confine ourselves to case for which R_0^0 is positive.

Under these conditions, equations (71-6), which may be written:

(92-3)
$$\operatorname{div}\mathbf{h} = -\xi R_0^0,$$

gives a good extension of *the classical Gauss theorem* to the theory of stationary spacetimes.

93. – Singularities of the gravitational field interior to and exterior to masses. – Consider a world-tube that is bounded by a hypersurface *S*, filled with a stationary interior gravitational field, and generated by timelines. This field induces a stationary gravitational field in a neighborhood of *S* that satisfies the Einstein equations of the exterior case and agrees with them on *S*. We propose to show that this latter field cannot be assumed to be regular in the interior of *S*.

Therefore, suppose that this exterior field is regular in the interior of *S*. Let W_3^i be an arbitrary spatial section relative to the interior field that is spatially oriented; it determines a two-dimensional domain D_2 on *S*. One may construct a hypersurface W_3^e that passes through D_2 and has a second-order contact with W_3^i on D_2 and is transversal to the timelines of the exterior field. We may take W_3^e to be a spatial section for the exterior field in *S* and adopt local coordinates in a neighborhood of *S* that make this field satisfy (91-1).

The vectors **h** that relate to the two fields will be identical on D_2, since they depend upon only the potentials and their first derivatives. Upon applying formula (92-3) to the interior field, one will get:

$$\operatorname{flux}_{D_2}\mathbf{h} = -\iiint_{W_3^i}\xi R_0^0\,d\tau < 0.$$

This flux is strictly negative, whereas, when the same formula is applied to the exterior field, we get:

$$\text{flux}_{D_2}\,\mathbf{h} = 0.$$

We state the following:

THEOREM – *If one is given a stationary, interior field bounded by a hypersurface S that is generated by the timelines then any exterior, stationary gravitational field that agrees with it on S may be regularly extended to the entire interior of S.*

94. – Stationary universe with a compact, orientable space. – Consider a stationary spacetime V_4 that admits a compact, orientable spatial section W_3 that is spatially-oriented. We suppose that this spacetime defines a *world model,* i.e., that its metric satisfies the Einstein equations for the interior and exterior cases with agreement on hypersurfaces S that are generated by timelines. We shall show that such a world model may not exist in the actual presence of energy distributions.

One deduces from the relation:

$$\text{div}\,\mathbf{h} = -\xi R_0^0,$$

by integrating over W_3 that:

$$\iiint_{W_3} \xi R_0^0\, d\tau = 0,$$

so ξR_0^0 will be strictly positive in certain domains of W_3 and zero on their complements. We have the:

THEOREM – *There cannot exist a stationary world model that has a compact, orientable, spatial section that is spatially-oriented.*

95. – Stationary universe with a non-compact space. – Now consider a stationary world model that admits a complete spatial section W_3 and has asymptotically-Euclidean behavior in the domain at infinity of W_3, with the energetic distributions assumed to be at a finite distance and occupy domains D_u on W_3.

Apply the relation (74-5) to B_r (with the notation of sec. 89) for a sufficiently large ρ. One obtains:

$$\iiint_{B_\rho} \xi\,\varphi_i\, R_0^i\, d\tau = \iiint_{B_\rho} \frac{\xi^2}{2} H^2 d\tau - \text{flux}_{S_\rho}\,\mathbf{p};$$

namely, since R_0^i is null outside of D_u:

$$\sum_u \iiint_{D_u} \xi\,\varphi_i\, R_0^i\, d\tau = \iiint_{R_\rho} \frac{\xi^2}{2} H^2 d\tau - \text{flux}_{S_\rho}\,\mathbf{p}.$$

When $\rho \to \infty$, the flux of **p** goes to zero. As a result, the first integral on the right-hand side will converge, and one will have:

$$\iiint_{W_3} \frac{\xi^2}{2} H^2 d\tau = \sum_u \iiint_{D_u} \xi \varphi_i R_0^i \, d\tau . \tag{95-1}$$

One may deduce an important consequence from this relation. Suppose that the matter is schematized in the form of a perfect fluid, and suppose that the streamlines of the various masses coincide with the timelines. One will then have:

$$u^i = 0, \qquad R_0^i = S_0^i = \chi [(\rho + p) u^i u_0 - p g_0^i] = 0$$

in the adapted coordinates that are being used, and from (95-1), one will deduce that $H^2 \equiv 0$ on W_3. The world-model will necessarily be a static spacetime in the Levi-Civita sense. We then state the following:

THEOREM – *Any stationary word-model with asymptotic-Euclidian behavior for which the matter streamlines (which are assumed to be schematized in the form of a perfect fluid) coincide with the timelines is everywhere static in the Levi-Civita sense.*

In particular, this theorem permits us to reduce the postulates – which one assumes are independent – that lead to the construction of the Schwarzschild world-model.

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