BOOK II

UNITARY THEORIES

Since they first appeared, the classical relativistic theories have profoundly modified our understanding of the different physical fields and our notion of field itself.

The special theory of relativity was based on electromagnetism. In the general geometric case of an improperly Euclidian, four-dimensional, Minkowski space the concept of an *electromagnetic field* that is represented by an antisymmetric tensor $F_{\lambda\mu}$ appeared, thanks to a detailed analysis of Maxwell's equations. We know the fundamental example of the unification of fields: The electric and magnetic field vectors, which were represented in the old spatial framework by two vectors that varied in time, appear – depending on the observer - to be two aspects of one and the same physical field, which is represented by the tensor field $F_{\lambda\mu}$ in the Minkowski framework. One might almost say that this example and this success defined the dream of physicists for the last fifty years.

By contrast, if dynamics and gravitation have acquired a new form in special relativity, then this new form does not contribute to any progress in the explanation of the phenomenon of gravitation. The gravitational field always appears as a singular phenomenon that is superposed upon, but foreign to, the Euclidian structure of Minkowski space and which does not interfere with the electromagnetic field.

On the contrary, the classical general theory of relativity is based on the gravitational field. This field is represented with the aid of a symmetric tensor $g_{\lambda\mu}$ – viz., the gravitational potential tensor – on the spacetime manifold, and it thus provides this manifold with a fundamental Riemannian metric of the hyperbolic normal type. Dynamics, in the classical sense, disappears completely, and the gravitational phenomena acquire a subtle and satisfactory explanation that is based upon the consideration of regular global metrics, which we sought to detail in the first part of this course.

In the framework of general relativity, the electromagnetic field continues to be represented by an antisymmetric tensor field, and one is led to adopt both the established concepts of general relativity and equations that are deduced by reasonable inductions from the ones of special relativity for electromagnetism. Without appealing to the concept of regularity, one may say that the fundamental equations of electromagnetism in general relativity are the following ones:

a) The Maxwell-Lorentz equations that govern the electromagnetic field $F_{\lambda\mu}$; however, one will note that only one of the equations involves the spacetime metric. On the other hand, the electric current appears as a notion that is foreign to that of field.

b) The Einstein equations, in which the Einstein tensor $S_{\lambda\mu}$ is found to be equal to the energy-momentum tensor $T_{\lambda\mu}$ up to a factor. The electromagnetic field intervenes in the right-hand side of this equation by way of an expression that is prudently induced by starting with special relativity, although it not deduced from the axioms of general relativity, and remains largely foreign.

There is interference between the gravitational field and the electromagnetic field. On the one hand, as we have seen, the propagation laws of the two fields are identical.

If such a theory of electromagnetism is mostly an advance on the former state then one will see that what remains in it is unsatisfactory to the spirit of our theory and merits the name that we give to it of the "provisional" theory of electromagnetism. To say that the difficulties undoubtedly lead to quantum mechanics and that it is not convenient to pursue them further in the classical context is an impotent attitude to impose on relativity.

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Very early on, theoreticians were therefore led to attempt to elaborate a *unitary theory* that would realize the unification of the gravitational and electromagnetic field into a single hyperfield whose data are equivalent to those of some geometrical structure for the universe. To a certain degree, the "mesonic" field has been combined with the problem in recent years.

Since 1919, which was the year when Hermann Weyl developed the first attempt at such a theory, the efforts have multiplied, but, one after the other, they have revealed aspects that are not strictly satisfactory to either viewpoint. The first – rather maladroit – sketches have nevertheless great importance in the development of contemporary differential geometry in general. Without a doubt, these are the efforts of Hermann Weyl and Eddington ([1]), which led Élie Cartan to construct his general concept of a space with a connection in a Lie group, starting in 1924. The geometrical concepts of Élie Cartan that were recently restated and developed in a global context by Ehresmann, Chern, André Weil, and myself actually constitute the general case of differential geometry.

By contrast, the latter attempts becoming much more interesting and refined from the physical viewpoint, and the actual discussion about the interpretation in quantum mechanics have made them increasingly important.

Grosso modo, one may classify the different theories that have been proposed into two broad categories:

1. The *five-dimensional* theories, which are also sometimes called *projective* theories, somewhat improperly. It was Kaluza ([2]), who presented the first inkling of such a theory in 1921, which was a theory that was reprised in 1926 by O. Klein ([3]). As we confirm, this theory, in the form of O. Klein, leads only to an *explanation* for the provisional theory of electromagnetism, and suggests no new equations. Here, the formalism is essentially one that is appropriate to a five-dimensional Riemannian manifold that admits a one-parameter group of isometries.

The formalism that is called *projective* was introduced by Veblen ([4]) in 1933, developed by Pauli ([5]) and Schouten, and used again quite recently by Jordan ([6]) in 1947, and leads to essentially equivalent results. The only differences are in their choices of mathematical representation. One may account for that by stating that identical field equations have been simultaneously elaborated by Jordan and his school in the projective formalism by Yves Thiry and myself in the formalism of a five-dimensional Riemannian manifold.

([1]) H. WEYL, *Raum, Zeit, Materie*, Eddington.
([2]) KALUZA, "Zum Unitäts problem der Physik," Sitzungsber. Preuss. Akad. Wiss., (1921), 966.
([3]) O, KLEIN, Z. Physik, **73** (1926), . 895.
([4]) O. VEBLEN, *Projective Relativitäts theorie*, Berlin, Springer (1933).
([5]) W. PAULI, Ann. Physik, **18** (1933), 305.
([6]) P. JORDAN, Ann. Physik,(1947), 219; P. JORDAN-MULLER, Z. Naturforschg., **2**a (1947), 1. See also I. BERG.

2. The theories that one may call theories *with affine connection.* It is appropriate to include the attempts of Weyl and Eddington, the numerous attempts of Einstein in collaboration with various students of his, and finally those of Schrödinger, as well as Einstein's last theory, which produced restricted results ([1]).

In the second part of this course, we will study an example of each category, and naturally we will choose theories that are recent and to-the-point for examples. For the first category, this will be the theory that I have proposed to call *the Jordan-Thiry theory* ([2]). We will study it in the formalism of moving frames, as I have suggested here, and which seems to me, at least for the projective formalism, adapted to understanding the profound geometrical and physical realities of the theory. For the second category, this will be the *Einstein-Schrödinger theory,* in a form that is appropriate to the one they actually used.

We are forced to point out advantages and disadvantages for each theory. *Grosso modo*, one may say that the first category leads to elegant theories in which the physical interpretation is clear, but which can be criticized for being insufficiently unitary. On the contrary, the second category amounts to a theory that seems to be as unitary as one may demand; however, this unitary character itself complicates the comparison with the theory of electromagnetism in classical general relativity, i.e., it complicates the correct physical interpretation of the geometric quantities that were introduced.

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As far as this mysterious unitary character I spoke of is concerned, I would like to make several remarks.

First of all, as Thiry observed, it is less ambiguous to employ the epithet of "unitary" in a negative proposition than in an affirmative one. What one intends by saying that the provisional theory of electromagnetism in classical general relativity is *not unitary* is sufficiently clear; the gravitational field is defined by the geometric structure defined for the universe, and the electromagnetic field intervenes only by making a sufficiently arbitrary modification of the purely gravitational theory, thanks to a contribution from the energy-momentum tensor.

In the positive sense, we may hope to realize in the state of our conceptions a physical fusion that is just as complete as the one that is realized by the electric and magnetic fields. In order to do this, it is possible to transmute the one into the other in a simple manner by a suitable choice. As far as the gravitational and electromagnetic fields are concerned, these things are certainly less simple.

One may agree to say that a theory is unitary *in the large sense* if it makes the gravitational and electromagnetic fields play symmetric roles in the representation of the fields and the formation of the equations; in particular, since the gravitational field is related to the geometrical structure of the universe in the concepts of general relativity, it is convenient to choose a structure such that the two fields arise from the same geometry.

([1]) See the corresponding bibliography in the second part.

([2]) BERGMANN, Ann. Math., **49** (1948), 255, which contains a grave error. On the other hand, see: A. LICHNEROWICZ and Y. THIRY, C.R. Acad. Sc., **224** (1947), 529; Y. THIRY, C. R. Acad. Sc., **226** (1948), pp. 216 and 1881, and *Thèse Paris* (1950).

A theory will be called unitary *in the strict sense* in the case where the rigorous equations regulate a non-decomposable hyperfield, and may be divided into propagation equations for the gravitational field and the electromagnetic field only approximately when the physical conditions are such that one of the fields dominates the other. One may think that only the theories of the second categories may be clearly unitary in the strict sense.

I. – THE JORDAN-THIRY THEORY

FIRST CHAPTER

THE TRAJECTORIES OF A CHARGED PARTICLE AND THE INTRODUCTION OF A FIVE-DIMENSIONAL SPACE

I. – ELECTROMAGNETISM IN CLASSICAL GENERAL RELATIVITY

1. – The equations of electromagnetism. – We recall the equations that are satisfied by electromagnetism in classical general relativity. To facilitate our ultimate comparisons with these equations, we suppose that the spacetime manifold V_4 is referred to systems of local coordinates (x^i) (i, any Latin index = 1, 2, 3, 4) and that the index 4 has a temporal character. The metric of V_4 is of hyperbolic normal type and is written:

(1-1) $$ds^2 = g_{ij}\,dx^i\,dx^j.$$

If R_{ij} is its Ricci tensor then the associated Einstein tensor will be:

(1-2) $$S_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R\,.$$

Having said this, in the presence of an electromagnetic field F_{ij} the *Einstein equations* take the form:

(1-2) $$S_{ij} = \chi\,T_{ij}\,,$$

in which the energy-momentum tensor T_{ij} contains the terms that relate to the electromagnetic field:

(1-4) $$\tau_{ij} = \tfrac{1}{4} g_{ij} F_{rs} F^{rs} - F_{ir} F^{r}{}_{j}\,,$$

which are terms that come from special relativity.

The electromagnetic field satisfies the *Maxwell-Lorentz equations*:

(1-5) $$\nabla_k F^{ki} = J^i,$$

in which ∇_k is the covariant derivative operator of the Riemannian connection, and J^i denotes the electric vector-current. The second group of equations may be written:

(1-6) $$\tfrac{1}{2}\varepsilon^{jkli}\partial_j F_{kl} = 0,$$

or in the invariant form:

(1-7) $$\tfrac{1}{2}\eta^{jkli}\nabla_j F_{kl} = 0,$$

and expresses the local existence of a vector-potential φ_i such that:

(1-8) $$F_{ij} = \partial_i\varphi_j - \partial_j\varphi_i .$$

Indeed, in the sequel we will assume the existence of a global vector-potential φ_i such that (1-8) is satisfied. Equations (1-3) [with (1-4)], (1-5), and (1-8) may then be considered to be the equations of the provisional theory of electromagnetism. In the case of a "pure electromagnetic" schema T_{ij} will reduce to τ_{ij} and $J^i = 0$.

2. – The trajectories of charged particle. – In (I, sec. **33**), we established that the motion of a charged particle, whose charge to mass ratio is:

$$k = \frac{e}{m} = \text{const.},$$

in the presence of an electromagnetic field is given by the equations:

(2-1) $$\frac{d^2x^i}{ds^2} + \Gamma^i_{jl}\frac{dx^k}{ds}\frac{dx^l}{ds} = k\,F^{\ i}_j\frac{dx^j}{ds}.$$

From (I, sec. 56), the trajectories of our particle are time-oriented lines that realize the extremum of the integral:

(2-2) $$\bar{s} = \int_{x_0}^{x_1}(ds + k\varphi) = \int_{x_0}^{x_1}[(g_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}} + k\varphi\dot{x}^i)]du \qquad \left(\dot{x}_i = \frac{dx^i}{du}\right),$$

in which φ is the vector-potential form and u is an arbitrary parameter.

These equations suggest various remarks:

a) In the purely gravitational case, the trajectories of a material particle are provided by the time-oriented geodesics of a Riemannian manifold. It seems desirable to extend this "geodesic principle" to the context of a unitary theory and to choose a geometric structure such that the motion of a charged particle is closely related to the geodesics of the geometric structure that represents the field.

b) One may remark that equations (1-2) say that the trajectories envisioned are the geodesics of the Finslerian manifold that admits the metric:

$$L(x^i, x^j) = (g_{ij} x^i x^j)^{\frac{1}{2}} + k\,\varphi_i\,\dot{x}^i .$$

It may appear during the study of the geometry of such a space. However, we know that there exist particles in our universe for which the ratio *k* takes different values. We are therefore led to envision a family of Finsler spaces, which seems much less satisfying.

We may then demand that it is possible to interpret the preceding trajectories, which correspond to different values of *k*, by making recourse to a unique, 5-dimensional, Finsler space whose metric is independent of *k*, and the new local coordinate x^0 is related in some way to the variable *k*. That question, or an equivalent question, is actually encountered – in various forms – in Classical Mechanics, as in relativity. In 1947, in collaboration with Thiry, I posed the corresponding general problem in the calculus of variations whose very simple solution assures the synthesis of well-known results. Without a doubt, the corresponding general mathematical procedure suggests, at best, the introduction of a penta-dimensional manifold.

II. – A PROBLEM IN THE CALCULUS OF VARIATIONS

3. – Finslerian manifold. – Let V_{n+1} be a differentiable manifold of class C^{p+1}, and let $W_{2(n+1)}$ be the fiber bundle of tangent vectors toV_{n+1}. A point of $W_{2(n+1)}$ consists of the combination of a point *x* of V_{n+1} (namely, its projection onto V_{n+1}) and a tangent vector $\dot{x}$ that is tangent to V_{n+1} at *x*. If (x^α) (α and any Greek index = 0, 1, …, *n*) denote a system of local coordinates on V_{n+1} then the union of the (x^α) and the components $(\dot{x}^\alpha)$ of the vector in the natural frame that is associated with the (x^α) provide a system of local coordinates on $W_{2(n+1)}$; that space is therefore naturally endowed with the structure of a differentiable manifold of class C^p.

We give ourselves a function $\mathcal{L}$ with scalar values and class C^p on $W_{2(n+1)}$, such that if *x* remains fixed and $\lambda\dot{x}$ is substituted for $\dot{x}$ then one will have:

$$\mathcal{L}(x, \lambda\dot{x}) = \lambda\mathcal{L}(x, \dot{x}) .$$

Locally, $\mathcal{L}$ is therefore a function of the (x^α) and the $(\dot{x}^\alpha)$ that is homogenous and of degree one in the (x^α).

When this is true, we will say that the given of the function $\mathcal{L}$ endows $W_{2(n+1)}$ with a *Finslerian structure*. A Finslerian manifold will be called *regular* if the function $\mathcal{L}$ leads to a regular problem in the calculus of variations on $W_{2(n+1)}$.

4. – Lie derivatives of $\mathcal{L}$. – Suppose we are given a vector field $\xi \neq 0$, of class C^p on a neighborhood of V_{n+1}; this field is the generator of a one-parameter "group" of local transformations on the manifold. These transformations are defined in local coordinates by integrating the differential system:

(4-1) $$\frac{dx^\alpha}{dt} = \xi^\alpha .$$

The trajectories of the vector field ξ will be the trajectories of the group.

We give t a sufficiently small value; integrating (4-1) for the initial conditions (x_0^α) gives:

(4-2) $$x^\beta = f^\beta(x_0^\alpha, t),$$

and the transformation that corresponds to that value of t takes the point whose local coordinates are (x_0^α) to a point x that belongs to the domain of the local coordinates that we envision and has local coordinates (x^β). This transformation may be extended to $W_{2(n+1)}$ in an intrinsic manner; it will suffice to make the vector $\dot{x}_0$ at x_0 correspond to the vector, $\dot{x}$, at x with the components:

$$\dot{x}^\beta = \frac{\partial f^\beta}{\partial x_0^\rho} \dot{x}_0^\rho ,$$

since this correspondence is obviously invariant under a change of local coordinates.

We associate the geometric object that is defined by the scalar field $\mathcal{L}$ with the transformed object $\overset{t}{\mathcal{L}}$, which is, by definition:

$$\overset{t}{\mathcal{L}}(x_0, \dot{x}_0) = \mathcal{L}(x, \dot{x}) .$$

One will note that:

$$\overset{t}{\mathcal{L}}(x_0, \lambda\dot{x}_0) = \lambda\mathcal{L}(x_0, \dot{x}_0) .$$

The Lie derivative of $\mathcal{L}$ with respect to the field ξ at $(x_0, \dot{x}_0)$ is, by definition:

$$X\mathcal{L}_0 = \lim_{t\to 0} \frac{\overset{t}{\mathcal{L}} - \mathcal{L}_0}{t} .$$

This is easy to evaluate upon finding the principal part of $\overset{t}{\mathcal{L}} - \mathcal{L}_0$; (4-2) may be written:

$$x^\beta = x_0^\beta + t(\xi_0^\beta + \varepsilon_0^\beta) \qquad (\varepsilon_0^\beta \to 0 \text{ when } t \to 0).$$

From this, one deduces:

$$\overset{t}{\mathcal{L}}(x_0, \dot{x}_0) = \mathcal{L}(x, \dot{x}) = \mathcal{L}(x_0, \dot{x}_0) + t[\xi_0^\alpha\, \partial_\alpha \mathcal{L}_0 + \partial_\rho \xi_0^\beta\, \dot{x}_0^\rho\, \partial_{\dot{\beta}} \mathcal{L}_0] + t\eta \qquad (\eta \to 0 \text{ when } t \to 0).$$

It results from this, upon suppressing the 0 indices, that:

(4-3) $$X\mathcal{L} = \xi_0^{\alpha}\,\partial_\alpha \mathcal{L}_0 + \partial_\rho \xi_0^{\beta}\,\dot{x}_0^{\beta}\,\partial_{\dot\beta}\mathcal{L}_0 \qquad \left(\partial_\alpha = \frac{\partial}{\partial x^\alpha}, \partial_\beta = \frac{\partial}{\partial x^\beta}\right).$$

If $\mathcal{L}$ is invariant under the transformations that are generated by ξ then we say that ξ is the generator of a local group of isometries of the Finslerian manifold. In order for ξ to be the generator of a local isometry group, it is necessary and sufficient that $X\mathcal{L} = 0$.

5. – Quotient manifold. The problem. – We consider a Finslerian manifold V_{n+1} and suppose that it admits a connected one-parameter group of global isometries that leaves no point of V_{n+1} invariant. The trajectories of the group will be designated by z; V_{n+1} is therefore generated by the z. We further assume that:

a) The z are homeomorphic to the real line $\mathbb{R}$ or the circle T^1;

b) One may find a differentiable manifold V_n of class C^{p+1} such that there exists a differentiable homeomorphism of class from the manifold V_{n+1} to the topological product, $V_n \times z$, under which z is mapped to the linear factor.

We say that the manifold V_n is the *quotient manifold* of V_{n+1} by the equivalence relation that the group defines.

Consider a system of local coordinates (x^i) (i, any Latin index = 1, 2, …, n). We may define local coordinates (x^α) in V_{n+1} in the following manner: The given of (x^i) determines a trajectory z. In order to determine a point on that trajectory, we choose the manifold x^0 = const. to which it belongs, since these manifolds will be the manifolds homeomorphic to V_n that defined by the homeomorphism *b*). Let ξ be the infinitesimal generator of the group of isometries; since no point of V_{n+1} is invariant, ξ is $\neq 0$ at any point of V_{n+1}. In the preceding local coordinates, the trajectories of ξ will be the lines, x = const.; as a result, the components of ξ will be:

$$\xi^i = 0, \qquad \xi^0 \neq 0,$$

and it will always be possible ([1]) to modify the homeomorphism of *b*) and the manifolds x^0 = const. in such a way that $\xi^0 = 1$. The systems of coordinates (x^0, x^i) such that $\xi^i = 0$, $\xi^0 = 1$ will be called *adapted* to the one-parameter group of isometries. The coordinate changes will take us from an adapted system to another one of the form:

(5-1) $$x^{i'} = \psi^{i'}(x^j), \qquad x^{0'} = x^0 + \psi(x^j),$$

in which the ψ are arbitrary functions of the (x^j).

In an adapted system of coordinates, formula (4-3) obviously reduces to:

([1]) See I, sec. **44**.

$$X\mathcal{L} = \partial_0 \mathcal{L} .$$

From this, it results that the function $\mathcal{L}$ is then expressed as a function $\mathcal{L}(x^i, \dot{x}^j, \dot{x}^0)$, which is homogenous and of first degree with respect to the $\dot{x}^\alpha$. Unless stated to the contrary, we will use adapted coordinated in all of the following sections.

We may then pose the following *problem*: Is it possible to endow the quotient manifold V_n with the structure of a Finslerian manifold by means of functions $\mathcal{L}$ in such a way that the geodesics of V_n, which are the extremals of the integral:

(5-2) $$\int_{x_0}^{x_1} \mathcal{L}(x, \dot{x})\, du \qquad \left(\dot{x} = \frac{dx}{du}\right)$$

correspond to the extremals of:

(5-3) $$\int_{z_0}^{z_1} L(z, \dot{z})\, du \qquad \left(\dot{z} = \frac{dz}{du}\right)$$

by projection onto V_n.

In the following sections, I will confine myself to *exclusively local* considerations.

6. – Determination of the function $\mathcal{L}$ (1). – 1. Suppose we are given an extremal of (5-2) with the parametric representation $x(u)$, in which u denotes an arbitrary parameter. It is well-known that the differential system for the extremals:

(6-1) $$\frac{dx^\alpha}{du} = \dot{x}^\alpha ,$$

in which $\dot{x}^\alpha$ satisfies:

(6-2) $$\frac{d}{du}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0,$$

admits the relative integral invariant (2):

(6-3) $$\omega = \sum_\alpha \frac{\partial L}{\partial \dot{x}^\alpha} dx^\alpha .$$

The converse is true, and may be verified directly in the following manner: Suppose that the differential system (6-1) admits the relative integral invariant (6-3), or – what would be equivalent – the absolute integral invariant:

(1) Cf. A. LICHNEROWICZ and Y. THIRY, C.R. Acad. Sc., **224** (1947), 529.

(2) See E. CARTAN, *Leçons sur les invariants intégraux*, chap. XVIII and *Espaces de FINSLER*, pp. 8-9.

$$(6\text{-}4)\qquad \begin{cases} d\omega = \sum_{\alpha} d\,\dfrac{\partial \mathcal{L}}{\partial \dot{x}^{\alpha}} \wedge dx^{\alpha} \\ \quad = \sum_{\alpha,\beta}\left[\dfrac{\partial^2 \mathcal{L}}{\partial \dot{x}^{\alpha}\partial \dot{x}^{\beta}}\, d\dot{x}^{\beta} \wedge dx^{\alpha} + \dfrac{1}{2}\left(\dfrac{\partial^2 \mathcal{L}}{\partial \dot{x}^{\alpha}\partial x^{\beta}} - \dfrac{\partial^2 \mathcal{L}}{\partial \dot{x}^{\beta}\partial x^{\alpha}}\right) dx^{\beta} \wedge dx^{\alpha}\right] \end{cases}$$

The characteristic system of $d\omega$ is obtained by annulling the coefficient of the term in $\wedge\, dx^{\alpha}$ in the last expression of (6-4). That will give:

$$\sum_{\beta}\left[\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^{\lambda}\partial \dot{x}^{\beta}}\, d\dot{x}^{\beta} + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^{\lambda}\partial x^{\beta}}\, dx^{\beta} - \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^{\beta}\partial x^{\lambda}}\, dx^{\beta}\right] = 0.$$

Namely, upon dividing by du:

$$(6\text{-}5)\qquad \sum_{\beta}\left[\frac{\partial}{\partial \dot{x}^{\beta}}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\lambda}}\right)\frac{d\dot{x}^{\beta}}{du} + \frac{\partial}{\partial x^{\beta}}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\lambda}}\right)\frac{dx^{\beta}}{du}\right] - \sum_{\beta}\frac{\partial}{\partial \dot{x}^{\beta}}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\lambda}}\right)\dot{x}^{\beta} = 0.$$

On account of the homogeneity of $\partial \mathcal{L}/\partial x^{\lambda}$, (6-5) may be written:

$$\frac{d}{du}\frac{\partial \mathcal{L}}{\partial \dot{x}^{\lambda}} - \frac{\partial \mathcal{L}}{\partial x^{\lambda}} = 0,$$

which proves the property.

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which proves the property.

2. We return to the abbreviated notation and the hypotheses of sec. **5**; in what follows, I will assume that:

$$(6\text{-}6)\qquad \partial_{\dot{0}\dot{0}}\mathcal{L} \neq 0.$$

The differential system of the extremals of (5-2) may therefore be characterized by the fact that it admits the relative integral invariant:

$$\omega = \partial_{\dot{k}}\mathcal{L}\, dx^{k} + \partial_{\dot{0}}\mathcal{L}\, dx^{0}.$$

Moreover, under the hypotheses we made, $\partial_0 \mathcal{L} = 0$, and one will have the first integral:

$$(6\text{-}7)\qquad \partial_{\dot{0}}\mathcal{L} = h.$$

When this relation is solved for $\dot{x}^0$, it will be locally equivalent to the relation:

(6-8) $$\dot{x}^0 = \varphi(x^k, \dot{x}^l, h),$$

in which φ is a function that is homogenous and of first degree in $\dot{x}^l$, and depends on h essentially.

Consider the family (E_h) of extremals of (5-2) that correspond to a definite value of h. For this family, the last term of ω will have the value $h\, dx^0$, and will defines a relative invariant. It will result from this that this family of extremals admits the relative integral invariant:

(6-9) $$\partial_{\dot{k}} \mathcal{L}\, dx^k .$$

Now, from the homogeneity of $\mathcal{L}$:

$$\dot{x}^k \partial_{\dot{k}} \mathcal{L} + \dot{x}^0 \partial_{\dot{0}} \mathcal{L} = \mathcal{L} .$$

As a result, for any solution of (6-7) or (6-8), the quantity $\dot{x}^k \partial_{\dot{k}} \mathcal{L}$ may be expressed as a function $\mathcal{L}$ of the variables $(x^k, \dot{x}^l, h)$:

(6-10) $$L(x^k, \dot{x}^l, h) = L[x^k, x^l, \varphi(x^k, \dot{x}^l, h)] - h\,\varphi(x^k, \dot{x}^l, h)$$

and one will have:

$$\partial_i L = \partial_i \mathcal{L} + \partial_{\dot{0}} \mathcal{L}\, \partial_i \varphi - h\, \partial_i \varphi = \partial_i \mathcal{L} .$$

Therefore, from (6-9), the projections of the (E_h) onto V_n are defined by a differential system that admits the relative integral invariant:

$$\varpi = \partial_{\dot{k}} L\, dx^k .$$

In other words, these projections are extremals of the integral:

(6-11) $$\int_{z_0}^{z_1} L(x^k, \dot{x}^l, h)\, du \qquad \left(\dot{x}^l = \frac{dx^l}{du}\right),$$

in which h has the chosen value. We state:

THEOREM – *For any function* $\mathcal{L}(x^k, \dot{x}^l, h)$ *that is homogenous and of degree* 1 *with respect to the* x^λ, *and is such that* $\partial_{\dot{0}\dot{0}} \mathcal{L} \neq 0$, *the extremals of the integral* (5-2) *on* V_{n+1} *that correspond to the value h of the first integral:*

(6-7) $$\partial_{\dot{0}} \mathcal{L} = h,$$

project onto V_n *along the extremals of the integral* (6-11), *in which h has the same value, and L is given by:*

(6-10) $$L(x^k, \dot{x}^l, h) = L[x^k, x^l, \varphi(x^k, \dot{x}^l, h)] - h\varphi(x^k, \dot{x}^l, h),$$

in which φ denotes the function that is obtained by solving (6-7) *with respect to x^0.*

7. – The inverse problem. – We call the correspondence that makes the function $\mathcal{L}(x^k, \dot{x}^l, h)$ correspond to the function $L(x^k, \dot{x}^l, h)$ that we just determined a *descent*.

Conversely, if we are locally given a function $L(x^k, \dot{x}^l, h)$ on V_n that is homogenous and of degree 1 with respect to the $\dot{x}^l$ then we will propose to find out whether there exists a function $\mathcal{L}(x^k, \dot{x}^l, h)$ that comes back to L by descent.

1) If there exists a solution $\mathcal{L}$ to this problem then it will be easy to determine it. Indeed, upon differentiating (6-10), one will necessarily get:

$$\frac{\partial L}{\partial h} = \partial_{\dot{0}}\mathcal{L}\frac{\partial \varphi}{\partial h} - h\frac{\partial \varphi}{\partial h} - \varphi,$$

which will reduce to:

$$\frac{\partial L}{\partial h} = -\varphi(x^k, \dot{x}^l, h),$$

for any solution to (6-7) or (6-8). The function φ is therefore known whenever L is known, and since φ depends essentially on h, $\dfrac{\partial^2 L}{\partial h^2}$ will be non-zero. If one solves the relation:

$$\dot{x}^0 = \varphi(x^k, \dot{x}^l, h),$$

with respect to h then it will become:

(7-1) $$h = \psi(x^k, \dot{x}^l, \dot{x}^0),$$

in which ψ is homogenous and of degree 0 with respect to $(\dot{x}^\lambda)$, and depends essentially on $(\dot{x}^0)$. Therefore, from (6-10), the function $\mathcal{L}$ will necessarily be expressed in the variables $(x^k, \dot{x}^l, \dot{x}^0)$ by the relation:

(7-2) $$\mathcal{L}(x^k, \dot{x}^l, \dot{x}^0) = L[x^k, \dot{x}^l, \psi(x^k, \dot{x}^l, \dot{x}^0)] + \dot{x}^0\psi(x^k, \dot{x}^l, \dot{x}^0).$$

2) If one is given a function L such that $\partial^2 L / \partial h^2 \neq 0$ then consider the function $\mathcal{L}$ that is defined by (7-2), in which ψ is obtained by solving the relation:

(7-3) $$\dot{x}^0 = -\frac{\partial L}{\partial h}(x^k, \dot{x}^l, h)$$

with respect to h. One will thus have:

$$\dot{x}^0 \equiv -\frac{\partial L}{\partial h}[x^k, \dot{x}^l, \psi(x^k, \dot{x}^l, \dot{x}^0)].$$

We apply the descent procedure to $\mathcal{L}$. One first obtains:

$$\partial_{\dot{0}}\mathcal{L} \equiv \frac{\partial L}{\partial h}[x^k, \dot{x}^l, \psi]\partial_{\dot{0}}\psi + \dot{x}^0 \partial_{\dot{0}}\psi + \psi,$$

namely, from the preceding identity:

$$\partial_{\dot{0}}\mathcal{L} \equiv \psi.$$

One deduces from this that the function φ that is associated with $\mathcal{L}$ is essentially:

$$\varphi = -\frac{\partial L}{\partial h}.$$

When we pass to the variables $(x^k, \dot{x}^l, h)$, we will get the result:

$$L(x^k, \dot{x}^l, h) = L[x^k, \dot{x}^l, \psi(x^k, \dot{x}^l, \dot{x}^0)] + \dot{x}^0 \psi(x^k, \dot{x}^l, \dot{x}^0)$$

from (7-2), in which ψ is obtained by solving the equation:

$$\dot{x}^0 = -\frac{\partial L}{\partial h}$$

with respect to *h*.

We shall call the correspondence that makes the function $\mathcal{L}$ that was defined by the preceding theorem correspond to a function *L* an *ascent.* We note two circumstances that occur frequently when one wants to use the ascent procedure.

a) One is interested only in *different families of curves* (ε_h) that depend on a parameter *h* and may be defined as the extremals of a function:

$$L(x^k, \dot{x}^l, h).$$

These curves may also be considered as extremals of the function:

$$\chi(h)\, L(x^k, \dot{x}^l, h),$$

in which $\chi(h)$ is an arbitrary function, and the ascent procedure that relates to these different functions *L* leads naturally to different functions $\mathcal{L}$.

b) It may also be the case that one is interested only in the extremals (ε) of a function:

$$L_0(x^k, \dot{x}^l)$$

that does not depend on any parameter h. One may then introduce a function $L(x^k, \dot{x}^l, h)$ such that for a definite value $h = h_0$:

$$L(x^k, \dot{x}^l, h) = L_0(x^k, \dot{x}^l)\,,$$

and apply the ascent procedure to L. The curves that are considered will then be interpreted as projections of the extremals that correspond to the value h_0 of the function $\mathcal{L}$. Naturally, there will be a great degree of arbitrariness when the ascent procedure is applied in these cases.

A case that is particularly interesting – in Mechanics, as well as in relativity – is the one in which the ascent procedure allows us to pass from the geodesics of a Finslerian metric to those of a Riemannian metric, since such a metric is much easier to work with. Conversely, we shall therefore apply the descent procedure to a Riemannian metric in order to characterize the problems that may be encountered with this procedure.

8. – Case in which $\mathcal{L}$ defines a Riemannian metric. First case. – Consider the function $\mathcal{L}$ that is defined by the relation:

$$\mathcal{L}^2 = \gamma_{\lambda\mu}\dot{x}^\lambda\dot{x}^\mu \qquad (\lambda, \mu = 0, 1, \ldots, n),$$

in which the $\gamma_{\lambda\mu}$ are the components of a symmetric tensor of V_{n+1} that does not depend upon the variable x_0 in adapted coordinates. The descent procedure will lead to two different results, depending on whether γ_{00} is zero or non- zero.

We first suppose that γ_{00} *is not annulled in the domain in question.* If $\gamma_{00} > 0$ then we will restrict ourselves to values of the parameter for which:

$$h^2 < \min \gamma_{00}$$

for any x that belongs to that domain.

We suppose that the form $\mathcal{L}^2$ is non-degenerate – so [$g = \det(\gamma_{\lambda\mu}) \neq 0$] – but we do not assume that it is positive-definite. At each point x, we confine ourselves only to values of $\dot{x}$ for which the right-hand side is positive. It is well known that it suffices that a geodesic of the Riemannian manifold should render this right-hand side positive at one point in order for it that to be true all along the geodesic ([1]).

Having posed that, the descent procedure will lead us to form the equation:

$$\text{(8-1)} \qquad \tfrac{1}{2}\partial_{\dot{0}}\mathcal{L}^2 \equiv \gamma_{00}\dot{x}^0 + \gamma_{0\mu}\dot{x}^\mu = h\mathcal{L} \qquad (i, j = 1, 2, \ldots, n)$$

([1]) For example, see I, sec. **16**.

and eliminate $\dot{x}$ between this equation and:

$$L = \mathcal{L} - h\dot{x}^0 . \tag{8-2}$$

It is possible to next express $\mathcal{L}$ as a function of the variables $(x^i, \dot{x}^j, h)$ with the aid of (8-1). Upon decomposing $\mathcal{L}^2$ into squares, beginning with the directrix variable $\dot{x}^0$, we will obtain:

$$\mathcal{L}^2 = \frac{1}{\gamma_{00}}\left[\tfrac{1}{2}\partial_{\dot{0}}\mathcal{L}^2\right]^2 + \Phi^2 , \tag{8-3}$$

with

$$\Phi^2 = g_{ij}\dot{x}^i\dot{x}^j , \qquad g_{ij} = \gamma_{ij} - \frac{\gamma_{0i}\gamma_{0j}}{\gamma_{00}} . \tag{8-4}$$

If $\gamma_{00} \neq 0$ then since the form $\mathcal{L}^2$ is non-degenerate, the form Φ^2 will be non-degenerate, and $g = \det(g_{ij}) \neq 0$. Conversely, if $\gamma_{00} \neq 0$ and $g \neq 0$ then the form $\mathcal{L}^2$ will be non-degenerate. One deduces from (8-1) and (8-3) that:

$$\mathcal{L}^2 = \frac{h^2}{\gamma_{00}}\mathcal{L}^2 + \Phi^2 ,$$

and Φ^2 will be positive for the values of the variables that we envisioned. It results from this that:

$$\mathcal{L}\sqrt{1 - \frac{h^2}{\gamma_{00}}} = \Phi , \tag{8-5}$$

which gives us $\mathcal{L}$ as a function of the desired variables.

One then obtains $\dot{x}^0$ as a function of $\mathcal{L}$ and the $(x^i, \dot{x}^j)$ from (8-1):

$$\dot{x}^0 = \frac{h}{\gamma_{00}}\mathcal{L} - \frac{\gamma_{0i}\dot{x}^i}{\gamma_{00}} . \tag{8-6}$$

one then deduces from this and (8-2) that:

$$L = \left(1 - \frac{h^2}{\gamma_{00}}\right)\mathcal{L} + h\frac{\gamma_{0i}\dot{x}^i}{\gamma_{00}} ,$$

and from (8-5), that will give:

$$L = \sqrt{\left(1 - \frac{h^2}{\gamma_{00}}\right)g_{ij}\dot{x}^i\dot{x}^j} + h\frac{\gamma_{0i}\dot{x}^i}{\gamma_{00}} . \tag{8-7}$$

Conversely, any function of the type (8-7) (with $\gamma_{00} \neq 0$, $g = \det(g_{ij}) \neq 0$) corresponds a non-degenerate Riemannian metric of the first case by ascent. Conforming to remark *b*) of sec. **7**, we note that L presents itself, relative to the variables $\dot{x}^i$, as *the sum of the square root of a non-degenerate quadratic form and a linear form.* Only functions L_0 of this type may lead to Riemannian metrics of the first case by ascent.

9. – Case in which L defines a Riemannian metric. Second case. – We now put ourselves in the situation where $\gamma_{00} = 0$. One then has:

(9-1)
$$\mathcal{L}^2 = 2\gamma_{0i}\dot{x}^i\dot{x}^j + \gamma_{ij}\dot{x}^i\dot{x}^j .$$

We assume that $\gamma_{0i}\dot{x}^i \neq 0$, $h \neq 0$. The descent procedure leads us to eliminate $\mathcal{L}$ and $\dot{x}^0$ from the relations (9-1):

(9-2)
$$\gamma_{0i}\dot{x}^i = h\,\mathcal{L} ,$$

and:

(9-3)
$$L = \mathcal{L} - h\dot{x}^0 .$$

One infers from (9-2) that:

$$\mathcal{L} = \frac{\gamma_{0i}\dot{x}^i}{h} .$$

If one refers to (9-1) then this will become:

$$\frac{(\gamma_{0i}\dot{x}^i)^2}{h^2} = 2\gamma_{0i}\dot{x}^i\dot{x}^j + \gamma_{ij}\dot{x}^i\dot{x}^j .$$

One deduces from this that:

$$\dot{x}^0 = \frac{\gamma_{0i}\dot{x}^i}{2h^2} - \frac{\gamma_{ij}\dot{x}^i\dot{x}^j}{2\gamma_{0i}\dot{x}^i} .$$

Therefore, the function L will be given by:

$$L = \frac{\gamma_{0i}\dot{x}^i}{h} - \frac{\gamma_{0i}\dot{x}^i}{2h} + h\frac{\gamma_{ij}\dot{x}^i\dot{x}^j}{2\gamma_{0i}\dot{x}^i} ,$$

namely:

(9-4)
$$L = \frac{\gamma_{0i}\dot{x}^i}{2h} + h\frac{\gamma_{ij}\dot{x}^i\dot{x}^j}{2\gamma_{0i}\dot{x}^i} .$$

Conversely, any function of the type (9-4) corresponds to a Riemannian metric of the second case by ascent. Conforming to remark *b*) of sec. **7**, we note that L presents itself as *the quotient of a quadratic form by a linear form* with respect to the variables $\dot{x}^i$. Only functions of this type may lead to Riemannian metrics of the second case by ascent.

10. – Examples from classical dynamics. Hamilton's principle. – In order to familiarize ourselves with these procedures and results, we shall study several examples from classical dynamics. We will then confirm that they indeed reduce to well-known procedures in some interesting particular cases.

Consider a dynamical system with bilateral perfect holonomic constraints and r degrees of freedom. Suppose, moreover, that this system is conservative. The possible configurations of this system will depend on time, in general, and the differentiable manifold that we agree to introduce in the general case will be the *configuration spacetime* of the system E_{r+1}, which will be the set of possible configurations at various instants.

For a specific choice of axes, the configuration of the system may be defined locally at each instant by means of r parameters q^i (i, any Latin index = 1, 2, …, r). Indeed, the set of (q^i) and time t define a local system of coordinates for the configuration spacetime of the system E_{r+1} .

One usually determines the motion of the system by seeking to determine the q^i as a function of time t. In terms of Lagrange variables $(q^i, \dot{q}^i, t)$, the differential equations of motion can be written:

$$\frac{dq^i}{dt} = \dot{q}^i , \qquad\qquad \frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}^i}\right) - \frac{\partial \Lambda}{\partial q^i} = 0 ,$$

in which Λ denotes the Lagrangian ($T + U$) of the system, with the classical notations. The differential equations of motion may be considered to define the extremals of the Hamiltonian action:

$$W = \int_{t_0}^{t_1} \Lambda(q^i, t, \dot{q}^i)\, dt ,$$

and they may also be characterized by the existence of Cartan's relative integral invariant:

$$\omega = \sum_i \frac{\partial \Lambda}{\partial \dot{q}^i} dq^i - H dt ,$$

in which H designates the Hamiltonian ($T_2 - T_0 - U$) of the system.

This type of procedure presents the inconvenience of not being invariant with respect to the changes of local coordinates that are permitted by mechanics on the configuration spacetime. As far as the parameters q^i are concerned, one must transform them by:

(10-1) $$q'^i = f'^i(q^j, t) .$$

As far as time – which presents an intrinsic character in classical mechanics – is concerned one has only:

(10-2) $$t' = \lambda t + \mu ,$$

in which λ and μ are constants ($\lambda \neq 0$).

The preceding technique is not invariant with respect to the changes of local coordinates, and as a result, it does not recommend itself to theoretical studies. It is preferable to introduce t as essentially the $(r + 1)^{\text{th}}$ coordinate of E_{r+1}:

$$t = q^{r+1},$$

and to define the motion by looking for a parametric representation $q^\alpha(u)$ (α, any Greek index = 1, 2, …, r + 1) as a function of an arbitrary parameter u. The (q^α) will therefore be a system of local coordinates on E_{r+1}, and the $\dot{q}^\alpha$ will be the components of a tangent vector relative to the natural frame of these coordinates. If:

$$\frac{dq^\alpha}{du} = \dot{q}^\alpha \tag{10-3}$$

then one will have:

$$\dot{q}^i = \frac{dq^i}{dt} = \frac{\dot{q}^i}{\dot{q}^{r+1}}.$$

Upon adopting the variable u as the integration variable in the Hamiltonian action, one will obtain:

$$W = \int_{q_0}^{q_1} L(q^\alpha, \dot{q}^\alpha)\, du\,, \tag{10-4}$$

in which we have introduced the function:

$$L(q^\alpha, \dot{q}^\alpha) = \Lambda\left(q^i, t, \frac{\dot{q}^i}{\dot{q}^{r+1}}\right)\dot{q}^{r+1}, \tag{10-5}$$

which is homogenous and of degree 1 with respect to $\dot{q}^\alpha$. Upon differentiating (10-5), one will immediately have:

$$\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial \Lambda}{\partial q'^i}, \qquad \frac{\partial L}{\partial q^{r+1}} = \Lambda - \sum_i \frac{\partial \Lambda}{\partial q'^i} q'^i = -H\,,$$

and the Cartan integral invariant will be none other than:

$$\omega = \sum_\alpha \frac{\partial \Lambda}{\partial \dot{q}^\alpha} dq^\alpha\,.$$

It is easy to specify L by starting with U and the expression for the *vis viva*:

$$2T = a_{ij} q'^i q'^j + 2b_i q'^i + 2T_0\,,$$

in which:

$$a = \det(a_{ij}) \neq 0.$$

It becomes:

$$\Lambda\left(q^i, q^{r+1}, \frac{\dot q^i}{\dot q^{r+1}}\right)\dot q^{r+1} = \tfrac{1}{2}\frac{a_{ij}\,\dot q^i\,\dot q^j}{\dot q^{r+1}} + b_i\,\dot q^i + (T_0 + U)\,\dot q^{r+1}.$$

One deduces from this that:

$$L(q^\alpha, \dot q^\alpha) = \frac{a_{ij}\,\dot q^i\,\dot q^j + 2b_i\,\dot q^i\,\dot q^{r+1} + 2(T_0 + U)(\dot q^{r+1})^2}{2\dot q^{r+1}}. \tag{10-6}$$

Therefore, the trajectories of our dynamical system in configuration spacetime will be defined as intrinsically extremals of the integral (10-4) in E_{r+1}, in which the function L is given by (10-5) or (10-6). In this manner, the formalism will be invariant under the changes of local coordinates on configuration spacetime, (10-1) and (10-2), that are permitted by mechanics.

11. – The ascent from Hamilton's principle to Eisenhart's ds^2. – As a function of $(\dot q^\alpha)$, L is the quotient of a quadratic form with a linear form. It is therefore possible to interpret the trajectories of our dynamical system as the projections of certain geodesics of an $(r+2)$-dimensional Riemannian manifold V_{r+2}, for which $\gamma_{00} = 0$ in adapted coordinates. Since L is devoid of any parameter that would play the role of h, we must identify this function with the value that the function (9-4) becomes for a definite value of the parameter h; for example, 1. We must write the function (9-4) by means of the $(r+1)$ coordinates q^α. From the looks of (10-6), we limit ourselves to denominators with a linear form $\gamma_{\dot 0\overline{r+1}}\dot q^{r+1}$ that consists of only one term. We will then get:

$$L(q^\alpha, \dot q^\alpha, 1) = \frac{\gamma_{\alpha\beta}\dot q^\alpha \dot q^\beta + (\gamma_{\dot 0\overline{r+1}}\dot q^{r+1})^2}{2\gamma_{\dot 0\overline{r+1}}\dot q^{r+1}}.$$

When we identify this expression with (10-6), we will get:

$$a_{ij} = \frac{\gamma_{ij}}{\gamma_{0\overline{r+1}}}, \qquad b_i = \frac{\gamma_{i\overline{r+1}}}{\gamma_{0\overline{r+1}}},$$

$$2(T_0 + U) = \frac{\gamma_{\overline{r+1}\,\overline{r+1}} + (\gamma_{0\overline{r+1}})^2}{\gamma_{0\overline{r+1}}}.$$

One may choose the function of the q^α, $\gamma_{0\overline{r+1}} = \psi$, of the q^α arbitrarily. One will then have:

$$\gamma_{ij} = \psi\, a_{ij}, \qquad \gamma_{\overline{i\,r+1}} = \psi\, b_i, \quad \gamma_{\overline{r+1}\,\overline{r+1}} = 2\psi\,(T_0 + U) - \psi,$$
$$\gamma_{00} = \gamma_{0i} = 0, \quad \gamma_{0\,\overline{r+1}} = \psi.$$

One will deduce from this that for V_{r+1} the ascent envisioned leads to the following ds^2:

(11-1) $$\begin{cases} ds^2 = \mathcal{L}^2(q^\alpha, dq^\alpha, dq^0) \\ \quad = \psi\{a_{ij}\, dq^i dq^j + 2b_i\, dq^i dq^{r+1} + [2(T_0 + U) - \psi](dq^{r+1})^2 + 2\, dq^0 dq^{r+1}\}. \end{cases}$$

This ds^2 generalizes a ds^2 that was obtained by a direct study by Eisenhart ([1]). One will obtain Eisenhart's ds^2 by starting with ψ = const. = k^2, which will lead to:

(11-2) $$\begin{cases} ds^2 = \mathcal{L}^2(q^\alpha, dq^\alpha, dq^0) \\ \quad = k^2\{a_{ij}\, dq^i dq^j + 2b_i\, dq^i dq^{r+1} + 2(T_0 + U_1)(dq^{r+1})^2 + 2\, dq^0 dq^{r+1}\}, \end{cases}$$

in which one has set:

(11-3) $$U_1 = U - \frac{k^2}{2}.$$

The geodesics of (11-2) coincide with those of:

(11-4) $$ds_E^2 = a_{ij} dq^i dq^j + 2b_i dq^i dq^{r+1} + 2(T_0 + U_1)(dq^{r+1})^2 + 2dq^0 dq^{r+1},$$

which is Eisenhart's ds^2. One immediately verifies that the determinant of the quadratic form that appears in the right-hand side of (11-4) is nothing but $a \neq 0$; as a result, this ds^2 will be non-degenerate. It is not obviously positive-definite.

We propose to evaluate the variation of the parameter q^0 along a trajectory of the motion in the configuration spacetime. First of all, the equation $\partial_{\dot 0}\mathcal{L} = h = 1$ may be written:

$$\tfrac{1}{2}\partial_{\dot 0}\mathcal{L}^2 = \mathcal{L}$$

here.

We suppose that the trajectory envisioned is considered to be the projection of a geodesic along which the ds^2 of (11-2) is non- zero. One will explicitly obtain:

$$k^2\, dq^{r+1} = ds,$$

namely:

(11-5) $$ds^2 = k^4\, dt^2.$$

Along the geodesic envisioned, the ds^2 that is defined by (11-2) is certainly positive. The expression for ds^2 that is given by (11-2) may be recast in the form:

([1]) EISENHART, Ann. of Math., **30** (1939), 591.

(11-6) $$\frac{ds^2}{dt^2} = 2k^2\left(\Lambda_1 + \frac{dq^0}{dt}\right)$$

in which one has set:

$$\Lambda_1 = T + U = L - \frac{k^2}{2}.$$

One infers from (11-5) and (11-6) that:

$$\frac{dq^0}{dt} = \frac{k^2}{2} - \Lambda_1 = k^2 - \Lambda .$$

One deduces from this that:

(11-7) $$q^0 = k^2 t + C - \int_0^t \Lambda dt .$$

q^0 is therefore related to the Hamiltonian action directly. Suppose we choose the constants k^2 and C; one may evaluate the function $q^0(t)$ along any trajectory of the motion in the configuration manifold by means of (11-7). The expressions for $q^0(t)$ and $q^i(t)$ that correspond to the motion will provide a parametric representation of a geodesic of (11-2) (or (11-4, for that matter), along which the ds^2 considered is positive, as a function of $t = q^{r+1}$. We state:

THEOREM – *The trajectories of a conservative holonomic dynamical system with bilateral perfect constraints on a configuration spacetime E_{r+1} may be obtained as follows: if one is given two constants k^2 and C then consider the topological product $V_{r+2} = E_{r+1} \times \mathbb{R}$, which is endowed with Eisenhart's Riemannian metric* (11-4), *in which q^0 is the abscissa of a point of $\mathbb{R}$.* *These trajectories will be the projections onto E_{r+1} of the geodesics of that Riemannian manifold along which ds_E^2 is positive and satisfies:*

$$ds_E^2 = k^2 dt^2 .$$

Conversely, if one associates each point of such a trajectory with that point of V_{r+2} *that projects onto it and is defined by* (11-7) *then that point will describe a geodesic that satisfies the preceding conditions.*

This theorem permits us to recover the Hamilton-Jacobi theory as a simple consequence of some well-known facts of Riemannian geometry. It also permits us to treat questions of stability in the Riemannian context by means of the technique of the "geodesic chart."

12. – The descent from Hamilton's principle to de Maupertuis's principle. – Suppose that the Finslerian manifold E_{r+1} admits a one-parameter group of isometries (in the sense of sec. **5**) such that for an adapted coordinate system, the manifolds that are defined by the homeomorphism b) of that section may be defined to be the manifolds t = constant.

Let E_r be the quotient manifold. In the sequel, we shall call it the *configuration space* of the system. Let (q^i) (i, any Latin index = 1, 2, …, r) be a system of local coordinates on E_r . From the hypotheses made, the set (q^i, t) will define a system of adapted coordinates on E_{r+1}; because of the restrictions on time, the adapted systems of coordinates that answer the question are defined up to a change of coordinates here by:

$$q'^i = \psi'^i(q^j)\,, \qquad t' = t + \mu\,,$$

in which the ψ are arbitrary functions and μ is a constant. In such a system of adapted coordinates, one will have the isometry property:

(12-1) $$\frac{\partial L}{\partial t} = 0\,.$$

Now, upon differentiating (10-5), one will get:

$$\frac{\partial L}{\partial t} = \dot{q}^{r+1}\frac{\partial \Lambda}{\partial t}(q^i, t, q'^i)\,.$$

One will deduce from this that (12-1) is equivalent to:

$$\frac{\partial \Lambda}{\partial t} = 0\,.$$

The Lagrangian Λ does not depend on time explicitly. We say that we are working under *the Painlevé hypothesis.*

Under these hypotheses, it is possible to apply the descent procedure with respect to the variable $t = q^{r+1}$ to the function Λ that is defined by (10-6). One obtains a first integral:

(12-2) $$\frac{\partial L}{\partial \dot{q}^{r+1}} = -\frac{a_{ij}\dot{q}^i\dot{q}^j}{2(\dot{q}^{r+1})^2} + (T_0 + U) = h\,,$$

which is nothing but the well-known first integral of energy:

$$H \equiv T_2 - T_0 = -h = E.$$

One infers from this that:

(12-3) $$(\dot{q}^{r+1})^2 = \frac{a_{ij}\,\dot{q}^i\,\dot{q}^j}{2(T_0 U - h)}.$$

Since the form $a_{ij}\,\dot q^i\,\dot q^j$ is positive-definite, one must restrict oneself to values of h for which the denominator is positive. One must therefore eliminate $\dot q^{r+1}$ between (12-3) and the relation:

$$L_1 = L - h\,\dot q^{r+1},$$

with:

$$L = \frac{a_{ij}\,\dot q^i\,\dot q^j}{2\dot q^{r+1}} + b_i\,\dot q^i + (T_0 + U)\,\dot q^{r+1}.$$

From (12-3), this gives:

$$L_1 = (T_0 + U - h)\,\dot q^{r+1} + b_i\,\dot q^i + (T_0 + U - h)\,\dot q^{r+1} = 2(T_0 + U - h)\,\dot q^{r+1} + b_i\,\dot q^i .$$

Hence:

(12-4) $$L_1 = \sqrt{(T_0 + U - h)\,a_{ij}\,\dot q^i\,\dot q^j} + b_i\,\dot q^i .$$

We state:

THE PRINCIPLE OF De MAUPERTUIS – *In the case where the dynamical system envisioned satisfies the Painlevé hypothesis, the trajectories of the motion in the configuration manifold that correspond to the total energy E are projected onto the configuration manifold E_r along extremals of the Maupertuisian action integral:*

$$\int_{z_0}^{z_1} [\sqrt{2(T_0 + U + E)\,a_{ij}\,dq^i dq^j} + b_i\,dq^i\,],$$

in which z_0 and z_1 are two points of E_r. Conversely, if one associates each point of such an extremal of E_r with the point of E_{r+1} that projects onto it and moves according to the law (12-3) ($h = - E$) *then that point will describe a motion in the configuration spacetime that corresponds to the energy.*

In the case where $b_i = 0$, the Mauperuisian action given on E_r gives a Riemannian metric for each value of the energy E.

13. – The ascent from the principle of de Maupertuis to a Riemannian ds^2. – The function L_1 that defines the Maupertuisian action presents itself as not only a function of the $\dot q^i$, but also as the sum of the square roots of a quadratic form and a linear form. For a fixed value of E, it is therefore possible to interpret the trajectories of our dynamical system in the configuration manifold as the projections onto E_r of geodesics of an $(r + 1)$-dimensional Riemannian manifold V_{r+1} for which $\gamma_{00} \neq 0$ in adapted coordinates. It is not E that will play the role of this parameter, because that would only lead back to Hamilton's principle. Since E has a fixed value, we will identify L_1 with the function that (8-6) reduces to for a definite value of the constant h – for example, 1.

Therefore, consider the function:

(13-1) $$L_1(q^i, \dot{q}^i, 1) = \sqrt{\alpha_{ij}\, \dot{q}^i\, \dot{q}^j} + b_i\, \dot{q}^i\,,$$

in which:

(13-2) $$\alpha_{ij} = 2\,(T_0 + U + E)\, a_{ij}\,.$$

On the other hand, from (8-6):

$$L_1(q^i, \dot{q}^i, h) = \sqrt{\left(1 - \frac{h^2}{\gamma_{00}}\right) g_{ij}\, \dot{q}^i \dot{q}^j} + h\frac{\gamma_{0i}\dot{q}^i}{\gamma_{00}}\,.$$

By identification, one obtains:

$$\alpha_{ij} = \left(1 - \frac{h^2}{\gamma_{00}}\right) g_{ij} \qquad b_i = \frac{\gamma_{0i}}{\gamma_{00}}\,.$$

It is therefore possible to choose the function $\gamma_{00} \neq 0$ arbitrarily, and one will have, conversely, that:

$$\gamma_{0i} = \gamma_{00}\, b_i, \qquad g_{ij} = \frac{\gamma_{00}}{\gamma_{00} - 1}\, \alpha_{ij}\,,$$

and as a result:

$$\gamma_{ij} = \frac{\gamma_{00}}{\gamma_{00} - 1}\, \gamma_{00}\alpha_{ij} + b_i b_j.$$

Upon substituting the values (13-2) for α_{ij}, one will then obtains the Riemannian metric:

$$ds^2 = \gamma_{00}\left\{\left[\frac{2(T_0\, U + E)}{\gamma_{00} - 1}\, a_{ij} + b_i\, b_j\right] dq^i dq^j + 2b_i\; dq^i dq^0 + (dq^0)^2\right\}$$

by ascent, namely:

$$ds^2 = \gamma_{00}\left\{\frac{2\,(T_0 U + E)}{\gamma_{00} - 1}\, a_{ij}\; dq^i dq^j + (dq^0 + b_i\, dq^i)^2\right\}.$$

In order to have a ds^2 that is as simple as possible, we may take $\gamma_{00} = 2$. We will thus get:

(13-3) $$ds^2 = 2\,[2\,(T_0 + U + E)\, a_{ij}\, dq^i dq^j + (dq^0 + b_i\, dq^i)^2\,] = \mathcal{L}^2\,(q^i, dq^i, dq^0),$$

whose geodesics will coincide with those of:

(13-4) $$ds^2 = 2\,(T_0 + U + E)\, a_{ij}\, dq^i dq^j + (dq^0 + b_i\, dq^i)^2\,.$$

We propose to evaluate the variation of the parameter q^0 along a trajectory in the configuration manifold. From (13-3), the equation $\partial_{\dot{0}}\mathcal{L} = h = 1$ may always be written:

(13-5) $$2\,(dq^0 + b_i\, dq^i) = ds.$$

Now, again from (13-3):

$$\frac{ds^2}{dt^2} = 4(T_0 + U + E)\,2T_2 + 2\frac{(dq^0 + b_i dq^i)^2}{dt^2}.$$

One deduces from this relation and (13-5) (with the classical mechanical notations) that:

$$\frac{(dq^0 + b_i dq^i)^2}{dt^2} = 2(T_0 + U + E)\cdot 2T_2.$$

Now, from the Painlevé integral:

$$2T_2 = 2(T_0 + U + E).$$

We therefore obtain:

$$\frac{dq^0}{dt} + T_1 = 2(T_0 + U + E) = (T_2 + T_0 + U + E) = \Lambda - T_1 + E_1,$$

namely:

$$\frac{dq^0}{dt} = \Lambda - b_i\frac{dq^i}{dt} + E_1.$$

One deduces from this by integration that:

(13-6) $$q^0 = \int_0^t \Lambda dt - \int_0^t b_i dq^i + Et + C.$$

We suppose that the energy constant E and the constant C have been chosen; one may evaluate q^0(t) by means of (13-6) along any trajectory in the configuration space that corresponds a motion with energy E. The expressions for $q^0(t)$ and $q^i(t)$ that correspond to the motion provide a parametric representation of a geodesic of (13-3) (or (13-4), for that matter). We state, in a form that relates to (13-4):

THEOREM – *Under the Painlevé hypothesis, the trajectories of a dynamical system in a configuration space E_r may be obtained as follows: If the motion that corresponds to an energy E and a constant C has been chosen then consider the topological product V_{r+1} = $E_r \times \mathbb{R}$ endowed with the Riemannian metric* (13-4), *in which q^0 is the abscissa of a point of $\mathbb{R}$. These trajectories are the projections onto E_r of the geodesics of the Riemannian manifold along which:*

$$dq^0 + b_i dq^i = \frac{\sqrt{2}}{2}ds.$$

Conversely, if one associates each point of such a trajectory that corresponds to the energy E with the point of V_{r+1} that projects onto it and is defined by (13-6) *then that point will describe a geodesic that satisfies the preceding conditions.*

Despite the extent of the geometric research into dynamics, the ds^2 in (13-4) has not been brought to our attention. It may be put to use in the theory of dynamical systems that satisfy the Painlevé hypothesis.

III. – APPLICATION TO THE RELATIVISTIC TRAJECTORIES OF CHARGED PARTICLES.

THE PRIMARY POSTULATES OF THE UNITARY THEORY.

14. – The ds^2 of Kaluza-Klein. – We leave behind the examples that are implied by classical dynamics and return to the differential system of the trajectories of charged particles in spacetime V_4 . We have seen that there exists a global vector-potential φ whose trajectories may be defined to be the time-oriented extremals of the integral that is associated with the function:

(14-1) $$f(x^i, \dot{x}^j) = (g_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}} + k\,\varphi_i\,\dot{x}^i \qquad (i, j, \text{ any Latin index} = 1,2, 3, 4),$$

in which k denotes the charge-to-mass ratio:

$$\frac{e}{m}$$

of the particle.

As a function of $\dot{x}^i$, f takes the form of the sum of the square root of a quadratic form and a linear form. It is therefore possible to interpret the trajectories of charged particles in V_4 as the projections of geodesics of a five-dimensional Riemannian manifold V_5 for which γ_{00} is $\neq 0$ in adapted coordinates. When we started the descent process with the function:

(14-2) $$\mathcal{L}^2 = \gamma_{\alpha\beta}\dot{x}^\alpha\ddot{x}^\beta \qquad (\alpha, \beta, \text{ any Greek index} = 0, 1, 2, 3),$$

we saw that the geodesics that satisfy the first integral that were defined by:

$$\tfrac{1}{2}\partial_{\dot{0}}\mathcal{L}^2 = h\mathcal{L}$$

project onto the extremals of:

(14-3) $$L(x^i, \dot{x}^j, h) = \sqrt{\left(1 - \frac{h^2}{\gamma_{00}}\right) g_{ij}\dot{x}^i\dot{x}^j} + h\frac{\gamma_{0i}}{\gamma_{00}}\dot{x}^i,$$

in which:

$$g_{ij} = \gamma_{ij} - \frac{\gamma_{i0}\gamma_{0j}}{\gamma_{00}}.$$

If we compare formulas (14-1) and (14-2) then the form quadratic form $g_{ij}\dot{x}^i\dot{x}^j$ will be multiplied by a factor:

$$\left(1 - \frac{h^2}{\gamma_{00}}\right),$$

which depends upon both the parameter h and the x^i, in general, by the intermediary of γ_{00}. The same thing is not true for f. We may avoid this difficulty by choosing:

(14-4) γ_{00} = constant (the Klein hypothesis),

which is a constant that we will, moreover, be led to assume to be *negative* in what follows. We then couple k and h by the relation:

(14-5)
$$k = \frac{\beta h}{\sqrt{1 - \dfrac{h^2}{\gamma_{00}}}},$$

in which β is a constant that we will allow to modify the numerical value of γ_{00} if the need arises. If that is true then we will see that the extremals of f are also the extremals of:

(14-6)
$$\sqrt{1 - \frac{h^2}{\gamma_{00}}}\, f(x^i, \dot{x}^j) = \sqrt{\left(1 - \frac{h^2}{\gamma_{00}}\right) g_{ij}\dot{x}^i\dot{x}^j} + h\beta\varphi_i\dot{x}^i.$$

If we identify the function (14-6) with $L(x^i, \dot{x}^j, h)$ then we will see that the symbol g_{ij} denotes the same quantities, and that $\gamma_{0i} = \beta\,\gamma_{00}\,\varphi_i$. We will therefore have:

$$\gamma_{00} = \text{const.} < 0 \qquad \gamma_{0i} = \beta\,\gamma_{00}\,\varphi_i, \qquad \gamma_{ij} = g_{ij} + \beta^2\,\gamma_{00}\,\varphi_i\,\varphi_j,$$

and the Riemannian metric that we envision is written:

(14-7)
$$d\sigma^2 = \mathcal{L}^2(x^i, dx^i, dx^0) = g_{ij}\,dx^i\,dx^j + \gamma_{00}\,(dx^0 + \beta\,\varphi_i\,dx^i)^2.$$

The quadratic form in the last expression is reducible to an algebraic sum of 5 squares, one of which – viz., the one that corresponds to the index, 4 – is positive, and the others of which are negative; the metric $d\sigma^2$ will therefore be defined by a non-degenerate form of the hyperbolic normal type.

We propose to evaluate the variation of x^0 along the trajectory of a charged particle in V_4. The necessary calculations for a geodesic of $\mathcal{L}^2$ that corresponds to the value h have been done before *and in the general case where γ_{00} may vary, moreover.* From (8-5), one has:

(14-8)
$$\mathcal{L}^2\left(1-\frac{h^2}{\gamma_{00}}\right)=g_{ij}\dot{x}^i\dot{x}^j\,,$$

and, from (8-6):

$$\dot{x}^0=\frac{h}{\gamma_{00}}\mathcal{L}-\frac{\gamma_{0i}}{\gamma_{00}}\dot{x}^i=\frac{h}{\sqrt{1-\dfrac{h^2}{\gamma_{00}}}}\frac{1}{\gamma_{00}}\sqrt{g_{ij}\dot{x}^i\dot{x}^j}-\beta\varphi_i\dot{x}^i\,,$$

and if we introduce k by way of (14-5) then we will get:

(14-9)
$$dx^0=\frac{1}{\beta\gamma_{00}}k\,ds-\beta\,\varphi\,.$$

One deduces by integration that:

(14-10)
$$x^0=\frac{1}{\beta}\int_{u_0}^{u}\frac{k}{\gamma_{00}}ds-\beta\int_{u_0}^{u}\varphi+C\,,$$

in which u designates an arbitrary parameter. Suppose that we choose the constant C; one evaluates the function $x^0(u)$ all along the trajectory of the charged particle in V_4 by means of (14-10). The expressions for $x^0(u)$ and $x^i(u)$ that correspond to that movement provide a parametric representation for a geodesic of (14-7); since γ_{00} is negative, it results from (14-8) that if the trajectory envisioned is oriented in time and $\mathcal{L}^2$ is essentially positive along the geodesic envisioned then $d\sigma^2$ will be positive ([1]). We state:

THEOREM – *The trajectories of a charged particle in the spacetime V_4 of general relativity may be obtained as the follows: Given a constant C consider a manifold V_5 that is homeomorphic to a topological product $V_4\times\mathbb{R}$ and endowed with the Riemannian metric* (14-7), *and in which γ_{00} is a constant (which we assume to be negative), and x^0 is the abscissa of a point of $\mathbb{R}$. The trajectories considered are the projections of geodesics of this Riemannian manifold along which $d\sigma^2$ is positive onto V_4, and:*

([1]) The preceding argument shows – along the way – a result that we did not insist upon in I: If an extremal of (14-1) is time-oriented ($ds^2>0$) at a point in V_4 then the same thing will be true all along this extremal. Indeed, for constants $\gamma_{00}<0$ and C, the extremal will be the projection of a well-defined geodesic in V_5. From (14-8), if the extremal is time-oriented at a point then $\mathcal{L}^2$ will be positive at that point, and the geodesic in V_5 will give a positive value to ds^2 at the corresponding point; hence, at every point. As a result, from (14-7), $ds^2>0$ at every point of the extremal.

$$dx^0 = \frac{1}{\beta\gamma_{00}} k\, ds - \beta\,\varphi .$$

Conversely, if one associates each point of the trajectory of a charged particle with ratio e/m = k with the point of V_5 *that projects onto it and is defined by* (14-10) *then that point will describes a geodesic of* V_5 *that satisfies the preceding conditions.*

15. – The postulates of a unitary theory. – 1. The preceding reasoning and results lead us to introduce a five-dimensional Riemannian manifold V_5 that is endowed with a metric of hyperbolic normal type and to suppose that this manifold admits a connected, one-parameter group of isometries (in the sense of sec. 5) whose trajectories are oriented in such a way that $d\sigma^2$ is negative along any one of them. These hypotheses translate into what is called the *cylindricality hypothesis.* The spacetime V_4 must be identified with the quotient manifold of V_5 by the equivalence relation that the group defines.

Conforming to the paper of Oskar Klein, in the course of sec. **14** we supposed, moreover, that:

$$\gamma_{00} = \text{constant.}$$

What is the intrinsic significance of this hypothesis as far as the isometry group is concerned? One immediately sees that it expresses the idea that the *trajectories of the isometry group are the geodesics of* V_5. Indeed, in order for the differential system of the geodesics of V_5, which may be written:

$$\frac{d^2x^\alpha}{d\sigma^2} + \Gamma^\alpha_{\lambda\mu}\frac{dx^\lambda}{d\sigma}\frac{dx^\mu}{d\sigma} = 0,$$

to admit the solution x^i = const. it is necessary and sufficient that $\Gamma^\alpha_{00} = 0$ (since $\dfrac{dx^0}{d\sigma}$ is constant along these lines). Now:

$$\Gamma^\alpha_{00} = \gamma^{\alpha\beta}[00,\beta] = -\tfrac{1}{2}\gamma^{\alpha\beta}\partial_\beta\gamma_{00} .$$

One deduces from this that it is necessary and sufficient that:

$$\partial_\beta\gamma_{00} = 0\,;$$

i.e., that γ_{00} = const. on V_5.

One sees that the two hypotheses we made – viz., the cylindricality postulate and the hypothesis that γ_{00} = const. – are completely geometrically distinct. The first hypothesis was suggested to us by the form itself of the problem of the calculus of variations that we solved and the need to obtain a geometric structure that is independent of the factor $k = e/m$. The second one, which was introduced in sec **14**, merely constitutes the simplest means of remaining in plain accord with the provisional theory for the longest possible time.

2. In this provisional theory, in which we assume the existence of a global electromagnetic vector-potential φ_i, the potentials of the fields are 14 in number, namely, 10 components for the tensor potential of gravitation g_{ij} and 4 components for the electromagnetic vector-potential φ_i. On the other hand, there exist 14 field equations for these potentials, 10 of which are provided by the Einstein equations and 4 more that come from the Maxwell-Lorentz equations. The left-hand sides of these equations are, moreover, coupled by "conditions" – or "conservation identities" – which we have studied in detail ([1]) and which number five, here.

In a unitary theory that is based on a penta-dimensional manifold, we are tempted to introduce the natural extension of the Einstein equations to V_5 as the field equations; these equations involve the symmetric tensor $S_{\alpha\beta}$, and we must compare them to the equations of the provisional theory. However, it is convenient to observe that we thus obtain 15 field equations, which is the number of independent components of a symmetric tensor on V_5, and not exactly 14, since the left-hand sides of these equations will be, moreover, coupled by 5 conservation identities that are provided by the Bianchi identities. There is therefore a difference of unity between the numbers of field equations. We are therefore led to abandon the hypothesis that γ_{00} = constant and introduce 15 potentials for our unitary field – viz., the 15 components of the tensor $\gamma_{\alpha\beta}$ – and to compare the equations that they suggest at an instant with equations of the provisional theory of electromagnetism. Of course, we must study the physical interpretation of the supplementary potential that is introduced and see if such an interpretation is compatible with experiment.

A difficulty presents itself when we reject the hypothesis that γ_{00} = constant, and it is a difficulty that will not be completely resolved in what follows. It concerns the problem of specifying the relativistic trajectories of charged particles that have served as the basis for our study up till now.

If the manifold V_5 that satisfies the cylindricality postulate is given, and V_4 is identified with the quotient manifold, then we will be tempted to obtain the trajectories of charged particles by proceeding in the following manner: *We consider the geodesics of V_5 that give $d\sigma^2$ a positive value and satisfy the first integral that translates into the relation:*

$$\tfrac{1}{2}\partial_{\dot{0}}\mathcal{L}^2 = h\mathcal{L},$$

in which h is a definite constant. The projection of such a geodesic onto V_4 must define the spatio-temporal trajectory of a charged particle. In what follows, we shall prove that this is essentially true as a consequence of the field equations and the matching conditions that we adopted.

Such a projection is an extremal in V_4 of the integral that is associated with the function:

$$L(x^i, \dot{x}^j, h) = \sqrt{\left(1 - \frac{h^2}{\gamma_{00}}\right) g_{ij}\dot{x}^i\dot{x}^j} + h\frac{\gamma_{0i}}{\gamma_{00}}\dot{x}^i = \sqrt{\left(1 - \frac{h^2}{\gamma_{00}}\right) g_{ij}\dot{x}^i\dot{x}^j} + h\beta\,\varphi_i\dot{x}^i,$$

so if we set:

([1]) See I, sec. **13**, **14**, **15**, and **21**, **22**, **23**, for example.

$$k = \frac{\beta h}{\sqrt{1 - \dfrac{h^2}{\gamma_{00}}}}$$

then it will be an extremal of the integral that is associated with the function:

(15-2) $$\frac{1}{k}\sqrt{g_{ij}\dot{x}^i\dot{x}^j} + \varphi_i \dot{x}^i .$$

One notes that h is constant along the trajectory envisioned in (15-1), but that γ_{00} may vary. Therefore, (15-2) is not rigorously equivalent to f from the standpoint of extremals in the general case. In fact, we confirm that γ_{00} varies very slightly; under these conditions, if the ratio $k = e / m$ must be considered to be variable for a specific particle then we will confirm that this theoretical variation is very small in practice and inaccessible to experiment or observation.

If the trajectories of a charged particle are thus defined then it will result from the manner by which we carried out the calculations that the significance of the coordinate x^0 is always given by the formula:

$$x^0 = \frac{1}{\beta}\int_{u_0}^{u} \frac{k}{\gamma_{00}} ds - \int_{u_0}^{u} \varphi + C ,$$

in which k / γ_{00} must be considered to be variable.

The theory thus-described will be satisfied to the extent that the variations that were introduced are sufficiently small for a generic point, and our equations approach those of the provisional theory for very weak variations of γ_{00} . Finally, we confirm that if one starts with the field equations then one may establish the "principle" itself for the geodesics of charged particles that we just stated; at any point, the situation will therefore be analogous that of general relativity in the purely gravitational case.

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CHAPTER II

THE FIELD EQUATIONS OF THE JORDAN-THIRY THEORY

I. – THE RIEMANNIAN MANIFOLD V_5 AND SPACETIME

16. – The Riemannian manifold V_5 . – In this chapter, we propose to specify the general principles of the Jordan-Thiry theory, as they are suggested by the variational problems that relate to the motion of charged particles that was studied in detail in the first chapter.

a) The primitive element in the Jordan-Thiry theory is defined by a five-dimensional differentiable manifold that satisfies the same differentiability hypotheses as the spacetime manifold of general relativity ([1]): In the intersection of the domains of two admissible coordinate systems, the coordinates of a point x of V_5 in one of the systems must be 4-times differentiable functions with non-zero Jacobian of the coordinates of x in the other system, such that the first and second derivatives are continuous, and the third and fourth derivatives may present discontinuities of the Hadamard type.

We suppose that a Riemannian metric ds^2, which is *everywhere of hyperbolic normal type*, is defined on this manifold. The local expression for this metric in a system of admissible coordinates is:

(16-1) $$d\sigma^2 = \gamma_{\alpha\beta}(x^\lambda)\, dx^\alpha dx^\beta \qquad (\alpha, \beta, \text{ all Greek indices} = 0,1, 2, 3, 4).$$

The fundamental tensor $\gamma_{\alpha\beta}$ will determine the elementary unitary phenomenon, i.e., the motion of a charged material particle. Its components are called *potentials* for the system of coordinates envisioned. We suppose that this tensor admits components of class C^1 on V_5, and that the derivatives $\partial_\gamma \gamma_{\alpha\beta}$ are functions of class piecewise-C^2.

The hypotheses we made on the type of metric amounts to saying that $d\sigma^2$ may be put into the form:

(16-2) $$d\sigma^2 = (\omega^4)^2 - \sum_{A=0}^{3} (\omega^A)^2 \qquad (A, \text{ any Latin capital} = 0, 1, 2, 3)$$

at each point of V_5, in which the ω^α are a linearly independent system of local Pfaff forms. The frame (x, $\mathbf{e}_\alpha$) that is associated with the dual basis is called *orthonormal* in V_5.

One has:

(16-3) $$\mathbf{dx} = \omega^\alpha \mathbf{e}_\alpha ,$$

([1]) See I, sec. **1** and **2**.

and the scalar products of the frame vectors are such that:

(16-4) $$\mathbf{e}_\alpha \mathbf{e}_\beta = 0 \qquad \text{for } \alpha \neq \beta; \qquad \mathbf{e}_A^2 = -1; \qquad \mathbf{e}_4^2 = 1.$$

b) We suppose, moreover, that the Riemannian manifold V_5 admits a global, connected, one-parameter group of isometries of V_5 whose trajectories z are oriented in such a way that $d\sigma^2 < 0$ and that leaves no point of V_5 invariant, and the family of these trajectories satisfy the following hypotheses:

a) The trajectories are homeomorphic to a circle T_1 .

b) One may find a four-dimensional, differentiable manifold V_4, which satisfies the same differentiability hypotheses as V_5, such that there exists differentiable homeomorphism of class C^2 on the manifold V_5 onto the topological product $V_4 \times T^1$, in which the z applies to the circular factor. This homeomorphism is, moreover, supposed to be piecewise-continuous up to order 4.

We may naturally identify V_4 with the space whose points z are trajectories. We have called V_4 the *quotient manifold* of V_5 by the equivalence relation that the group of isometries defines.

We have seen that there consequently exist local coordinates in V_5 that are called *adapted* to the group and enjoy the following properties ([1]):

1. The (x^i) (i, any lowercase Latin index = 1, 2, 3, 4) are an arbitrary system of local coordinates on V_4. The manifolds x^0 = const. are globally-defined manifolds in V_5 and are homeomorphic to V_4 . The homeomorphism of *b*) may be assumed to apply to the manifolds x^0 = const. on the manifolds that are homeomorphic to V_4 in the product $V_4 \times T^1$.

2. Relative to the adapted coordinates, the potentials $\gamma_{\alpha\beta}$ are independent of the variable x^0. The vector $\boldsymbol{\xi}$, which is the infinitesimal generator of the isometry group, admits the contravariant components:

(16-5) $$\xi^i = 0, \qquad \xi^0 = 1.$$

The square of this vector is:

$$\boldsymbol{\xi}^2 = \gamma_{00} < 0.$$

We set:

(16-6) $$\xi = \sqrt{|\boldsymbol{\xi}^2|} > 0 \qquad (\gamma_{00} = -\xi^2).$$

3. These coordinates are defined, up to a change of coordinates, by:

(16-7) $$x'^i = \psi'^i(x^j) \qquad x'^0 = x^0 + \psi(x^j),$$

([1]) See I, sec. **44** and **62**, II, sec. **5**.

in which ψ denotes the restriction to a local chart of an arbitrary function $\Psi(x)$ that is defined on V_4.

In all of what follows, we will introduce only *adapted* local coordinates. The manifolds x^0 = const. of such a system of coordinates are called the *sections* of V_5 that are associated with the system. They are preserved by the transformation:

$$x'^i = \psi'^i(x^j), \qquad x'^0 = x^0. \tag{16-8}$$

We call the change of adapted coordinates:

$$x'^i = x^j, \qquad x'^0 = x^0 + \psi(x^j) \tag{16-9}$$

a change of the system of section, or a *change of gauge.*

Let W_4 be a specific section of V_5. It is a manifold that is homeomorphic to V_4 for which the (x^i) define local coordinates. On each W_4, the metric of V_4, along with its group of isometries, defines tensors; it is therefore true that the γ_{ij} define a symmetric tensor, the γ_{0i}, a covariant vector, and γ_{00} or ξ, a scalar, since these quantities will transform according to the tensorial law under the transformation (16-8).

Among these tensors, certain ones (of which, the scalars γ_{00} or ξ are the simplest examples) must be applied to the same tensor of V_4 in all of the maps that are induced by the homeomorphisms of the sections of the different systems onto V_4. The image tensors are said to be *intrinsically-defined* on the quotient manifold V_4. In order for a tensor of W_4 to generate an intrinsically-defined tensor on V_4, it is necessary and sufficient that it must be invariant under a change of gauge (16-9).

17. – Spacetime V_4 and its sections. – We associate each point x of a neighborhood on V_5 with an orthonormal frame whose first vector $\mathbf{e}_0$ is a tangent vector at x to the trajectory $z(x)$ that passes through x and has square 1. Such a frame will be called *adapted.* Relative to the adapted frame, the metric is expressed with the aid of local Pfaff forms ω^0, ω^j, where the ω^j are zero along the trajectories:

$$d\sigma^2 = -(\omega^0)^2 + ds^2, \tag{17.1}$$

in which:

$$\omega^0 = -\frac{1}{\xi}(\gamma_{00}\,dx^0 + \gamma_{0i}\,dx^i) \tag{17-2}$$

and:

$$ds^2 = (\omega^4)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2 = \left(\gamma_{ij} - \frac{\gamma_{0j}\gamma_{0j}}{\gamma_{00}}\right) dx^i dx^j. \tag{17-3}$$

It results from this that the quadratic form ds^2 determines a Riemannian metric on V_4 that is of hyperbolic normal type. The quantities:

$$g_{ij} = \gamma_{ij} - \frac{\gamma_{0j}\gamma_{0j}}{\gamma_{00}} \tag{17-4}$$

are the components of an intrinsically-defined tensor on V_4. The associated contravariant tensor is $g_{ij} = \gamma_{ij}$.

In what follows, we will always assume that the quotient manifold V_4 and the section W_4 are endowed with the structure of a Riemannian manifold that is defined by ds^2 (17-3).

18. – The electromagnetic field tensor. – In V_5, we consider the vector ξ_λ and the vector φ_λ, which is collinear to it and is defined by:

$$\beta\,\varphi_\lambda = -\frac{\xi_\lambda}{\xi^2} = \frac{\gamma_{0\lambda}}{\gamma_{00}} \qquad (\beta\,\varphi_0 = 1), \tag{18-1}$$

in which β denotes a numerical constant whose value we shall ultimately fix. The φ_i define a covariant vector field on W_4, and, up to a change of gauge, these quantities transform according to the formula:

$$\beta\,\varphi_i = \beta\,\varphi_i' + \partial_i\psi\,. \tag{18-2}$$

The rotation $F_{\lambda\mu}$ of φ_λ is such that:

$$F_{0\lambda} = \partial_0\varphi_\lambda - \partial_\lambda\varphi_0 = 0, \tag{18-3}$$

and, from (18-2), the F_{ij} are invariant under change of gauge, so they intrinsically define a tensor on V_4. The vanishing of this tensor says that the trajectories $z(x)$ are orthogonal trajectories to the local sections.

In summation, we find an intrinsically-defined scalar ξ, a metric of hyperbolic normal type:

$$ds^2 = g_{ij}\,dx^i\,dx^j, \tag{18-4}$$

and an antisymmetric tensor F_{ij} on the manifold V_4. Other than these elements, a vector field φ_i is defined on each W_4 (which is canonically homeomorphic to V_4) such that one one has:

$$g_{ij} = \gamma_{ij} - \beta^2\,\gamma_{00}\,\varphi_i\,\varphi_i = \gamma_{ij} + \beta^2\,\xi^2\,\varphi_i\,\varphi_j\,, \qquad F_{ij} = \partial_i\varphi_j - \partial_j\varphi_i\,. \tag{18-5}$$

We are therefore led to identify the quotient manifold V_4 endowed with the metric (18-4) with *the spacetime of general relativity*, so the tensor g_{ij} becomes the *gravitational tensor*. The non-canonical reciprocal image on V_4 of the covariant vector field φ_i that is associated with a section of V_5 may be interpreted as the *electromagnetic vector-potential*; under a change of the system of sections, this image is found to be transformed according to (18-2), which is none other than a change of gauge in the initial sense [1]. By abuse of language, we say that the vector φ_i is the vector-potential. The tensor F_{ij}, which is intrinsically-defined on V_4, must then be interpreted as the *electromagnetic field.*

II. – THE FIELD EQUATIONS IN V_5

19. – **The system of field equations.** – Having specified the geometrical context of our theory, the next step for us to make consists of choosing a system of "field equations" – i.e., a tensorial system of partial differential equations that refer to the potentials $\gamma_{\alpha\beta}$ – and to relate these potentials to the mass and charge distributions in spacetime.

The most natural idea consists of formally generalizing the Einstein equations of general relativity, and to set:

$$S_{\alpha\beta} = \Theta_{\alpha\beta}\,, \tag{19-1}$$

in which $S_{\alpha\beta}$ and $\Theta_{\alpha\beta}$ are two symmetric tensors. The tensor $\Theta_{\alpha\beta}$ must describe, at best, the state of the distribution of masses and charges at the points of V_5 (interior, unitary case), so that in the regions of V_5 that do not contain any such distribution, it must be identically zero (exterior, unitary case). Naturally, $\Theta_{\alpha\beta}$ is zero in the region that is envisioned when one finds oneself in the presence of a gravitational and electromagnetic field, but in the absence of masses and charges; that will define an essential difference with the energy-momentum tensor of classical general relativity, which is non-zero in the presence of an electromagnetic field. It is the pure electromagnetic field schema of general relativity that corresponds to the exterior, unitary case here. We shall return to the choice of this right-hand tensor $\Theta_{\alpha\beta}$ in a later part of this chapter.

As for the tensor $S_{\alpha\beta}$ that depends only upon the structure of the Riemannian manifold V_5, it is natural to restrict it with the same conditions as in general relativity.

1. *The components $S_{\alpha\beta}$ do not depend on their potentials and their derivatives of the first two orders, and are linear with respect to the second order derivatives.*

2. *The tensor $S_{\alpha\beta}$ satisfies the "conservation conditions or identities:"*

$$\gamma^{\alpha\beta} D_\alpha S_{\alpha\beta} = 0, \tag{19-2}$$

in which D_α denotes the covariant derivative operator for the Riemannian connection of V_5 (*we reserve the notation* ∇ *for the covariant derivative operator for the connection on the spacetime* V_4).

The existence of such conservation identities is, as we saw in detail apropos of general relativity, intimately related to the fact that the arbitrary local coordinate changes make it possible to restrict five of the conveniently-chosen potentials to take given local values, and that the system (19-1) must not be over-determined under these conditions.

[1] See I, sec. **20**.

Indeed, such conservation identities, which are related to the possibility of adopting arbitrary local coordinates, must exist in any field theory.

Cartan's theorem ([1]) shows that the preceding conditions imply that we must take $S_{\alpha\beta}$ to be a tensor of the form:

$$S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\gamma_{\alpha\beta}(R + k),$$

in which k is a constant that generalizes the cosmological constant. In the sequel, we shall set $k = 0$. Now, as we know, the equations in which k is a non-zero constant do not possess the gauge invariance that we introduced, due to the equations that we adopted ([2]). Henceforth, we set:

(19-3) $$S_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2}\gamma_{\alpha\beta}R,$$

and with these notations, the equations of the exterior, unitary case may be written:

(19-4) $$S_{\alpha\beta} = 0.$$

Like the homologous quantities in general relativity, the quantities $S_{\alpha\beta}$ in the left-hand side of the field equations can be constructed by starting with *simple variational processes*. Since such processes play a fundamental role in the Einstein-Schrödinger theory, and in order to permit easy comparisons, we shall, in the course of the following two sections, specify the variational considerations that provide one of the points of departure for the Einstein-Schrödinger theory.

20. Variations of the curvature tensor. – Suppose that the Riemannian metric $\gamma_{\alpha\beta}$ in a domain of V_5 is varied in the sense of the calculus of variations, and denote the variation of $\gamma_{\alpha\beta}$ by $\delta\gamma_{\alpha\beta}$; the $\delta\gamma_{\alpha\beta}$ are obviously the components of a symmetric tensor. This variation of the metric tensor results in a variation of the components $\Gamma^{\lambda}_{\mu\nu}$ of the Riemannian connection, which are variations that we shall denote by $\delta\Gamma^{\lambda}_{\mu\nu}$.

It is well known that under a change of local coordinates $x'^{\alpha} = x'^{\alpha}(x^{\beta})$ the components $\Gamma^{\lambda}_{\mu\nu}$ of a connection transform according to the formula:

(20-1) $$\Gamma^{\lambda}_{\mu\nu} = A^{\lambda}_{\alpha'}A^{\beta'}_{\mu}A^{\gamma'}_{\nu}\Gamma^{\alpha'}_{\beta'\gamma'} + A^{\lambda}_{\rho'}\partial_{\mu}A^{\rho'}_{\nu},$$

in which one has set:

$$A^{\lambda}_{\alpha'} = \frac{\partial x^{\lambda}}{\partial x^{\alpha'}}, \qquad A^{\beta}_{\mu} = \frac{\partial x^{\beta'}}{\partial x^{\mu}}.$$

One deduces from (20-1) that:

([1]) See I, sec. **3**.

([2]) Nevertheless, it is not completely without interest for us to study the structure of the equations with a non-zero constant k on W_4.

(20-2) $$\delta\Gamma^{\lambda}_{\mu\nu} = A^{\lambda}_{\alpha'} A^{\beta'}_{\mu} A^{\gamma'}_{\nu} \delta\Gamma^{\alpha'}_{\beta'\gamma'} ,$$

and therefore the $\delta\Gamma^{\lambda}_{\mu\nu}$ are components of tensor that is once contravariant and twice covariant. Moreover, one will note that the operator δ commutes with the ordinary partial derivative with respect to a local coordinate.

It is easy to obtain the corresponding variations of the curvature tensor and the Ricci tensor. One deduces from the explicit expression:

$$R^{\lambda}{}_{\alpha,\mu\beta} = \partial_{\mu}\Gamma^{\lambda}_{\alpha\beta} - \partial_{\beta}\Gamma^{\lambda}_{\alpha\mu} + \Gamma^{\lambda}_{\rho\mu}\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\lambda}_{\rho\beta}\Gamma^{\rho}_{\alpha\mu} ,$$

by variation, that:

$$\delta R^{\lambda}{}_{\alpha,\mu\beta} = \partial_{\mu}\delta\Gamma^{\lambda}_{\alpha\beta} - \partial_{\beta}\delta\Gamma^{\lambda}_{\alpha\mu} + \Gamma^{\lambda}_{\rho\mu}\delta\Gamma^{\rho}_{\alpha\beta} + \Gamma^{\rho}_{\alpha\beta}\delta\Gamma^{\lambda}_{\rho\mu} - \Gamma^{\lambda}_{\rho\beta}\delta\Gamma^{\rho}_{\alpha\mu} - \Gamma^{\rho}_{\alpha\mu}\delta\Gamma^{\lambda}_{\rho\beta} .$$

We calculate the covariant derivatives of the tensor $\delta\Gamma^{\lambda}_{\mu\nu}$. They are:

$$D_{\mu}\delta\Gamma^{\lambda}_{\alpha\beta} = \partial_{\mu}\delta\Gamma^{\lambda}_{\alpha\beta} + \Gamma^{\lambda}_{\rho\mu}\delta\Gamma^{\rho}_{\alpha\beta} - \Gamma^{\lambda}_{\rho\beta}\delta\Gamma^{\rho}_{\alpha\mu} - \Gamma^{\rho}_{\beta\mu}\delta\Gamma^{\lambda}_{\alpha\rho} ,$$

and similarly:

$$D_{\beta}\delta\Gamma^{\lambda}_{\alpha\mu} = \partial_{\beta}\delta\Gamma^{\lambda}_{\alpha\mu} + \Gamma^{\lambda}_{\rho\beta}\delta\Gamma^{\rho}_{\alpha\mu} - \Gamma^{\rho}_{\alpha\beta}\delta\Gamma^{\lambda}_{\rho\mu} - \Gamma^{\rho}_{\beta\mu}\delta\Gamma^{\lambda}_{\alpha\rho} ,$$

if we take the symmetry of Γ in its lower indices into account. Upon subtracting term-by-term, we will get([1]):

(20-3) $$\delta R^{\lambda}{}_{\alpha,\mu\beta} = D_{\mu}\delta\Gamma^{\lambda}_{\alpha\beta} - D_{\beta}\delta\Gamma^{\lambda}_{\alpha\mu} .$$

Upon contracting over the indices λ and μ, one obtains the variation of the Ricci tensor:

(20-4) $$\delta R_{\alpha\beta} = D_{\rho}\delta\Gamma^{\rho}_{\alpha\beta} - D_{\beta}\delta\Gamma^{\lambda}_{\alpha\lambda} .$$

Multiplying by $\gamma^{\alpha\beta}$ leads to an interesting relation. Since $D_\rho\, \gamma^{\alpha\beta} = 0$, we first obtain:

$$\gamma^{\alpha\beta}\delta R_{\alpha\beta} = D_{\rho}(\gamma^{\alpha\beta}\delta\Gamma^{\rho}_{\alpha\beta}) - D_{\rho}(\gamma^{\rho\alpha}\delta\Gamma^{\lambda}_{\alpha\lambda}) .$$

We are then led to introduce the vector:

$$A^{\rho} = \gamma^{\alpha\beta}\delta\Gamma^{\rho}_{\alpha\beta} - \gamma^{\rho\alpha}\delta\Gamma^{\lambda}_{\alpha\lambda} .$$

We will thus obtain:

(20-5) $$\gamma^{\alpha\beta}\delta R_{\alpha\beta} = D_{\rho}A^{\rho} ,$$

and the scalar that appears in the left-hand side is expressed as the divergence of a vector.

([1]) The preceding calculations are much simpler in normal coordinates. However, we are ultimately led to make analogous calculations in the absence of normal coordinates.

21. – The variational principle. – Consider a five-dimensional, differentiable chain C in the manifold, and vary the metric in such a way that the variations of the potentials and their first covariant derivatives are zero on the boundary ∂C of the chain. As a result, the $\delta\Gamma^{\alpha}_{\beta\gamma}$ will also be zero on this boundary. We propose to study the corresponding variation of the integral:

$$I = \int_C \mathcal{L}\, dx^0 \wedge \cdots \wedge dx^4 , \tag{21-1}$$

in which $\mathcal{L}$ is the tensor density:

$$\mathcal{L} = R_{\alpha\beta}\, \gamma_{\alpha\beta} \sqrt{|\gamma|} . \tag{21-2}$$

The differential element is therefore a 5-form $R\, d\tau$ that is the product of the volume element $d\tau$ on V_5 with the scalar Riemannian curvature R. By variation, one obtains:

$$\delta I = \int_C \delta R_{\alpha\beta}\, \gamma^{\alpha\beta} \sqrt{|\gamma|}\, dx^0 \wedge \cdots \wedge dx^4 + \int_C R_{\alpha\beta}\, \delta(\gamma^{\alpha\beta} \sqrt{|\gamma|}\,)\, dx^0 \wedge \cdots \wedge dx^4 .$$

Now, from formula (20-5), one has:

$$\int_C \delta R_{\alpha\beta}\, \gamma^{\alpha\beta} \sqrt{|\gamma|}\, dx^0 \wedge \cdots \wedge dx^4 = \mathrm{flux}_{\partial C}\mathbf{A} = 0 ,$$

since $A^{\rho} = 0$ on the boundary of ∂C. On the other hand:

$$\frac{\delta\sqrt{|\gamma|}}{\sqrt{|\gamma|}} = \tfrac{1}{2}\gamma^{\lambda\mu}\, \delta\gamma_{\lambda\mu} = -\tfrac{1}{2}\gamma_{\lambda\mu}\, \delta\gamma^{\lambda\mu} .$$

One deduces from this that:

$$R_{\alpha\beta}\, \delta(\gamma^{\alpha\beta}\sqrt{|\gamma|}) = [R_{\alpha\beta}\, \delta\gamma^{\alpha\beta} - \tfrac{1}{2} R_{\alpha\beta}\, \gamma^{\alpha\beta}\, \gamma_{\lambda\mu}\, \delta\gamma^{\lambda\mu}]\sqrt{|\gamma|} ,$$

so, upon changing the names of the indices in the second term in the brackets, we will get:

$$R_{\alpha\beta}\, \delta(\gamma^{\alpha\beta}\sqrt{|\gamma|}) = (R_{\alpha\beta} - \tfrac{1}{2} R_{\alpha\beta})\, \delta\gamma^{\alpha\beta}\sqrt{|\gamma|} .$$

Therefore, the variation in question becomes:

$$\delta I = \int_C S_{\alpha\beta}\, \delta\gamma^{\alpha\beta} \sqrt{|\gamma|}\, dx^0 \wedge \cdots \wedge dx^4 . \tag{21-3}$$

On the other hand, from a well known formula concerning the differentiation of $\gamma^{\alpha\beta}$:

$$\delta\gamma^{\alpha\beta} = -\gamma^{\alpha\lambda}\gamma^{\beta\mu}\delta\gamma_{\lambda\mu}.$$

If we substitute this in (21-3), we obtain:

(21-4) $$\delta I = -\int_C S^{\lambda\mu}\,\delta\gamma_{\lambda\mu}\sqrt{|\gamma|}\,dx^0 \wedge \cdots \wedge dx^4.$$

If $\mathcal{L}$ is expressed with the aid of the $\gamma^{\alpha\beta}$ then the tensor $S_{\alpha\beta}$ will be related to the variational derivative of $\mathcal{L}$ for the variations envisioned by the relation:

(21-5) $$S_{\alpha\beta} = \frac{1}{\sqrt{|\gamma|}}\frac{\delta\mathcal{L}}{\delta\gamma^{\alpha\beta}}.$$

Similarly, if $\mathcal{L}$ is expressed with the aid of the $\gamma_{\alpha\beta}$ then:

(21-6) $$S^{\alpha\beta} = -\frac{1}{\sqrt{|\gamma|}}\frac{\delta\mathcal{L}}{\delta\gamma_{\alpha\beta}}.$$

In order for I to be an extremum for such variations, it is necessary and sufficient that $S_{\alpha\beta} = 0$. We may state:

THEOREM – *In the exterior, unitary case, the field equations may be characterized by the following variational principle: They define an extremum for the integral of the variations of the potentials over a five-dimensional, differentiable chain C, and their first derivatives will be zero on the boundary of C.*

III. – DEFINING THE EQUATIONS IN SPACETIME V_4

22. – Case of a V_{n+1} with a positive-definite metric. Passing from an orthonormal frame to a natural frame. – Apropos of the theory of stationary spacetimes in general relativity, we have been led to study a Riemannian manifold of arbitrary dimension V_{n+1} that admits a 1-parameter group of isometries and satisfies the same hypotheses as V_5. We have denoted the manifolds that are homologous to V_4 and W_5 by V_n and W_n. Since the calculations are purely local, they have been carried out while supposing that V_n is reduced to a neighborhood. Finally, in order for these calculations to be easily adaptable to the various hypotheses on the signature of the metric of V_{n+1} and the orientation of the trajectories of the isometry, they have been carried out while supposing that V_{n+1} is endowed with a *positive-definite* metric. We have therefore set:

$$d\sigma^2 = \sum_\alpha (\omega^\alpha)^2 = (\omega^0)^2 + \sum_i (\omega^i)^2 \qquad (\alpha = 0, 1, \ldots, n;\ i = 1, \ldots, n)$$

in which the ω^i are annulled along the trajectories, and perform the calculations in the adapted, orthonormal frame that corresponds to these (ω^α) ([1]).

$$d\mathbf{x} = dx^\alpha\,\mathbf{e}_\alpha = \omega^\alpha\,\mathbf{e}_\alpha .$$

Since the ω^i are local Pfaff forms with respect to the dx^i, one will have:

(22-1) $$\varphi^0 = \xi\,(dx^0 + \beta\,\varphi_i\,dx^i), \qquad \omega^j = A^i_j dx^j$$

upon substituting the quantities $\beta\varphi^i$ for the φ^i in I, sec. **65**.

We denote the inverse matrix to (A^i_j) by $(\bar{A}^i_j)$. One has:

(22-2) $$dx^i = \bar{A}^i_j \omega^i ,$$

and the passage from local coordinates to the orthonormal frames that are associated with the forms (ω^i) on V_n or W_n is performed with the aid of the matrices (A^i_j) and $(\bar{A}^i_j)$.

If φ^i represents the components of the vector-potential relative to the ω^i, then one has:

(22-3) $$dx^0 = \frac{\omega^0}{\xi} - \beta\,\varphi_i\,\omega^i .$$

Therefore, the passage from the natural frame to orthonormal frame in V_{n+1} is performed with the aid of the matrices (A^α_β), $(\bar{A}^\alpha_\beta)$ whose purely Latin part was just introduced and whose other elements are given by:

$$A^0_0 = \xi , \qquad A^0_i = \xi\beta\varphi_i , \qquad A^i_0 = 0 ,$$

and:

$$\bar{A}^0_0 = \frac{1}{\xi} , \qquad \bar{A}^0_i = -\beta\varphi_i , \qquad \bar{A}^i_0 = 0 .$$

23. – Components of the Ricci tensor and the Einstein tensor for V_{n+1} in an orthonormal frame. – Let $\hat{R}_{ij}$ be the Ricci tensor of the manifold V_n and let $\hat{S}_{ij}$ be its Einstein tensor. We denote the covariant derivative operator on V_n by ∇_i and the Laplacian of ξ in V_n by Δ. Under these conditions, if one substitutes βF_{ij} for F_{ij} in formulas I (69-5) then they will take the form:

([1]) See I, sec. **66** and following.

(23-1)
$$\begin{cases} R_{ij} = \hat{R}_{ij} - \dfrac{\beta^2 \xi^2}{2} F_i^{\ k} F_{jk} - \dfrac{1}{\xi} \nabla_j (\partial_i \xi), \\ R_{i0} = \dfrac{\beta}{2\xi^2} \nabla_j (\xi^3 F_i^{\ j}), \\ R_{00} = -\dfrac{1}{\xi} \Delta\xi + \dfrac{\beta^2 \xi^2}{2} F^2, \end{cases}$$

with:

(23-2)
$$F^2 = \frac{1}{2} F_{kl} F^{kl},$$

in which the different components that were introduced are defined relative to the orthonormal frame and the ^ sign is defined relative to V_n .

From formulas (23-1), one easily deduces the expression for the components of the Einstein tensor $S_{\alpha\beta}$ in an orthonormal frame with the aid of tensors that are intrinsically defined on V_n . One has:

$$\bar{S}_{\alpha\beta} = \bar{R}_{\alpha\beta} - \tfrac{1}{2} \delta_{\alpha\beta} R,$$

and, on the other hand:

$$R = \delta^{ij} \bar{R}_{ij} + \delta^{00} \bar{R}_{00} .$$

Therefore, from (23-1), one has:

(23-3)
$$R = \hat{R} - \frac{\beta^2 \xi^2}{2} F^2 - \frac{2}{\xi} \Delta\xi .$$

From this, one deduces the expression for the different components of $\bar{S}_{\alpha\beta}$:

(23-4)
$$\begin{cases} \bar{S}_{ij} = \hat{S}_{ij} + \dfrac{\beta^2 \xi^2}{2} \left[\tfrac{1}{2} g_{ij} F_{kl} F^{kl} - F_i^{\ k} F_{jk} \right] - \dfrac{1}{\xi} [\nabla_j (\partial_i \xi) - g_{ij} \Delta\xi], \\ \bar{S}_{i0} = \dfrac{\beta}{2\xi^2} \nabla_j (\xi^3 F_i^{\ j}), \\ \bar{S}_{00} = -\dfrac{1}{\xi} \hat{R} + \dfrac{3}{4} \beta^2 \xi^2 F^2. \end{cases}$$

We note that the left-hand sides of (23-1) and (23-4) involve only tensors that are defined intrinsically on V_n, and as a result, they themselves will define intrinsic tensors on V_n . We let P_{ij} denote the symmetric tensor field on V_n whose components in the orthonormal frame are $P_{ij} = S_{ij}$; similarly, we let Q_i designate the covariant vector field on V_n whose components are in the orthonormal frame are $Q_i = S_{i0}$. Finally, the left-hand side of the last equation in (23-4) is a scalar L that is defined on V_n , and in the orthonormal frame it will equal: $L = S_{00}$.

24. – Divergence formulas on a section W_n . – Suppose that V_{n+1} is referred to an adapted system of local coordinates, and let W_n be a section that is referred to this system. With the aid of formulas (23-1), it is possible to evaluate the components R_0^0 of the Ricci tensor in the natural frame that is associated with these local coordinates. In the difference of the tensors that appear in (23-1) and (23-4), the quantity R_0^0 presents a scalar character precisely on W_n, but not gauge invariance, and, as a result, it will not constitute an intrinsically-defined scalar on V_n .

One deduces from the classical tensorial transformation formulas that:

$$R_0^0 = \bar{A}_\alpha^0 \, A_0^\beta \, \bar{R}_\beta^\alpha ,$$

in which the A are given by the results of sec. **22**. Since $\bar{A}_0^i = 0$, $\bar{A}_0^0 = \xi$, we will get:

$$R_0^0 = \xi \, \bar{A}_\alpha^0 \, \bar{R}_0^\alpha ,$$

or, more specifically:

(24-1) $$R_0^0 = \bar{R}_0^0 - \xi\beta \, \varphi_i \, \bar{R}_0^i .$$

Since the metric on V_{n+1} is positive-definite, one will have $\bar{R}_0^0 = \bar{R}_{00}, \bar{R}_0^i = \bar{R}_{i0}$. It will then be possible to evaluate the left-hand side of (24-1) with the aid of formulas (23-1). A calculation that does not differ from the one in I, sec. **71** will give:

(24-2) $$R_0^0 = -\frac{1}{\xi} g^{ij} \nabla_j [\partial_i \xi + \frac{\beta^2 \xi^2}{2} \varphi_k F^k{}_i] .$$

We are therefore led to introduce the vector field $\boldsymbol{h}$ on W_n, which is defined by:

(24-3) $$h_i = \partial_i \xi + \frac{\beta^2 \xi^2}{2} \varphi_k F^k{}_l ,$$

and we will get:

(24-4) $$R_0^0 = -\frac{1}{\xi} \operatorname{div} \mathbf{h} ,$$

in which the divergence is evaluated on the Riemannian manifold W_n that is endowed with the metric ds^2.

It is possible to deduce another interesting divergence formula from this formula, which does not differ from the one in I, sec. **74**. One has:

$$R_{00} = A_0^0 A_0^0 \bar{R}_{00} = \xi^2 \bar{R}_{00} = \gamma_{00} \bar{R}_{00}$$

for the component R_{00} of the Ricci tensor relative to the natural frame.

Upon evaluating the difference:

$$\frac{R_{00}}{\gamma_{00}} - R_0^0 = \beta\varphi_i R_0^i ,$$

with the aid of the last formulas of (23-1) and (24-2), one obtains:

(24-5) $$\xi\,\varphi_i R_0^i - \frac{\beta\xi^2}{2} F^2 = -g^{ij}\,\nabla_j\left[\frac{\beta\xi^2}{2}\varphi_k F_i^{\;k}\right].$$

If we introduce the vector field $\boldsymbol{p}$ on W_n, with the components:

(24-6) $$p_i = \frac{\beta\xi^3}{2}\varphi_k F_i^{\;k} ,$$

then we will get the formula:

(24-7) $$\xi\,\varphi_i R_0^i - \frac{\beta\xi^2}{2} F^2 = -\operatorname{div}\mathbf{p} ,$$

in which the divergence is again evaluated on the Riemannian manifold W_n.

25. – Applications to the manifold V_5 with isometry trajectories that are oriented so that $d\sigma^2 < 0$. – Recall the manifold V_5, which admits the metric of the hyperbolic normal type:

(25-1) $$d\sigma^2 = -(\omega^0)^2 + [(\omega^0)^2 - (\omega^1)^2 - (\omega^2)^2 - (\omega^3)^2],$$

(in adapted orthonormal frames), in which the (ω^i) are zero along the trajectories of the isometries:

(25-2) $$\omega^i = \bar{A}_j^i\, dx^j ,$$

and in which, from (17-2) and (18-1):

(25-3) $$\omega^0 = -\frac{1}{\xi}(\gamma_{00}\,dx^0 + \gamma_{0i}\,dx^i) = -\frac{\gamma_{00}}{\xi}(dx^0 + \beta\,\varphi_i dx^i) = \xi(dx^0 + \beta\,\varphi_i dx^i) .$$

Without modifying the local coordinates, we deduce a quadratic form from (25-1) that is the sum of five squares by performing the following transformation on the Pfaff forms:

(25-4) $$\varpi^A = i\,\omega^A \qquad (A = 0, 1, 2, 3), \qquad \varpi^4 = \omega^4,$$

and we express the metric in the new form:

$$d\sigma^2 = \sum_\alpha (\varpi^\alpha)^2 .$$

In this new form, we may apply the formulas that were recalled or established in the course of the last three sections. Now, the components relative to the elliptic form of the metric (i.e., relative to the ϖ^α) are deduced from those relative to the hyperbolic form (i.e., relative to the ω^α) by the following rule: Any contravariant index A = 0, 1, 2, 3, corresponds to multiplication by i, and any covariant index A corresponds to multiplication by – i; the index 4 corresponds to multiplication by 1. Moreover, since:

$$\varpi^0 = \bar{\xi}(dx^0 + \beta\varphi_i\,dx^i), \qquad \omega^0 = \xi(dx^0 + \beta\varphi_i\,dx^i),$$

one deduces that $\bar{\xi} = i\xi$ and, as a result, $\bar{\xi}^2 = -\xi^2 = \gamma_{00}$.

We are therefore led to write the formulas of sec. **22**, **23**, **24** in barred notation and, thanks to the preceding rule, to transform them in such a way that we obtain formulas that are valid for V_5 and its isometry trajectories that are oriented so that $d\sigma^2 < 0$. One first confirms that the formulas of sec. **22** that relate to the passage from an adapted, orthonormal frame to an adapted, natural frame and its inverse transformation do not suffer any modification.

When formulas (23-1) are transformed by the preceding rule, they will take a form that we will use from now on:

$$(25\text{-}5)\qquad \left\{\begin{aligned} \bar{R}_{ij} &= \hat{R}_{ij} + \frac{\beta^2\xi^2}{2}F_i^{\ k}F_{jk} - \frac{1}{\xi}\nabla_j(\partial_i\xi), \\ \bar{R}_{i0} &= -\frac{\beta}{2\xi^2}\nabla_j(\xi^2 F_i^{\ j}), \\ \bar{R}_{00} &= \frac{1}{\xi}\Delta\xi + \frac{\beta^2\xi^2}{2}F^2, \end{aligned}\right.$$

with:

$$F^2 = \tfrac{1}{2}F_{kl}\,F^{kl}.$$

We note that F^2 is no longer necessarily positive-definite or zero.

Similarly, when formulas (23-4) are transformed by our rule, they will become:

$$(25\text{-}6)\qquad \left\{\begin{aligned} \bar{S}_{ij} &= \hat{S}_{ij} - \frac{\beta^2\xi^2}{2}\left[\tfrac{1}{4}g_{ij}F_{kl}F^{kl} - F_i^{\ k}F_{jk}\right] - \frac{1}{\xi}[\nabla_j(\partial_i\xi) - g_{ij}\Delta\xi] \\ \bar{S}_{i0} &= -\frac{\beta}{2\xi^2}\nabla_j(\xi^3 F_i^{\ j}) \\ \bar{S}_{00} &= \frac{1}{2}\hat{R} + \frac{3}{4}\beta^2\xi^2F^2. \end{aligned}\right.$$

Finally, we transform the divergence formulas that were given in sec. **24**. After transformation, formula (24-2) will become:

(25-7) $$R_0^0 = -\frac{1}{\xi} g^{ij} \nabla_j [\partial_i \xi - \frac{\beta^2 \xi^2}{2} \varphi_k \overline{F}^k{}_i].$$

One deduces from this that upon introducing the vector ***h*** of W_4 with the covariant components:

(25-8) $$h_i = \partial_i \xi - \frac{\beta^2 \xi^3}{2} \varphi_k F^k{}_l ,$$

they will become:

(25-9) $$R_0^0 = -\frac{1}{\xi} \operatorname{div} \mathbf{h} ,$$

in which the divergence is evaluated on the Riemannian manifold of hyperbolic normal type W_4.

Similarly, when the transformation is performed on formula (24-5), it will give:

$$i\xi \varphi_i R_0^i + i \frac{\beta \xi^2}{2} F^2 = i\, g^{ij} \nabla_j \left[\frac{\beta \xi^2}{2} \varphi_k F_i^k \right];$$

namely, upon dividing by i and introducing the vector field **p** on W_4 whose covariant components are:

(25-10) $$p_i = \frac{\beta \xi^3}{2} \varphi_k F_i^k ,$$

the relation:

(25-11) $$\xi \varphi_i R_0^i + \frac{\beta \xi^3}{2} F^2 = \operatorname{div} \mathbf{p} ,$$

in which the divergence is evaluated under the same conditions.

26. – Formulas in local coordinates. – Conforming to a remark in sec. **23**, the left-hand sides of equations (25-6) define a symmetric tensor field P, a covariant vector field Q, and a scalar L on V_4, respectively. The components of the vector field and tensor field in local coordinates are immediately expressed with the aid of their components in orthonormal frames, i.e., $\overline{S}_{ij}$ and $\overline{S}_{i0}$. They become:

(26-1) $$P_{ij} = \overline{A}_i^k \, \overline{A}_j^l \, \overline{S}_{kl} ,$$

and:

(26-2) $$Q_i = \overline{A}_i^k \, \overline{S}_{k0} .$$

Upon referring all of the tensors that we have defined on V_4 to local coordinates, one will deduces from this that:

(26-3)
$$\begin{cases} P_{ij} = \hat{S}_{ij} - \dfrac{\beta^2\xi^2}{2}\left[\tfrac{1}{4} g_{ij}F_{kl}F^{kl} - F_i^{\ k}F_{jk}\right] - \dfrac{1}{\xi}[\nabla_j(\partial_i\xi) - g_{ij}\Delta\xi], \\ Q_i = -\dfrac{\beta}{2\xi^2}\nabla_j(\xi^3 F_i^{\ j}), \\ L = \dfrac{1}{2}\hat{R} + \dfrac{3}{4}\beta^2\xi^2 F^2. \end{cases}$$

with:

$$F^2 = \tfrac{1}{2}F_{kl}F^{kl}.$$

One will note that $F^k{}_i = g^{kl}F_{li} = \gamma^{kl}F_{li} = \gamma^{k\lambda}F_{\lambda i}$, since $g^{kl} = \gamma^{kl}$ and $F_{0i} = 0$. It is therefore pointless to indicate whether the components of the electromagnetic field tensor are mixed or contravariant and whether this tensor field is defined on V_5 or V_4.

Similarly, the vectors $\boldsymbol{h}$ and $\boldsymbol{p}$ that appear in the divergence formulas may be defined by their components in local coordinates:

(26-4)
$$h_i = \partial_i\xi + \frac{\beta^2\xi^3}{2}\varphi_k F_i^{\ k},$$

and:

(26-5)
$$p_i = \frac{\beta\xi^3}{2}\varphi_k F_i^{\ k}.$$

27. The field equations in the exterior, unitary case. – In the exterior, unitary case, the field equations on V_5 translate into the fifteen equations: $S_{\alpha\beta} = 0$. It is possible to specify them with the aid of the tensors that were defined on V_4 or W_4 in several ways that might possibly be interesting.

a) First of all, in orthonormal frames, one has the system:

(27-1)
$$\bar{S}_{ij} = 0, \qquad \bar{S}_{i0} = 0, \qquad \bar{S}_{00} = 0,$$

or, in an equivalent manner, in adapted, local coordinates:

(27-2)
$$P_{ij} = 0, \qquad Q_i = 0, \qquad L = 0,$$

in which the values of the left-hand sides are given by (25-6) and (26-3). One will note that P_{ij} is distinct from the components of S_{ij} and $S_{\alpha\beta}$ in local coordinates. On the contrary, one will have $P^{ij} = S^{ij}$, since:

(27-3)
$$P^{ij} = \bar{A}^i_k\,\bar{A}^j_l\,\bar{P}^{kl} = \bar{A}^i_k\,\bar{A}^j_l\,\bar{S}^{kl} = S^{ij}.$$

On the other hand, upon performing a change of frame, one will get:

$$S^{i0} = \bar{A}^i_j\,\bar{A}^0_0\,\bar{S}^{j0} + \bar{A}^i_j\,\bar{A}^0_k\,\bar{S}^{jk} = \frac{1}{\xi}Q^i - \beta\varphi_k S^{ik} \;.$$

One deduces from this that:

(27-4) $$Q^i = \xi\,(S^{i0} + \beta\,\varphi_k\,S^{ik}).$$

The introduction of the components S_{ij}, S_{i0}, S_{00} on V_4 would give only complicated expressions that are of no interest.

b) One may substitute the following system for the system (27-1):

(27-5) $$\bar{S}_{ij} = 0\,, \qquad \bar{R}_{i0} = 0\,, \qquad \bar{R}_{00} = 0\,,$$

which is equivalent to it. As far as notation is concerned, this system differs from (27-1) only in its fifteenth equation. In order to establish the equivalence of (27-5) with the system (27-1), we first evaluate the scalar S, which is the contraction of $S_{\alpha\beta}$, as a function of R.

One deduces from the relation:

$$\bar{S}_{\alpha\beta} = \bar{R}_{\alpha\beta} - \tfrac{1}{2}\bar{\gamma}_{\alpha\beta}R\,,$$

the relation:

(27-6) $$S = R - \tfrac{5}{2}R = -\tfrac{3}{2}R\,.$$

Now we seek to evaluate $\bar{S}_{00}$ as a function of $\bar{R}_{00}$ for a solution of the first ten equations $\bar{S}_{ij} = 0$ that are common to the two systems. For such a solution:

$$S = \bar{\gamma}^{ij}\bar{S}_{ij} + \bar{\gamma}^{00}\bar{S}_{00} = -\bar{S}_{00}\,.$$

Now:

$$\bar{S}_{00} = \bar{R}_{00} - \tfrac{1}{2}\bar{\gamma}_{00}R = \bar{R}_{00} + \tfrac{1}{2}R = \bar{R}_{00} - \tfrac{1}{3}S = \bar{R}_{00} + \tfrac{1}{3}\bar{S}_{00}\,.$$

One deduces from this that:

$$\bar{R}_{00} = \tfrac{2}{3}\bar{S}_{00}$$

for any solution of the ten equations that we envision, which assures the equivalence of (27-1) and (27-5).

In adapted, local coordinates, the system (27-5) will correspond to the system:

(27-7) $$P_{ij} = 0, \qquad Q_i = 0,$$

and:

(27-8) $$\frac{1}{\xi}\Delta\xi + \frac{\beta^2\xi^2}{2}F^2 = 0\,.$$

c) Finally, it is possible to substitute the equation $R_0^0 = 0$ for the fifteenth equation of system (27-5), namely (27-8), relative to an adapted system of local coordinates. In equation (24-1), which is valid without any hypothesis on the signature:

$$R_0^0 = \overline{R}_0^0 - \xi\beta\overline{\varphi}_i\overline{R}_0^i,$$

and which may be written:

$$R_0^0 = -R_0^0 - \xi\beta\varphi_i R_0^i$$

with the adopted signature, one sees that for any solution of the equations $\overline{R}_{i0} = 0$ the equations $R_0^0 = 0$ and $\overline{R}_{00} = 0$ will be equivalent. One deduces from this that one may substitute the equation:

(27-9) $$\text{div}\,\mathbf{h} = 0$$

for (27-8), which is an equation that is equivalent for a solution of $\overline{R}_{i0} = 0$.

Finally, one will note that one will have the relation:

(27-10) $$\text{div}\,\mathbf{p} = \frac{\beta\xi^2}{2}F^2,$$

as a consequence of the field equations in the exterior, unitary case.

28. – The equations of the Kaluza-Klein theory. – The equations of the Kaluza-Klein theory are deduced from the preceding calculations immediately. In that theory, the geometric context is the one that we specified in first part of this chapter, but with the supplementary hypothesis that the trajectories of the isometry group of V_5 must be geodesics of this manifold that are oriented such that $d\sigma^2 < 0$, and that as a result γ_{00} = constant in local coordinates, or, in an equivalent manner, ξ = constant. We may choose the infinitesimal generator $\boldsymbol{\xi}$ in such a way that $\xi = 1$. As a result, $\gamma_{00} = -1$.

Conforming to the variational processes that were indicated in sec. **21**, the field equations will be obtained by annulling the variation of the integral:

$$I = \int_C R_{\alpha\beta}\gamma^{\alpha\beta}\sqrt{|\gamma|}\,dx^0\wedge\cdots\wedge dx^4$$

for the variations of the potentials that preserve the fixed value $\gamma_{00} = -1$, and therefore annul their first derivatives on the boundary of C. Since $\delta\gamma_{00} = 0$, one obtains from (21-4):

$$\begin{aligned}\delta I &= \int_C S^{\alpha\beta}\delta\gamma_{\alpha\beta}\sqrt{|\gamma|}\,dx^0\wedge\cdots\wedge dx^4\\ &= -\int_C S^{ij}\delta\gamma_{ij}\sqrt{|\gamma|}\,dx^0\wedge\cdots\wedge dx^4 - 2\int_C S^{i0}\delta\gamma_{i0}\sqrt{|\gamma|}\,dx^0\wedge\cdots\wedge dx^4.\end{aligned}$$

From this, one deduces that in the Kaluza-Klein theory, the field equations may be written:

(28-1) $$S^{ij} = 0, \qquad S^{i0} = 0.$$

By virtue of (27-3) and (27-4), this system is equivalent to the system of 14 equations:

(28-2) $$P_{ij} = 0, \qquad Q_i = 0.$$

If one sets $\xi = 1$ in the expressions (26-3) for the left-hand sides then one will obtains explicitly:

(28-3) $$\begin{cases} \hat{S}_{ij} = \dfrac{\beta^2}{2}\left(\frac{1}{4} g_{ij} F_{kl} F^{kl} - F_i^{\ k} F_{jk}\right), \\ \nabla_j F^j_{\ i} = 0, \end{cases}$$

which are equations that will coincide rigorously with those of the provisional theory of electromagnetism for the pure electromagnetic field (sec. I) if one sets:

(28-4) $$\frac{\beta^2}{2} = \chi_0 ,$$

in which χ_0 will henceforth denote Einstein's gravitational constant (a constant that was denoted by χ in I). Except for the formalism that was employed, the "projective" theories that were due to Veblen, Hoffman, and Pauli ([1]), respectively, do not differ from the preceding penta-dimensional theory, which was due to Kaluza and Klein, in substance.

One therefore confirms that in order to deduce the equations of electromagnetism in classical general relativity from the Jordan-Thiry equations, it will suffice to suppress the fifteenth equation (27-8), and to give the value 1 to ξ in the remaining 14 equations (27-7).

IV. – THE EQUATIONS OF THE INTERIOR CASE. PHYSICAL INTERPRETATION OF THE THEORY

29. – The equations of the interior, unitary case and conservation conditions in V_5 **.** – The tensor on the right-hand side of the equation $\Theta_{\alpha\beta}$ that we will now introduce is intended to represent the distribution of the masses and charges at the spacetime point envisioned. We suppose that this tensor formally generalizes the energy-momentum tensor of classical general relativity in the absence of an electromagnetic field. Although it might be interesting to take the "perfect fluid" schema as our point of departure, here we confine ourselves to generalizing the "pure matter" schema of general relativity by setting:

([1]) VEBLEN, *Projektive Relativitätstheorie,* Springer (1933); W. PAULI, Ann. d. Physik, **18** (1933), 305-337.

(29-1) $$\Theta_{\alpha\beta} = r\, v_\alpha\, v_\beta,$$

in which r denotes a positive scalar, and v_α denotes a unitary vector on V_5 (which is a vector that is oriented so that $d\sigma^2 > 0$ and has square +1). Moreover, one will note that the passage from the tensor (29-1) to the more complete tensor:

(29-2) $$\Theta_{\alpha\beta} = r\, v_\alpha\, v_\beta - p\, \gamma_{\alpha\beta}$$

is effected in the various equations that follow in a simple manner, and will lead to an analysis that is analogous to that of Part I, Chapter VI.

Once we have adopted the tensor (29-1), by reasoning that is analogous to that in I, sec. **45**, we will easily establish that if X denotes the infinitesimal transformation that is generated by the isometry group then ee will necessarily have:

$$Xr = 0, \qquad X\mathbf{v} = 0,$$

in which r and $\mathbf{v}$ are invariant under the transformations of the group; in adapted, local coordinates, r and the components v_α of $\mathbf{v}$ will be independent of the variable x_0. The trajectories in V_5 of the vector field $\mathbf{v}$ will be called the *penta-dimensional streamlines.*

One immediately deduces some simple consequences from the equations of the interior, unitary case:

$$S_{\alpha\beta} = \Theta_{\alpha\beta},$$

in which $\Theta_{\alpha\beta}$ is provided by (29-1). Since the left-hand side tensor $S_{\alpha\beta}$ satisfies the conservation identities, one will necessarily have:

(29-2) $$D_\alpha \Theta^\alpha{}_\beta = D_\alpha (r\, v^\alpha v_\beta) = 0.$$

Upon developing (29-2), one will obtain:

(29-3) $$D_\alpha (r\, v^\alpha)\, v_\beta + r\, v^\alpha D_\alpha v_\beta = 0.$$

Now, since v_α is unitary, one will have:

(29-4) $$v^\beta D_\alpha\, v_\beta = 0.$$

Upon multiplying both sides of (29-3) by v_β and taking (29-4) into account, one will obtain the relation:

(29–5) $$D_\alpha (r\, v^\alpha) = 0,$$

and (29-3) will reduce to:

(29-6) $$v^\alpha D_\alpha\, v_\beta = 0.$$

Equation (29-5) presents the aspect of a continuity equation; we shall interpret this equation in a moment. As for equations (29-6), they say that the penta-dimensional streamlines are the geodesics of V_5, when they are oriented so that $d\sigma^2 > 0$. We state:

THEOREM – *The streamlines in the manifold V_5 are the geodesics of that Riemannian manifold, which are oriented so that $d\sigma^2 > 0$.*

30. – The vector v and the unitary velocity vector of V_5 . – From cylindricality, and by virtue of (8-1), the penta-dimensional streamlines, along which:

$$\frac{dx^\alpha}{d\sigma} = v^\alpha,$$

will be such that along one of them:

$$\gamma_{00}\, v^0 + \gamma_{0i}\, v^i = \text{const.} = h,$$

namely:

(30-1) $$v_0 = h.$$

Since $\gamma_{00} = -\xi^2$ is negative, it will result from the considerations of sec. **8** and (8-5) that these penta-dimensional streamlines project onto V_4 along lines – which are called *spacetime streamlines* – that are oriented in time such that along one of these lines:

(30-2) $$\left(1 + \frac{h^2}{\xi^2}\right) d\sigma^2 = ds^2,$$

which is an extremal of the integral:

(30-3) $$\int_{z_0}^{z_1} \left(\sqrt{1 + \frac{h^2}{\xi^2}}\, ds + \beta h \varphi\right),$$

in which φ designates the vector-potential form, for a choice of vector-potential.

Let u^i be the *unitary velocity vector*, i.e., the unitary vector in the metric ds^2 of V_4 that is tangent to the spacetime streamlines. Once one knows this vector, the scalar ξ, and the constant h, it is easy to determine the vector v^α on V_5 . Indeed, along a streamline one has:

$$v^i = \frac{dx^i}{d\sigma} = \frac{dx^i}{ds}\frac{ds}{d\sigma} = u^i\,\frac{ds}{d\sigma} = \sqrt{1 + \frac{h^2}{\xi^2}}\, u^i .$$

If one passes to adapted, orthonormal frames then from the form of the relations for the frame change one will obtain:

$$\bar{v}^i = \sqrt{1 + \frac{h^2}{\xi^2}}\, \bar{u}^i,$$

which is equivalent to:

(30-4) $$\bar{v}_i = \sqrt{1 + \frac{h^2}{\xi^2}}\, \bar{u}_i .$$

On the other hand, one obviously has:

$$\bar{v}_0 = \bar{A}_0^0 \, v_0 = \frac{1}{\xi} v_0 ,$$

namely:

(30-5) $$\bar{v}_0 = \frac{h}{\xi} .$$

We have therefore determined the components $\bar{v}_\alpha$ of the vector we just introduced by starting with h, ξ, and the unitary velocity vector on V_4.

31. – The electromagnetic tensors and the variation of the gravitational factor. – In an adapted, orthonormal frame, the equations of the interior, unitary case may be put into the following form:

(31-1) $$\hat{S}_{ij} - \frac{\beta^2 \xi^2}{2} \left[\tfrac{1}{4} \bar{g}_{ij} \bar{F}_{kl} \, \bar{F}^{kl} - \bar{F}_i^k \, \bar{F}_{jk} \right] - \frac{1}{\xi} (\nabla_j \partial_i \xi - \bar{g}_{ij} \, \Delta \xi) = r \, \bar{v}_i \, \bar{v}_j ,$$

(31-2) $$\nabla_j (\xi^2 \bar{F}_i^j) = \frac{2\xi^2}{\beta} r \, \bar{v}_i \, \bar{v}_0 ,$$

(31-3) $$\bar{S}_{00} = r (\bar{v}_0)^2 .$$

We examine equations (31-2), to which we must add the constraint that the tensor $\bar{F}_{ij}$ must be derived from a vector-potential. The form of the equations (31-2) leads us to introduce inductions into the vacuum that are distinct from the field. The tensor $\bar{F}_{ij}$ may be interpreted as representing the two space vectors of the magnetic induction **B** and the electric field **E** for the space and time that is associated with the orthonormal frame envisioned (I, sec. **3**). From Maxwell's equations, one knows that the corresponding tensor always has zero exterior derivative. Under the same conditions, the tensor $\bar{H}_{ij} = \xi^2 \bar{F}_{ij}$ will then be interpreted as representing the magnetic field **H**, and the electric induction **D**. In order for this to be the case ([1]), it will be necessary and sufficient that we attribute a *dielectric constant* ε and a *magnetic permeability* τ such that:

(31-4) $$\varepsilon = \xi^3, \qquad \tau = \frac{1}{\xi^3} \qquad (\varepsilon \tau = 1),$$

which are quantities that vary slightly, in accordance with the field equations. With these conditions, equations (31-2) take the form:

(31-5) $$\nabla_j (\bar{H}^j{}_i) = \bar{J}_i ,$$

([1]) Cf. BECKER, *Théorie des électrons*, Alcan, (1938), pp. 358-365.

in which the electric current vector is currentlly given by:

(31-6) $$\bar{J}_i = \frac{2\xi^2}{\beta} r\,\bar{v}_i\,\bar{v}_0\,.$$

One knows ([1]) that a purely electromagnetic energy-momentum tensor corresponds to $\bar{F}_{ij}$ and $\bar{H}_{ij}$, and it is defined by:

$$\bar{\tau}_{ij} = \tfrac{1}{4}\bar{g}_{ij}\bar{H}_{kl}\bar{F}^{kl} - \tfrac{1}{2}(\bar{H}_i{}^k\bar{F}_{jk} + \bar{F}_i{}^k\bar{H}_{jk}),$$

namely, since $\bar{H}_{ij} = \xi^2\bar{F}_{ij}$:

(31-7) $$\bar{\tau}_{ij} = \xi^3\left(\tfrac{1}{4}\bar{g}_{ij}\bar{F}_{kl}\bar{F}^{kl} - \bar{F}_i{}^k\bar{F}_{jk}\right).$$

We are therefore led to put equations (31-1) in the form:

(31-8) $$\hat{S}_{ij} = \frac{\beta^2}{2\xi}[\bar{\tau}_{ij} + \frac{2}{\beta^2}(\nabla_j\partial_i\xi - \bar{g}_{ij}\Delta\xi)] + r\,\bar{v}_i\,\bar{v}_j\,,$$

and to interpret the factor:

(31-9) $$\chi = \frac{\beta^2}{2\xi} = \frac{\chi_0}{\xi}\,,$$

as a "gravitational factor," a variable that reduces to the value χ_0 of Einstein's gravitational constant for $\xi = 1$.

32. – Matter density and charge density. – A tensor on V_4 that is proportional to $\bar{v}_i\,\bar{v}_j$ – hence, from (30-4), to $\bar{v}_i\,\bar{v}_j$ – appears in the last term of the right-hand side of (31-8). We are thus led to set:

(32-1) $$r\,\bar{v}_i\,\bar{v}_j = \chi\rho\;\bar{v}_i\,\bar{v}_j\,,$$

in which ρ is the *proper matter density*. Upon substituting the values of $\bar{v}_i$ and $\bar{v}_j$ that are inferred from (30-4), one will get:

(32-2) $$\chi\rho = r\left(1 + \frac{h^2}{\xi^2}\right);$$

namely, upon introducing χ_0, instead of χ:

([1]) SCHOUTEN, *Tensor Analysis for Physicists,* Oxford, (1951), pp. 225-226.

(32-3) $$\chi_0 \rho = \xi\, r \left(1 + \frac{h^2}{\xi^2}\right).$$

Similarly, the expression (31-6) for the electric current vector leads us to set:

(32-4) $$\bar{J}_i = \frac{2\xi^2}{\beta} r\, \bar{v}_i\, \bar{v}_0 = \mu\, \bar{u}_i\,,$$

in which μ is the proper charge density. Upon substituting the values of $\bar{v}_i$ and $\bar{v}_0$ that are inferred from (30-4) and (30-5), one will obtain:

(32-5) $$\mu = \frac{2\xi}{\beta} hr \sqrt{1 + \frac{h^2}{\xi^2}}\,,$$

namely:

(32-6) $$\chi\mu = \beta hr \sqrt{1 + \frac{h^2}{\xi^2}}\,.$$

Upon dividing (32-6) by (32-2), one will obtain:

(32-7) $$k = \frac{\mu}{\rho} = \frac{\beta h}{\sqrt{1 + \dfrac{h^2}{\xi^2}}}\,,$$

which is a formula that does not differ from (15-1), in essence. With the notation k thus-introduced, one sees that the spacetime streamlines – which are extremals of the integral (30-3) – will be extremals of the integral:

(32-8) $$\int_{z_0}^{z_1} \left(\frac{1}{k} ds + \varphi\right).$$

One will note that the sign of the constant h determines the sign of the charge density μ. In the presence of matter and the absence of charge, one will have $h = 0$.

In summation, the first 14 equations of the interior, unitary case may be put into the form:

(32-9) $$\hat{S}_{ij} = \frac{\chi_0}{\xi}[\bar{\tau}_{ij} + \frac{2}{\beta^2}(\nabla_j \partial_i \xi - g_{ij} \Delta\xi) + \rho u_i u_j\,],$$

(32-10) $$\nabla_j H^j_{\ i} = \mu\, u_{\mathrm{i}}\,,$$

in which $H_{ij} = \xi^3 F_{ij}$, and $\bar{\tau}_{ij}$ is given by (31-7).

As for the fifteenth equation, we may put it into a form that is convenient for calculating the value of $\overline{R}_{00}$ when starting with the right-hand sides. From (27-6), one has:

$$\Theta = -\frac{3}{2}R.$$

One deduces from this that:

$$\overline{R}_{00} = \overline{S}_{00} + \tfrac{1}{2}\overline{\gamma}_{00}R = \overline{S}_{00} - \tfrac{1}{2}R = \overline{S}_{00} + \tfrac{1}{3}\Theta .$$

Now:

$$\overline{S}_{00} = r(\overline{v}_0)^2 = r\frac{h^2}{\xi^2}, \qquad \Theta = r.$$

Therefore, one obtains:

$$\overline{R}_{00} = r\left(\frac{1}{3} + \frac{h^2}{\xi^2}\right),$$

and from the expression(32-3) for r:

$$\overline{R}_{00} = \frac{\chi_0\rho}{3\xi}\frac{1+3\dfrac{h^2}{\xi^2}}{1+\dfrac{h^2}{\xi^2}}.$$

This results in the equation:

(32-11) $$\Delta\xi + \chi_0\xi^2F^2 = \frac{1}{3}\chi_0\rho\frac{1+3\dfrac{h^2}{\xi^2}}{1+\dfrac{h^2}{\xi^2}}.$$

One may deduce from this equation and the corresponding equation in the exterior, unitary case by an approximation method (upon whose details we shall not insist) that the variation due to ξ in spacetime remains numerically quite small, and that the same will be true for the variation due to k.

33. – Conservation conditions in V_4 **.** – The conservation conditions (29-5) and (29-6) in V_5 may be easily translated into spacetime V_4. Equations (29-6) are expressed in V_4 by the fact that the spacetime streamlines are extremals of (30-3) and by the supplementary condition:

(33-1) $$v_0 = h.$$

(33-2)

We now propose to translate the condition:

(33-3) $$D_\alpha(r\,v^\alpha) = 0$$

into V_4. To that effect, we establish the following lemma ([1]), in which we systematically use adapted, orthonormal frames:

LEMMA – *If the vector field* $\boldsymbol{\eta}$ *on a manifold* V_{n+1} *with a positive-definite metric that admits a one-parameter isometry group, is invariant under that isometry group, then the divergence* $D_\alpha \bar{\eta}^\alpha$ *of that vector field on* V_{n+1} *can be expressed by starting with the divergence of the vector* $\xi \bar{\eta}^i$ *in* V_n *by the formula:*

$$D_\alpha \bar{\eta}^\alpha = \tfrac{1}{3} \nabla_j (\xi \, \bar{\eta}^j) .$$

Indeed, in coordinates and an adapted frame, one has:

$$\bar{\eta}_0 = \bar{A}_0^0 \eta_0 = \tfrac{1}{3} \eta_0 ,$$

and since $\boldsymbol{\eta}$ is invariant under the isometry group, $\bar{\eta}_0$ will be independent of the coordinate x^0. One deduces from this that:

$$\partial_0 \bar{\eta}_0 = 0.$$

Having said that, one obtains from the expressions in I, (67-5) for the coefficients of rotation $\gamma_{\lambda\mu\nu}$ that:

$$D_0 \bar{\eta}^0 = D_0 \bar{\eta}_0 = \partial_0 \bar{\eta}_0 - \gamma_{0j0} \bar{\eta}^j = \frac{\partial_j \xi}{\xi} \bar{\eta}^j .$$

On the other hand, from the same formulas I, (67-5):

$$D_j \bar{\eta}^j = D_j \bar{\eta}_j = \partial_j \bar{\eta}_j - \gamma_{jkj} \bar{\eta}^k - \gamma_{j0j} \bar{\eta}^0 = \partial_j \bar{\eta}_j - \gamma_{jkj} \bar{\eta}^k = \nabla_j \bar{\eta}^j .$$

One will also obtain:

$$D_\alpha \bar{\eta}^\alpha = \nabla_j \bar{\eta}^j + \frac{\partial_j \xi}{\xi} \bar{\eta}^j ;$$

namely:

(33-3) $$D_\alpha \bar{\eta}^\alpha = \tfrac{1}{3} \nabla_j (\xi \bar{\eta}^j) ,$$

which proves the lemma.

This formula (33-3), which was established for a positive-definite signature, remains valid by the rule of sec. **25** that permits us to translate it onto the manifold V_5 that has a hyperbolic normal signature whose isometry trajectories are oriented so that $d\sigma^2 < 0$. In order to translate (33-2), we may thus apply this formula to the vector $\bar{\eta}^\alpha = r \, \bar{v}^\alpha$ on V_5. Therefore, on V_4, (33-2) translates into the relation:

([1]) This lemma is completely related to the result that was established in I, sec. **72**.

$$\nabla_i (\xi r \bar{v}^i) = 0 ;$$

namely, upon substituting the values of ξr and $\bar{v}^i$ that one infers from (32-3) and (30-4), one will get:

$$\nabla_i \left(\frac{\chi_0 \rho \bar{u}^i}{\sqrt{1 + \frac{h^2}{\xi^2}}} \right) = 0 ,$$

which is a relation that may be written in local coordinates in V_4 as:

$$\nabla_i \left(\frac{\rho u^i}{\sqrt{1 + \frac{h^2}{\xi^2}}} \right) = 0 . \tag{33-4}$$

On the other hand, if one takes the contracted covariant derivative of both sides of (32-10) then one obtains the condition of the conservation of electricity:

$$\nabla_i (\mu\, u^i) = 0. \tag{33-5}$$

On account of (33-4), this equation is equivalent to the fact that along a spacetime streamline:

$$\frac{\mu}{\rho} \sqrt{1 + \frac{h^2}{\xi^2}} = \text{const.},$$

which is in accord with (32-7), since the value of the constant is βh. On account of (33-4), equation (33-5) will be, moreover, deducible from the condition $v_0 = h$ along a streamline. Therefore, we have the fact that the spacetime streamlines are extremals of (32-8), and the two equations (33-4) and (33-5) form a set that is equivalent to the set of the five conservation conditions $D_\alpha \Theta^\alpha{}_\beta$ in V_5 .

One will note that for $\xi = 1$ equations (32-9) and (32-10) reduce to:

$$\hat{S}_{ij} = \chi_0 (\tau_{ij} + \rho\, u_i\, u_j), \tag{33-6}$$

$$\nabla_j F^j{}_i = \mu\, u_i , \tag{33-7}$$

in which τ_{ij} is the classical energy-momentum tensor of the electromagnetic field. As for equations (33-4) and (33-5), they may be written:

$$\nabla_i (\rho\, u^i) = 0, \qquad\qquad \nabla_i (\mu\, u^i) = 0.$$

For $\xi = 1$ – i.e., in the case of the Kaluza-Klein theory – one sees that the equations that were described reduce to the equations of electromagnetism in general relativity for the "pure electromagnetic field-matter" schema in the case for which one admits the equations of Lorentz transport (see I, sec**. 24**). Upon adopting a more complete tensor $\Theta^{\alpha\beta}$, one may, moreover, obtain the equations of the "charged, perfect fluid" schema that were developed in I, Chap. VI.

The physical interpretation of the Jordan-Thiry equations that we just gave here differs very appreciably from the one that was suggested by those authors, and seems more satisfactory to us. In the interpretation that was given by Jordan and Thiry, the introduction of a dielectric constant and a magnetic permeability, with the aid of ξ, was not envisioned, and the law of variation for the gravitational factor as a function of x was different. The present interpretation, which seems to be in precise analogy with the viewpoint that was developed directly in electrodynamics by Born and Infeld, seems more interesting. Here, we touch upon one of the essential difficulties that presents itself in the study of any unitary theory; it consists in the multiplicity of physical interpretations that may be assigned to "field equations" that were deduced from mathematical conditions, *a priori.*

V. – THE CAUCHY PROBLEM AND THE GEODESIC PRINCIPLE

34. – The Cauchy problem in the exterior, unitary case. – In this section and the ones that follow, we will confine ourselves to local considerations. We may, in turn, assume that the spacetime manifold V_4 has been reduced to a neighborhood.

The Cauchy problem in spacetime that relates to the Jordan-Thiry equations (27-7) and (27-8) in the exterior, unitary case may be stated in the following manner:

PROBLEM – *If we are given the gravitational potentials g_{ij}, a vector-potential φ_i, and the scalar ξ, as well as their first derivatives, on a hypersurface S in V_4 then how do we determine these quantities outside of S, assuming that they satisfy the Jordan-Thiry equations* (27-7) *and* (27-8) *for the exterior unitary case.*

We assume that S is not tangent to the elementary cone of V_4 and that if (x^i) denotes a system of local coordinates of V_4 then it will be represented locally by the equation $x^4 = 0$. One will then have $g^{44} \neq 0$ (here the variable x^4 is not assumed to correspond to a specific orientation in space and time). We assume that the values of g_{ij}, φ_i, ξ, and their derivatives $\partial_4 g_{ij}, \partial_4\varphi_i, \partial_4\xi$ are known on S. These quantities, which are the "Cauchy data" of our problem, are assumed to be at least three times and twice-continuously differentiable with respect to the variables x^u (u, v = 1, 2, 3), respectively. One determines the values of $\partial_u g_{ij}, \partial_u\varphi_i, \partial_u\xi$ on S by differentiation on that hypersurface.

Having said that, we assume that we are given the manifold V_5 as the topological product $V_4 \times T^1$. If (x^i) is a system of local coordinates on V_4, and x^0 is the canonical coordinate on the circle T^1 then a point of V_5 will admit the local coordinates (x^i, x^0), and the manifolds x^0 = const. will be the factor manifolds W_4 of V_5; the "trajectories" will be the factor lines x^i = const.

Consider the hypersurface Σ in V_5 that is generated by the projections of the trajectories onto the points of S. Σ is defined locally in V_5 by the equation $x^4 = 0$ in the local coordinates envisioned. From (18-1) and (18-3), the preceding Cauchy data on Σ will provide us with the values of the quantities:

$$\gamma_{00} = -\xi^2, \qquad \gamma_{0i} = -\beta\,\varphi_i\,\xi^2, \qquad \gamma_{ij} = -\beta^2\,\xi^2\,\varphi_i\,\varphi_j, \tag{34-1}$$

and their first derivatives, since these quantities are functions of only (x^0). One will note, moreover, that:

$$\gamma^{44} = g^{44} \neq 0. \tag{34-2}$$

The following problem in V_5 will therefore correspond to the problem that we initially posed:

PROBLEM – *If we are given the potentials $\gamma_{\alpha\beta}(x^u)$ and their first derivatives $\partial_\lambda\,\gamma_{\alpha\beta}(x^u)$ on a hypersurface Σ in V_5 that is generated by the trajectories then how do we determine the values of these potentials outside of Σ if we assume that they satisfy the Jordan-Thiry equations $S_{\alpha\beta} = 0$.*

From (34-2), the hypersurface Σ is assumed to be *non-tangent to the elementary cones* of V_5. The values $\gamma_{\alpha\beta}(x^u)$ and $\partial_4\,\gamma_{\alpha\beta}(x^u)$ on Σ are the new Cauchy data. The study that we made in general relativity, (I, sec. **14**), leads us to replace the system $S_{\alpha\beta} = 0$ with the equivalent system that is formed by the union of the following two systems:

$$R_{AB}\,\eta - \tfrac{1}{2}\,\gamma^{44}\,\partial_{44}\,\gamma_{AB} + F_{AB} = 0 \qquad (A, B = 0, 1, 2, 3), \tag{34-3}$$

and

$$S_\lambda^4 = 0, \tag{34-4}$$

in which the F_{AB} and the S_λ^4 do not contain any derivative of x^4 with the index 2 and, in turn, admit values on Σ that are deduced from the Cauchy data by algebraic operations and differentiations on Σ.

We propose to study the values of the derivatives of order higher than the first on Σ. First of all, since $\gamma^{44} \neq 0$, equations (34-3) will provide the values of the $\partial_{44}\,\gamma_{AB}(x^u)$ on Σ. No equation contains the derivatives $\partial_{44}\,\gamma_{\lambda 4}$. This fact is related to the existence of changes of local coordinates that conserve the numerical values of the coordinates at any point of Σ, as well as the Cauchy data, and they will be coordinate changes of the form:

$$x'^\lambda = x^\lambda + \frac{(x^4)^3}{6}[\varphi^\lambda(x^4) + \varepsilon^\lambda(x^4)] \qquad (\lambda' = \lambda\text{, numerically}), \tag{34-5}$$

in which ε^λ goes to 0 when x^4 goes to 0. The derivatives,$(\partial_{44}\,\gamma_{AB})_\Sigma$ will not be modified by such a change of coordinates, whereas the $(\partial_{44}\,\gamma_{\lambda 4})_\Sigma$ can pick up an arbitrary function of (x^u). Upon using an adapted, local coordinate transformation of the type (34-5), in which the φ^j are different on either side of Σ, which is permitted by the structure of V_5, one will see that one can make the possible discontinuities of these second derivatives appear or disappear, which are discontinuities that are devoid of any physical sense then. In particular, one may restrict the $(\partial_{44}\,\gamma_{\lambda 4})_\Sigma$ to be continuous when they cross Σ for the admissible systems of adapted coordinates.

Up to this restriction, one sees that the second derivatives of the potentials are *continuous upon traversing* Σ. The same conclusions may possibly be extended to the successive derivatives of the potentials by differentiating equations (34-3) with respect to x^4. At the conclusion of that operation, one will see only equations (34-3), to the exclusion of equation (34-4).

Having said that, we consider Cauchy data that satisfy the five conditions:

$$(S_\lambda^4)_\Sigma = 0,$$

and assume that we know a $d\sigma^2$ that corresponds to these Cauchy data and satisfies equations (34-3). It results from the conservation identities:

$$D_\alpha S^\alpha{}_\lambda = 0$$

that equations (34-4) are then satisfied outside of Σ. The problem of integrating the Jordan-Thiry equations thus divides into two problems here:

a) The search for Cauchy data that satisfy $S_\lambda^4 = 0$ on S.

b) The integration of the system $R_{AB} = 0$ for Cauchy data that satisfy the conditions of *a*).

Suppose for the moment (which is an abuse, of sorts) that all of the data real-analytic. With the aid of the Cauchy-Kowalewska theorem, one thus easily establishes that, up to a change of local coordinates and assuming that (34-5) preserves the coordinates and the Cauchy data at any point of Σ, the system (34-3) will admit one and only one *cylindrical* analytical solution $\gamma_{\alpha\beta}(x^i)$. The coordinate change permits us to give arbitrary values to the $\gamma_{\lambda 4}(x^i)$ outside of Σ that are compatible with the Cauchy data. The corresponding "physical" theorem of existence and uniqueness has been established by Mme. Fourés under simple differentiability hypotheses.

It results from this study that the *characteristic manifolds S* of the Jordan-Thiry equations on V_4 will always be manifolds that are tangent to the elementary cones of V_4. Ihe characteristic manifolds S_c in V_5 are the manifolds that are generated by the projections of the trajectories onto one S_c and are, in turn, tangent to the elementary cones of V_5. In V_4 and V_5, these manifolds play the roles of *wave surfaces* of the unitary field, respectively. It is the traversing of Σ that can produce discontinuities in the second derivatives of these potentials.

In our penta-dimensional formalism, the phenomena that correspond physically to the pure electromagnetic field schema are completely analogous to the ones that appear in general relativity in the exterior case.

35. – The Cauchy problem in the interior, unitary case. – The Cauchy problem may also be posed in the presence of a non-zero right-hand side. On the manifold V_5, we are therefore led to the following problem:

PROBLEM – *If we are given the potentials $\gamma_{\alpha\beta}(x^i)$ and their first derivatives $\partial_4 \gamma_{\alpha\beta}(x^i)$ on a hypersurface Σ in V_5 then how do we determine the values of these potentials, as well as those of r and v_α, in a neighborhood of Σ, assuming that they satisfy the Jordan-Thiry equations $S_{\alpha\beta} = r\, v_\alpha v_\beta$.*

The hypersurface Σ is always assumed to be non-tangent to the elementary cone of V_5 ($\gamma^{44} \neq 0$). From the relation:

$$R - \tfrac{2}{3} S = -\tfrac{2}{3} r\,,$$

one deduces that:

$$R_{\alpha\beta} = S_{\alpha\beta} + \tfrac{1}{2}\gamma_{\alpha\beta} R = r(v_\alpha v_\beta - \tfrac{1}{3}\gamma_{\alpha\beta})\,.$$

The study we made in general relativity (I, sec. **18**) led us to replace the initial system of the interior, unitary case with an equivalent system that is composed of the union of the following two systems:

(35-1) $$R_{AB} = -\tfrac{1}{2}\gamma^{44}\partial_{44}\gamma_{AB} + F_{AB} = r(v_A v_B - \tfrac{1}{3}\gamma_{AB})$$

and:

(35-2) $$S^4_\lambda = r v^4 v_\lambda\,,$$

in which F_{AB} and S^4_λ do not contain derivatives with respect to x^4 with the index 2. We agree to add the equation:

(35-3) $$\gamma^{\lambda\mu} v_\lambda v_\mu = 1$$

and the inequality $r > 0$.

Any solution ($\gamma_{\alpha\beta}$, v_λ, r) of this system will also satisfy equations (29-5) and (29-6), which express the conservative character of the tensor $\Theta_{\alpha\beta}$. These equations may be put into the form:

(35-4) $$v^\alpha D_\alpha v^\beta = v^4 \partial_4 v_\beta + \Phi_\beta(\text{C.d.}, v_\lambda, \partial_A v_\lambda) = 0 \quad (\text{C.d.} = \text{Cauchy data}),$$

(35-5) $$D_\alpha (r\, v^\alpha) \equiv v^4 \partial_4 r + r\partial_4 v^4 + F(\text{C.d.}, v_\lambda,\ \partial_A v_\lambda, r, \partial_A r) = 0.$$

Having said that, we assume that the Cauchy data $\gamma_{\alpha\beta}(x^i)$ and $\partial_4 \gamma_{\alpha\beta}(x^i)$ are three and two-times continuously differentiable on Σ, respectively. The values of the S^4_λ will be determined on Σ once they are given. It will then be possible to determine the values of r and the v_λ on Σ. From (35-2) and (35-3), one will first have:

$$(r\,v^4)^2 = \gamma^{\lambda\mu}\,S_\lambda^4\,S_\mu^4 .$$

The right-hand side must therefore be positive. We set:

$$\gamma^{\lambda\mu} S_\lambda^4 S_\mu^4 = (\Omega^4)^2 > 0,$$

in such a way that:

(35-6) $$r\,v^4 = \Omega^4.$$

With the aid of (35-2), one deduces from this that:

(35-7) $$v_\lambda = \frac{S_\lambda^4}{\Omega^4}, \qquad v^4 = \frac{S^{44}}{\Omega^4}, \qquad r = \frac{(\Omega^4)^2}{S^{44}}.$$

One will note that the right-hand sides – and, in turn, the left-hand sides as well – depend only upon the variables (x^u), which confirms the *invariance of r and the vector* **v** *with respect to the group of isometries of* V_5 . Since the scalar r must be positive, one must also have $S^{44} > 0$. On the other hand, the Cauchy data determine the v_λ up to sign, namely, the indeterminacy in the sign of Ω^4. We assume that this sign has been chosen once and for all.

Equations (35-1) then provide values for the derivatives $\partial_{44}\,\gamma_{\alpha\beta}$ on Σ; Since $S^{44} > 0$ – and, in particular, it is non-zero – we have that $v^4 \neq 0$. As a result, equations (35-4) and (35-5) will provide values for the derivatives $\partial_4 v_\lambda$ and $\partial_4 r$ on Σ, respectively. It results from this that the quantities v_λ, r, $\partial_{44}\,\gamma_{AB}$, r, $\partial_4 v_\lambda$, $\partial_4 r$ will have well-defined values on a hypersurface S that satisfies the hypotheses we made and *cannot be discontinuous when traversing* Σ. The same conclusions will possibly extend to the values of the higher derivatives of a solution ($\gamma_{\alpha\beta}$, v_λ, r) on S by differentiating either (35-1) or (35-5) and (35-5) with respect x^4.

One deduces the values of the quantities u^i, ρ, μ on the hypersurface S in V_4 immediately from the values of r and v_λ . Since $h = v_0$, the formulas of sec. **30** and **32** will imply that:

$$u^i = \frac{v^i}{\sqrt{1+\dfrac{v_0^2}{\xi^2}}}, \qquad \chi\rho = r\left(1+\frac{v_0^2}{\xi^2}\right), \qquad \chi\mu = \beta v_0 r\sqrt{1+\frac{v_0^2}{\xi^2}} .$$

Having said that, consider a set ($\gamma_{\alpha\beta}$, v_λ, r) that satisfies equations (35-1), (35-4), and (35-5) *in a neighborhood* of Σ and equations (35-2) and (35-3) *on* Σ. Because of the conservation conditions, an argument that is identical to the one that was made in I, sec. **18** will show that (35-2) and (35-3) are then satisfied outside of Σ. From the viewpoint of integrating equations (35-1), (35-4), and (35-5) that has preoccupied us, this fact will therefore suffice for us to establish that there again exists a "physically" unique solution to the Cauchy problem, provided that our are hypotheses are satisfied, namely:

$$\gamma^{44} \neq 0, \qquad (\Omega^4)^2 = \gamma^{\lambda\mu} S^4_\lambda S^4_\mu > 0, \qquad S^{44} > 0.$$

We now examine what sort of hypersurface Σ might produce discontinuities when one traverses it *with a given interior, unitary field* (r finite ≠ 0). One will observe this phenomenon:

a) When Σ is tangent to the elementary cone of V_5 or is a *characteristic manifold* S_c ($\gamma^{44} = 0$).

b) When $\Omega^4 = 0$, which, from (35-6), entails that $v^4 = 0$, and, in turn, $S^4_\lambda = 0$. The surface Σ will be tangent to a penta-dimensional streamline or *will be generated by such streamlines*.

If $S^{44} = 0$, with r remaining finite, then one will have $\Omega^4 = 0$, and one will come back to the preceding case. Therefore, other than Σ_c, the exceptional hypersurfaces here will consist of the hypersurfaces Σ_1 that are generated by the penta-dimensional streamlines. They will have corresponding hypersurfaces S_1 in V_4, along which $u^4 = 0$; i.e., they will be generated by spacetime streamlines.

The phenomena that were studied here that correspond physically to the electromagnetic field-pure matter schema are therefore completely analogous in V_5 to the ones that appeared in general relativity in the pure matter case.

36. – Matching conditions and the prolongation of the interior to the exterior. – We propose to present a model on a manifold V_5 that involves several distributions that are connected to charged matter. Each distribution generates a domain that is bounded by a hypersurface Σ. On one side of Σ, there exists a cylindrical metric $d\sigma^2$ that satisfies the Jordan-Thiry equations for the interior, unitary case. In each of these domains, the potentials are continuous relative to an admissible coordinate system, as well as their first derivatives.

What happens when we traverse a hypersurface Σ? Conforming to the general axioms of the theory (see sec. **16**, *a*), we must impose the following conditions, which generalize the Schwarzchild conditions of general relativity.

MATCHING CONDITIONS. – *For any point x of S, there exists an admissible coordinate system whose domain contains x, and is such that the potentials that are defined by $d\sigma^2$ relative to this system are continuous, as well as their first derivatives, when one traverses Σ.*

Since the differentiable manifold V_5, is twice-continuously differentiable, the potentials and their first derivatives are, of course, continuous for any admissible coordinate system and any point of V_5. On the contrary, since the field equations take different forms on either side of Σ, the second derivatives of the potentials will be discontinuous upon traversing Σ.

We give ourselves a $d\sigma^2$ that corresponds to a domain that is bounded by a hypersurface Σ that is generated by the trajectories in the interior unitary case. We propose to find under what condition there would exist an exterior $d\sigma^2$ on the other side of Σ that agrees with the given interior $d\sigma^2$, in the preceding sense, on Σ. We say that we are treating the problem of *prolonging the interior to the exterior.*

Therefore, suppose that there exists such an exterior $d\sigma^2$. Since the hypersurface Σ is defined locally by $x^4 = 0$ in an admissible adapted coordinate system, the quantities S_λ^4 that are associated with the exterior $d\sigma^2$ are identically zero. Now, their values on S depend upon only the potentials, their first derivatives, and their second derivatives with respect to x^4 with an index that is at most 1. From the matching of the interior $d\sigma^2$ and the exterior $d\sigma^2$, the quantities S_λ^4 that are associated with the interior field must also be zero on Σ, and one will have:

$$\Theta_\lambda^4 = r\, v^4 v_\lambda = 0$$

on Σ. One deduces from this that Σ is necessarily such that:

$$v^4 = 0 ;$$

i.e., that Σ, which is generated by trajectories, is also generated by the penta-dimensional streamlines of the interior field.

Conversely, assume that this is true. Since the hypersurface Σ is generated by streamlines, it will admit tangent plane that cuts the elementary cone of V_5 ; we say that it is time-oriented. On the other hand, on Σ ($x^4 = 0$), the quantities S_λ^4 that are associated with the interior field are zero. An exterior $d\sigma^2$ that agree with the interior $d\sigma^2$ on Σ is therefore a solution of the exterior Cauchy problem relative to Σ and the Cauchy data that are provided on Σ by the interior field, which are data that satisfy the conditions:

$$S_\lambda^4 = 0$$

on Σ. One knows that under these conditions this problem will admit a physically unique solution locally.

We state:

THEOREM – *In order for the prolongation of the interior to the exterior when one traverse a hypersurface Σ that is generated by the trajectories of the interior field to admit a solution, it is necessary that Σ be generated by the penta-dimensional streamlines of the interior field. That condition is sufficient for the local existence of an exterior solution.*

37. – Geodesic principle in the Jordan-Thiry theory. – Suppose that there exists an interior, unitary field and an exterior, unitary field that agree on a hypersurface Σ, which is generated by trajectories. The hypersurface Σ will then be generated by the penta-dimensional streamlines of the interior field, which are streamlines that are geodesics of the interior field, and which are oriented so that $d\sigma^2 > 0$, and from the matching conditions, they will also therefore be geodesics of the exterior field that are oriented such that $d\sigma^2 > 0$. Thus, *the hypersurface* Σ *will necessarily be generated by geodesics of the exterior field that are oriented so that* $d\sigma^2 > 0$.

We consider a small charged particle in a given exterior, unitary field. In V_5, this particle will describe a domain that is bounded by Σ and has a very small section in V_4 that is generated, on the one hand, by trajectories of V_5, and, on the other hand, by exterior geodesics that are oriented such that $d\sigma^2 < 0$. If one passes to the limit and neglects the section in V_4 then one will sees that S reduces to a two-dimensional surface in V_5 that is always generated, on the one hand, by trajectories of V_5, and, on the other hand, by geodesics of the exterior $d\sigma^2$ that are oriented such that $d\sigma^2 > 0$. One will note that in order for this condition to be satisfied, it will suffice that the trajectories that generate Σ sweep out a geodesic Γ of the given exterior $d\sigma^2$ that is oriented such that $d\sigma^2 > 0$. Indeed, let x be a point of Σ, and let x_Γ be the point of Γ that projects onto the same point as x on V_4. Apply the canonical isometry of V_5 that takes x_Γ to x to Σ and Γ. Σ is invariant under this isometry, and Γ is transformed into a geodesic of the same type that passes through x. One sees that in order to determine the motion of a charged particle in V_5, it will suffice to start with a geodesic of V_5 that is oriented such that $d\sigma^2 > 0$ and project it onto V_4, conforming to the considerations of the preceding chapter. We state:

GEODESIC PRINCIPLE – *The geodesics in* V_5 *of an exterior unitary field that are oriented so that* $d\sigma^2 > 0$ *may be interpreted as the penta-dimensional trajectories of charged particles in this unitary field.*

One sees that, as in general relativity, this principle is a consequence of, on the one hand, the conservation conditions – i.e., the field equations – and, on the other hand, the matching conditions. We therefore find ourselves in complete agreement with the statements of sec. **15**.

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CHAPTER III

GLOBAL STUDY OF UNITARY FIELDS

38. – Global propositions in unitary theory. – In the Jordan-Thiry theory, a *spacetime model* consists of a Riemannian manifold V_5 that the hypotheses of secs. **16** and **19**. In particular, we note the following circumstances:

a) In the domains of V_5 that are swept out by a matter distribution – charged or not – and bounded by the frontier hypersurfaces Σ that are generated by the trajectories, the metric that describes the field will satisfy the Jordan-Thiry equations of the interior unitary case.

b) In the domains of V_5 that are not swept out by any matter distribution, the metric will satisfy the Jordan-Thiry equations of the exterior unitary case, viz., $S_{\alpha\beta} = 0$.

c) The potentials and their first derivatives will be continuous upon traversing a hypersurface Σ, in accord with the matching conditions.

Upon starting with this notion of a spacetime model, one will be led to look for hypotheses under which the following propositions are valid, and for the same reasons as in general relativity:

PROPOSITION (AU) – *The introduction of a matter distribution – charged or not – into a given exterior, unitary field may be performed only in domains in which that field is not regular.*

PROPOSITION (BU) – *An everywhere-regular, exterior, unitary field is trivial.*

By *trivial*, we mean a field that is described by a metric that is locally-Euclidean on V_5, so its electromagnetic field will be zero, as well as the locally-Euclidian metric on V_4 (in such a way that one may say that the gravitational field is locally zero).

In this chapter, we propose to establish the preceding propositions in the case of stationary fields and under hypotheses that will be specified in detail along the way. We commence by occupying ourselves with proposition (BU), which is both the more interesting one and the more delicate to achieve.

I. – STATIONARY, EXTERIOR, UNITARY FIELDS

39. – Notion of a stationary, unitary field. – In accord with the considerations that were developed in general relativity, a stationary, unitary field will be described by a Riemannian manifold V_5 that satisfies the following hypotheses:

a) The manifold V_5 is homeomorphic to the topological product of a manifold V_4 of class C^4 with a circle T^1 under a homeomorphism of class C^4. It is endowed with a Riemannian metric of class C^3 that is of the hyperbolic normal type:

$$d\sigma^2 = \gamma_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad (\alpha, \beta = 0, 1, 2, 4) \tag{39-1}$$

and admits a global, one-parameter group of isometries whose trajectories, which are oriented so that $d\sigma^2 < 0$, are the images of the factor lines. One uses adapted coordinates (x^0, x^i) ($i, j, \ldots = 1, 2, 3, 4$) such that the Killing vector $\boldsymbol{\xi}$, which is the infinitesimal generator of this isometry group admits the components:

$$\xi^0 = 1, \qquad \xi^i = 0, \qquad (i = 1, 2, 3, 4),$$

and the sections x^0 = const. will be images of the factor manifolds and, as a result, they will be homeomorphic to V_4. The $\gamma_{\alpha\beta}$ will be functions of only the (x^i) in these coordinates.

The quotient manifold V_4 is endowed with the structure of a Riemannian manifold of class C^3 and hyperbolic normal type by way of the quotient metric:

$$ds^2 = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1, 2, 3, 4), \tag{39-2}$$

that was defined in sec. **17**. We find the intrinsically-defined scalar ξ and the antisymmetric tensor F_{ij} on V_4 . If W_4 denotes an arbitrary section that is associated with $\boldsymbol{\xi}$ in a system of adapted coordinates then this manifold can be considered to be a Riemannian manifold with the metric (39-2). Other than the preceding elements (or rather, their images), one finds the vector-potential φ^i defined it.

b) Let W_4 ($x^0 = 0$) be a section of a well-defined system of adapted, local coordinates. We suppose that W_4 *admits a global, one-parameter group of isometries that leave the scalar* ξ *and the vector-potential* φ_i *invariant.* The manifold W_4 is assumed to be homeomorphic to the topological product of a manifold V_3 of class C^4 and the real line $\mathbb{R}$ by a homeomorphism of class C^4 in such a way that the group trajectories refer to the linear factors. For a system of local coordinates (x^i) on W_4 (and, as a result, on V_4), we choose a system of coordinates (x^u, x^4) ($u, v = 1, 2, 3$) that are adapted to the group action on W_4. The manifold V_3 is endowed with the structure of a Riemannian manifold by the quotient metric of class C^3:

$$d\dot{s}^2 = \dot{g}_{uv}\, dx^u dx^v \qquad (u, v = 1, 2, 3). \tag{39-3}$$

c) Under these conditions, the manifold V_5, which is homeomorphic to the product $V_4 \times T^1$, will be homeomorphic to $V_3 \times \mathbb{R} \times T^1$, and, as a result, it will be homeomorphic to $V_4' \times \mathbb{R}$, in which V_4' denotes the topological product $V_3 \times T^1$. If (x^0, x^u, x^4) denotes the system of coordinates on V_5 that just introduced then the lines x^0 = const., x^u = const. are the ones that must refer to the linear factors in this latter homeomorphism. One will note that in this coordinate system one has:

$$(\partial_4 \gamma_{\alpha\beta})_{x^0=0} = 0$$

on W_4, and that, on the other hand, from the existence of the canonical isometries that were described in *a*):

$$\partial_{04} \gamma_{\alpha\beta} = 0 .$$

One deduces from this that:

$$\partial_4 \gamma_{\alpha\beta} = 0$$

on V_5 . Thus, the lines x^A = const. (A, B, …, = 0, 1, 2, 3) will be the trajectories of a global isometry group of V_5 . We assume that these trajectories are *oriented so that* $d\sigma^2 > 0$, and call them the *time lines* of V_5 . One sees that since:

$$g_{44} = \gamma_{44} + \beta^2 \xi^2 (\varphi_4)^2,$$

the projections of these lines on V_4 are oriented so that $ds^2 > 0$ (i.e., in time). They are called the spacetime timelines. By an argument that is based in the hyperbolic normal type of W_4, it will immediately result that the metric (39-3) on V_3 is negative-definite.

d) Let $\boldsymbol{\xi}$ be the Killing vector field that is the infinitesimal generator of this new one-parameter isometry group on V_5, whose trajectories are oriented so that $d\sigma^2 > 0$. In the coordinates (x^0, x^u, x_4) that we introduced, this vector admits the components:

(39-4) $$\zeta^0 = 0, \qquad \zeta^u = 0, \qquad \zeta^4 = 0,$$

while the vector $\boldsymbol{\xi}$ admits the components:

(39-5) $$\xi^0 = 1, \qquad \xi^u = 0, \qquad \xi^4 = 0.$$

The manifolds x^0 = const. are images of the factor-manifolds that are homeomorphic to V_4 under the homeomorphism of V_5 with $V_4 \times T^1$. The manifolds x^4 = const. are images of the factor-manifolds that homeomorphic to V_4 under the homeomorphism of V_5 with $V_4' \times \mathbb{R}$.

We call systems of coordinates that enjoy the preceding two properties *totally-adapted* coordinates. A change of local coordinates that preserves the values (39-4) and (39-5) of the components of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ will necessarily have the form:

(39-6) $$x'^u = f'^u(x^v), \qquad x'^0 = x^0 + \psi(x^u), \qquad x'^4 = x^4 + \theta(x^u).$$

As a result, totally-adapted coordinates will be defined up to the change (39-6), in which we let ψ and θ be the restrictions of the functions that were defined on V_3 to the neighborhoods. It is these coordinates, which are simultaneously adapted to both isometry groups, that shall use in the sequel. The $\gamma_{\alpha\beta}$ depend upon only the (x^u) in these coordinates.

e) As we have seen (see I, sec. **2** and **63**), the metric of the quotient manifold may be defined conveniently in adapted coordinates with the aid of the associated contravariant tensor. This is why one may define the metric ds^2 of hyperbolic normal type on V_4 by the condition:

$$g^{ij} = \gamma^{ij} \qquad (i, j = 1, 2, 3, 4),$$

and hence the negative-definite metric $(d\dot{s})^2$ on V_3 by the condition:

$$\dot{g}^{uv} = g^{uv} = \gamma^{uv} \qquad (u, v = 1, 2, 3).$$

Consider the manifold V_4, which is the quotient manifold of V_5 by an isometry group with trajectories that are oriented so that $d\sigma^2 > 0$. It is endowed with the structure of a Riemannian manifold with a negative-definite metric:

$$ds^2 = \dot{g}_{AB}\, dx^A\, dx^B \qquad (A, B = 0, 1, 2), \tag{39-7}$$

in which the associated contravariant tensor is such that:

$$\dot{g}^{AB} = \gamma^{AB},$$

and it admits a global, one-parameter group of isometries $x^u \to x^u$, $x^0 \to x^0 + h$, whose trajectories are homeomorphic to T^1. It is clear that the metric $(d\dot{s})^2$ on V_3 can also be defined by starting with the Riemannian manifold V'_4 and its isometry group, since:

$$\dot{g}^{uv} = \gamma^{uv} = g'^{uv}.$$

40. – Complete, stationary, exterior, unitary fields. – The stationary, unitary field that was just described will be called *spatially-complete* – or, more briefly, *complete* – if the Riemannian manifold V_3 is a complete manifold.

In the sections that follow, we imagine that the stationary fields are assumed to be *complete* and *exterior*, in the sense of the Jordan-Thiry theory; i.e., they satisfy the equations $S_{\alpha\beta} = 0$ or $R_{\alpha\beta} = 0$ everywhere.

Thanks to the existence of the group of isometries on V_5 with trajectories that are oriented so that $d\sigma^2 > 0$ (the stationary character of V_5), we can begin the task of writing down the equations by means of the tensors that are defined on V_4. As we have remarked (I, sec. **70**), equations [I, (69-5)] are not modified in form when one passes from the positive-definite signature to the hyperbolic normal signature with trajectoties that are oriented so that $d\sigma^2 > 0$. We adopt the notations of I, sec. **69** by referring to the elements that relate to V'_4 and its metric ds'^2 by a ' and not a ˙. We substitute the notations φ'_A and H'_{AB} for φ^i and H_{ij}, and we must replace ξ with ζ. Upon annulling the components of the Ricci tensor of V_5 relative to orthonormal frames that are adapted to the stationary character of V_5, from (I, 69-5), one will thus obtain:

(40-1) $$\bar{R}'_{AB} - \frac{1}{\zeta}\nabla'_{AB}(\partial_A \zeta) - \frac{\zeta^2}{2}\bar{H}'^{\,C}_{A}\,\bar{H}'_{AB} = 0\,,$$

(40-2) $$\nabla'_B(\zeta^2 \bar{H}'^{\,B}_{A}) = 0,$$

(40-3) $$-\frac{1}{\zeta}\Delta'\zeta + \frac{\zeta^2}{2}H'^2 = 0\,, \qquad (H'^2 = \frac{1}{2}\bar{H}'_{AB}\,\bar{H}'^{AB} \geq 0)\,,$$

in which the tensors that were introduced on V'_4 are defined by their components in orthornormal frames.

Let W'_4 be a section x^4 = const. of V_5 . On this section, one has, moreover, as a consequence of the field equations and from (24-6) and (24-7) (in which one sets $\beta = 1$):

(40-4) $$\mathrm{div}'\mathbf{p}' = \frac{\zeta^3}{2}H'^2\,,$$

with:

(40-5) $$\bar{p}'_A = \frac{\zeta^3}{2}\bar{\varphi}'_B\,\bar{H}'^{\,B}_{A}\,.$$

41. – Another form for equation (40-3). – In totally adapted local coordinates, the scalars ζ and H'^2 depend only on the variables (x^u). They are therefore intrinsically defined on V_3 as functions with scalar values. We propose to transform equation (40-3) in such a way that the only operations that are involved will be the ones that are defined on V_3 . In order to avoid sign difficulties in the course of calculation, we consider a Riemannian manifold with a positive-definite metric.

We shall therefore establish the following lemma:

LEMMA – *Let V'_4 be a Riemannian manifold with a positive-definite metric ds'^2 that admits a one-parameter group of isometries, and let V_3 be the quotient manifold with the metric $(d\dot{s})^2$. If ζ is a scalar on V'_4 that is invariant under the group of isometries then the Laplacian $\Delta'\zeta$ of* z *on V'_4 is related to the Laplacian $\dot{\Delta}\zeta$ on V_3 by the relation:*

(41-1) $$\Delta'\zeta = \dot{\Delta}\zeta + \frac{1}{\zeta}\,\bar{g}^{uv}\,\partial_u\zeta\,\partial_v\zeta\,.$$

We prove this in an orthonormal frame of V'_4 that is adapted to the group of isometries. The gradient vector of ζ on V'_4 ($\partial_0\zeta = 0,\ \partial_u\zeta$) is invariant under the group of isometries. As a result, its divergence on V'_4 will be given by the lemma of sec. **33**. One will therefore have:

$$\Delta'\zeta = \frac{1}{\zeta}\,\bar{g}^{uv}\,\dot{\nabla}_u(\zeta\,\partial_v\zeta)\,.$$

One deduces from this, upon expanding the derivative that appears in the right-hand side, that:

$$\Delta'\zeta = \bar{g}^{uv}\,\dot{\nabla}_u(\zeta\,\partial_v\zeta) + \frac{1}{\zeta}\,\bar{g}^{uv}\,\partial_u\zeta\,\partial_v\zeta\,,$$

namely:

$$\Delta'\zeta = \dot{\Delta}\zeta + \frac{1}{\zeta}\,\bar{g}^{uv}\partial_u\zeta\,\partial_v\zeta\,,$$

and our lemma is proved.

In order to effect the passage from a positive-definite metric to a negative-definite metric, one must substitute $i\zeta$ for ζ and perform multiplications by i or $-i$ on all of the indices, depending on whether they are contravariant or covariant, resp. One will see that formula (41-1) is not modified by that rule. Upon accounting for the expression for $\Delta'\zeta$ in (40-3), one may put this equation into the form:

(41-2) $$-\dot{\Delta}\zeta = \frac{1}{\zeta}\,\bar{g}^{uv}\partial_u\zeta\,\partial_v\zeta - \frac{\zeta^3}{2}H'^2\,.$$

One will note that since z is positive the right-hand side of (41-2) is negative or null for the signature envisioned.

42. – Case in which the space V_3 is compact. – Having said this, consider a stationary field that is described on a Riemannian manifold V_5, *whose space V_3 is compact*. On this compact manifold, the function ζ is such that $-\dot{\Delta}\zeta \geq 0$. It attains its minimum at a point of V_3, and as a result, it will necessarily reduce to a constant. One therefore has:

$$\zeta = \text{const.} = \zeta_0\,, \qquad \bar{H}'_{AB} = 0\,,$$

and, from (40-1), the field equations will reduce to:

$$\bar{R}'_{AB} = 0\,.$$

Therefore, there locally exist adapted coordinates such that the metric on V_5 takes the form:

$$d\sigma^2 = \zeta_0^2(dx^4)^2 + ds'^2\,,$$

and the manifold V'_4 with metric ds'^2 admits a zero Ricci tensor. Thanks to the existence of a group of isometries on V'_4, we can express this last condition be means of tensors that are defined on V_3. Here, we once more use equations of the type [I, 969-5)]; in our notations, we substitute η ($\eta^2 = -\dot{g}_{00}$) for ξ and the antisymmetric tensor K for the antisymmetric tensor H. Since the signature is negative-definite and, as a result, the trajectories are oriented so that $ds'^2 < 0$, these equations will give:

(42-1) $$\dot{\overline{R}}_{uv} - \frac{1}{\eta}\dot{\nabla}_v(\partial_u \eta) + \frac{\eta^2}{2}\overline{K}_u{}^w\overline{K}_{vw} = 0,$$

(42-2) $$\dot{\nabla}_v(\eta^3\overline{K}_u{}^v) = 0,$$

(42-3) $$-\frac{1}{\eta}\dot{\Delta}\eta = \frac{\eta^2}{2}K^2 \qquad (K^2 = \tfrac{1}{2}\overline{K}_{uv}\,\overline{K}^{uv} \geq 0).$$

in an orthonormal frame on V_3.

Equation (42-3) shows us that the function η is such that $-\dot{\Delta}\eta \geq 0$ on V_3.

It necessarily results from this that this function reduces to a constant and, in turn, that $\overline{K}_{uv} = 0$. From (42-1), the manifold V_3 will therefore have a zero Ricci tensor and since it is three-dimensional, it will be locally Euclidean. Since $\overline{K}_{uv} = 0$, there will exist adapted coordinates on V_4' locally, for which:

$$ds'^2 = -\eta_0^2(dx^0)^2 + (d\dot{s})^2,$$

and, as a result, they will exist locally on the manifold V_5 over a neighborhood of V_3 in totally-adapted coordinates such that:

$$d\sigma^2 = \zeta_0^2(dx^4)^2 - \eta_0^2(dx^0)^2 + (d\dot{s})^2,$$

and one will see that the manifold V_5 is locally Euclidean. The electromagnetic field that is described by V_5 is zero, and the metric on V_4 may be locally written:

$$ds^2 = \zeta_0^2(dx^4)^2 + (d\dot{s})^2,$$

is locally Euclidean. The exterior unitary field that is envisioned is necessarily trivial. We state:

THEOREM – *In the case of a compact space V_3 an everywhere regular stationary exterior unitary field is necessarily trivial.*

43. – Asymptotically-Euclidean behavior of a stationary, unitary field. – We now assume that the complete, Riemannian manifold V_3 *admits a domain at infinity*.

Consider a three-dimensional Euclidean space $\mathcal{E}_3$ that admits a negative-definite metric $d\bar{s}^2$. We assume that they are referred to a privileged coordinate system (y^u), for which:

$$d\bar{s}^2 = \delta_{uv}\, dy^u\, dy^v,$$

in which:

$$\delta_{uv} = 0 \text{ for } u \neq v, \qquad \delta_{uu} = -1.$$

We say that the stationary field envisioned admits asymptotically-Euclidean behavior when, for a point a of V_3 and a sufficiently large number:

1. There exists a homeomorphism h of class C^2 from the domain $d(a, x) > \mathbb{R}$ of V_3 onto a domain of $\mathcal{E}_3$ whose complement is homeomorphic to a closed ball (this homeomorphism thus defines the structure of a Euclidean space on the domain of V_3 envisioned);

2. One may find sections x^0 = const., x^4 = const. of V_5 such that for the privileged system of totally-adapted coordinates $(\bar{y}^\alpha)$ that are defined on the domain of V_5 over the domain $d(a, x) > \mathbb{R}$ of V_3, the potentials and their first derivatives relative to this system satisfy the inequalities:

$$(43\text{-}1) \qquad |\, \gamma_{\alpha\beta} - \delta_{\alpha\beta} \,| < \frac{M}{r} \qquad |\, \partial_\gamma \gamma_{\alpha\beta} \,| < \frac{M}{r^2} \qquad [r = d(a, x);\, x \in V_3],$$

in which M is a positive number and in which $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$, $\delta_{00} = -1$, $\delta_{44} = +1$.

It is clear that the point a plays only an auxiliary role here. If x is a point of V_3 for which $r = d(a, x) > \mathbb{R}$, and if y is its image in $\mathcal{E}_3$ under h then one will establish, as in (I, sec. **88**), that, by virtue of the inequalities (43-1) between r and the ordinary distance ρ from y to the origin O in $\mathcal{E}_3$:

$$\rho = \sum_u (y^u)^2 ,$$

one will have the inequality:

$$(43\text{-}2) \qquad \rho < Kr,$$

in which K denotes a fixed number.

44. – Study of the flux vector $\mathbf{p}'$ of W_4. – Let r_0 be a fixed number that is greater than R, and let S_{r_0} be the set of points of V_3 such that $d(a, x) > r_0$. Consider a sphere Σ_p of center O and radius ρ in $\mathcal{E}_3$, and take ρ sufficiently large that Σ_p will contain the image of S_{r_0} under h. We denote the image of the sphere Σ_p in V_3 by $\mathcal{S}_p$.

Consider the compact subset B_ρ of V_3 that is defined by:

a) The points x of V_3 for which $d(a, x) \leq r_0$;

b) The points x for which $d(a, x) \geq r_0$, whose image in $\mathcal{E}_3$ is interior to Σ_p or that sphere.

The boundary of B_ρ is $\mathcal{S}_p$, oriented outward.

Having said that, consider one of the sections x^4 = const. that was introduced in 2. of the preceding section, which is a section that we will denote by W_4'. The points of W_4' that project onto the points of B_ρ in V_3 define a compact subset that is homeomorphic to the product $B_\rho \times T^1$, and whose boundary ∂C_ρ is homeomorphic to the product $\mathcal{S}_p \times T^1$. Let $\mathbf{p}'$ be the vector on W_4 that was introduced in sec. **40**, whose components are:

$$\bar{p}_A' = \frac{\zeta^3}{2}\,\bar{\varphi}_B'\,\bar{H}_A'^{\,B}.$$

We propose to study the behavior of the flux of $\mathbf{p}'$ upon traversing ∂C_ρ when $\rho \to \infty$.

As far as the modulus p' of the vector $\mathbf{p}'$ is concerned, it results from the inequalities (43-1), by an argument that is identical to the one in (I, sec. **89**), that there exists a fixed number C_1 such that:

$$p' < \frac{C_1}{r^3} \qquad (r = d(a, x);\ x = \text{projection of the point in } W_4'),$$

and, as a result, from (43-2), that there will exist a finite number C_2 such that:

(44-1) $$p' < \frac{C_2}{r^3}.$$

On the other hand, the area of ∂C_ρ under the metric ds'^2 on W_4', namely:

$$\int_{\partial C_\rho} d\Sigma_3\,,$$

in which $d\Sigma_3$ is the area element of this metric, satisfies the relations:

(44-2) $$\left|\int_{\partial C_\rho} d\Sigma_3\right| = \left|\int_{\partial C_\rho}\left[1 + O\left(\frac{1}{r}\right)\right] dx^0 \wedge d\bar{\Sigma}\right| < C_3 \int_{T^1} dx^0 \cdot \int_{\Sigma_\rho} d\bar{\Sigma} < C_4 \cdot \rho^2,$$

in which C_3 and C_4 denote a fixed number and $d\bar{\Sigma}$ is the Euclidean area element for the metric $-d\bar{s}^2$ on $\mathcal{E}_3$. One deduces from (44-1) and (44-2):

(44-3) $$\lim_{\rho\to\infty} \text{flux}_{\partial C_\rho} \mathbf{p}' = 0,$$

since this flux is evaluated with the metric ds'^2 on W_4'.

45. – Case in which the space V_3 admits a domain at infinity. – For the case in which the complete manifold V_3 admits a domain at infinity, we consider a stationary, exterior field with asymptotically-Euclidean behavior. On the manifold W_4' that was introduced in sec. **44**, one has:

$$\text{div}'\mathbf{p}' = \frac{\zeta^3}{2} H'^2 .$$

By integrating over C_ρ when ρ is sufficiently large, one will obtain:

$$\text{flux}_{\partial C_\rho} \mathbf{p}' = \int\iiint_{C_\rho} \frac{\zeta^3}{2} H'^2 d\tau' \qquad (H'^2 \geq 0),$$

in which $d\tau'$ denotes the volume element of W_4'. Suppose that H'^2 is positive at a point of W_4'. It will then be positive in a certain neighborhood D of that point, and for a sufficiently large ρ, one will have:

$$\text{flux}_{\partial C_\rho} \mathbf{p}' \geq \int\iiint_{D} \frac{\zeta^3}{2} H'^2 d\tau' .$$

Now, the left-hand side goes to zero when $\rho \to \infty$, which is in contradiction with the preceding inequality. One therefore has $H'^2 = 0$. Equation (41-2) may then be written:

$$L(\zeta) \equiv -\dot{\Delta}\zeta - \left(\frac{1}{\zeta}\dot{g}^{uv}\,\partial_v\zeta\right)\partial_u\zeta = 0,$$

and the first theorem of (I, sec. **82**) may be applied to the function ζ on the complete Riemannian manifold V_3, which tends to 1 in the domain at infinity. One therefore necessarily has $\zeta = 1$, and from (40-1), $\bar{R}'_{AB} = 0$. Therefore, there will locally exist adapted coordinates such that the metric on V_4 takes the form:

$$d\sigma^2 = (dx^4)^2 + ds'^2,$$

and the manifold W_4' with the metric ds'^2 admits a zero Ricci tensor. We find ourselves being within the scope of the conditions in sec. **42**, and we write equations (42-1), (42-2), (42-3). Consider a section W_3' of W_4' that is defined by x^0 = const. There exists a vector $\overline{q}_u$ on this section, with components:

$$\overline{q}_u = \frac{\eta^3}{2}\overline{\psi}_u \overline{K}_u{}^v ,$$

in which ψ plays the role of φ, which is such that:

$$\operatorname{div}\mathbf{q} = \frac{\eta^3}{2}K^2 .$$

If B_ρ' denotes the compact subset of W_3' that projects onto V_3 along B_ρ, and if $\mathcal{S}_\rho'$ is its boundary then one will establish, as in (I, sec. **89**), that:

$$\lim_{\rho\to\infty} \operatorname{flux}_{\mathcal{S}_\rho'} \mathbf{q}' = 0 .$$

One immediately deduces from this that $\overline{K}_{uv}$ = 0. Since equation (42-3) reduces to $\dot{\Delta}\eta = 0$, and η goes to 1 uniformly in the domain at infinity in V_3, one will necessarily have $\eta = 1$, and that result will be obtained as in sec. **42**. We state:

THEOREM – *An everywhere-regular, complete, stationary, exterior, unitary field with asymptotically-Euclidean behavior is necessarily trivial.*

II. – A THEOREM ON STATIONARY SPACETIME MODELS

46. – Matching stationary, unitary fields. – We now propose to study proposition (AU). As in general relativity, an exterior, unitary field that prolongs a stationary field upon traversing a hypersurface Σ that is generated by the timelines of that field is itself locally stationary. Here, we therefore concern ourselves with unitary fields that are interior or exterior stationary.

In each domain where the field satisfies the Jordan-Thiry equations of a particular case – interior or exterior – the differentiability hypotheses will be the ones that we made before. Upon traversing the hypersurface Σ that separates a domain of V_5 that is swept out by a matter distribution from a domain that is not, the admissible coordinate systems will now be only (C^2, piecewise-C^4), and the metric $d\sigma^2$ is of class (C^1, piecewise-C^3).

Let (x^α) be a system of coordinates that are totally adapted to the interior field; for example, ones with Σ defined locally by the equation $x^1 = 0$. For the exterior, unitary field, one may obtain a system of adapted, local coordinates in a neighborhood of Σ that presents a second-order contact with the preceding ones along Σ; i.e., they satisfy:

$$x'^{\alpha} = x^{\alpha} + \frac{(x^1)^3}{6}[\varphi^{\lambda}(x^J) + \varepsilon^{\lambda}(x^{\mu})] \qquad (J = 0, 2, 3, 4), \tag{46-1}$$

in which $\varepsilon^{\lambda} \to 0$ when $x^1 = 0$. One may say that equations (46-1) express the agreement of the isometries for the two fields on Σ.

47. – Study of R_4^4 for a section W'_4 that is oriented so that $d\sigma^2 < 0$. – Consider a stationary, interior, unitary field, and let W'_4 be an arbitrary section of the timelines that are *oriented so that* $d\sigma^2 < 0$. The datum of this section determines the data of the system of sections that are deduced from it by isometries of V_5 whose trajectories are timelines. In an associated system of totally adapted coordinates, one will have:

$$\gamma^{44} > 0. \tag{47-1}$$

One recalls, moreover, that since the timelines are oriented in time one has $\gamma_{44} > 0$, and, as a result, that the quadratic form of the coefficients $\gamma^{AB} = g'^{AB}$ (which is none other than the form that is associated with $ds'^2 = g'_{AB}\, dx^A dx^B$) is negative-definite. *For such a system of sections, R_4^4 will be strictly positive for the Ricci tensor of the field.*

In order to establish this, we first observe that:

$$R_4^4 = S_4^4 + \tfrac{1}{2} R.$$

However:

$$S_4^4 = r v^4 v_4 \qquad\qquad R = -\frac{2}{3}\Theta = -\frac{2}{3} r.$$

One deduces from this that:

$$R_4^4 = r\left(v^4 v_4 - \frac{1}{3}\right) = r\left(v^4 v_4 - \frac{1}{2}\right) + \frac{r}{6}.$$

We evaluate the quantity:

$$v^4 v_4 - \tfrac{1}{2} = \gamma^{44}(v_4)^2 + \gamma^{4A} v_4 v_A - \tfrac{1}{2}[\gamma^{44}(v_4)^2 + 2\gamma^{4A} v_4 v_A + \gamma^{AB} v_A v_B],$$

in which we have accounted for the unitary character of the vector v_α in the right-hand side. One thus obtains:

$$v^4 v_4 - \tfrac{1}{2} = \tfrac{1}{2}[\gamma^{44}(v_4)^2 - \gamma^{AB} v_A v_B],$$

and, as a result:

(47-2) $$R_4^4 = \frac{r}{2}[\gamma^{44}(v_4)^2 - \gamma^{AB} v_A v_B] + \frac{r}{6}.$$

Since r is strictly positive for an interior field, one sees that the component R_4^4 is strictly positive.

Having said that, we may express ζR_4^4 by means of the divergence that one evaluates on W_4'. That equation will not be modified upon transforming equation (24-2) by the rule that corresponds to the hyperbolic normal signature for trajectories that are oriented so that $d\sigma^2 > 0$, and one may write an equation that relates to V_5 and W_4' that differs only by the notations:

(47-3) $$\text{div}'\mathbf{h}' = -\zeta R_4^4,$$

in which the vector $\mathbf{h}'$ on W_4' has the components in adapted coordinates:

(47-4) $$h'_A = \partial_A \zeta + \frac{\zeta^2}{2}\varphi'_B H'^B_A.$$

Formula (47-3) provides the equivalent of Gauss's theorem in our theory of stationary fields.

48. – Existence of singularities for the transition from the exterior to the interior for the unitary field of a matter distribution. – Consider a domain D of V_5 that is bounded by a hypersurface Σ, contains a stationary interior field, and is generated by the time lines. In a neighborhood of Σ, this field will induce a stationary, unitary field that satisfies the Jordan-Thiry equations of the exterior case, and agrees with them on Σ. We propose to show that this last case may not be assumed to be regular in D.

We therefore assume that this field is regular in D and let $d\sigma_i^2$ and $d\sigma_e^2$ be the two corresponding metrics on D that correspond to the interior and exterior field, respectively. Let $W_4'^{(i)}$ be a section relative to the interior field and orient it so that $d\sigma_i^2 < 0$; it will determine a three-dimensional domain D_3 on Σ. One may construct a hypersurface $W_4'^{(e)}$ that passes through D_3 that has a second-order contact with $W_4'^{(i)}$ on D_3 and is transversal to the timelines of the exterior field in D_3. We may adopt $W_4'^{(e)}$ as a section for the exterior field in D and adopt local adapted coordinates for the two fields in a neighborhood that have a second-order contact.

The vectors $\mathbf{h}'$ on D_3 that relate to the two fields are identical since they depend only on the potentials and their first derivatives. Upon applying formula (47-3) to the interior unitary field, one will obtain:

$$\text{flux}_{D_3}\mathbf{h}' = -\int\iiint_{W_4'^{(i)}} \zeta R_4^4 \, d\tau' < 0,$$

in which $d\tau'$ is the volume element relative to ds'^2. This flux is therefore strictly negative, whereas the same formula, when applied to the exterior field, will give:

$$\mathrm{flux}_{D_3}\mathbf{h}' = 0,$$

which leads us to a contradiction. We state:

THEOREM – *If we are given a stationary, interior, unitary field that is bounded by a hypersurface S that is generated by timelines then the stationary, exterior, unitary field that agrees with it on S may not be prolonged regularly to the entire domain of the interior field.*

III. – GLOBAL PROBLEMS IN THE KALUZA-KLEIN THEORY

49. – Global propositions in the Kaluza-Klein theory. – Propositions that are completely analogous to the ones that were stated in sec. **38** present themselves in the theory of electromagnetism in general relativity, or, equivalently, in the penta-dimensional Kaluza-Klein representation of that theory. We are therefore led to seek the hypotheses under which the following propositions are valid:

PROPOSITION (AK) – *The introduction of a matter distribution – charged or not – into a gravitational and electromagnetic field that satisfies the relativistic equations of the pure electromagnetic field schema may be accomplished only in domains where that field is not regular.*

PROPOSITION (BK) – *If a gravitational and electromagnetic field satisfies the relativistic equations of the pure electromagnetic field schema everwhere then that field will be trivial.*

This latter proposition does not differ from the one that we pointed out in general relativity in I, sec. **33**. We begin by specifying the various equations that we need in order to study proposition (BK). In order to permit useful comparisons with Book I, we adopt, not the penta-dimensional viewpoint in the hypotheses and statements, but the same viewpoint as in the relativistic theory of electromagnetism.

50. – The field equations for the pure electromagnetic field schema. – Consider a spacetime on which we have defined a metric of hyperbolic normal type:

$$ds^2 = g_{ij}(x^k)\, dx^i\, dx^j \qquad (i, j = 1, 2, 3, 4),$$

and a global vector-potential φ_i, whose rotation F_{ij} represents the electromagnetic field. We assume that the gravitational field and the electromagnetic field thus-introduced are stationary; i.e., we assume that spacetime admits a global group of isometries whose trajectories are oriented so that $ds^2 > 0$, and which leaves the vector-potential φ_i invariant. The quotient manifold will be designated by V_3.

We introduce a manifold V_5 that is the topological product of spacetime with a circle T^1. If (x) is a system of local coordinates on spacetime and x^0 is the canonical coordinate on T^1 then a point of V_5 will admit local coordinates (x^0, x^i), and the variables x^0 = const. will be the factor manifolds of V_5. The linear factors will be represented by x^i = const.

We identify spacetime with one of the sections W_4 of W_5, and we endow V_5 with the metric that is defined in the chosen local coordinates by:

$$\gamma_{00} = -\xi_0^2, \qquad \gamma_{0i} = -\beta\varphi_i\,\xi_0^2, \qquad \gamma_{ij} = g_{ij} - \beta^2\xi_0^2\varphi_i\varphi_j,$$

in which ξ_0 designates a constant; since g_{44} is strictly positive in adapted coordinates on spacetime, we may choose ξ_0 to be sufficiently small that γ_{44} is strictly positive. We thus obtain a Riemannian manifold for V_5 that has the hyperbolic normal metric:

$$d\mathrm{s}^2 = \gamma_{\alpha\beta}\,dx^\alpha dx^\beta,$$

which enjoys properties that are identical to the ones that were analyzed in sec. **39**. In what follows, we shall use notations that are identical to the ones in that section (with $\xi = \xi_0$). For a convenient choice of β, the equations of the Kaluza-Klein theory for the exterior unitary case relative to orthonormal frames on V_5 that are adapted to the isometry trajectories x^i = const. can be put into the form:

(50-1) $$\bar{S}_{ij} = 0, \qquad \bar{S}_{i0} = 0.$$

Let W_4' be an x^4 = const. section of the manifold V_5. We propose to establish that equation (40-4) is again a consequence of our new field equations on that section. Indeed, one obtains:

$$S_4^A = \bar{A}_\alpha^A\,\bar{A}_4^\beta\,\bar{S}_\beta^\alpha$$

for the components S_4^A (A = 0, 1, 2, 3) of the Einstein tensor of V_5 in totally-adapted coordinates; namely, from (50-1):

$$S_4^A = \bar{A}_0^A\,\bar{A}_4^0\,\bar{S}_0^0.$$

Now $\bar{A}_0^A = 0$. One deduces from this that equations (50-1) for the field entail that:

$$S_4^A = R_4^A = 0.$$

Now, from (24-7), when it translated into the form that corresponds to a negative-definite metric, one will have:

$$\zeta\,\varphi_A'\,R_4^A + \frac{\zeta^3}{2}H'^2 = \mathrm{div}'\,\mathbf{p}' \qquad (\zeta^2 = \gamma_{44}),$$

identically on W_4', with:

(50-2)
$$\bar{p}'_A = \frac{\zeta^3}{2}\bar{\varphi}'_A \bar{H}'^{\,B}_A .$$

One deduces from this that equations (50-1) entail that:

(50-3)
$$\mathrm{div}'\mathbf{p}' = \frac{\zeta^3}{2} H'^2$$

on W_4 .

The field equations entail another interesting consequence that one may establish in penta-dimensional formalism, but which is simpler to establish directly. Let W_4 be an x^0 = const. section of the manifold V_5, and let φ_i be the vector-potential on W_4 . From equations (50-1), one arrives at the Maxwell equations:

$$\nabla_j F^{ji} = 0.$$

One deduces from this that:

$$\nabla_j(\varphi_i F^{ji}) = \tfrac{1}{2} F_{ji} F^{ji} = F^2.$$

on W_4 . Therefore, if $\boldsymbol{\psi}$ denotes the vector on W_4 whose components are:

(50-4)
$$\psi_j = \varphi_i F^{ji}$$

then one will obtain:

(50-5)
$$\nabla_j \psi^j = F^2.$$

Since the vector $\boldsymbol{\psi}$ is invariant under the isometries of W_4, one may express the divergence that appears in the left-hand side of (50-5) by means of a divergence on V_3 with the aid of the lemma of sec. **33**. If $\bar{\psi}^j$ denotes the components of $\boldsymbol{\psi}$ relative to an orthonormal frame on W_4 that is adapted to the group of isometries on that manifold, then one must have:

(50-6)
$$\frac{1}{\eta}\dot{\nabla}_\nu(\eta\bar{\psi}^\nu) = F^2 \qquad (\eta^2 = g_{44}).$$

51. – Case in which the space V_3 is compact and orientable. – Consider a gravitational and electromagnetic field that satisfies the relativistic equations of the pure electromagnetic field schema everywhere on a spacetime that has a compact space V_3 . Since the manifold W_4' is homeomorphic to the product $V_3 \times T^1$, it will also be compact and, from (50-3), one will get:

$$\int_{W_3'} \frac{\zeta^3}{2} H'^2 = 0 .$$

One deduces from this that $\bar{H}'_{AB} = 0$. Therefore, there locally exist adapted coordinates such that the metric on V_5 may be put into the form:

$$d\sigma^2 = \zeta^2 (dx^4)^2 + ds'^2 .$$

In these coordinates, $\gamma_{04} = 0$, $\varphi_4 = 0$, and, as a result:

$$F_{u4} = \partial_u \varphi_4 - \partial_4 \varphi_u = 0$$

in these coordinates. It will then result from a simple calculation (see I, sec. **92**) that:

$$F^2 = \tfrac{1}{2} F_{ji} F^{ji} = \tfrac{1}{2}\, g^{uw} g^{vi} F_{uv} F_{wi} ,$$

and as a result, the scalar F^2 will be *positive or zero, and it will be zero only for a zero electromagnetic field.*

Now, since the manifold V_3 is compact and orientable, one will deduce from (50-6) that:

$$\int_{V_3} F^2 = 0 .$$

We deduces from this that the electromagnetic field is zero, and we come down to the same problem as in the absence of the electromagnetic field, which is a problem that was solved in I, sec. **86**. We state:

THEOREM – *If a stationary, electromagnetic and gravitational field satisfies the equations that relate to the pure electromagnetic field schema then that field will necessarily be trivial in the case of a compact orientable space* V_3 .

52. – Asymptotically-Euclidean behavior. – Now assume that the Riemannian manifold V_3 is *complete and admits a domain at infinity.*

If $\mathcal{E}_3$ and (y^u) denote the same elements as in sec. **43** then we will say that a stationary, gravitational and electromagnetic field admits asymptotically-Euclidean behavior when, for a point a of V_3 and a sufficiently large number R:

1. There exists a homeomorphism of class C^2 of the domain $d(a, x) > R$ in V_3 onto a domain of $\mathcal{E}_3$ whose complement is homeomorphic to closed ball.

2. One can find x^4 = const. sections of W_4 such that for the privileged system of adapted coordinates (y^i) that we defined in the domain W_4 over the domain $d(a, x) > R$ in V_3 by (y^u) and $y^4 = x^4$, the potentials and their first derivatives, the potentials φ_i, and their first derivatives relative to this system will satisfy the inequalities:

$$| g_{ij} - \partial_{ij} | < \frac{M}{r} , \qquad | \partial_k g_{ij} | < \frac{M}{r^2} ,$$

and:

$$|\varphi_i| < \frac{M}{r}, \qquad\qquad |\partial_k \varphi_i| < \frac{M}{r^2},$$

in which the notations are identical to the ones in sec. **43**.

It immediately results from this that the stationary unitary field on the manifold V_5 that is described by the metric $\gamma_{\alpha\beta}$ (in which, one may assume that $\gamma_{00} = -1$ by taking $y^0 = \xi_0\, x^0$), admits asymptotically-Euclidean behavior, in the sense of sec. **43**.

53. – Case in which V_3 **admits a domain at infinity.** – Therefore, consider a stationary gravitational and electromagnetic field that admits asymptotically-Euclidean behavior in the case of a complete space V_3 and presents a domain at infinity. One deduces from equation (50-3), as in sec. **44** and **45**, that:

$$H'^2 = 0 \;.$$

One deduces from this that:

$$F^2 \geq 0,$$

in which equality is attained only for a zero electromagnetic field. Since the notations B'_ρ and $\mathcal{S}'_\rho$ are identical to the ones in sec. **44**, one will deduce from (50-6) that:

$$\mathrm{flux}_{\mathcal{S}'_\rho} \boldsymbol{\pi} = \iiint_{B'_\rho} \eta\, F^2 d\tau',$$

in which $d\tau'$ is the volume element of $(d\dot{s})^2$, and $\pi^\mu = \eta\psi^\mu$. On the other hand, from the asymptotically-Euclidean behavior:

$$\lim_{\rho\to\infty} \mathrm{flux}_{\mathcal{S}'_\rho} \boldsymbol{\pi} = 0.$$

One deduces from this that $F^2 = 0$, and the result will be obtained as before.

THEOREM – *If a stationary gravitational and electromagnetic field satisfies the relativistic equations of the pure electromagnetic field schema for a complete manifold* V_3 *that admits a domain at infinity and presents asymptotically-Euclidean behavior at infinity then that field will necessarily be trivial.*

54. – Proposition (AK) **for stationary fields**. – In the Jordan-Thiry theory, establishing the proposition envisioned for stationary fields will result from the consideration of the R_4^4 component of the Ricci tensor on V_5 for certain systems of totally-adapted coordinates. This component will be zero for an exterior, unitary field and strictly positive for an interior, unitary field.

That will not always be the case in the Kaluza-Klein theory. Indeed, by an easy, but somewhat lengthy, calculation one will obtain:

$$R_4^4 = \tfrac{1}{2}\chi_0 F^2 + \chi_0 \rho \left(u^4 u_4 - \tfrac{1}{2}\right) + \chi_0 \rho\, u^4 u_4 \,.$$

As a result, this component will not be zeroin the absence of a matter distribution, and will not have a well-defined sign in the presence of such a distribution. An argument that is analogous to the one in sec. **48** cannot be carried out then.

Therefore, from this viewpoint, the Jordan-Thiry theory presents a coherence that is better than that of the Kaluza-Klein theory. We shall see that the proposition envisioned may nevertheless be proved for the two theories under the hypothesis of a charged matter distribution for which the charge has *well-defined sign* over the entire distribution.

55. – Proposition (A) **for matter distributions whose charge has a definite sign**. **–** In this last section of the chapter, *we shall no longer assume that the field envisioned is stationary*.

In either of the two theories envisioned here, we consider the Maxwell equations, which we write on spacetime as:

(55-1) $$\nabla_j H^{ji} = \mu\, u^i.$$

We assume that we are carrying out our proof in a four-dimensional domain D_4 that is homeomorphic to the topological product $D_3 \times I$ of a tri-dimensional domain D_3 with an interval I. Let (x^u) be a system of local coordinates in D_3, and let x^4 be the canonical coordinate on I; (x^u, x^4) defines a system of local coordinates for D_4 that we shall use. For $i = 4$, (35-1) may be written:

(55-2) $$\nabla_j H^{j4} = \mu\, u^4.$$

Consider the vector **k** that has components:

$$k^j = H^{j4}$$

in the coordinate system envisioned. One has:

$$\nabla_j H^{j4} = \nabla_j k^j + \Gamma^4_{jk} H^{jk} = \nabla_j k^j,$$

from the symmetry of Γ in its lower indices and the antisymmetry of H. Equation (55-2) may therefore be put into the form:

(55-3) $$\nabla_j k^j = \mu\, u^4,$$

in which the vector **k** has the components:

(55-4) $$k^u = H^{u4}, \qquad k^4 = 0.$$

Consider an interior, unitary field that corresponds to *a charge matter distribution for which the charge density* μ *admits a well-defined sign* and which is bounded by a hypersurface S in spacetime, upon which it agrees with an exterior, unitary field. We assume that one may bound the domain that is swept out by the distribution by two sections of the neighboring streamlines in such a manner as to obtain a domain D_4, which we refer to the local coordinates (x^u, x^4) that we introduced, since the lines x^u = const. will be segments of the streamlines in D_4 and the coordinate changes will be furthermore admissible and have the form:

$$x'^u = f'^u(x^v), \qquad x'^4 = x^4 + \text{const.}$$

The boundary of D_4 is composed of Σ ($x^4 = 0$), Σ' ($x^4 = h$), and a subset T of S. Let $\mathbf{k}$ be the vector that is defined by (55-4) in these coordinates. One will have:

$$\text{flux}_\Sigma\, \mathbf{k} = 0, \qquad \text{flux}_{\Sigma'}\mathbf{k} = 0,$$

and, in turn, from (55-3):

$$\text{flux}_T\mathbf{k} = \int\iiint_{D_4} \frac{\mu}{\sqrt{g_{44}}} \sqrt{|g|}\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 .$$

This flux is therefore essentially non-zero. If the exterior unitary field is regular in D_4 then one will establish, as before, that:

$$\text{flux}_T\, \mathbf{k} = 0,$$

which implies a contradiction. We state:

THEOREM – *If we are given a gravitational and electromagnetic field in a domain* D_4 *that corresponds to a distribution whose charge has a well-defined sign then an exterior, unitary field that it agrees with cannot be regular in* D_4 .

This theorem is just as valid in the Jordan-Thiry theory as it is in the relativistic theory of electromagnetism.

BIBLIOGRAPHY FOR PART ONE

P. BERGMANN, - Ann. Math., **49** (1948), 255.
P. JORDAN. – *Ann. Physik* (1947), 219.
KALUZA. – Sitz. Preuss. Akad. Wiss. (1921), 1966.
O. KLEIN. – Z. Physik, **37** (1926), 895.
A. LICHNEROWICZ and Y. THIRY. – Comptes rendus Acad. Sc., **224** (1947), 529.
W. PAULI. – Ann. Physik, (1933), 305.
Y. THIRY. – Comptes rendus Acad. Sc., **226** (1948), 216 and 1881, and *Thése Paris* (1950).
O. VEBLEN. – *Projektive Relativitätstheorie,* Berlin, Springer, (1933).

H. WEYL. – *Raum, Zeit, Materie*.

CHAPTER IV

II. – THE EINSTEIN-SCHRÖDINGER THEORY

NOTIONS ON SPACES WITH AFFINE CONNECTIONS

56. – Definition of an affine connection. – a) Consider a differentiable manifold V_n of dimension n and class C^r ($r \geq 2$). At each point x of V_n, one can define a vector space T_x of vectors tangent to V_n at x and the vector space T'_x of linear forms at x that is dual to it. One calls an ordered set of n linearly independent vectors of T_x – namely, $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ – a *frame with origin x*, and one denotes it by R^x. Two frames, R^x, with vectors $\mathbf{e}_\alpha$ and R'^x, with vectors $\mathbf{e}_\beta$, may be deduced from each other by way of:

(56-1)
$$\mathbf{e}_{\beta'} = A^\alpha_{\beta'}\,\mathbf{e}_\alpha ,$$

in which the matrix:

$$A = (A^\alpha_{\beta'})$$

is an arbitrary, regular $n\times n$ matrix, and therefore, an element of the linear group G of n real variables. We may translate (56-1) into the abbreviated form:

(56-2)
$$R'^x = R^x A .$$

By duality, every such a frame R^x in T_x corresponds to a frame in T'_x, or *co-frame*, namely θ_x, which is an ordered set of n linearly independent linear forms $\theta^1, \theta^2, \ldots, \theta^n$ at the point x. If θ_x and θ'_x are the co-frames that are dual to R^x and R'^x then one will have:

(56-3)
$$\theta_x = A\theta'_x .$$

b) Having said this, consider an arbitrary covering of V_n by open neighborhoods U. In each U, we are given an ordered set of n linearly independent Pfaff forms $(\theta^\alpha(x))$ of class C^{r-1}. For each x in U, these forms will define a co-frame θ^U_x, and, by duality, a frame R^r_U. In other words, over each admissible U we may choose local sections of the fiber space of coframes of V_n whose structure group is G, or the fiber space of frames.

If U and V are two neighborhoods of V_n, and if $x \in U \cap V$ then there will exist a regular matrix $A^U_V(x)$ of class C^{r-1} such that:

(56-4)
$$\theta^U_x = A^U_V\,\theta^V_x \qquad (x \in U \cap V)$$

and:

(56-5)
$$R^x_V = R^x_U\,A^U_V$$

and one will obviously have:

$$(A^{-1})_V^U = A_U^V .$$

If U, V, W are three neighborhoods of V_n, and if $x \in U \cap V \cap W$ then we will have:

$$\theta_x^U = A_V^U \,\theta_x^V = A_V^U A_W^V \,\theta_x^W .$$

As a result, we will have:

(56-6) $$A_V^U = A_V^U A_W^V \qquad (x \in U \cap V \cap W) .$$

In the sequel, it will be convenient to introduce the $n\times n$ matrix of Pfaff forms that are defined for each $x \in U \cap V$ by the relation:

(56-7) $$A_{UV} = (A^{-1})_V^U \,dA_V^U = A_U^V \,dA_V^U = -dA_U^V \,A_V^U .$$

Suppose that $x \in U \cap V \cap W$. By differentiating (56-6), one will get:

$$dA_W^U = dA_V^U A_W^V + A_V^U dA_W^V .$$

If we multiply the sides of this relation by the matrix:

$$(A^{-1})_V^U = (A^{-1})_V^V \,(A^{-1})_W^U$$

then it will become:

(56-8) $$\Lambda_{UW} = (A^{-1})_W^V \,\Lambda_{UV} A_W^V + \Lambda_{VW} .$$

c) In any neighborhood that is endowed with frames, an *affine connection* on V_n is defined by the data of a matrix ω of Pfaff forms of class C^{r-2} such that for $x \in U \cap V$ one has:

(56-9) $$\omega_V = (A^{-1})_V^U \,\omega_U A_V^U + \Lambda_{UV} .$$

Let U, V, W be three neighborhoods of V_n. For $x \in U \cap V \cap W$, one has three matrices: ω_U, ω_V, ω_W. We seek to determine whether these matrices satisfy relations of the type (56-9) pair-wise. To that effect, we assume that one has (56-9) and:

$$\omega_W = (A^{-1})_W^V \,\omega_V A_W^V + \Lambda_{VW} .$$

We replace ω_V in this latter relation by its value from (56-9). It becomes:

$$\omega_W = (A^{-1})_W^V \,(A^{-1})_V^U \,\omega_U A_V^U A_W^V + (A^{-1})_W^V \,\Lambda_{UV} A_W^V + \Lambda_{VW}$$

so, from (56-6) and (56-8):

$$\omega_W = (A^{-1})_W^U \,\omega_U A_W^U + \Lambda_{UW}$$

and our definition is therefore self-consistent.

There exist an infinitude of affine connections on a differentiable manifold V_n . Starting with a denumerable covering of V_n, one may construct one directly by induction on the neighborhoods with the aid of the preceding result. In a moment, we shall confirm that one affine connection on V_n may be deduced from another. A differentiable manifold of class C^r that is endowed with an affine connection of class C^{r-2} is called *a space with affine connection* of class C^{r-2}.

57. – Explicit formulas. – We now propose to be more specific about some of the preceding formulas that may be useful to us in the sequel. To that effect, we introduce the coframe matrices:

$$\theta_x^{\mathrm{U}} = (\theta^{\alpha}) \qquad \theta_x^{\mathrm{U}} = (\theta^{\beta'}) .$$

If $x \in U \cap V$ then the matrices A_V^U and A_U^V have the elements:

$$A_V^U = (A_{\beta'}^{\alpha}) \qquad A_U^V = (A_{\alpha}^{\beta'}) .$$

With these notations, (56-4) translates into:

(57-1) $$\theta^{\alpha} = A_{\beta'}^{\alpha}\, \theta^{\beta'} .$$

The matrix Λ_{UV} of differential forms has the elements:

(57-2) $$\Lambda_{\mu'}^{\lambda'} = A_{\sigma}^{\lambda'}\, dA_{\mu}^{\sigma} .$$

We denote the elements of the connection matrices ω_U and ω_V by:

$$\omega_U = (\omega_{\beta}^{\alpha}) \qquad \omega_V = (\omega_{\mu'}^{\lambda'}) .$$

Relation (56-9) then implies:

(57-3) $$\omega_{\mu'}^{\lambda'} = A_{\alpha}^{\lambda'}\, \omega_{\beta}^{\alpha} A_{\mu'}^{\beta} + A_{\sigma}^{\lambda'}\, dA_{\mu'}^{\sigma} \qquad (x \in U \cap V).$$

In (57-3), we recognize the transformation law for the local forms that define a Riemannian connection under a change of frame. In what follows, we will set:

(57-4) $$\omega_{\beta}^{\alpha} = \gamma_{\beta\gamma}^{\alpha}\theta^{\gamma} \qquad (x \in U).$$

The $\gamma_{\beta\gamma}^{\alpha}$ are called the *coefficients of the affine connection* envisioned at the point x with the chosen frames θ_x and R^x. From (57-3), these coefficients will transform according to the rule:

(57-5) $$\gamma_{\mu'\rho'}^{\lambda'} = A_{\alpha}^{\lambda'}\, A_{\mu'}^{\beta}\, A_{\rho'}^{\alpha}\, \gamma_{\beta\gamma}^{\alpha} + A_{\sigma}^{\lambda'}\, \partial_{\rho'} A_{\mu'}^{\sigma}$$

in which $\partial_{\rho'}$ denotes the Pfaffian derivatives of A with respect to $\theta^{\rho'}$.

58. – Passing from one affine connection to another. – Consider two affine connections on V_n with coefficients $\gamma^{\alpha}_{\beta\gamma}$ and $\bar{\gamma}^{\alpha}_{\beta\gamma}$, respectively, with respect to the same frame. By passing to another system of frames, one will have (57-5) and:

$$\bar{\gamma}^{\lambda'}_{\mu'\rho'} = A^{\lambda'}_{\alpha}\, A^{\beta}_{\mu'}\, A^{\alpha}_{\rho'}\, \bar{\gamma}^{\alpha}_{\beta\gamma} + A^{\lambda'}_{\sigma}\, \partial_{\rho'} A^{\sigma}_{\mu'}\,.$$

Upon subtracting both equations one gets:

$$\bar{\gamma}^{\lambda'}_{\mu'\rho'} - \gamma^{\lambda'}_{\mu'\rho'} = A^{\lambda'}_{\alpha}\, A^{\beta}_{\mu'}\, A^{\alpha}_{\rho'}\, (\bar{\gamma}^{\alpha}_{\beta\gamma} - \gamma^{\alpha}_{\beta\gamma})\,.$$

It results from this that the quantities:

$$T^{\alpha}_{\beta\gamma} \doteq \bar{\gamma}^{\alpha}_{\beta\gamma} - \gamma^{\alpha}_{\beta\gamma}$$

are the components of a tensor of rank 3 that is once contravariant and twice covariant. Conversely, by adding the components of such a tensor to the coefficients of an affine connection one obviously obtains the coefficients of an affine connection. We state the:

THEOREM – *Given an affine connection on V_n, one obtains all of the other ones by adding the coefficients of an arbitrary tensor of rank three that is once contravariant and twice covariant to its coefficients.*

59. – Torsion of an affine connection. – If Θ and $\bar{\Theta}$ are matrices whose elements Θ^{α}_{β} and $\bar{\Theta}^{\alpha}_{\beta}$ are differential forms then we will set:

$$\Theta \wedge \bar{\Theta} = (\Theta^{\alpha}_{\beta} \wedge \bar{\Theta}^{\beta}_{\gamma}),$$

and let $d\Theta$ denote the matrix:

$$d\Theta = (d\Theta^{\alpha}_{\beta}).$$

One obviously has $d(d\Theta)) = 0$. We still use the same notation if one of the matrices considered has only one row.

Having said this, if $x \in U \cap V$ then one will have:

$$\theta^{V} = A^{V}_{U}\, \theta^{U},$$

and, as a result, by taking the exterior derivative of both sides:

$$d\theta^{V} = A^{V}_{U}\, d\theta^{U} + dA^{V}_{U} \wedge \theta^{U} \;=\; A^{V}_{U}\, d\theta^{U} + dA^{V}_{U}\, A^{U}_{V} \wedge \theta^{V},$$

namely:

(59-1) $$d\theta^V = A_U^V\, d\theta^U - \Lambda_{UV} \wedge \theta^U .$$

On the other hand, from (56-9), after exterior multiplying by θ^V one will have:

$$\omega_V \wedge \theta^V = A_U^V \omega_V \wedge A_V^U \theta^V + \Lambda_{UV} \wedge \theta^V,$$

namely:

(59-2) $$\omega_V \wedge \theta^V = A_U^V\, \omega_V \wedge \theta^U + \Lambda_{UV} \wedge \theta^V.$$

By (59-1) and (59-2) one obtains:

(59-3) $$d\theta^V + \omega_V \wedge \theta^V = A_U^V\, (d\theta^U + \omega_U \wedge \theta^U) .$$

In each neighborhood *U*, consider the matrix with one row whose elements are local exterior quadratic differential forms:

$$\Sigma^U = d\theta^V + \omega_V \wedge \theta^V.$$

(59-3) expresses the fact that for $x \in U \cap V$:

(59-4) $$\Sigma^V = A_U^V\, \Sigma^U .$$

One may translate the result by saying that the Σ define a *vector-valued* exterior quadratic differential form. If $\Sigma^U = (\Sigma^\alpha)$ and $\Sigma^V = (\Sigma^{\beta'})$ then, from (59-4), one will have:

(59-5) $$\Sigma^{\beta'} = \mathrm{A}_\alpha^{\beta'} \Sigma^\alpha ,$$

in which Σ^α is given by:

(59-6) $$\Sigma^\alpha = d\theta^\alpha + \omega_\beta^\alpha \wedge \theta^\beta.$$

If we set:

(59-7) $$\Sigma^\alpha = -S_{\beta\gamma}^\alpha\, \theta^\beta \wedge \theta^\gamma. \qquad (S_{\beta\gamma}^\alpha = -S_{\gamma\beta}^\alpha)$$

then it will result from equation (59-5) that $S_{\beta\gamma}^\alpha$ are the components of a tensor of rank three that is antisymmetric with respect to the lower indices. This tensor is called the *torsion tensor* of the connection.

60. – Curvature of an affine connection. – We now study the exterior derivative of the connection matrix. By differentiating (56-9), one gets:

$$d\omega_V = A_U^V\, d\omega_U\, A_V^U + dA_U^V \wedge \omega_U\, A_V^U - A_U^V\, \omega_U \wedge dA_V^U + d\Lambda_{UV} ,$$

namely, from (56-7):

(60-1) $$d\omega_V = A_U^V\, d\omega_U A_V^U + dA_U^V \wedge \omega_U A_V^U - A_U^V \omega_U \wedge dA_V^U + dA_V^U \wedge dA_V^U .$$

On the other hand, we evaluate:

$$\omega_V \wedge \omega_V = A_U^V \omega_U \wedge \omega_U A_V^U + \omega_U A_U^V \wedge A_V^U \Lambda^{UV} + \Lambda^{UV} A_U^V \wedge \omega_U A_V^U + \Lambda_{UV} \wedge \Lambda_{UV} .$$

One gets from (56-7) that:

(60-2) $$\omega_V \wedge \omega_V = A_U^V \omega_U \wedge \omega_U A_V^U + A_U^V \omega_U \wedge dA_V^U - dA_U^V \wedge \omega_U A_V^U - dA_U^V \wedge dA_V^U .$$

By adding (60-1) and (60-2), one will obtain:

(60-3) $$d\omega_V + \omega_V \wedge \omega_V = A_U^V (d\omega_U + \omega_U \wedge \omega_U)\, A_V^U .$$

In each neighborhood U, consider the $n\times n$ matrix whose elements are quadratic exterior differential forms:

$$\Omega_U = d\omega_U + \omega_U \wedge \omega_U .$$

One has:

(60-4) $$\Omega_V = A_U^V\, \Omega_U\, A_V^U .$$

One may interpret this result by saying that the Ω define a tensor-valued quadratic exterior differential form of type (1, 1). If $\Omega_U = (\Omega_\beta^\alpha)$ and $\Omega_V = (\Omega_{\mu'}^{\lambda'})$ then one will obtain:

(60-5) $$\Omega_{\mu'}^{\lambda'} = A_\alpha^{\lambda'} A_{\mu'}^\beta\, \Omega_\beta^\alpha$$

in which Ω_β^α is given by:

(60-6) $$\Omega_\beta^\alpha = d\omega_\beta^\alpha + \omega_\rho^\alpha \wedge \omega_\beta^\rho .$$

If we set:

(60-7) $$\Omega_\beta^\alpha = \tfrac{1}{2} R^\alpha{}_{\beta,\lambda\mu}\, \theta^\lambda \wedge \theta^\mu$$

then it will result from (60-5) that the $R^\alpha{}_{\beta,\lambda\mu}$ are the components of a tensor of rank four that is anti-symmetric with respect to λ and μ. It is the *curvature tensor* of the connection.

61. – The Bianchi identities for an affine connection. – We start with the formulas that define the torsion and curvature of a connection. From now on, we shall write them by suppressing the index U whenever the presence of that index is irrelevant to the calculations being performed. One will then have:

(61-1) $$\Sigma = d\theta + \omega \wedge \theta$$

and:

(61-2) $$\Omega = d\omega + \omega \wedge \omega .$$

Take the exterior derivatives of both sides of (61-1). Since $d(d\theta) = 0$, one obtains:

$$d\Sigma = d\omega \wedge \theta - \omega \wedge d\theta;$$

namely, upon using the values of $d\omega$ and $d\theta$ from relations (61-1) and (61-2):

$$d\Sigma = (\Omega - \omega \wedge \omega) \wedge \theta - \omega \wedge (\Sigma - \omega \wedge \theta).$$

One deduces from this, after simplifying, that:

(61-3) $$d\Sigma = \Omega \wedge \theta - \omega \wedge \Sigma.$$

Similarly, take the exterior derivatives of both sides of (61-2). Since $d(d\omega) = 0$, one will obtain:

$$d\Omega = d\omega \wedge \omega - \omega \wedge d\omega.$$

so by replacing $d\omega$ with its expression in (61-2):

$$d\Omega = (\Omega - \omega \wedge \omega) \wedge \omega - \omega \wedge (\Omega - \omega \wedge \omega).$$

After simplifying, one will thus have:

(61-4) $$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega.$$

Formulas (61-3) and (61-4) are called the *Bianchi identities* for an affine connection. In explicit form, they may be written:

(61-5) $$d\Sigma^\alpha = \Omega^\alpha_\beta \wedge \theta^\beta - \omega^\alpha_\beta \wedge \Sigma^\beta$$

(61-6) $$d\Omega^\alpha_\beta = \Omega^\alpha_\rho \wedge \omega^\rho_\beta - \omega^\alpha_\rho \wedge \Omega^\rho_\beta .$$

62. – Absolute differential and covariant derivative for an affine connection. – *a*) Consider a contravariant vector field. Its components in a neighborhood U are defined by the matrix with one row v^U and by the analogous matrix v^V in a neighborhood V, and for $x \in U \cap V$ one has:

$$v^V = A^V_U \, v^U .$$

By differentiating, one obtains:

(62-1) $$dv^V = A^V_U \, dv^U + dA^V_U \, v^U .$$

On the other hand, from (56-9) one has:

$$\omega_V v^V = A^V_U \, \omega_U \, v^U + \Lambda_{UV} v^U = A^V_U \, \omega_U v^U - dA^V_U \, A^U_V \, v^V ;$$

namely:

(62-2) $$\omega_V v^V = A^V_U \, \omega_U v^U - dA^V_U \, v^U .$$

By adding (62-1) and (62-2), term-by term, one will get:

$$dv^V + \omega_V v^V = A_U^V (dv^U + \omega_U v^U). \tag{62-3}$$

It results from this that the quantities:

$$Dv = dv + \omega v \tag{62-4}$$

define a contravariant vector-valued, linear, exterior differential form. The contravarinat components are given explicitly by the Pfaff forms:

$$Dv^\alpha = dv^\alpha + \omega_\rho^\alpha v^\rho . \tag{62-5}$$

The form (62-4) is called the *absolute differential* of the vector field relative to the connection. If one sets:

$$Dv^\alpha = D_\beta v^\alpha \theta^\alpha$$

then one will see from (62-3) that $D_\beta v^\alpha$ are the components of a tensor for which the index β is a covariant index and which is called the covariant derivative of v for the connection. From (62-5), one will have:

$$D_\beta v^\alpha = \partial_\beta v^\alpha + \gamma_{\rho\beta}^\alpha v^\rho ,$$

in which ∂ represents the Pfaffian derivative.

One likewise establishes that if one considers a covariant vector field that is defined in each neighborhood U by the matrix with one column w_U then the quantities:

$$Dw = dw - w\omega$$

will define a covariant vector-valued, linear, differential form whose components are Pfaff forms:

$$Dw_\alpha = dw_\alpha - w_\rho \omega_\alpha^\rho$$

and which defines the absolute differential of this field. The corresponding covariant derivative is:

$$D_\beta w_\alpha = \partial_\beta w_\alpha - \gamma_{\alpha\beta}^\rho w_\rho .$$

b) Let T be a finite-dimensional vector space that serves as the space of values for a tensor of some particular type, and let $R(G)$ be a linear representation of G in T. If one is given a certain set of neighborhoods V_n then a tensor will be defined in each neighborhood U by the data of a function $t_U(x)$ ($x \in U$) with values in T such that for $x \in U \cap V$:

$$t_U(x) = R(A_V^U)\ t_V(x). \tag{62-6}$$

For such a tensor field, by considering the linear representation $\bar{R}$ of the Lie algebra of G that is induced by R, one may establish that one may construct a tensor-valued, linear, differential form:

(62-7) $$Dt_U = \bar{R}(\omega_U)\, t_U$$

of type $R(G)$, i.e., it is such that:

(62-8) $$Dt_U = R(A_V^U)\, Dt_V ,$$

and which is called the absolute differential of the tensor for the affine connection envisioned.

For example, if we consider a tensor field that is once covariant and once contravariant then formula (62-6) may be written explicitly:

$$t^{\alpha}{}_{\beta} = A^{\alpha}_{\lambda'} A^{\mu'}_{\beta} t^{\lambda'}{}_{\mu'}$$

namely, in matrix notation:

$$t_U = \bar{A}_U^V\, t_V A_U^V = \operatorname{adj}(A_U^V)\, t_V$$

in which the linear representation R envisioned is made specific. One will then have:

$$Dt_U = dt_U + \omega_U t - t\,\omega_U$$

or, in explicit form:

(62-9) $$Dt^{\alpha}{}_{\beta} = dt^{\alpha}{}_{\beta} + \omega^{\alpha}_{\rho}\, t^{\rho}{}_{\beta} - \omega^{\rho}_{\beta}\, t^{\alpha}{}_{\rho} ,$$

and the general form for the absolute differential will appear easily from (62-9). One immediately passes to the covariant derivative. One also sees that as far as the absolute differential is concerned, the sum, tensor product, and contracted product satisfy the usual rules of differentiation.

63. – Formulas in the natural frame of local coordinates. – We say that we refer a connection or the various tensors that we have introduced to a natural frame when we adopt the co-frame θ_x^U that is defined in each local coordinate neighborhood by the n differentials (dx^1, dx^2, …, dx^n) of the local coordinates (x^{α}); the associated frames R_x^U are called the *natural frames* that are associated with the local coordinates. If U and V are the domains of the local coordinates (x^{α}) and $(x^{\beta'})$, respectively, then one will have:

$$A^{\alpha}_{\beta'} = \frac{\partial x^{\alpha}}{\partial x^{\beta'}} , \qquad A^{\beta'}_{\alpha} = \frac{\partial x^{\beta'}}{\partial x^{\alpha}}$$

for $x \in U \cap V$. With these values, one will always have (57-3) for the connection matrices, namely:

$$\omega_{\mu'}^{\lambda'} = A_{\alpha}^{\lambda'}\,\omega_{\beta}^{\alpha}\,A_{\mu'}^{\beta} + A_{\sigma}^{\lambda'}\,dA_{\mu'}^{\sigma}.$$

In natural frames, we introduce the special notation:

$$\omega_{\beta}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} dx^{\gamma}$$

in order to denote the coefficients of the affine connection. Under a change of natural frame, these coefficients will always transform according to the formula:

(63-1) $$\Gamma_{\mu'\rho'}^{\lambda'} = A_{\alpha}^{\lambda'}\,A_{\mu'}^{\beta}\,A_{\rho'}^{\gamma}\,\Gamma_{\beta\gamma}^{\alpha} + A_{\sigma}^{\lambda'}\,\partial_{\rho'}A_{\mu'}^{\sigma},$$

but one will note that $\partial_{\rho'}$ now denotes the ordinary partial derivative of $A_{\mu'}^{\sigma}$ with respect to $x^{\rho'}$. From this, it results that:

(63-2) $$\partial_{\rho'}A_{\mu'}^{\sigma} = \partial_{\mu'}A_{\rho'}^{\sigma},$$

i.e., that the last term in the right-hand side of (63-1) is symmetric in the indices.

Since $d(dx^{\alpha}) = 0$, formula (59-6), which defines the torsion form, reduces in a natural frame to:

$$\Sigma^{\alpha} = \omega_{\beta}^{\alpha} \wedge dx^{\beta}.$$

One thus has:

$$\Sigma^{\alpha} = \Gamma_{\beta\gamma}^{\alpha}\,dx^{\alpha} \wedge dx^{\beta} = -\tfrac{1}{2}(\Gamma_{\beta\gamma}^{\alpha} - \Gamma_{\gamma\beta}^{\alpha})\,dx^{\beta} \wedge dx^{\gamma}.$$

One deduces from this that the torsion tensor that is defined by (59-7) has the components:

(63-3) $$S^{\alpha}{}_{\beta\gamma} = \tfrac{1}{2}(\Gamma_{\beta\gamma}^{\alpha} - \Gamma_{\gamma\beta}^{\alpha}) \qquad (S^{\alpha}{}_{\beta\gamma} = -S^{\alpha}{}_{\gamma\beta}).$$

One may verify immediately with the aid of (63-1) and (63-2) that under a change of local coordinates the quantities $S^{\alpha}{}_{\beta\gamma}$ that are defined by (63-3) will essentially transform according to tensor laws. One will note, moreover, that if the $\Gamma_{\beta\gamma}^{\alpha}$ are the coefficients of an affine connection in a natural frame then the same thing will be true for the quantities:

$$\bar{\Gamma}_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha}$$

since, from (63-3), the $\bar{\Gamma}_{\beta\gamma}^{\alpha}$ are deduced from the $\Gamma_{\beta\gamma}^{\alpha}$ by adding a tensor of rank three that is once contravariant and twice covariant.

Finally, from formulas (60-6) and (60-7), it is easy to deduce the explicit expression for the curvature tensor as a function of the coefficients $\Gamma_{\beta\gamma}^{\alpha}$ of the connection. Indeed, one has:

$$d\omega_{\beta}^{\alpha} = d(\Gamma_{\beta\mu}^{\alpha}dx^{\mu}) = d\Gamma_{\beta\mu}^{\alpha} \wedge dx^{\mu} = \partial_{\lambda}\Gamma_{\beta\mu}^{\alpha}dx^{\lambda} \wedge dx^{\mu} = \tfrac{1}{2}(\partial_{\lambda}\Gamma_{\beta\mu}^{\alpha} - \partial_{\mu}\Gamma_{\beta\lambda}^{\alpha})dx^{\lambda} \wedge dx^{\mu}.$$

On the other hand:

$$\omega_\rho^\alpha \wedge \omega_\beta^\rho = \Gamma^\alpha_{\rho\lambda}\Gamma^\rho_{\beta\mu}\, dx^\lambda \wedge dx^\mu = \tfrac{1}{2}(\Gamma^\alpha_{\rho\lambda}\Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\beta\lambda})\, dx^\lambda \wedge dx^\mu .$$

One deduces from (60-7) that:

(63-4) $$R^\alpha{}_{\beta,\lambda\mu} = \partial_\lambda \Gamma^\alpha_{\beta\mu} - \partial_\mu \Gamma^\alpha_{\beta\lambda} + \Gamma^\alpha_{\rho\lambda}\Gamma^\rho_{\beta\mu} - \Gamma^\alpha_{\rho\mu}\Gamma^\rho_{\beta\lambda} .$$

64. – Tensors deduced by contraction. – *a*) One may deduce the following covariant tensor From the torsion tensor $S^\alpha{}_{\beta\gamma}$ by contraction:

(64-1) $$S_\alpha = S^\rho{}_{\alpha\rho} .$$

From (63-3), one will thus have:

(64-2) $$2S_\alpha = \Gamma^\rho_{\alpha\rho} - \Gamma^\rho_{\rho\alpha}$$

in the natural frame that is associated with local coordinates.

From the anti-symmetry of the torsion tensor in its lower indices, the other possible contraction will lead to the opposite covariant vector.

b) One may obtain two covariant tensors of rank 2 by contracting the curvature tensor that are essentially distinct. One of them constitutes the generalization of the Ricci tensor of Riemannian geometry, so we also call it the *Ricci tensor.* It is defined by:

(64-3) $$R_{\lambda\mu} = R^\alpha{}_{\lambda,\alpha\mu} .$$

Its components in a natural frame are given explicitly as functions of the coefficients of the connection by the formula:

(64-4) $$R_{\lambda\mu} = \partial_\sigma \Gamma^\sigma_{\lambda\mu} - \partial_\mu \Gamma^\sigma_{\lambda\sigma} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\lambda\mu} - \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\lambda\sigma} .$$

This tensor does not have any particular symmetry, in general.

The second tensor is obtained by contracting α and β in relation (63-4), and as a result, it is anti-symmetric in the remaining lower indices λ and μ. We set:

(64-5) $$V_{\lambda\mu} = R^\alpha{}_{\alpha,\lambda\mu} .$$

This tensor has the explicit expression:

(64-6) $$V_{\lambda\mu} = \partial_\lambda \Gamma^\alpha_{\alpha\mu} - \partial_\mu \Gamma^\alpha_{\alpha\lambda} .$$

It obviously reduces to zero in the case of a Riemannian connection. In the case of an arbitrary affine connection, $V_{\lambda\mu}$ will be the *rotation of a vector field.* Indeed, if we give the manifold V_n a Riemannian structure with a positive definite metric – which is always possible – and let $\Pi^{\alpha}_{\beta\gamma}$ be the coefficients in a natural frame of the corresponding Riemannian connection then one will have:

$$\Gamma^{\alpha}_{\beta\gamma} = \Pi^{\alpha}_{\beta\gamma} + T^{\alpha}_{\beta\gamma},$$

in which T is a tensor. By introducing the covariant vector:

$$T_{\mu} = T^{\alpha}_{\alpha\mu},$$

one will obtain:

$$\Gamma^{\alpha}_{\alpha\mu} = \Pi^{\alpha}_{\alpha\mu} + \mathrm{T}^{\alpha}_{\alpha\mu}.$$

One deduces from this that:

$$V_{\lambda\mu} = \partial_{\lambda}\mathrm{T}_{\mu} - \partial_{\mu}\mathrm{T}_{\lambda},$$

which expresses the stated property.

Suppose that the manifold V_n is *orientable.* It is easy to obtain a geometric interpretation for the condition $V_{\lambda\mu} = 0$. Suppose that there exists an exterior differential n-form η on V_n that is not annulled at any point. The components of such a form in local coordinates may be written:

$$\eta_{\lambda_1\lambda_2\cdots\lambda_n} = \sqrt{|g|}\,\varepsilon_{\lambda_1\lambda_2\cdots\lambda_n},$$

in which $\sqrt{|g|}$ is the strict component of the form η, which is a component that one may always assume to be positive, and in which $\varepsilon_{\lambda_1\lambda_2\cdots\lambda_n}$ is the classical indicator of the permutation. Under a direct change of local coordinates, $\sqrt{|g|}$ will transform according to the formula:

$$\sqrt{|g'|} = \sqrt{|g|}\det\left(A^{\alpha}_{\beta'}\right).$$

The covariant derivative of η has the components:

$$D_{\mu}\eta_{\lambda_1\lambda_2\cdots\lambda_n} = \partial_{\mu}\eta_{\lambda_1\lambda_2\cdots\lambda_n} - \Gamma^{\alpha}_{\alpha\mu}\,\eta_{\lambda_1\lambda_2\cdots\lambda_n},$$

namely:

$$\mathrm{D}_{\mu}\eta_{\lambda_1\lambda_2\cdots\lambda_n} = (\partial_{\mu}\sqrt{|g|} - \Gamma^{\alpha}_{\alpha\mu}\sqrt{|g|})\varepsilon_{\lambda_1\lambda_2\cdots\lambda_n}.$$

If this derivative is zero then one will have:

$$\Gamma^{\alpha}_{\alpha\mu} = \frac{\partial_{\mu}\sqrt{|g|}}{\sqrt{|g|}},$$

and it follows from (64-6) that $V_{\lambda\mu} = 0$.

Conversely, if $V_{\lambda\mu} = 0$ then, starting with the transformation formulas for $\Gamma^{\alpha}_{\alpha\mu}$ under a local coordinate change, one easily proves that one may construct an n-form on V_n that is not annulled at any point and which has a zero covariant derivative under the affine connection considered.

65. – Symmetric connection associated with an affine connection. Einstein tensor. – From now on, we will use the symbols () and [] as the symbols of symmetrization and anti-symmetrization, respectively. In particular, if $\varphi_{\lambda\mu\dots}$ denotes a system of quantities that depend on the two indices λ and μ, and possibly other indices, we have:

$$\varphi_{(\lambda\mu)\dots} = \tfrac{1}{2}(\varphi_{\lambda\mu\dots} + \varphi_{\mu\lambda\dots}) \qquad \varphi_{[\lambda\mu]\dots} = \tfrac{1}{2}(\varphi_{\lambda\mu\dots} - \varphi_{\mu\lambda\dots}).$$

One immediately deduces that:

$$\varphi_{\lambda\mu\dots} = \varphi_{(\lambda\mu)\dots} + \varphi_{[\lambda\mu]\dots} \tag{65-1}$$

Now that we have introduced this notation, consider the coefficients $\Gamma^{\alpha}_{\beta\gamma}$ of an affine connection in a natural frame. From (65-1), one has:

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{(\beta\gamma)} + \Gamma^{\alpha}_{[\beta\gamma]}.$$

Now, by virtue of (63-3):

$$\Gamma^{\alpha}_{[\beta\gamma]} = S^{\alpha}{}_{\beta\gamma}.$$

One deduces from this that:

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{(\beta\gamma)} + S^{\alpha}{}_{\beta\gamma}. \tag{65-2}$$

Since the $S^{\alpha}{}_{\beta\gamma}$ are the components of a tensor, it will results that in a natural frame the $\Gamma^{\alpha}_{(\beta\gamma)}$ are the coefficients of an affine connection whose torsion is obviously zero. We say that this connection is the *symmetric affine connection* that is associated with given affine connection. It is naturally possible to express the various tensors that were introduced relative to the affine connection as functions of the torsion tensor and elements relative to the symmetric connection. However, the corresponding formulas are not particularly interesting.

In elaborating his theory, Einstein introduced a tensor that we shall not use as a basis, but which it is still convenient to point out. This tensor $E_{\lambda\mu}$ is expressed as a function of the coefficients $\Gamma^{\alpha}_{\beta\gamma}$ of the affine connection by:

$$E_{\lambda\mu} = \partial_{\sigma}\Gamma^{\sigma}_{\lambda\mu} - \tfrac{1}{2}[\partial_{\mu}\Gamma^{\sigma}_{(\lambda\sigma)} + \partial_{\lambda}\Gamma^{\sigma}_{(\mu\sigma)}] + \Gamma^{\sigma}_{(\rho\sigma)}\Gamma^{\rho}_{\lambda\mu} - \Gamma^{\sigma}_{\rho\mu}\Gamma^{\rho}_{\lambda\sigma}. \tag{65-3}$$

We shall relate the quantities $E_{\lambda\mu}$ to the preceding tensors that we introduced, and having done this, we shall then establish that the $E_{\lambda\mu}$ essentially define a tensor.

Start with the Ricci tensor of the connection:

(65-4) $$R_{\lambda\mu} = \partial_\sigma \Gamma^\sigma_{\lambda\mu} - \partial_\mu \Gamma^\sigma_{\lambda\sigma} + \Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\lambda\mu} - \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\lambda\sigma}$$

and add the Ricci tensor $\bar{R}_{\mu\lambda}$ of the affine connection $\bar{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ to it. One has:

$$\bar{R}_{\mu\lambda} = \partial_\sigma \bar{\Gamma}^\sigma_{\lambda\mu} - \partial_\mu \bar{\Gamma}^\sigma_{\lambda\sigma} + \bar{\Gamma}^\sigma_{\rho\sigma}\bar{\Gamma}^\rho_{\lambda\mu} - \bar{\Gamma}^\sigma_{\rho\mu}\bar{\Gamma}^\rho_{\lambda\sigma}$$

or, by reverting to the $\Gamma^\alpha_{\beta\gamma}$:

(65-5) $$\bar{R}_{\mu\lambda} = \partial_\sigma \Gamma^\sigma_{\lambda\mu} - \partial_\lambda \Gamma^\sigma_{\sigma\mu} + \Gamma^\sigma_{\sigma\rho}\Gamma^\rho_{\lambda\mu} - \Gamma^\sigma_{\lambda\rho}\Gamma^\rho_{\sigma\mu} .$$

By adding (65-4) and (65-5) term-by-term, one will obtain:

$$\tfrac{1}{2}(R_{\lambda\mu} + \bar{R}_{\lambda\mu}) = E_{\lambda\mu} - \tfrac{1}{2}(\partial_\mu S_\lambda - \partial_\lambda S_\mu) .$$

We have thus obtained:

(65-6) $$E_{\lambda\mu} = \tfrac{1}{2}(R_{\lambda\mu} + \bar{R}_{\lambda\mu}) - \tfrac{1}{2}(\partial_\lambda S_\mu - \partial_\mu S_\lambda) ,$$

which establishes that $E_{\lambda\mu}$ essentially defines a tensor.

On the other hand, one has:

$$E_{\lambda\mu} - R_{\lambda\mu} = \partial_\sigma \Gamma^\sigma_{\lambda\mu} - \tfrac{1}{2}[\partial_\mu \Gamma^\sigma_{(\lambda\sigma)} + \partial_\lambda \Gamma^\sigma_{(\mu\sigma)}] + \Gamma^\rho_{\lambda\mu} S_\rho$$

so, upon replacing the first term of the right-hand side with $\partial_\mu S_\lambda$:

$$E_{\lambda\mu} - R_{\lambda\mu} = \partial_\sigma \Gamma^\sigma_{\lambda\mu} - \tfrac{1}{2}[\partial_\mu \Gamma^\sigma_{(\lambda\sigma)} + \partial_\lambda \Gamma^\sigma_{(\mu\sigma)}] + \Gamma^\rho_{\lambda\mu} S_\rho .$$

If $W_{\lambda\mu}$ denotes the contracted anti-symmetric curvature tensor of the connection $\Gamma^\alpha_{(\beta\gamma)}$ then one will have:

(65-7) $$E_{\lambda\mu} - R_{\lambda\mu} = D_\mu S_\lambda - W_{\lambda\mu} ,$$

in which the covariant derivative D_μ always corresponds to the initial affine connection.

66. – Parallelism and geodesics. **–** Let l be a continuously differentiable path in V_n that is defined parametrically by $x = x(t)$. In the local coordinate domain (x^α), we set $x^\alpha = dx^\alpha / dt$. If one is given a tangent direction to V_n at each point of l (i.e., an equivalence class of vectors whose origin is at x that is obtained by considering two non-zero collinear vectors as equivalent) then one knows ([1]) that this is what one intends by saying that *these directions are parallel relative to l and the connection*, and that this notion does not depend on the parametric representation that is chosen for *l*.

In order for this to be true, it is necessary and sufficient that one must have:

$$\left(\frac{dv^\alpha}{dt} + \Gamma^\alpha_{\rho\sigma} v^\rho \dot{x}^\sigma\right) v^\beta - \left(\frac{dv^\beta}{dt} + \Gamma^\beta_{\rho\sigma} v^\rho \dot{x}^\sigma\right) v^\alpha = 0 \tag{66-1}$$

on *l* and for every pair α, β, and for an arbitrary vector **v** that admits this direction at each point of *l*.

We take the directions tangent to *l* to be a system of directions along *l*, which we may define by the vector x^α. If this system of directions is parallel relative to *l* then the path *l* will be called a *geodesic* arc of the connection. One will then have:

$$\left(\frac{dv^\alpha}{dt} + \Gamma^\alpha_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma\right) \dot{x}^\beta - \left(\frac{dv^\beta}{dt} + \Gamma^\beta_{\rho\sigma} \dot{x}^\rho \dot{x}^\sigma\right) \dot{x}^\alpha = 0, \tag{66-2}$$

and one will see that the geodesics of the affine connection $\Gamma^\alpha_{\beta\gamma}$ depend upon only the associated symmetric connection $\Gamma^\alpha_{(\beta\gamma)}$.

Consider two affine connection on V_n with the coefficients $\dot{\Gamma}^\alpha_{\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma}$, and look for the conditions under which parallelism along any path is the same for both connections. Starting from (66-1), one will easily establish that the most general change of connection that preserves parallelism is:

$$\dot{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + 2\delta^\alpha_\beta p_\gamma, \tag{66-3}$$

in which p_γ denotes an arbitrary covariant vector. (66-3) obviously translates into the relations:

$$\dot{\Gamma}^\alpha_{(\beta\gamma)} = \Gamma^\alpha_{(\beta\gamma)} + \delta^\alpha_\beta p_\gamma + \delta^\alpha_\gamma p_\beta \tag{66-4}$$

and:

$$\dot{S}^\alpha{}_{\beta\gamma} = S^\alpha{}_{\beta\gamma} + \delta^\alpha_\beta p_\gamma + \delta^\alpha_\gamma p_\beta .$$

Such a change naturally preserves geodesics, *a fortiori*. An analogous argument shows that the most general change of connection that preserves geodesics is obtained by performing the change (66-4) on the symmetric connection and modifying the torsion tensor arbitrarily.

67. – Variational formulas for the curvature tensors. – Suppose that the connection envisioned is varied in a domain of the manifold V_n, in the sense of the

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([1]) Cf. EISENHART, *Non-Riemannian Geometry*, Amer. Math. Soc. Colloquium, pp. 12-13 and pp. 30-31.

calculus of variations, and let $\delta\Gamma^{\alpha}_{\beta\gamma}$ be the variation of the coefficient $\Gamma^{\alpha}_{\beta\gamma}$ of the affine connection. It is clear that $\delta\Gamma^{\alpha}_{\beta\gamma}$ defines a tensor that is once-contravariant and twice-covariant: one may either reason as in sec **20** or observe that the definition of the $\delta\Gamma^{\alpha}_{\beta\gamma}$ involves the difference of two connections, i.e., the components of a tensor of the indicated type ([1]). Moreover, one recalls that the operator δ commutes with the ordinary partial derivative with respect to a local coordinate.

In what follows, we shall need the variations of the curvature tensor and the Ricci tensor that corresponds to such a variation of the connection. We thus propose to extend and adapt the calculations that were done in sec. **20** for a Riemannian connection to the case of an affine connection that has torsion. We start with the explicit expression for the curvature tensor:

$$R^{\alpha}{}_{\beta,\lambda\mu} = \partial_\lambda \Gamma^{\alpha}_{\beta\mu} - \partial_\mu \Gamma^{\alpha}_{\beta\lambda} + \Gamma^{\alpha}_{\rho\lambda}\Gamma^{\rho}_{\beta\mu} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\rho}_{\beta\lambda} .$$

One deduces from this by variation that:

$$\delta R^{\alpha}{}_{\beta,\lambda\mu} = \partial_\lambda \delta\Gamma^{\alpha}_{\beta\mu} - \partial_\mu \delta\Gamma^{\alpha}_{\beta\lambda} + \Gamma^{\alpha}_{\rho\lambda}\delta\Gamma^{\rho}_{\beta\mu} + \Gamma^{\rho}_{\beta\mu}\delta\Gamma^{\alpha}_{\rho\lambda} - \Gamma^{\alpha}_{\rho\mu}\Gamma^{\rho}_{\beta\lambda} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\alpha}_{\rho\mu} .$$

We then calculate the covariant derivative of the tensor $\delta\Gamma^{\alpha}_{\beta\mu}$. We get:

$$D_\lambda \delta\Gamma^{\alpha}_{\beta\mu} = \partial_\lambda \delta\Gamma^{\alpha}_{\beta\mu} + \Gamma^{\alpha}_{\rho\lambda}\delta\Gamma^{\rho}_{\beta\mu} - \Gamma^{\rho}_{\beta\lambda}\delta\Gamma^{\alpha}_{\rho\mu} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\alpha}_{\beta\rho}$$

and:

$$D_\mu \delta\Gamma^{\alpha}_{\beta\lambda} = \partial_\mu \delta\Gamma^{\alpha}_{\beta\lambda} + \Gamma^{\alpha}_{\rho\mu}\delta\Gamma^{\rho}_{\beta\lambda} - \Gamma^{\rho}_{\beta\mu}\delta\Gamma^{\alpha}_{\rho\lambda} - \Gamma^{\rho}_{\lambda\mu}\Gamma^{\alpha}_{\beta\rho} .$$

By subtracting term-by-term, one obtains:

$$\delta R^{\alpha}{}_{\beta,\lambda\mu} = D_\lambda \delta\Gamma^{\alpha}_{\beta\mu} - D_\mu \delta\Gamma^{\alpha}_{\beta\lambda} - 2S^{\rho}{}_{\lambda\mu}\, \delta\Gamma^{\sigma}_{\lambda\rho} . \tag{67-1}$$

These formulas differ from the ones that were obtained in sec. **20** by the presence of torsion terms.

68. – Local transformations on a differentiable manifold. – Let U be a neighborhood of V_n and let f be a differentiable homeomorphism of U onto a neighborhood V. This differentiable homeomorphism induces an isomorphism of the vector space T_y that is tangent to $y \in U$ onto the vector space $T_{f(y)}$, and, more generally, an isomorphism of the space of tensors of a definite type at y onto the corresponding space at y. We denote this isomorphism by $\overline{f}_y$.

([1]) Of course, the $\delta\gamma^{\alpha}_{\beta\gamma}$ are the components of the "variation of the connection" tensor in an arbitrary frame. However, since we shall have recourse to an explicit expression for the curvature tensor, it is more convenient to reason in local coordinates.

Suppose that one is given a tensor field Θ_y on a domain *D* that covers *U*. We refer to the transform of this tensor field by *f*, which we denote by $(\bar{f}\Theta)_x$, when we mean the tensor field on *V* that is defined by:

(68-1) $$(\bar{f}\Theta)_x = \bar{f}^{-1}_{f(x)}\,\Theta^{-1}_{f(x)} \quad \text{in which } x \in V, \quad f^{-1}(x) \in U.$$

Let (x^α) be local coordinates whose domain covers *V* and (y^β) local coordinates whose domain covers *U*. The map *f* may be defined in these local coordinates by the relations:

(68-2) $$x^\alpha = f^\alpha(y^\beta) \qquad\qquad y^\alpha = g^\beta(x^\alpha).$$

It is then easy to modify the local coordinates of *V* in such a way that $\bar{f}$ is exhibited in the simplest manner. It suffices to take new local coordinates for *x*:

(68-3) $$x^{\beta'} = g^\beta(x^\alpha).$$

The map *f* is then described by the relations:

$$x^{\beta'} = y^\beta$$

between the coordinates (y^β) of $y \in U$ and the coordinates $(x^{\beta'})$ of $x \in V$; $\bar{f}$ is then the map that makes any tensor at *y* that is referred to the natural frame for the coordinates (y^β) correspond to the tensor at *x* that has the same components relative to the natural frame for the coordinates $(x^{\alpha'})$.

By the same procedure, an affine connection Γ_y defined on *U* may be transformed into an affine connection that is defined on *V* that will have the same coefficients with respect to the natural frame relative to the coordinates $(x^{\alpha'})$.

69. – Lie derivative. – Consider a non-zero vector field ξ^α in a neighborhood *U* to which local coordinates are referred. One knows that the integration of the differential system:

$$\frac{dx^\alpha}{dt} = \xi^\alpha,$$

with the initial condition that the point *y* must has the coordinates (y^α) at $t = 0$, defines a local transformation group of one parameter *t* that makes the point $y(y^\alpha)$ correspond to the point $x = f_t(y)$ whose coordinates are (x^α) in the same local coordinate system. Suppose that one defines a field of geometric objects – tensor or connections – on *U*. For a sufficiently small *t*, one thus finds the object $(\bar{f}_t\Phi)_x$ defined at *x*. The Lie derivative of Φ relative to the vector field ξ, or the corresponding infinitesimal transformation $X = \xi^\alpha\partial_\alpha$ is defined by:

(69-1) $$X\Phi_x = \lim_{t\to 0}\frac{1}{t}[\Phi_x - (\bar{f}_t\Phi)_x].$$

One sees immediately that for the geometric objects considered here $X\Phi_x$ will always be a tensor. One has:

$$(\bar{f}_t\Phi)_x = \Phi_x - t\,X\,\Phi_x + tO,$$

in which the notation O will denote a term that goes to zero with t in the sequel. The relation $X\Phi_x = 0$ say that the field F is invariant under the local transformation group considered.

In order to evaluate the Lie derivative in the various cases, we must seek the principal part at t of the bracket that appears in the right-hand side of (69-1). At the point $y(y^\alpha)$, the transformation f_t makes the point with coordinates:

$$x^\alpha = y^\alpha + t\,\xi^\alpha(y) + t\,O$$

in the same local coordinate systems. As a result:

$$y^\alpha = x^\alpha - t\xi^\alpha(x) + tO$$

and, conforming to (68-3), we perform the change of local coordinates:

(69-2) $$x^{\alpha'} = x^\alpha - t\xi^\alpha + tO \qquad (= a, \text{ numerically}).$$

If $\varphi(x)$ denotes a scalar field then one will have:

$$(\bar{f}_t\varphi)(x) = \varphi(y) = \varphi(x) - t\,\xi^\rho\partial_\rho\varphi + t\mathrm{O},$$

and, as a natural result:

$$X\varphi = \xi^\rho\partial_\rho\varphi.$$

We now evaluate the Lie derivative of an arbitrary covariant tensor $\gamma_{\alpha\beta}$ of rank two. It follows from the considerations of sec. **68** that:

$$(\bar{f}_t\gamma)_{\lambda'\mu'}(x) = \gamma_{\lambda\mu}(y) = \gamma_{\lambda\mu}(x) - t\xi^\rho\partial_\rho\gamma_{\lambda\mu}(x) + tO.$$

One deduces from this, as well as reverting to the coordinates (x^α), with the aid of (69-2) that:

$$(\bar{f}_t\,\gamma)_{\alpha\beta}(x) = A_\alpha^{\lambda'}A_\beta^{\mu'}(\bar{f}_t\,\gamma)_{\lambda'\mu'}(x)$$

$$= (\delta_\alpha^\beta - t\,\partial_\alpha\,\xi^\lambda + t\,O)(\delta_\beta^\mu - t\,\partial_\beta\,\xi^\mu + t\,O)(\gamma_{\lambda\mu} - t\,\xi^\rho\,\partial_\rho\gamma_{\lambda\mu} + t\,O).$$

As a result, by taking the coefficient of the terms in t in the right-hand side, and subtracting from $\gamma_{\alpha\beta}(x)$, one will obtain:

$$X\gamma_{\alpha\beta} = \xi^\rho \partial_\rho \gamma_{\alpha\beta} + \gamma_{\rho\beta}\partial_\alpha \xi^\rho + \gamma_{\alpha\rho}\partial_\beta \xi^\rho . \tag{69-3}$$

One establishes in an identical manner that if $\gamma^{\alpha\beta}$ is a contravariant tensor of rank two then one will has:

$$X\gamma^{\alpha\beta} = \xi^\rho \partial_\rho \gamma^{\alpha\beta} - \gamma^{\rho\beta}\partial_\rho \xi^\alpha - \gamma^{\alpha\rho}\partial_\rho \xi^\beta , \tag{69-4}$$

and the rule that gives the Lie derivative of a tensor will become apparent.

Finally, we propose to evaluate the Lie derivative of an affine connection $\Gamma^\alpha_{\beta\gamma}$. If one affects the transformed connection with index t then one will first obtain:

$$\overset{t}{\Gamma}{}^{\lambda'}_{\mu'\nu'} = \Gamma^\lambda_{\mu\nu}(y) = \Gamma^\lambda_{\mu\nu}(x) - t\,\xi^\rho \partial_\rho \Gamma^\lambda_{\mu\nu} + tO.$$

The coefficients of the transformed connection in the initial local coordinates will be given by:

$$\left(\overset{t}{\Gamma}{}^{\alpha}_{\beta\gamma}\right) = A^\alpha_{\lambda'} A^{\mu'}_\beta A^{\nu'}_\gamma \left(\overset{t}{\Gamma}{}^{\lambda'}_{\mu'\nu'}\right)(x) + A^\alpha_{\lambda'}\partial_\beta A^{\nu'}_\gamma$$

or, explicitly:

$$\begin{aligned}\left(\overset{t}{\Gamma}{}^{\alpha}_{\beta\gamma}\right) = &= (\delta^\beta_\lambda + t\,\partial_\lambda \xi^\alpha + t\,O)(\delta^\mu_\beta - t\,\partial_\beta \xi^\mu + t\,O)(\delta^\nu_\gamma - t\,\partial_\gamma \xi^\nu + t\,O) \\ &\quad (\Gamma^\lambda_{\mu\nu} - t\,\xi^\rho \partial_\rho \Gamma^\lambda_{\mu\nu} + t\,O) + (\delta^\alpha_\nu + t\,\partial_\nu \xi^\alpha + t\,O)(-t\,\partial_\beta \partial_\gamma \xi^\nu + t\,O) .\end{aligned}$$

One deduces from this that:

$$X\Gamma^\alpha_{\beta\gamma} = \xi^\rho \partial_\rho \Gamma^\alpha_{\beta\gamma} - \Gamma^\rho_{\beta\gamma}\,\partial_\rho \xi^\alpha + \Gamma^\alpha_{\rho\gamma}\,\partial_\beta \xi^\rho + \Gamma^\alpha_{\beta\rho}\,\partial_\gamma \xi^\rho + \partial_\beta \partial_\gamma \xi^\alpha . \tag{69-5}$$

One will observe that one can express the right-hand sides of the preceding formulas as the sums of tensors with the aid of an affine connection. For example, by substituting covariant derivatives for the ordinary partial derivatives in (69-3), one will obtain:

$$X\gamma_{\alpha\beta} = \xi^\rho (D_\rho \gamma_{\alpha\beta} + \Gamma^\lambda_{\alpha\rho}\gamma_{\lambda\beta} + \Gamma^\lambda_{\beta\rho}\gamma_{\alpha\lambda}) + \gamma_{\lambda\beta}(D_\alpha \xi^\lambda - \Gamma^\lambda_{\rho\alpha}\xi^\rho) + \gamma_{\alpha\lambda}(D_\beta \xi^\lambda - \Gamma^\lambda_{\rho\beta}\xi^\rho) .$$

By introducing the torsion tensor of the connection, one will then deduce that:

$$X\gamma_{\alpha\beta} = \xi^\rho D_\rho \gamma_{\alpha\beta} + \gamma_{\lambda\beta} D_\alpha \xi^\lambda + \gamma_{\alpha\lambda} D_\beta \xi^\lambda + 2\xi^\rho\,(\gamma_{\lambda\beta} S^\lambda{}_{\alpha\rho} + \gamma_{\alpha\lambda} S^\lambda{}_{\beta\rho}) . \tag{69-6}$$

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CHAPTER V

THE FIELD EQUATIONS OF EINSTEIN'S THEORY

I. – STUDY OF THE TENSOR $g_{\alpha\beta}$

70. – The tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$. – Starting with the four-dimensional vector space T_4, we give ourselves a real covariant tensor of rank two *with no particular symmetry properties* that satisfies some hypotheses that we shall specify. We propose to first give a certain number of elementary results that concern the tensors that one may deduce by symmetrization, anti-symmetrization, and passing to an associated tensor.

For the tensor $g_{\alpha\beta}$, we suppose that:

a) $g = \det(g_{\alpha\beta}) \neq 0$;

b) The quadratic form $\Phi(X) = g_{\alpha\beta} X^\alpha X^\beta$ is a non-degenerate form of hyperbolic normal type with one positive square and three negative ones.

Since $g \neq 0$, the matrix $(g_{\alpha\beta})$ will always be invertible and admit an inverse matrix, which we denote by $(g^{\alpha\beta})$, such that:

(70-1) $$g_{\alpha\rho} g^{\beta\rho} = g_{\rho\alpha} g^{\rho\beta} = \delta_\alpha^\beta \qquad (\delta_\alpha^\beta = 0 \text{ for } \alpha \neq \beta, = 1 \text{ for } \beta = \alpha).$$

The $g^{\alpha\beta}$ are obviously the components of a contravariant tensor of rank two. The tensors $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are called the *associated* tensors. One obviously has:

$$\det(g^{\alpha\beta}) = \frac{1}{g} \neq 0.$$

In the sequel, we let the same root letter – g, here – denote both tensors, one of which is covariant and the other of which is contravariant.

We also introduce the tensorial density:

(70-2) $$\underline{g}^{\alpha\beta} = g^{\alpha\beta} \sqrt{|g|}.$$

One will note that:

$$|\det(\underline{g}^{\alpha\beta})| = |g|^2 \frac{1}{|g|} = |g|.$$

It will then result that one may substitute the given of the tensorial density $\underline{g}^{\alpha\beta}$ for the given of a tensor $g_{\alpha\beta}$ or the associated tensor $g^{\alpha\beta}$; for example, one has:

$$g^{\alpha\beta} = \underline{g}^{\alpha\beta} \frac{1}{\sqrt{|\det(\underline{g}^{\alpha\beta})|}} .$$

71. – The tensors that are deduced by symmetrization and anti-symmetrization. – Henceforth, we set:

(71-1) $$g_{\alpha\beta} = h_{\alpha\beta} + k_{\alpha\beta}; \qquad g^{\alpha\beta} = l^{\alpha\beta} + m^{\alpha\beta},$$

in which:

(71-2) $$h_{\alpha\beta} = g_{(\alpha\beta)}, \qquad k_{\alpha\beta} = g_{[\alpha\beta]}; \qquad l^{\alpha\beta} = g^{(\alpha\beta)}, \qquad m^{\alpha\beta} = g^{[\alpha\beta]};$$

are symmetric (anti-symmetric, resp.) tensor.

We propose to study the relations that exist between the various tensors that were just introduced.

First observe that, from (71-1), the quadratic form Φ(X) may be written:

$$\Phi(X) = h_{\alpha\beta} X^\alpha X^\beta.$$

From hypothesis *b*, one will thus have:

$$h = \det(h_{\alpha\beta}) < 0,$$

and in particular it will be non-zero. We shall also introduce the associated tensor $h^{\alpha\beta}$.

From (70-1), one has the obvious relation:

(71-3) $$g^{\alpha\beta} = g_{\lambda\mu} g^{\lambda\alpha} g_{\mu\beta}.$$

Upon symmetrizing this, one will obtain:

$$l^{\alpha\beta} = \tfrac{1}{2} (g_{\lambda\mu} g^{\lambda\alpha} g_{\mu\beta} + g_{\lambda\mu} g^{\lambda\alpha} g_{\mu\beta}) = \tfrac{1}{2} (g_{\lambda\mu} + g_{\mu\lambda}) g^{\lambda\alpha} g^{\mu\beta}.$$

One will thus have:

(71-4) $$l^{\alpha\beta} = h_{\lambda\mu} g^{\lambda\alpha} g^{\mu\beta},$$

and one will likewise establish that:

(71-5) $$l^{\alpha\beta} = h_{\lambda\mu} g^{\alpha\lambda} g^{\beta\mu}.$$

Upon anti-symmetrizing (71-3), one will obtain:

(71-6) $$m^{\alpha\beta} = k_{\lambda\mu} g^{\lambda\alpha} g^{\mu\beta} = k_{\lambda\mu} g^{\alpha\lambda} g^{\beta\mu}.$$

On the other hand, set:

$$Y_\alpha = g_{\lambda\alpha} X^\lambda, \qquad X^\lambda = g^{\lambda\alpha} Y_\alpha .$$

For X^λ and Y_α that are related in this way, one sees that:

$$\Psi(Y) = l^{\alpha\beta} Y_\alpha Y_\beta = h_{\lambda\mu} X^\lambda X^\mu = \Phi(X).$$

It results from this that the quadratic form:

(71-7) $$\Psi(Y) = g^{\alpha\beta} Y_\alpha Y_\beta = l^{\alpha\beta} Y_\alpha Y_\beta$$

will also be non-degenerate and of hyperbolic normal type. In particular: $\det(l^{\alpha\beta}) < 0$.

One will likewise establish that:

(71-8) $$h_{\alpha\beta} = l^{\lambda\mu} g_{\lambda\alpha} g_{\mu\beta}\,, \qquad k_{\alpha\beta} = m^{\lambda\mu} g_{\lambda\alpha} g_{\mu\beta} = m^{\lambda\mu} g_{\alpha\lambda} g_{\beta\mu}\,.$$

72. – Explicit expression for g with the aid of the $h_{\alpha\beta}$ and $k_{\alpha\beta}$. – Let G, H, K denote the matrices whose general elements are $g_{\alpha\beta}$, $h_{\alpha\beta}$, $k_{\alpha\beta}$, respectively. If l denotes a scalar constant then one will obviously have:

(72-1) $$K + \lambda H = (KH^{-1} + \lambda I)\, H.$$

By making $\lambda = 1$ in (72-1) and multiplying by H^{-1}, it will follow that:

(72-2) $$GH^{-1} = KH^{-1} + I.$$

We propose to evaluate the determinant in the right-hand side of (72-2), and, to that effect, to evaluate:

$$\psi(\lambda) = \det\,(KH^{-1} + \lambda I).$$

One first sees that:

$$\begin{aligned}\det(K + \lambda H) &= \det\,(k_{\alpha\beta} + \lambda h_{\alpha\beta}) = \det\,(k_{\beta\alpha} + \lambda h_{\alpha\beta}) = \det\,(-k_{\alpha\beta} + \lambda h_{\alpha\beta}) \\ &= \det\,(-K + \lambda H).\end{aligned}$$

Since the matrices envisioned are 4×4 one thus has:

$$\det\,(K + \lambda H) = \det\,(K - \lambda H).$$

As a result, by virtue of (72-1), $\psi(\lambda)$ will be an even function of λ. One will thus have:

$$\psi(\lambda) = \lambda^4 + c\lambda^2 + \frac{k}{h}\,,$$

so it will suffice for us to evaluate the coefficient c. It will be the sum of the diagonal minors of order two in the matrix:

$$KH^{-1} = (a_\alpha{}^\beta) \qquad\qquad a_\alpha{}^\beta = k_{\alpha\rho}\, h^{\beta\rho}.$$

One will note that:

$$\sum_\alpha a_\alpha{}^\alpha = 0.$$

One of these minors corresponds to any pair (α, β) $(\alpha \neq \beta)$:

$$a_\alpha^{\ \alpha} a_\beta^{\ \beta} - a_\alpha^{\ \beta} a_\beta^{\ \alpha} \qquad \text{(no summation).}$$

Twice the sum of these minors is therefore:

$$2c = \sum_{\alpha\neq\beta} a_\alpha^{\ \alpha} a_\beta^{\ \beta} - \sum_{\alpha\neq\beta} a_\alpha^{\ \beta} a_\beta^{\ \alpha} = -\sum_\alpha \left(a_\alpha^{\ \alpha}\right)^2 - \sum_{\alpha\neq\beta} a_\alpha^{\ \beta} a_\beta^{\ \alpha} = -\sum_{\alpha,\beta} a_\alpha^{\ \beta} a_\beta^{\ \alpha}\,.$$

One then deduces that:

$$2c = -\,k_{\alpha\rho}\, h^{\beta\rho}\, k_{\beta\sigma}\, h^{\alpha\sigma} = k_{\alpha\beta}\; k_{\rho\sigma} h^{\alpha\rho}\, h^{\beta\sigma}.$$

By taking the determinants of both sides of (72-2), one obtains:

$$\frac{g}{h} = \psi(1) = 1 + \tfrac{1}{2}\, k_{\alpha\beta}\, k_{\rho\sigma} h^{\alpha\rho}\, h^{\beta\sigma} + \frac{k}{h},$$

and, as a result:

(72-1) $$g = h + \frac{h}{2} k_{\alpha\beta}\, k_{\rho\sigma} h^{\alpha\rho}\, h^{\beta\sigma} + k.$$

73. – Explicit expressions for $l^{\alpha\beta}$ and $m^{\alpha\beta}$ as a function of the tensors $h_{\alpha\beta}$ and $k_{\alpha\beta}$. – In order to evaluate $g^{\alpha\beta}$ one may remark that:

$$g\, g^{\alpha\beta} = \frac{\partial g}{\partial g_{\alpha\beta}}\,.$$

As a result:

$$g\, l^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g}{\partial g_{\alpha\beta}} + \frac{\partial g}{\partial g_{\beta\alpha}}\right) \qquad g\, m^{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g}{\partial g_{\alpha\beta}} - \frac{\partial g}{\partial g_{\beta\alpha}}\right).$$

Consider g to be a function of $h_{\alpha\beta}$ and $k_{\alpha\beta}$ by the intermediary of the $g_{\alpha\beta}$. One will get:

$$dg = \frac{\partial g}{\partial g_{\alpha\beta}}(dh_{\alpha\beta} + dk_{\alpha\beta}) = \frac{1}{2}\left(\frac{\partial g}{\partial g_{\alpha\beta}} + \frac{\partial g}{\partial g_{\beta\alpha}}\right) dh_{\alpha\beta} + \frac{1}{2}\left(\frac{\partial g}{\partial g_{\alpha\beta}} - \frac{\partial g}{\partial g_{\beta\alpha}}\right) dk_{\alpha\beta}.$$

One deduces from this that:

(73-1) $$g\, l^{\alpha\beta} = \frac{\partial g}{\partial h_{\alpha\beta}}, \qquad g\, m^{\alpha\beta} = \frac{\partial g}{\partial k_{\alpha\beta}},$$

which provides a convenient way of evaluating the $l^{\alpha\beta}$ and $m^{\alpha\beta}$.

We begin with the $m^{\alpha\beta}$, whose expression is simplest. One deduces from (72-1) by derivation that:

$$\frac{\partial g}{\partial k_{\lambda\mu}} = h\, k_{\rho\sigma} h^{\lambda\rho} h^{\mu\sigma} + \frac{\partial k}{\partial k_{\lambda\mu}}.$$

If $K^{\lambda\mu}$ denotes the minor of k relative to the element $k_{\lambda\mu}$ then one will have:

$$\frac{\partial k}{\partial k_{\lambda\mu}} = K^{\lambda\mu},$$

and one will obtain:

(73-2) $$m^{\lambda\mu} = \frac{h}{g} k_{\rho\sigma} h^{\lambda\rho} h^{\mu\sigma} + \frac{1}{g} K^{\lambda\mu}.$$

In the case where $k \neq 0$, $k_{\alpha\beta}$ will admit an associated tensor $k^{\alpha\beta}$, and one will have:

(73-3) $$m^{\lambda\mu} = \frac{h}{g} k_{\rho\sigma} h^{\lambda\rho} h^{\mu\sigma} + \frac{k}{g} k^{\lambda\mu}.$$

From (72-1), in order to evaluate the $l^{\lambda\mu}$, we shall need to evaluate the derivatives $\dfrac{\partial h^{\alpha\beta}}{\partial h_{\rho\sigma}}$. It follows from the relation:

$$h^{\alpha\lambda} h_{\beta\lambda} = \delta^\alpha_\beta,$$

by derivation and inversion that:

$$\frac{\partial h^{\alpha\beta}}{\partial h_{\rho\sigma}} = - h^{\alpha\lambda} h^{\beta\mu} \frac{\partial h_{\lambda\mu}}{\partial h_{\rho\sigma}} = -\tfrac{1}{2} (h^{\alpha\lambda} h^{\beta\mu} + h^{\alpha\mu} h^{\beta\lambda}) \frac{\partial h_{\lambda\mu}}{\partial h_{\rho\sigma}},$$

namely:

(73-4) $$\frac{\partial h^{\alpha\beta}}{\partial h_{\rho\sigma}} = -\tfrac{1}{2} (h^{\alpha\rho} h^{\beta\sigma} + h^{\alpha\sigma} h^{\beta\rho}).$$

By differentiating (72-1) with respect to $h_{\lambda\mu}$ one will obtains, from (73-4):

$$\frac{\partial g}{\partial h_{\lambda\mu}} = \frac{\partial h}{\partial h_{\lambda\mu}} (1 + \tfrac{1}{2} k_{\alpha\beta}\, k_{\rho\sigma}\, h^{\alpha\rho} h^{\beta\sigma}) - \frac{h}{2} k_{\alpha\beta}\, k_{\rho\sigma}\, (h^{\alpha\lambda} h^{\rho\mu} + h^{\alpha\mu} h^{\rho\lambda})\, h^{\beta\sigma}.$$

There naturally exist inverse formulas that are analogous to (73-3) and (73-5) and express $h_{\alpha\beta}$ and $k_{\alpha\beta}$ as functions of $l^{\lambda\mu}$ and $m^{\lambda\mu}$.

II. – DEFINING THE FIELD EQUATIONS

74. – The fundamental manifold. – The primitive element of our formula consists of a four-dimensional spacetime manifold V_4 that is endowed with the same structure of a differentiable manifold that one finds in general relativity. V_4 is first assumed to be of class C^2. One denotes an admissible coordinate system by (x^α) (α, any Greek index = 0, 1, 2, 3). In the intersection of the domains of two admissible coordinate systems, one assumes, moreover, that the second derivatives of the coordinate change are functions of class piecewise-C^2. This is what we mean when we say that the manifold V_4 is a differentiable manifold of class (C^2, piecewise-C^4).

We assume that two geometric elements are defined on this manifold V_4:

1. A *tensor field* $g_{\alpha\beta}$ of class (C^1, piecewise-C^3), i.e., whose components are continuously differentiable and whose derivatives $\partial_\gamma g_{\alpha\beta}$ are functions of class piecewise-C^2. At each point x of V_4, the tensor $g_{\alpha\beta}$ satisfies the hypotheses of sec. **70**; in particular, the determinant g is non-zero. The tensor $g_{\alpha\beta}$ is called the *fundamental tensor.*

2. An arbitrary *affine connection* whose coefficients $\Gamma^\alpha_{\beta\gamma}$ are continuous and have class piecewise-C^2.

These are the elements that we shall restrict with the "field equations," which we shall derive from a variational principle, by analogy with general relativity or with the Jordan-Thiry theory (see sec. **21**)

75. – Several derivation formulas. – Before we specify this variational principle, we propose to point out several elementary derivation formulas that relate to the tensor $g_{\alpha\beta}$, the associated tensor $g^{\alpha\beta}$, and the determinant g.

Suppose that the $g_{\alpha\beta}$ are differentiable functions of one variable u. One deduces from the relation:

$$g^{\lambda\sigma} g_{\rho\sigma} = \delta^\lambda_\rho ,$$

by derivation, that:

$$\frac{dg^{\lambda\sigma}}{du} g_{\rho\sigma} + g^{\lambda\sigma} \frac{dg_{\rho\sigma}}{du} = 0. \tag{75-1}$$

By multiplying both sides of the preceding relation by $g^{\rho\mu}$, one will obtain:

$$\frac{dg^{\lambda\mu}}{du} = -g^{\lambda\sigma} g^{\rho\mu} \frac{dg_{\rho\sigma}}{du} . \tag{75-2}$$

Likewise, one will deduce from (75-1) that:

(75-3) $$\frac{dg_{\rho\sigma}}{du} = -g_{\lambda\sigma}\, g_{\rho\mu}\, \frac{dg^{\lambda\mu}}{du}.$$

We evaluate the logarithmic derivative of $| g |$. From a well-known formula on the derivative of a determinant, one will get:

(75-4) $$\frac{d \log | g |}{du} = g^{\alpha\beta}\, \frac{dg_{\alpha\beta}}{du}.$$

By replacing $\frac{dg_{\alpha\beta}}{du}$ with its value from (75-3), one will get:

$$\frac{d \log | g |}{du} = -g^{\alpha\beta}\, g_{\alpha\mu}\, g_{\lambda\beta}\, \frac{dg^{\lambda\mu}}{du};$$

namely:

(75-5) $$\frac{d \log | g |}{du} = -g_{\lambda\mu}\, \frac{dg^{\lambda\mu}}{du}.$$

We shall use these formulas in what follows. In particular, we apply (75-4) and (75-5) to the case where $u = x^\rho$. We set:

(75-6) $$\gamma_\rho = \frac{\partial_\rho \sqrt{| g |}}{\sqrt{| g |}} = \tfrac{1}{2} \partial_\rho \log | g |.$$

It will thus follow that:

(75-7) $$\gamma_\rho = \tfrac{1}{2}\, g^{\alpha\beta}\, \partial_\rho g_{\alpha\beta} = -\tfrac{1}{2}\, g_{\alpha\beta}\, \partial_\rho g^{\alpha\beta}.$$

76. – The variational principle. – Let C be a four-dimensional, differential chain in the manifold and arbitrarily vary the fundamental tensor and the connection in such a fashion that the variations will be zero on the boundary ∂C of the chain envisioned. Consider the corresponding variation of the scalar-valued integral:

(76-1) $$I = \int_C g^{\alpha\beta} R_{\alpha\beta} \sqrt{| g |}\; dx^0 \wedge \ldots \wedge dx^3,$$

in which $R_{\alpha\beta}$ denotes the Ricci tensor of the affine connection $\Gamma^\alpha_{\beta\gamma}$.

The field equations of the theory are the ones that define the extremum of the integral I vis-à-vis all variations of the fundamental tensor and the connection that are restricted by only the requirement that they should vanish on the boundary of C.

We evaluate the variation of I, while distinguishing the contribution that is made by the variation of the connection, as well as the one that is made by the variation of the fundamental tensor. We obtain:

$$(76\text{-}2)\qquad \delta I = \int_C g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{|g|}\,dx^0\wedge\ldots\wedge dx^3 + \int_C g^{\alpha\beta}R_{\alpha\beta}\,\delta[g^{\alpha\beta}\sqrt{|g|}]\,dx^0\wedge\ldots\wedge dx^3.$$

We set:

$$(76\text{-}3)\qquad \delta_1 I = \int_C g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{|g|}\,dx^0\wedge\ldots\wedge dx^3$$

and:

$$(76\text{-}4)\qquad \delta_2 I = \int_C g^{\alpha\beta}R_{\alpha\beta}\,\delta[g^{\alpha\beta}\sqrt{|g|}]\,dx^0\wedge\ldots\wedge dx^3.$$

We first occupy ourselves with the evaluation of $\delta_1 I$ by making the variation $\delta\Gamma^\rho_{\alpha\beta}$ of the affine connection appear explicitly.

77. – First form of the field equations. – By evaluating the variation of the Ricci tensor as we did in sec. **67**, one has:

$$\delta R_{\alpha\beta} = D_\rho\delta\Gamma^\rho_{\alpha\beta} - D_\beta\delta\Gamma^\rho_{\alpha\rho} - 2S^\sigma{}_{\rho\beta}\,\delta\Gamma^\rho_{\alpha\beta}.$$

One deduces from this that:

$$(77\text{-}1)\quad \begin{cases} g^{\alpha\beta}\delta R_{\alpha\beta} = D_\rho(g^{\alpha\beta}\delta\Gamma^\rho_{\alpha\beta}) - D_\beta(g^{\alpha\beta}\delta\Gamma^\rho_{\alpha\rho}) & -D_\rho g^{\alpha\beta}\delta\Gamma^\rho_{\alpha\beta} \\ & +D_\rho g^{\alpha\beta}\delta\Gamma^\rho_{\alpha\beta} - 2S^\sigma{}_{\rho\beta}\delta\Gamma^\rho_{\alpha\sigma}g^{\alpha\beta}. \end{cases}$$

We will thus be led to introduce the vector:

$$(77\text{-}2)\qquad A^\rho = g^{\alpha\beta}\,\delta\Gamma^\rho_{\alpha\beta} - g^{\alpha\rho}\,\delta\Gamma^\sigma_{\alpha\sigma},$$

and we note that since the variation of the connection is zero on the boundary of C, *one will have* $A^\rho = 0$ *on that boundary*. Formula (77-1) will then become:

$$(77\text{-}3)\qquad g^{\alpha\beta}\,\delta R_{\alpha\beta} = D_\rho A^\rho - D_\rho g^{\alpha\beta}\,\delta\Gamma^\rho_{\alpha\beta} + D_\beta g^{\alpha\beta}\,\delta\Gamma^\rho_{\alpha\rho} - 2\,g^{\alpha\beta}S^\sigma{}_{\rho\beta}\,\delta\Gamma^\rho_{\alpha\sigma}.$$

We therefore propose to evaluate the integral:

$$J(\delta) = \int_C D_\rho A^\rho\cdot\sqrt{|g|}\,dx^0\wedge\ldots\wedge dx^3.$$

One has:

$$D_\rho A^\rho\cdot\sqrt{|g|} = (\partial_\rho A^\rho + \Gamma^\rho_{\lambda\rho}A^\lambda)\sqrt{|g|} = \partial_\rho(A^\rho\sqrt{|g|}) + (\Gamma^\sigma_{\rho\sigma} - \gamma_\rho)A^\rho\sqrt{|g|}$$

in which γ_ρ is defined by (75-6). Under these conditions, one obtains:

$$J(\delta) = \int_C \partial_\rho (A^\rho \sqrt{|g|})\ dx^0 \wedge \ldots \wedge dx^3 + \int_C (\Gamma^\sigma_{\rho\sigma} - \gamma_\rho) A^\rho \sqrt{|g|}\ dx^0 \wedge \ldots \wedge dx^3.$$

By applying Stokes's formula, the first integral in the right-hand side will be transformed into an integral that is taken over the boundary of C, which will be zero since A^ρ is zero on that boundary. One deduces from (76-3) and (77-3) that:

$$\delta_1 I = \int_C M \sqrt{|g|}\ dx^0 \wedge \ldots \wedge dx^3, \tag{77-4}$$

with:

$$M = D_\rho g^{\alpha\beta}\ \delta\Gamma^\rho_{\alpha\sigma} - D_\rho g^{\alpha\beta}\ \delta\Gamma^\rho_{\alpha\sigma} - (\Gamma^\sigma_{\rho\sigma} - \gamma_\rho)\, A^\rho + 2\, g^{\alpha\beta} S^\sigma{}_{\rho\beta}\, \delta\Gamma^\rho_{\alpha\sigma}\,, \tag{77-5}$$

in which A^ρ is given by (77-2). We propose to exhibit the coefficient of $\delta\Gamma^\rho_{\alpha\beta}$ in the various terms of the scalar M. One has:

$$M = [D_\rho g^{\alpha\beta} - \delta^\beta_\rho\, D_\lambda g^{\alpha\lambda} - (\Gamma^\sigma_{\rho\sigma} - \gamma_\rho)\, g^{\alpha\beta} + \delta^\beta_\rho\, (\Gamma^\sigma_{\lambda\sigma} - \gamma_\lambda)\, g^{\alpha\lambda} + 2 g^{\alpha\sigma} S^\beta{}_{\rho\sigma}] \cdot \delta\Gamma^\rho_{\alpha\beta}\,. \tag{77-6}$$

Given a scalar θ, which we shall choose in a moment, consider the quantities:

$$G^{\alpha\beta}{}_\rho(\theta) = D_\rho g^{\alpha\beta} - (\Gamma^\sigma_{\rho\sigma} - \gamma_\rho)\, g^{\alpha\beta} + 2\theta\ \delta^\beta_\rho\, g^{\alpha\sigma} S_\sigma.$$

By contraction, we will obtain:

$$G^{\alpha\lambda}{}_\lambda(\theta) = D_\lambda g^{\alpha\lambda} - (\Gamma^\sigma_{\lambda\sigma} - \gamma_\lambda)\, g^{\alpha\lambda} + 2 g^{\alpha\sigma} S_\sigma (4\theta - 1).$$

One deduces from this that:

$$[G^{\alpha\beta}{}_\rho(\theta) - \delta^\beta_\rho\, G^{\alpha\lambda}{}_\lambda(\theta)]\, \delta\Gamma^\rho_{\alpha\beta} = M + 2(\theta - 4\theta + 1)\, \delta^\beta_\rho\, g^{\alpha\sigma} S_\sigma\, \delta\Gamma^\rho_{\alpha\beta}\,.$$

We thus take $\theta = \frac{1}{3}$ and set:

$$G^{\alpha\beta}{}_\rho = D_\rho g^{\alpha\beta} - (\Gamma^\sigma_{\rho\sigma} - \gamma_\rho)\, g^{\alpha\beta} + 2 g^{\alpha\sigma} S^\beta{}_{\rho\sigma} + \tfrac{2}{3}\, \delta^\beta_\rho\, g^{\alpha\sigma} S_\sigma. \tag{77-7}$$

Thanks to the introduction of these quantities, one will have the simple formula:

$$M = [G^{\alpha\beta}{}_\rho - \delta^\beta_\rho\, G^{\alpha\lambda}{}_\lambda]\, \delta\Gamma^\rho_{\alpha\beta}\,.$$

Thus, *for any variation of the affine connection that vanishes on* ∂C one will have:

$$\delta_1 I = -\int_C \left[G^{\alpha\beta}{}_\rho - \delta^\beta_\rho G^{\alpha\lambda}{}_\lambda \right] \delta\Gamma^\rho_{\alpha\beta} \sqrt{|g|}\ dx^0 \wedge \ldots \wedge dx^3. \tag{77-8}$$

On the other hand, from the considerations of sec. **70**, it amounts to the same thing to impose arbitrary variations on $g_{\alpha\beta}$ or $g^{\alpha\beta}$, and on $\underline{g}_{\alpha\beta}$ or $\underline{g}^{\alpha\beta}$. Therefore, for an arbitrary variation of the tensorial density $\underline{g}^{\alpha\beta}$, one has:

$$\delta_2 I = \int_C R_{\alpha\beta}\, \delta \underline{g}^{\alpha\beta}\; dx^0 \wedge \ldots \wedge dx^3. \tag{77-9}$$

Suppose that we vary the coefficients of the affine connection without varying the fundamental tensor or the tensorial density $\underline{g}^{\alpha\beta}$. Conforming to the variational principle, one must have $\delta I = \delta_1 I = 0$ for any variation of the affine connection that is zero on the boundary of C. One deduces from (77-8) by a classical argument that one must have:

$$G^{\alpha\beta}{}_{\rho} - \delta^{\beta}_{\rho}\, G^{\alpha\lambda}{}_{\lambda} = 0. \tag{77-10}$$

One obtains by contraction that:

$$G^{\alpha\lambda}{}_{\lambda} - 4G^{\alpha\lambda}{}_{\lambda} = -\,3G^{\alpha\lambda}{}_{\lambda} = 0.$$

It results from this that the system (77-10) thus obtained will be equivalent to the system:

$$G^{\alpha\beta}{}_{\rho} = 0, \tag{77-11}$$

in which the $G^{\alpha\beta}{}_{\rho}$ are given by equations (77-7).

On the contrary, if we vary the $\underline{g}^{\alpha\beta}$ without varying the affine connection then we must have $\delta I = \delta_2 I = 0$ for any variation of the $g^{\alpha\beta}$ that is zero on the boundary of C. The following system of equations will result from this by the same classical argument:

$$R_{\alpha\beta} = 0. \tag{77-12}$$

The set of the two systems (77-11) and (77-12) constitutes "the system of field equations." In the following sections, we shall transform these systems – mainly (77-11) – in such a fashion that we will obtain a system that is infinitely more manageable.

78. – Introduction of a new connection. – In order to simplify the form of equations (77-11), we shall replace the original connection $\Gamma^{\alpha}_{\beta\gamma}$ with a new affine connection with coefficients $L^{\alpha}_{\beta\gamma}$, which will be defined by the following lemma:

LEMMA – *Given an arbitrary affine connection $\Gamma^{\alpha}_{\beta\gamma}$, there exists one and only one affine connection $L^{\alpha}_{\beta\gamma}$ that defines the same parallelism and whose covariant torsion vector is zero.*

Indeed, from (66-3), let:

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + 2\delta^{\alpha}_{\beta}\, p_{\gamma}$$

be an affine connection that defines the same parallelism as the connection Γ. Its torsion tensor is:

$$\Sigma^{\alpha}{}_{\beta\gamma} = S^{\alpha}{}_{\beta\gamma} + \delta^{\alpha}_{\beta}\, p_{\gamma} - \delta^{\alpha}_{\gamma}\, p_{\beta}\,,$$

and, as a result, its covariant torsion vector will be:

$$\Sigma_{\beta} = S_{\beta} + p_{\beta} - 4p_{\beta} = S_{\beta} - 3p_{\beta}.$$

In order for Σ_{β} to be zero, it is necessary and sufficient that:

$$p_{\beta} = \tfrac{1}{3} S_{\beta}.$$

One deduces from this that the only affine connection that answers the question is:

$$L^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \tfrac{2}{3}\delta^{\alpha}_{\beta} S_{\gamma}\,, \tag{78-1}$$

and that connection will be such that:

$$2\Sigma_{\beta} = L^{\alpha}_{\beta\gamma} - L^{\alpha}_{\rho\beta} = 0. \tag{78-2}$$

Having said that, we specify the $G^{\alpha}{}_{\beta\gamma}$ with the aid of the new connection $L^{\alpha}_{\beta\gamma}$. Upon specifying the covariant derivatives, we will first get:

$$G^{\alpha}{}_{\beta\gamma} = \partial_{\rho} g^{\alpha\beta} + \Gamma^{\alpha}_{\sigma\rho} g^{\sigma\beta} + \Gamma^{\beta}_{\sigma\rho} g^{\alpha\sigma} + 2g^{\alpha\sigma} S^{\beta}{}_{\rho\sigma} - (\Gamma^{\sigma}_{\rho\sigma} - \gamma_{\rho})\, g^{\alpha\beta} + \tfrac{2}{3}\delta^{\alpha}_{\beta}\; g^{\alpha\sigma} S_{\sigma}.$$

If we take into account that:

$$\Gamma^{\beta}_{\sigma\rho} - 2S^{\beta}{}_{\rho\sigma} = \Gamma^{\beta}_{\rho\sigma}$$

then it will follow that:

$$G^{\alpha\beta}{}_{\rho} = \partial_{\rho} g^{\alpha\beta} + \Gamma^{\alpha}_{\sigma\rho} g^{\sigma\beta} + \Gamma^{\beta}_{\rho\sigma} g^{\alpha\sigma} - (\Gamma^{\sigma}_{\rho\sigma} - \gamma_{\rho})\, g^{\alpha\beta} + \tfrac{2}{3}\delta^{\beta}_{\rho}\; g^{\alpha\sigma} S_{\sigma}.$$

By introducing the L instead of the Γ, one will deduce that:

$$G^{\alpha\beta}{}_{\rho} = \partial_{\rho} g^{\alpha\beta} + (L^{\alpha}_{\sigma\rho} - \tfrac{2}{3}\delta^{\alpha}_{\sigma} S_{\rho}) g^{\sigma\beta} + (L^{\beta}_{\rho\sigma} - \tfrac{2}{3}\delta^{\beta}_{\rho} S_{\sigma}) g^{\alpha\sigma} - (L^{\sigma}_{\rho\sigma} - \gamma_{\rho})\, g^{\alpha\beta}$$
$$+ \tfrac{2}{3}\, g^{\alpha\sigma} S_{\rho} + \tfrac{2}{3}\delta^{\beta}_{\rho}\, g^{\alpha\sigma} S_{\sigma}.$$

After simplification, one will obtain:

$$G^{\alpha\beta}{}_{\rho} = \partial_{\rho} g^{\alpha\beta} + L^{\alpha}_{\sigma\rho}\, g^{\sigma\beta} + L^{\beta}_{\rho\sigma}\, g^{\alpha\sigma} - (L^{\sigma}_{\sigma\rho} - \gamma_{\rho})\, g^{\alpha\beta}. \tag{78-3}$$

One will deduce an expression for the γ_ρ immediately from the field equations (77-11) – namely, $G^{\alpha\beta}{}_\rho = 0$, in which the $G^{\alpha\beta}{}_\rho$ may be provided by the formulas (78-3) – with the aid of L. Indeed, by multiplying the two sides of (78-2) by $g_{\alpha\beta}$, one will have, as a consequence of (77-11):

$$-2\gamma_\rho + L^\sigma_{\sigma\rho} + L^\sigma_{\rho\sigma} - 4(L^\sigma_{\sigma\rho} - \gamma_\rho) = 0,$$

namely:

$$\gamma_\rho = L^\sigma_{\sigma\rho},$$

from (78-2). It will result from this that equations (77-11) entail the equations:

(78-4) $$\partial_\rho g^{\alpha\beta} + L^\alpha_{\sigma\rho} g^{\sigma\beta} + L^\beta_{\rho\sigma} g^{\alpha\sigma} = 0.$$

Conversely, suppose that we are given an affine connection $L^\alpha_{\beta\gamma}$ with zero torsion vector and a covariant vector S_γ. If equations (78-4) are satisfied then upon multiplying both sides of (78-4) by $g_{\alpha\beta}$, one will get

(78-5) $$2\gamma_\rho = L^\sigma_{\sigma\rho} + L^\sigma_{\rho\sigma}.$$

If one now takes into account the fact that the torsion vector is zero then one will get:

(78-6) $$\gamma_\rho = L^\sigma_{\sigma\rho},$$

and for:

(78-7) $$\Gamma^\alpha_{\beta\gamma} = L^\alpha_{\beta\gamma} - \tfrac{2}{3}\delta^\alpha_\beta S_\gamma$$

one will see from (78-3) that equations (77-11) are satisfied.

Thus, *the first system of field equations* (77-11) *will be equivalent to* (78-4) *when the connection L is restricted to admit a zero torsion vector.*

79. – New form of the field equations. – From now on, we shall try to replace the connection $\Gamma^\alpha_{\beta\gamma}$ with the connection $L^\alpha_{\beta\gamma}$, which satisfies equations (78-4) and admits a zero torsion vector.

a) If one takes (78-4) into account then one may replace the four conditions (78-2), $\Sigma_\beta = 0$, with four interesting conditions that involve the fundamental tensor. Consider an affine connection $L^\alpha_{\beta\gamma}$ that satisfies (78-4), but we make no hypothesis on its torsion vector. One will then get (78-5). One deduces from (78-4) by contraction that:

(79-1) $$\partial_\rho g^{\rho\beta} + L^\rho_{\sigma\rho} g^{\sigma\beta} + L^\beta_{\rho\sigma} g^{\rho\sigma} = 0.$$

On the contrary, the other possible contraction will give:

(79-2) $$\partial_\rho g^{\beta\rho} + L^{\beta}_{\sigma\rho} g^{\sigma\rho} + L^{\rho}_{\rho\sigma} g^{\beta\sigma} = 0.$$

By subtracting the last two equations, one will obtain:

$$2\,\partial_\rho g^{[\rho\beta]} + L^{\sigma}_{\rho\sigma} g^{\rho\beta} - L^{\sigma}_{\sigma\rho} g^{\beta\rho} = 0.$$

Now:

$$L^{\sigma}_{\rho\sigma} = L^{\sigma}_{\sigma\rho} + \Sigma_\rho .$$

One deduces from this that:

$$L^{\sigma}_{\rho\sigma} g^{\rho\beta} - L^{\sigma}_{\sigma\rho} g^{\beta\rho} = (L^{\sigma}_{\sigma\rho} + \Sigma_\rho)(l^{\rho\beta} + m^{\rho\beta}) - L^{\sigma}_{\sigma\rho}(l^{\rho\beta} + m^{\rho\beta}) = \Sigma_\rho l^{\rho\beta} + (2\,L^{\sigma}_{\sigma\rho} + \Sigma_\rho)\, m^{\rho\beta} ;$$

namely, from (78-5):

$$L^{\sigma}_{\rho\sigma} g^{\rho\beta} - L^{\sigma}_{\sigma\rho} g^{\beta\rho} = \Sigma_\rho l^{\rho\beta} + 2\gamma_\rho m^{\rho\beta}.$$

One will thus obtain:

(79-3) $$\partial_\rho g^{[\rho\beta]} + \gamma_\rho g^{[\rho\beta]} = -\tfrac{1}{2}\,\Sigma_\rho l^{\rho\beta}.$$

Since $\det(l^{\rho\beta}) \neq 0$ (see sec. **71**), it results that in order for the torsion vector Σ_ρ of the connection envisioned to be zero, it is necessary and sufficient that one have:

(79-4) $$\partial_\rho g^{[\rho\beta]} + \gamma_\rho g^{[\rho\beta]} = 0 ;$$

namely:

$$\partial_\rho \underline{g}^{[\rho\beta]} = 0.$$

Therefore the search for a connection $\Gamma^{\alpha}_{\beta\gamma}$ that satisfies the first system of field equations is equivalent to the following problem: Find a connection $L^{\alpha}_{\beta\gamma}$ that satisfies the equations:

(79-5) $$\partial_\rho g^{\alpha\beta} + \mathrm{L}^{\alpha}_{\sigma\rho} g^{\sigma\beta} + \mathrm{L}^{\beta}_{\rho\sigma} g^{\alpha\sigma} = 0,$$

while the derivatives of the fundamental tensor are assumed to satisfy the four relations:

(79-6) $$\partial_\rho \underline{g}^{[\rho\beta]} = 0.$$

The formula:

(79-7) $$\Gamma^{\rho}_{\alpha\beta} = L^{\rho}_{\alpha\beta} - \tfrac{2}{3}\,\delta^{\rho}_{\alpha}\, S_\beta ,$$

in which S_β is an arbitrary covariant vector, will then gives the desired connection.

One may replace formulas (79-5) with the equivalent formulas that are obtained by multiplying both sides of (79-5) by $g_{\lambda\beta}\, g_{\alpha\mu}$. It will then follow from (75-3) that:

(79-8) $$\partial_\rho g_{\lambda\mu} - L^{\sigma}_{\lambda\rho} g_{\sigma\mu} - L^{\sigma}_{\rho\mu} g_{\lambda\sigma} = 0.$$

b) Let $P_{\alpha\beta}$ denote the Ricci tensor of the connection. It is easy to evaluate the tensor $R_{\alpha\beta}$ by starting with $P_{\alpha\beta}$. Indeed, one has:

$$P_{\alpha\beta} = \partial_\rho L^\rho_{\alpha\beta} - \partial_\beta L^\rho_{\alpha\rho} + L^\sigma_{\rho\sigma} L^\rho_{\alpha\beta} - L^\sigma_{\rho\beta} L^\rho_{\alpha\sigma}.$$

On the other hand, one gets from (79-7) that:

$$R_{\alpha\beta} = \partial_\rho (\Gamma^\rho_{\alpha\beta} - \tfrac{2}{3}\delta^\rho_\alpha S_\beta) - \partial_\beta (\Gamma^\rho_{\alpha\rho} - \tfrac{2}{3}\delta^\rho_\alpha S_\rho)$$
$$+ (\Gamma^\sigma_{\rho\sigma} - \tfrac{2}{3}\delta^\sigma_\rho S_\sigma)(\Gamma^\rho_{\alpha\beta} - \tfrac{2}{3}\delta^\rho_\alpha S_\beta) - (\Gamma^\sigma_{\rho\beta} - \tfrac{2}{3}\delta^\rho_\sigma S_\beta)(\Gamma^\rho_{\alpha\sigma} - \tfrac{2}{3}\delta^\rho_\alpha S_\sigma).$$

One deduces from this that:

$$R_{\alpha\beta} = P_{\alpha\beta} - \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha) - \tfrac{2}{3}(L^\rho_{\alpha\beta} S_\rho + L^\sigma_{\alpha\sigma} S_\beta - L^\sigma_{\alpha\beta} S_\sigma - L^\sigma_{\alpha\sigma} S_\beta + \tfrac{2}{3} S_\alpha S_\beta - \tfrac{2}{3} S_\alpha S_\beta);$$

namely:

(79-9) $$R_{\alpha\beta} = P_{\alpha\beta} - \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha).$$

It results from this that one may replace equations (77-12), which relate to the connection Γ, with the equations:

(79-10) $$P_{\alpha\beta} - \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha) = 0.$$

c) We are thus led to adopt as our new quantities that determine the unitary field, not the fundamental tensor $g_{\alpha\beta}$, but the affine connection $L^\alpha_{\beta\gamma}$, which is arbitrary *a priori*, and the covariant tensor S_α. The field equations will then be given by equations (79-6), (79-8), and (79-10).

The field quantities consist of the sixteen $g_{\alpha\beta}$, sixty-four $L^\alpha_{\beta\gamma}$, and four S_α, and we effectively have the four equations (79-6), the sixty-four equations (79-8), and the sixteen equations (79-10) at our disposal. We ultimately establish that these equations are not independent, but that there exist four "conservation identities" that insure the role that is played by the admissible coordinate changes, exactly as they do in general relativity.

One may remark that the system of equations (79-10) is equivalent to the set of the two systems:

(79-11) $$P_{(\alpha\beta)} = 0$$

and:

(79-12) $$P_{[\alpha\beta]} = \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha).$$

If P denotes the quadratic exterior differential form whose coefficients are $P_{[\alpha\beta]}$ then it clearly must follow as a consequence of (79-12) that:

(79-13) $$\partial_\alpha P_{[\beta\gamma]} + \partial_\beta P_{[\gamma\alpha]} + \partial_\gamma P_{[\alpha\beta]} = 0.$$

Conversely, equation (79-13) entails only the local existence of a vector-potential. That is why it is preferable to use equations (79-12) instead of (79-13).

80. – The contracted, anti-symmetric, curvature tensor and Einstein tensor of L. – Consider a solution $g_{\alpha\beta}$, $L^\rho_{\alpha\beta}$ of equations (79-6) and (79-8). As we saw in sec. **79**, from it results these equations that the torsion vector Σ_ρ of the connection L is zero, and, as a result, from (78-5), that one will have:

(80-1) $$\gamma_\rho = \frac{\partial_\rho \sqrt{|g|}}{\sqrt{|g|}} = L^\sigma_{\sigma\rho} = L^\sigma_{\rho\sigma}.$$

If the manifold V_4 is orientable then there will exist an exterior differential form on V_4 :

(80-2) $$\eta = \sqrt{|g|}\, dx^0 \wedge \ldots \wedge dx^3$$

that has zero covariant derivative for the connection $L^\rho_{\alpha\beta}$, as well as the associated symmetric connection $L^\rho_{(\alpha\beta)}$. In any event, from (80-1), the contracted, anti-symmetric, curvature tensors $V_{\lambda\mu}$ and $W_{\lambda\mu}$ relative to the connections $L^\rho_{\alpha\beta}$ and $L^\rho_{(\alpha\beta)}$ are identically null:

(80-3) $$V_{\lambda\mu} = 0, \qquad W_{\lambda\mu} = 0.$$

In what follows, we shall denote the covariant derivative relative to the connection $L^\rho_{\lambda\mu}$ by d_μ. If $E_{\lambda\mu}$ is the Einstein connection of this connection then one sees that (65-7) may be written here:

$$E_{\lambda\mu} = P_{\lambda\mu} + d_\mu \Sigma_\lambda - W_{\lambda\mu},$$

which reduces, with the conditions (79-6) and (79-8), to:

(80-4) $$E_{\lambda\mu} = P_{\lambda\mu}.$$

On the other hand, with the same conditions, (65-6) gives:

$$E_{\lambda\mu} = \tfrac{1}{2}(P_{\lambda\mu} + \overline{P}_{\mu\lambda}) - \tfrac{1}{2}(\partial_\lambda \Sigma_\mu - \partial_\mu \Sigma_\lambda) = \tfrac{1}{2}(P_{\lambda\mu} + \overline{P}_{\mu\lambda}).$$

In particular, one sees from (80-4) that one may replace equations (79-10) with the equations:

(80-6) $$E_{\alpha\beta} - \tfrac{1}{2}(\partial_\alpha S_\beta - \partial_\beta S_\alpha) = 0.$$

81. – A symmetry theorem. – We make the following tensor correspond to the fundamental tensor $g_{\alpha\beta}$:

(81-1) $$\bar{g}_{\alpha\beta} = g_{\alpha\beta} = h_{\alpha\beta} - k_{\alpha\beta}.$$

The tensor $\bar{g}_{\alpha\beta}$ obviously satisfies the same hypotheses as the tensor $g_{\alpha\beta}$; in particular, $\bar{g} = \det(\bar{g}_{\alpha\beta}) = g \neq 0$. The associated tensor is:

$$\bar{g}^{\alpha\beta} = g^{\alpha\beta} = l^{\alpha\beta} - m^{\alpha\beta}.$$

We have attached any affine connection $L^{\rho}_{\alpha\beta}$ to the affine connection with coefficients:

(81-2) $$\bar{L}^{\rho}_{\alpha\beta} = L^{\rho}_{\beta\alpha}.$$

If $\bar{\Gamma}^{\rho}_{\alpha\beta}$ denotes the affine connection that thus corresponds to $\Gamma^{\rho}_{\alpha\beta}$ then its torsion tensor will be:

$$\bar{S}^{\rho}{}_{\alpha\beta} = - S^{\rho}{}_{\alpha\beta}$$

and, as a result, its torsion vector will be:

(81-3) $$\bar{S}_{\alpha} = - S_{\alpha}.$$

Having said that, we propose to establish the following theorem:

THEOREM – If ($g_{\alpha\beta}$, $L^{\rho}_{\alpha\beta}$, S_{α}) *define a solution to the field equations* (79-6), (79-8), (79-10) *then the same is true for* $(\bar{g}_{\alpha\beta}, \bar{L}^{\rho}_{\alpha\beta}, \bar{S}_{\alpha})$.

The field equations are, as we have seen, defined by the set of three systems:

(81-4) $$\partial_{\rho} \underline{g}^{[\rho\beta]} = 0$$

(81-5) $$\partial_{\rho} g_{\lambda\mu} - L^{\sigma}_{\lambda\rho} g_{\sigma\mu} - L^{\sigma}_{\rho\mu} g_{\lambda\sigma} = 0$$

(81-6) $$P_{\alpha\beta} - \tfrac{2}{3}(\partial_{\alpha} S_{\beta} - \partial_{\beta} S_{\alpha}) = 0.$$

If $(\bar{g}_{\alpha\beta}, \bar{L}^{\rho}_{\alpha\beta}, \bar{S}_{\alpha})$ is the set of quantities that that are deduced from the solution ($g_{\alpha\beta}$, $L^{\rho}_{\alpha\beta}$, S_{α}) by the operations that we just introduced then one will first have, from (81-4), and since $\bar{g} = g$, that:

(81-7) $$\partial_{\rho} \underline{\bar{g}}^{[\rho\beta]} = \partial_{\rho} \underline{g}^{[\rho\beta]} = 0.$$

On the other hand:

$$\partial_{\rho} \bar{g}_{\lambda\mu} - \bar{L}^{\sigma}_{\lambda\rho}\, \bar{g}_{\sigma\mu} - \bar{L}^{\sigma}_{\rho\mu}\, \bar{g}_{\lambda\sigma} = \partial_{\rho} g_{\lambda\mu} - L^{\sigma}_{\lambda\rho}\, g_{\sigma\mu} - L^{\sigma}_{\rho\mu}\, g_{\lambda\sigma}.$$

Now, the right-hand side differs from the left-hand side of (81-5) only by the exchange of the letters λ and μ. One deduces from this that:

(81-8) $$\partial_\rho \bar{g}_{\lambda\mu} - \bar{L}^\sigma_{\lambda\rho}\, \bar{g}_{\sigma\mu} - \bar{L}^\sigma_{\rho\mu}\, \bar{g}_{\lambda\sigma} = 0.$$

Finally, since (80-5) gives us that:

$$\bar{P}_{\alpha\beta} = P_{\beta\alpha} ,$$

it will follow from (81-6) that:

(81-9) $$\bar{P}_{\alpha\beta} - \tfrac{2}{3}(\partial_\alpha \bar{S}_\beta - \partial_\beta \bar{S}_\alpha) = P_{\beta\alpha} - \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha) = 0.$$

Our theorem is thus proved. Einstein assigned the name of "pseudo-Hermiticity" to the property that this theorem suggests.

82. – Tensorial form of the field equations. - It is easy to exhibit the tensorial form of the first group of field equations (81-4) and (81-5). To that effect, we are thus led to introduce the tensor:

(82-1) $$F_{\lambda\mu} = \tfrac{1}{2}\varepsilon_{\lambda\mu\alpha\beta}\underline{g}^{[\alpha\beta]} = \tfrac{1}{2}\varepsilon_{\lambda\mu\alpha\beta}\sqrt{|g|}\, m^{\alpha\beta} ,$$

in which $\varepsilon_{\lambda\mu\alpha\beta}$ is the classical indicator of the permutation. We let F denote the quadratic exterior form that is defined by $F_{\lambda\mu}$. In order to facilitate the evaluation of its exterior differential dF, we calculate:

$$\tfrac{1}{2}\varepsilon^{\lambda\mu\nu\rho}\,\partial_\nu F_{\lambda\mu} = \tfrac{1}{4}\varepsilon^{\lambda\mu\nu\rho}\varepsilon_{\lambda\mu\alpha\beta}\,\partial_\nu \underline{g}^{[\alpha\beta]} = \tfrac{1}{4}\varepsilon^{\nu\rho}{}_{\alpha\beta}\,\partial_\nu \underline{g}^{[\alpha\beta]} ,$$

namely:

$$\tfrac{1}{2}\varepsilon^{\lambda\mu\nu\rho}\,\partial_\nu F_{\lambda\mu} = \tfrac{1}{2}\partial_\alpha \underline{g}^{[\alpha\rho]} .$$

One deduces from this that the system (81-4) is equivalent to:

(82-2) $$dF = 0.$$

On the other hand, we evaluate the covariant derivative $d_\rho g_{\lambda\mu}$ of the fundamental tensor for the connection L. We will get:

$$d_\rho g_{\lambda\mu} = \partial_\rho g_{\lambda\mu} - L^\sigma_{\lambda\rho} g_{\sigma\mu} - L^\sigma_{\rho\mu} g_{\lambda\sigma} .$$

One deduces from this that equations (81-5) may be put into the form:

$$d_\rho g_{\lambda\mu} + (L^\sigma_{\mu\rho} - L^\sigma_{\rho\mu}) g_{\lambda\sigma} = 0 ;$$

namely:

(82-3) $$d_\rho g_{\lambda\mu} = 2S^\sigma{}_{\rho\mu}\, g_{\lambda\sigma}.$$

III. – CONSERVATION IDENTITIES

83. – First form of the conservation identities. – The fact that we used a variational procedure that involves the integral of a scalar density in order to form the field equations offers two advantages: This process leads to equations that are invariant under admissible coordinate changes, and the left-hand sides automatically satisfy four conservation identities, which we shall form by a method that is due to Hermann Weyl, in principle.

Recall the integral:

$$I = \int_C R_{\alpha\beta} g^{\alpha\beta} \sqrt{|g|}\ dx^0 \wedge \ldots \wedge dx^3 = \int_C R\eta\,,$$

in which one has set:

$$R = R_{\alpha\beta} g^{\alpha\beta}, \qquad \eta = \sqrt{|g|}\, dx^0 \wedge \ldots \wedge dx^3.$$

Consider an arbitrary vector field ξ^ρ on a chain C that is *zero on the boundary of C.* That field will define an infinitesimal transformation. If $X(R\eta)$ denotes the Lie derivative of the form $R\eta$ then we will set:

$$XI = \int_C X(R\,\eta)\,.$$

From (69-4), one has the following expression for the Lie derivative of the tensor $g^{\alpha\beta}$:

$$Xg^{\alpha\beta} = \xi^\rho \partial_\rho g^{\alpha\beta} - g^{\lambda\beta}\partial_\lambda \xi^\alpha - g^{\alpha\lambda}\partial_\lambda \xi^\beta\,. \tag{83-1}$$

One deduces from this and the formula for the logarithmic derivative of g that:

$$\frac{X\sqrt{|g|}}{\sqrt{|g|}} = -\tfrac{1}{2}\, g_{\alpha\beta} X g^{\alpha\beta} = -\tfrac{1}{2}[\xi^\rho g_{\alpha\beta}\partial_\rho g^{\alpha\beta} - 2\partial_\lambda \xi^\lambda]\ ;$$

namely:

$$\frac{X\sqrt{|g|}}{\sqrt{|g|}} = \xi^\rho \gamma_\rho + \partial_\rho \xi^\rho = \frac{\partial_\rho(\xi^\rho\sqrt{|g|})}{\sqrt{|g|}}\,. \tag{83-2}$$

One deduces from this that:

$$X(R\ \eta) = (\sqrt{|g|}\, XR + RX\sqrt{|g|})\ dx^0 \wedge \ldots \wedge dx^3$$

$$= [\xi^\rho \sqrt{|g|} \cdot \partial_\rho R + R\partial_\rho(\xi^\rho\sqrt{|g|})]\ dx^0 \wedge \ldots \wedge dx^3\ ;$$

namely:

$$X(R\ \eta) = \partial_\rho(\xi^\rho R\sqrt{|g|})\ dx^0 \wedge \ldots \wedge dx^3.$$

One thus has:

$$XI = \int_C \partial_\rho(\xi^\rho R\sqrt{|g|})\ dx^0 \wedge \ldots \wedge dx^3.$$

By applying Stokes's formula to the integral in the right-hand side, one will transform that integral into an integral that is taken over the boundary of C, and it will be zero since $\xi^\rho = 0$ on ∂C. One will thus have $XI = 0$ for any field that satisfies the hypotheses that were made.

Let $(g_{\alpha\beta}, L^\rho_{\alpha\beta})$ be a solution of equations (81-4) and (81-5). If S_α denotes an arbitrary covariant vector then we have seen that one can deduce an affine connection:

(83-3) $$\Gamma^\rho_{\alpha\beta} = L^\rho_{\alpha\beta} - \tfrac{2}{3}\delta^\rho_\alpha S_\beta$$

that satisfies the equations:

(83-4) $$G^{\alpha\beta}{}_\rho = 0.$$

Suppose that the field x vanishes on the boundary of C, *along with its first-order derivatives.* From (69-5), one has on ∂C:

$$X\Gamma^\rho_{\alpha\beta} = 0.$$

With these hypotheses, one has, from (77-8), that:

$$\int_C XR_{\alpha\beta} \cdot g_{\alpha\beta}\sqrt{|\,g\,|}\; dx^0 \wedge \ldots \wedge dx^3 = 0,$$

and XI will then reduce to:

$$XI = \int_C R_{\alpha\beta}\, X\underline{g}^{\alpha\beta}\, dx^0 \wedge \ldots \wedge dx^3.$$

Now, by virtue of (83-1) and (83-2), it will follow that:

$$X\underline{g}^{\alpha\beta} = \xi^\rho \partial_\rho \underline{g}^{\alpha\beta} + \underline{g}^{\alpha\beta}\partial_\rho \xi^\rho - \underline{g}^{\lambda\beta}\partial_\lambda \xi^\alpha - \underline{g}^{\alpha\lambda}\partial_\lambda \xi^\beta,$$

and, as a result:

(83-5) $$R_{\alpha\beta}\; X\underline{g}^{\alpha\beta} = \xi^\rho R_{\alpha\beta}\,\partial_\rho \underline{g}^{\alpha\beta} + \partial_\rho \xi^\rho R_{\alpha\beta}\underline{g}^{\alpha\beta} - (R_{\rho\sigma}\underline{g}^{\lambda\sigma} - R_{\sigma\rho}\underline{g}^{\sigma\lambda})\,\partial_\lambda \xi^\rho.$$

In the sequel, we set:

(83-6) $$2L_\rho{}^\lambda = R_{\rho\sigma} g^{\lambda\sigma} + R_{\sigma\rho} g^{\sigma\lambda}, \qquad 2\,\underline{L}_\rho{}^\lambda = R_{\rho\sigma}\underline{g}^{\lambda\sigma} + R_{\sigma\rho}\underline{g}^{\sigma\lambda}.$$

One deduces from this, by contraction, that:

(83-7) $$L_\tau{}^\tau = R_{\alpha\beta}\, g^{\alpha\beta} \qquad\qquad \underline{L}_\tau{}^\tau = R_{\alpha\beta}\underline{g}^{\alpha\beta}.$$

One will thus obtain:

$$\partial_\rho \xi^\rho \cdot R_{\alpha\beta}\underline{g}^{\alpha\beta} = \partial_\rho(\xi^\rho \underline{L}_\tau{}^\tau) - \xi^\rho \partial_\rho \underline{L}_\tau{}^\tau.$$

The relation (83-5) may then be expressed in the form:

$$R_{\alpha\beta} X\ \underline{g}^{\alpha\beta} = \partial_\lambda(\xi^\lambda \underline{L}_\tau{}^\tau - 2\xi^\rho\, \partial\underline{L}_\rho{}^\lambda) + 2\xi^\rho \partial_\lambda(\underline{L}_\rho{}^\lambda - 2\delta^\lambda_\rho\, \underline{L}_\tau{}^\tau) + \xi^\rho R_{\alpha\beta}\, \partial_\rho \underline{g}^{\alpha\beta}\,.$$

We are thus led to introduce the tensor and tensorial density:

(83-8) $$M_\rho^\lambda \ = \ L_\rho^\lambda - \tfrac{1}{2}\delta^\lambda_\rho L_\tau{}^\tau\,, \qquad \underline{M}_\rho{}^\lambda = \underline{L}_\rho{}^\lambda - \tfrac{1}{2}\delta^\lambda_\rho\, \underline{L}_\tau{}^\tau\,.$$

One obtains:

$$R_{\alpha\beta} X\ \underline{g}^{\alpha\beta} = -2\partial_\lambda(\xi^\rho \underline{M}_\rho{}^\lambda) + 2\xi^\rho \partial_\lambda \underline{M}_\rho{}^\lambda + \xi^\rho R_{\alpha\beta} \partial_\rho \underline{g}^{\alpha\beta}\,.$$

Now, by applying the Stokes formula:

$$\int_C \partial_\lambda(\xi^\rho \underline{M}_\rho{}^\lambda)\ dx^0 \wedge \ldots \wedge dx^3 = 0.$$

It results from this that:

$$XI = \int_C \xi^\rho [\partial_\lambda \underline{M}_\rho{}^\lambda + \tfrac{1}{2} R_{\alpha\beta}\, \partial_\rho \underline{g}^{\alpha\beta}]\ dx^0 \wedge \ldots \wedge dx^3.$$

Since $XI = 0$ for any ξ^ρ that satisfy the hypotheses made, it will result from a classical argument that one must necessarily have:

(83-9) $$\partial_\lambda \underline{M}_\rho{}^\lambda + \tfrac{1}{2} R_{\alpha\beta}\, \partial_\rho \underline{g}^{\alpha\beta} \ = 0.$$

We have thus established that *the identities* (83-9) *are satisfied for any set* $(g_{\alpha\beta},\ L^\rho_{\alpha\beta})$ *that is deduced by* (83-3) *for a solution* $(g_{\alpha\beta},\ L^\rho_{\alpha\beta})$ *of the equations* (81-4), (81-5); M_ρ^λ is given by starting with $g^{\alpha\beta}$ and the Ricci tensor by (83-6) and (83-8).

84. – Second form of the conservation identities. – With $(g_{\alpha\beta},\ L^\rho_{\alpha\beta})$ always denoting a solution of the system (81-4), (81-5), we now start with the integral:

$$J = \int_C P_{\alpha\beta} g^{\alpha\beta} \sqrt{|\,g\,|}\ dx^0 \wedge \ldots \wedge dx^3.$$

Since the connection L admits a zero torsion vector, one will have moreover:

$$\int_C XP_{\alpha\beta}\, g^{\alpha\beta} \sqrt{|\,g\,|}\ dx^0 \wedge \ldots \wedge dx^3 = 0.$$

One deduces from this that:

$$XJ = \int_C P_{\alpha\beta} X\underline{g}^{\alpha\beta}\ \ dx^0 \wedge \ldots \wedge dx^3 = 0,$$

and an argument that is identical to the preceding one will lead to a new form for the conservation identities (83-9). We set:

(84-1) $$2H_\rho^\lambda = P_{\rho\sigma}g^{\lambda\sigma} + P_{\sigma\rho}g^{\sigma\lambda}, \qquad 2\underline{H}_\rho^{\ \lambda} = P_{\rho\sigma}\underline{g}^{\lambda\sigma} + P_{\sigma\rho}\underline{g}^{\sigma\lambda},$$

and:

(84-2) $$K_\rho^\lambda = H_\rho^\lambda - \tfrac{1}{2}\delta_\rho^\lambda H_\tau^{\ \tau}, \qquad \underline{K}_\rho^{\ \lambda} = \underline{H}_\rho^{\ \lambda} - \tfrac{1}{2}\delta_\rho^\lambda \underline{H}_\tau^{\ \tau}.$$

For any solution $(g_{\alpha\beta},\ L_{\alpha\beta}^\rho)$ of equations (81-4), (81-5), one will thus obtain the four identities:

(84-3) $$\partial_\lambda \underline{K}_\rho^{\ \lambda} + \tfrac{1}{2}P_{\alpha\beta}\partial_\rho \underline{g}^{\alpha\beta} = 0.$$

One may express H_ρ^λ in a more convenient way by writing:

$$2H_\rho^\lambda = [P_{(\rho\sigma)} + P_{[\rho\sigma]}][g^{(\lambda\sigma)} + g^{[\lambda\sigma]}] + [P_{(\rho\sigma)} - P_{[\rho\sigma]}][g^{(\lambda\sigma)} - g^{[\lambda\sigma]}].$$

After reduction, it will follow that:

(84-4) $$H_\rho^\lambda = P_{(\rho\sigma)}\,g^{(\lambda\sigma)} + P_{[\rho\sigma]}\,g^{[\lambda\sigma]}.$$

One will note that since:

$$P_{\alpha\beta} = R_{\alpha\beta} + \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha),$$

one will have:

$$H_\rho^\lambda = L_\rho^\lambda + \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha)\,g^{[\lambda\sigma]}.$$

One deduces from this that:

$$K_\rho^\lambda = M_\rho^\lambda + S_\rho^\lambda,$$

with:

$$S_\rho^\lambda = \tfrac{2}{3}[(\partial_\alpha S_\beta - \partial_\beta S_\alpha)\,g^{[\lambda\sigma]} - \tfrac{1}{2}\delta_\rho^\lambda(\partial_\alpha S_\beta - \partial_\beta S_\alpha)\,g^{[\alpha\beta]}].$$

One easily sees in a direct manner that for any vector S_α one will have the identity:

$$\partial_\lambda(S_\rho^{\ \lambda}\sqrt{|\,g\,|}) + \tfrac{1}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha)\,\partial_\rho \underline{g}^{[\alpha\beta]} = 0.$$

If $g_{\alpha\beta}$ is a symmetric tensor and $L_{\alpha\beta}^\rho$ is the associated Riemannian connection then one will verify immediately that the identities (84-3) reduce to the classical conservation identities of general relativity.

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CHAPTER VI

THE CAUCHY PROBLEM FOR THE FIELD EQUATIONS

85. – The equations that couple the fundamental tensor with the connection. – Among the explicit field equations in (81-4), (81-5), and (81-6), we first occupy ourselves with equations (81-5), namely:

$$\partial_\rho g_{\lambda\mu} - L^\sigma_{\lambda\rho} g_{\sigma\mu} - L^\sigma_{\rho\mu} g_{\lambda\sigma} = 0, \tag{85-1}$$

which couple the fundamental tensor $g_{\lambda\mu}$ and its first-order derivatives with the affine connection $L^\rho_{\alpha\beta}$. It is clear that these equations constitute an extension of the classical relations that determine the coefficients of a Riemannian connection by starting with the metric to the case of an asymmetric tensor $g_{\lambda\mu}$ and an asymmetric connection.

Given a tensor field $g_{\mu\nu}$ on a manifold V_n, consider the system of equations (85-1) as a system of equations with the coefficients of the connection as the unknowns. By very long-winded calculations (which we shall not detail), one may establish that this system admits a unique solution, except in some exceptional cases ([1]). In the case of our manifold V_4 and for a tensor $g_{\mu\nu}$ that satisfies the hypotheses that were made in sec. **70**, Hlavaty and Saenz ([2]) have shown that the only exceptional case is the one for which one has both:

$$k = \det(k_{\lambda\mu}) = 0 \qquad \text{and} \qquad g = 2h. \tag{85-2}$$

If we discard this case in what follows then we may confirm that the system (85-1), when given our hypotheses, admits one and only one solution. We will thus be led to introduce the quantities that define the field in the form of the fundamental tensor $g_{\mu\nu}$ and the covariant tensor S_α, which satisfy the equations:

$$\partial_\rho \underline{g}^{[\rho\beta]} = 0, \tag{85-3}$$

$$P_{\alpha\beta} - \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\alpha S_\beta) = 0, \tag{85-4}$$

in which the $L^\alpha_{\beta\gamma}$ are considered to be the functions of the $g_{\mu\nu}$ and their first derivatives that are defined by the unique solution to the system (85-1).

([1]) See HLAVATY, Journ. of Rat. Mech. and Anal., **2**, (1953), 2-52; see also M. A. TONNELAT., Journ. de Phys., **12** (1951), . 81-88.

([2]) HLAVATY and SAENZ, Journ. of Rat. Mech. and Anal., **2** (1953), 523-536.

86. – The Cauchy problem for equations (85-3), (85-4). – We are thus led to study the following purely local problem, which generalizes the Cauchy problem of general relativity (see I, sec. **14**).

CAUCHY PROBLEM – *Given the components of the fundamental tensor and their first derivatives on a hypersurface S, as well as the components of the covariant vector* S_α*, determine the fundamental tensor and the vector* S_α *in a neighborhood of S, assuming that they satisfy equations* (85-3) *and* (85-4).

We assume that the hypersurface S is represented locally by the equation $x^0 = 0$, and that it satisfies the relation:

(86-1) $$g^{00} \neq 0.$$

It is easy to interpret this hypothesis geometrically. If S is represented locally by the equation $f(x^0, x^1, x^2, x^3) = 0$ then that hypotheses will say that:

(86-2) $$\Delta_1 f = g^{\alpha\beta}\partial_\alpha f \partial_\beta f = l^{\alpha\beta}\partial_\alpha f \partial_\beta f$$

is *non-zero*. If $l_{\alpha\beta}$ represents the tensor that is associated with the tensor $l^{\alpha\beta}$ then thar relation will say that the hypersurface S̲ is not tangent to the cone C_x that has the equation:

$$l_{\alpha\beta}\, dx^\alpha dx^\beta = 0.$$

Moreover, $\Delta_1 f$ is nothing but the first-order differential parameter of the function f in the metric of hyperbolic normal type:

(86-3) $$ds^2 = l_{\alpha\beta}\, dx^\alpha dx^\beta.$$

In the sequel, we will make the following convention:

$$i, j, \text{ any Latin index} = 1, 2, 3.$$

Having said that, we establish the following theorem:

THEOREM – *In the neighborhood of a hypersurface S that is represented locally by* $x^0 = 0$ *and has the property that* $g^{00} \neq 0$*, knowing the system of quantities* g_{ij}, $g^{[0i]}$, $\underline{g}^{(0\lambda)}$ *is equivalent to knowing the fundamental tensor* $g_{\lambda\mu}$.

Consider the system of local coordinates for which S is represented locally by $x^0 = 0$. One has $g^{00} \neq 0$ in a certain neighborhood in which we shall place ourselves. First, observe that knowing the quantities g_{ij}, $\underline{g}^{[0i]}, \underline{g}^{(0\lambda)}$ will give us the quantities g_{ij}, $\underline{g}^{0j}, \underline{g}^{j0}, \underline{g}^{00} \neq 0$. We must therefore show that the knowledge of these quantities leads to the knowledge of the components g_{0j}, g_{j0}, g_{00}. One deduces from the relations:

$$g^{00}g_{0j} + g^{i0}g_{ij} = 0, \qquad g^{00}g_{j0} + g^{0i}g_{ji} = 0, \qquad g^{00}g_{00} + g^{0i}g_{0i} = 1,$$

upon multiplying by $\sqrt{|g|}$, that:

(86-4) $$\underline{g}^{00} g_{0j} + \underline{g}^{i0} g_{ij} = 0, \qquad \underline{g}^{00} g_{j0} + \underline{g}^{0i} g_{ji} = 0,$$

and:

(86-5) $$\underline{g}^{00} g_{00} + \underline{g}^{0i} g_{0i} = \sqrt{|g|}.$$

Relations (86-4), in which $\underline{g}^{00} \neq 0$, give us the values of g_{0j} and g_{j0}. On the other hand, one may observe that $\sqrt{|g|}$ is given by the relation:

$$\sqrt{|g|} = \left| \frac{\det(g_{ij})}{\underline{g}^{00}} \right| .$$

One may then deduce the value of g_{00} from (86-5). Our theorem is thus established.

We are thus led to take our Cauchy data on the hypersurface S to be the values of the following quantities on S:

(86-6) $$g_{ij},\ \underline{g}^{[0i]}, \underline{g}^{(0\lambda)}, \partial_0 g_{ij}, \partial_0 \underline{g}^{(0\lambda)},\ S_i,\ S_0 .$$

As we confirm, the values of the quantities $\partial_{00} S_0$, $\partial_{000} S_0$, etc., on S do not result from the field equations. This seems to suggest that we should restrict the vector S_α by an invariant auxiliary condition, for example:

(86-7) $$\partial_\alpha (g^{(\alpha\beta)} S_\beta \sqrt{|g|}) = 0.$$

That is what we shall do from now on.

87. – A theorem that is deduced from the conservation identities. – We shall deduce the following theorem from the conservation identities (84-3):

THEOREM – *For any solution of the system* (85-3), *the four quantities* $K_\rho^{\ 0}$ *will be expressed uniquely as functions of the quantities* g_{ij}, $\underline{g}^{[0i]}$, $\underline{g}^{(0\lambda)}$, $\partial_0 g_{ij}$, $\partial_0 \underline{g}^{(0\lambda)}$, *and their derivatives with respect to the variables* (x^k).

Indeed, one first has that the quantities $\partial_0 \underline{g}^{[0i]}$ may be expressed in terms of the $\partial_k \underline{g}^{[ki]}$ with the aid of equations (85-3). On the other hand, since the coefficients of the connection are functions of the components of the fundamental tensor and their first derivatives – or the quantities g_{ij}, $\underline{g}^{[0i]}$, $\underline{g}^{(0\lambda)}$ and their first derivatives – the components $\underline{K}_\rho^{\ \lambda}$ (like the components $P_{\alpha\beta}$) will be, *a priori*, functions of the following quantities:

$$(87\text{-}1)\qquad \begin{cases} \underline{K}_{\rho}{}^{\lambda} = \Phi_{\rho}{}^{\lambda}[g_{ij}, \underline{g}^{[0i]}, \underline{g}^{(0\lambda)}, & \partial_k g_{ij}, \partial_k \underline{g}^{[0i]}, \partial_k \underline{g}^{(0\lambda)}, \partial_{kl} g_{ij}, \partial_{kl} \underline{g}^{[0i]}, \partial_{kl} \underline{g}^{(0\lambda)}, \\ & \partial_0 g_{ij}, \partial_0 \underline{g}^{(0\lambda)}, \partial_{0k} g_{ij}, \partial_{0k} \underline{g}^{(0\lambda)}, \partial_{00} g_{ij}, \partial_{00} \underline{g}^{(0\lambda)}]. \end{cases}$$

Consider the four quantities $\underline{K}_{\rho}{}^{0}$. The conservation identities (84-3) may be put into the form:

$$\partial_0 \underline{K}_{\rho}{}^{0} = -\partial_k \underline{K}_{\rho}{}^{0} + \tfrac{1}{2} P_{\alpha\beta} \partial_{\rho} \underline{g}^{\alpha\beta}.$$

It results from this that the four quantities $\partial_0 \underline{K}_{\rho}{}^{0}$ depend upon only the arguments of the functions $\Phi_{\rho}{}^{\lambda}$ and the first derivatives of these arguments with respect to the x^k. Those quantities are thus independent of the arguments $\partial_{000} g_{ij}$ and $\partial_{000} \underline{g}^{(0\lambda)}$, whose numerical arguments will be arbitrary on S. One deduces the following identities from this:

$$\frac{\partial \Phi_{\rho}{}^{0}}{\partial(\partial_{00} g_{ij})} \equiv 0, \qquad \frac{\partial \Phi_{\rho}{}^{0}}{\partial(\partial_{000} g^{(0\lambda)})} \equiv 0.$$

The $\underline{K}_{\rho}{}^{0}$ and, as a result, the $K_{\rho}{}^{0}$ depend only upon the arguments that were indicated in our statement. Therefore, for any solution of (85-3), one will have:

$$(87\text{-}2)\begin{cases} \underline{K}_{\rho}{}^{0} = \Phi_{\rho}{}^{0}[g_{ij}, \underline{g}^{[0i]}, \underline{g}^{(0\lambda)}, & \partial_k g_{ij}, \partial_k \underline{g}^{[0i]}, \partial_k \underline{g}^{(0\lambda)}, \\ & \partial_{kl} g_{ij}, \partial_{kl} \underline{g}^{[0i]}, \partial_{kl} \underline{g}^{(0\lambda)}, \partial_0 g_{ij}, \partial_0 \underline{g}^{(0\lambda)}, \partial_{0k} g_{ij}, \partial_{0k} \underline{g}^{(0\lambda)}]. \end{cases}$$

88. – A theorem about coordinate changes upon crossing *S*. – Our purely local study is carried out in the domain of a certain coordinate system. However, being given the Cauchy data on S in the domain envisioned leaves open the possibility of coordinate changes that preserve the numerical values of the coordinates at any point of S, as well as the Cauchy data. As in general relativity, we will thus be led to consider the coordinate changes that are defined by the formula:

$$(88\text{-}1)\qquad x^{\lambda'} = x^{\lambda} + \frac{(x^0)^3}{6}[\varphi^{(\lambda)}(x^i) + \varepsilon^{\lambda}] \qquad (\lambda' = \lambda \text{ numerically}),$$

in which ε^{λ} goes to zero when x^0 goes to zero.

Recall that the partial derivatives of our new coordinates on S with respect to the old ones are such that:

$$(88\text{-}2)\qquad (A_{\mu}^{\lambda'})_S = \delta_{\mu}^{\lambda}, \qquad (\partial_0 A_{\mu}^{\lambda'})_S = (\partial_{\mu} A_0^{\lambda'})_S = 0,$$

$$(88\text{-}3)\qquad (\partial_{00} A_i^{\lambda'})_S = (\partial_{i0} A_0^{\lambda'})_S = 0,$$

in such a way that of the second derivatives of A, only the derivatives $\partial_{00} A_0^{\lambda'}$ are non-zero on S:

(88-4) $$(\partial_{00} A_0^{\lambda'})_S = \varphi^{(\lambda)}.$$

Under a coordinate change, one will obviously have:

(88-5) $$S_\mu = A_\mu^{\lambda'} S_{\lambda'}, \quad g_{\lambda\mu} = A_\lambda^{\alpha'} A_\mu^{\beta'} g_{\alpha'\beta'}, \qquad g^{\alpha'\beta'} = A_\lambda^{\alpha'} A_\mu^{\beta'} g^{\lambda\mu},$$

and, by derivation:

(88-6) $$\partial_0 g_{\lambda\mu} = A_\lambda^{\alpha'} A_\mu^{\beta'} A_0^{\rho'} \partial_{\rho'} g_{\alpha'\beta'} + \partial_0 A_\lambda^{\alpha'} \cdot A_\mu^{\beta'} g_{\alpha'\beta'} + \partial_0 A_\mu^{\beta'} A_\lambda^{\alpha'} g_{\alpha'\beta'},$$

(88-7) $$\partial_{0'} g^{\alpha'\beta'} = A_\lambda^{\alpha'} A_\mu^{\beta'} A_{0'}^{\rho} \partial_\rho g^{\lambda\mu} + A_{0'}^{\rho} \partial_\rho A_\lambda^{\alpha'} A_\mu^{\beta'} g^{\lambda\mu} + A_{0'}^{\rho} \partial_\rho A_\mu^{\beta'} A_\lambda^{\alpha'} g^{\lambda\mu}.$$

Having said that, we shall establish the following theorem:

THEOREM – *The coordinate change* (88-1):

a) *preserves the numerical values of the coordinates of any point of S, along with the Cauchy data.*

b) *preserves the numerical values of* $\partial_{00} g_{ij}$ *and* $\partial_{00} \underline{g}^{(0i)}$ *on S.*

c) *allows one to give arbitrary values to the* $\partial_{00} \underline{g}^{(0\lambda)}$ *on S.*

Property *a*) results from equations (88-2), (88-5), and (88-6) in an obvious way. In order to establish properties *b*) and *c*), we begin by differentiating (88-6) and (88-7). We thus obtain:

(88-8) $$\partial_{00} g_{\lambda\mu} = A_\lambda^{\alpha'} A_\mu^{\beta'} A_0^{\rho'} A_0^{\sigma'} \partial_{\rho'} \partial_{\sigma'} g_{\alpha'\beta'} + \partial_{00} A_\lambda^{\alpha'} \cdot A_\mu^{\beta'} g_{\alpha'\beta'} + \partial_0 A_\mu^{\beta'} A_\lambda^{\alpha'} g_{\alpha'\beta'} + O,$$

(88-9) $$\begin{cases} \partial_{0'0'} g^{\alpha'\beta'} = A_\lambda^{\alpha'} A_\mu^{\beta'} A_{0'}^{\rho} A_{0'}^{\sigma} \partial_\rho \partial_\sigma g^{\lambda\mu} & + A_{0'}^{\rho} A_{0'}^{\sigma} \partial_{\rho\sigma} A_\lambda^{\alpha'} A_\mu^{\beta'} g^{\lambda\mu} \\ & + A_{0'}^{\rho} A_{0'}^{\sigma} \partial_{\rho\sigma} A_\mu^{\beta'} A_\lambda^{\alpha'} g^{\lambda\mu} + O, \end{cases}$$

in which O denotes terms that contain first derivatives of A, and which will be, as a result, annulled on S. One first deduces from immediately (88-8) that:

(88-10) $$\partial_{00} g_{ij} = \partial_{0'0'} g_{i'j'}.$$

On the other hand, one will have:

(88-11) $$\partial_{00} g_{0i} = \partial_{0'0'} g_{0'i'} + \varphi^{(\rho)} g_{\rho i}, \qquad \partial_{00} g_{i0} = \partial_{0'0'} g_{i'0'} + \varphi^{(\rho)} g_{i\rho},$$

and:

(88-12) $$\partial_{00} g_{00} = \partial_{0'0'} g_{0'0'} + \varphi^{(\rho)} g_{\rho 0} + \varphi^{(\rho)} g_{0\rho}.$$

We now study how $\partial_{00}\sqrt{|g|}$ gets modified. One has:

$$\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} = \tfrac{1}{2}g^{\alpha\beta}\partial_0 g_{\alpha\beta}\,.$$

One deduces from this that:

$$\frac{\partial_{00}\sqrt{|g|}}{\sqrt{|g|}} - \left[\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\right]^2 = \tfrac{1}{2}g^{\alpha\beta}\partial_{00}g_{\alpha\beta} + \tfrac{1}{2}\partial_0 g^{\alpha\beta}\partial_0 g_{\alpha\beta}\,. \tag{88-13}$$

By writing relation (88-13) for the derivatives of $x^{0'}$ with respect to $\sqrt{|g|}$ and subtracting term-by-term, one will obtain:

$$D \equiv \frac{\partial_{00}\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_{0'0'}\sqrt{|g'|}}{\sqrt{|g'|}} = \tfrac{1}{2}(g^{\alpha\beta}\partial_{00}g_{\alpha\beta} - \tfrac{1}{2}g^{\alpha'\beta'}\partial_{0'0'}g_{\alpha'\beta'})\,;$$

namely, from (88-10):

$$D \equiv \tfrac{1}{2}(g^{0i}\partial_{00}g_{0i} - g^{0'i'}\partial_{0'0'}g_{0'i'} + g^{i0}\partial_{00}g_{i0} - g^{i'0'}\partial_{0'0'}g_{i'0'} + g^{00}\partial_{00}g_{00} - g^{0'0'}\partial_{0'0'}g_{0'0'})\,.$$

By virtue of (88-11) and (88-12), one will have:

$$D = \tfrac{1}{2}\,(g^{0i}\,g_{\rho i}\,\varphi^{(\rho)} + g^{i\,0}\,g_{i\rho}\,\varphi^{(\rho)} + g^{00}\,g_{0\rho}\,\varphi^{(\rho)})\;;$$

namely:

$$D = \tfrac{1}{2}\,(g^{0\lambda}\,g_{\rho\lambda}\,\varphi^{(\rho)} + g^{\lambda\,0}\,g_{\lambda\rho}\,\varphi^{(\rho)}) = \varphi^{(0)},$$

since $\sqrt{|g'|} = \sqrt{|g|}$, from which, one will deduce that:

$$\partial_{00}\sqrt{|g|} - \partial_{0'0'}\sqrt{|g'|} = \varphi^{(0)}\sqrt{|g|}\,. \tag{88-14}$$

Now consider formulas (88-4) for the coordinate change (88-1). It follows that:

$$\left\{\begin{aligned} \partial_{0'0'}g^{0'i'} &= \partial_{00}g^{0i} + \varphi^{(0)}g^{0i} + \varphi^{(i)}g^{00}, \\ \partial_{0'0'}g^{i'0'} &= \partial_{00}g^{i0} + \varphi^{(0)}g^{i0} + \varphi^{(i)}g^{00}, \\ \partial_{0'0'}g^{0'0'} &= \partial_{00}g^{00} + 2\varphi^{(0)}g^{00}. \end{aligned}\right.$$

One deduces from this that:

$$\begin{aligned} \partial_{00}g^{[0i]} &= \partial_{0'0'}g^{[0'i']} - \varphi^{(0)}g^{[0i]}\,, \\ \partial_{00}g^{(0\lambda)} &= \partial_{0'0'}g^{(0'\lambda')} - \varphi^{(\lambda)}g^{00}\,, \end{aligned}$$

and, from (88-14):

(88-15) $$\partial_{00}\underline{g}^{[0i]} - \partial_{0'0'}\underline{g}^{[0'i']} = -\varphi^{(0)}\underline{g}^{[0i]} + \underline{g}^{[0i]}\varphi^{(0)} = 0,$$

(88-16) $$\partial_{00}\underline{g}^{(0\lambda)} - \partial_{0'0'}\underline{g}^{(0'\lambda')} = -\varphi^{(0)}\underline{g}^{(0\lambda)} - \varphi^{(\lambda)}\underline{g}^{00} + \varphi^{(0)}\underline{g}^{(0\lambda)} = -\varphi^{(\lambda)}g^{00}.$$

(88-15) succeeds in establishing property *b*). From (88-16), it results that since $g^{00} \neq 0$, one may attribute arbitrary values to the $\partial_{0'0'}g^{(0'\lambda')}$ on S by choosing the functions $\varphi^{(\lambda)}$. Our theorem is thus established.

Recall once more the definition of the index of a derivative with respect to the local coordinates (x^α) to which S is referred, which will be the number of times that the index 0 appears in it. For any affine connection that is a solution to equations (85-1), it is clear that the Ricci tensor $P_{\alpha\beta}$ will be expressed as a function of the g_{ij}, $\underline{g}^{[0i]}$, $\underline{g}^{(0\lambda)}$, and their derivatives up to order two. On the other hand, under a change of coordinates (88-1), one will have:

$$P_{\alpha'\beta'} = P_{\alpha\beta}$$

on S, and the coordinate change allows us to arbitrarily modify the values of the $\partial_{00}g^{(0\lambda)}$ on S. One thus deduces from the preceding theorem that:

COROLLARY – *In order for any affine connection* $L^\alpha_{\beta\lambda}$ *to be a solution of* (85-1) *, the only derivatives of index two that may appear in the components of the Ricci tensor* $P_{\alpha\beta}$ *are the* $\partial_{00}g_{ij}$ *and* $\partial_{00}\underline{g}^{[0i]}$*, but not the* $\partial_{00}\underline{g}^{(0\lambda)}$.

89. – Decomposing the problem of integrating the field equations. – We first propose to show that the system (85-4) is equivalent to a system such that one part of it involves the quantities $K_\rho^{\ \lambda}$ that figure in the conservation identities.

THEOREM – *In the neighborhood of S the system* (85-4) *is equivalent to the system composed of the following equations:*

(89-1)$_a$ $$P_{[ij]} = 0,$$

(89-2)$_b$ $$P_{[ij]} - \tfrac{2}{3}(\partial_i S_j - \partial_j S_i) = 0,$$

(89-2)$_c$ $$P_{[i0]} - \tfrac{2}{3}(\partial_i S_0 - \partial_0 S_i) = 0,$$

and:

(89-2) $$M_\rho^{\ 0} \equiv K_\rho^{\ 0} - \tfrac{2}{3}[(\partial_\rho S_\sigma - \partial_\sigma S_\rho)\, g^{[0\sigma]} - \tfrac{1}{2}\delta^0_\rho(\partial_\alpha S_\beta - \partial_\beta S_\alpha)\; g^{[\alpha\beta]} = 0.$$

Indeed, set:

$$R_{\alpha\beta} = P_{\alpha\beta} - \tfrac{2}{3}(\partial_\alpha S_\beta - \partial_\beta S_\alpha)\,.$$

The system (85-4) is equivalent to the system:

$$R_{(\alpha\beta)} = 0, \qquad R_{[\alpha\beta]} = 0.$$

Now, equations (89-1) may be written:

(89-3) $$R_{(ij)} = 0, \qquad R_{[\alpha\beta]} = 0,$$

and it will suffice for us to show that for any solution of (89-3), the equations:

$$M_\rho{}^0 \equiv R_{(\rho\sigma)}\, g^{(0\sigma)} + R_{[\rho\sigma]}\, g^{[0\sigma]} - \tfrac{1}{2}\delta^0_\rho [R_{(\rho\sigma)}\, g^{(\rho\sigma)} + R_{[\rho\sigma]}\, g^{[\rho\sigma]}] = 0$$

imply the equations:

$$R_{(\rho 0)} = 0.$$

For $\rho = i$, one first has:

$$M_i^0 \equiv R_{(0\sigma)}\, g^{(0\sigma)} + R_{[0\sigma]}\, g^{[0\sigma]}.$$

For a solution of (89-3):

$$M_i^0 = R_{(i0)}\, g^{00},$$

and since $g^{00} \neq 0$, the equations $M_i^0 = 0$ will imply that $R_{(i0)} = 0$.

For $\rho = 0$, one has:

$$M_0{}^0 \equiv R_{(0\sigma)}\, g^{(0\sigma)} + R_{[0\sigma]}\, g^{[0\sigma]} - \tfrac{1}{2}[R_{(\rho\sigma)}\, g^{(\rho\sigma)} + R_{[\rho\sigma]}\, g^{[\rho\sigma]}],$$

and for a solution of (89-3), it will follow from the preceding analysis that:

$$M_0{}^0 = R_{00}\, g^{00} - \tfrac{1}{2} R_{00}\, g^{00} = \tfrac{1}{2} R_{00}\, g^{00},$$

and $M_0{}^0 = 0$ entails that $R_{00} = 0$, which proves our theorem.

One will observe that equations (89-2) may be explicitly written as:

(89-4) $$M_i^0 \equiv K_i^0 - \tfrac{2}{3}(\partial_i S_k - \partial_k S_i)\, g^{[0k]} = 0$$

and:

$$M_0{}^0 \equiv K_0{}^0 - \tfrac{2}{3}[(\partial_0 S_k - \partial_k S_0)\, g^{[0k]} - (\partial_0 S_k - \partial_k S_0)\, g^{[0k]} - \tfrac{1}{2}(\partial_i S_j - \partial_j S_i)\, g^{[ij]}] = 0\,;$$

namely:

(89-5) $$M_0{}^0 \equiv K_0{}^0 - \tfrac{2}{3}(\partial_i S_j - \partial_j S_i)\, g^{[ij]} = 0.$$

By virtue of the theorem in sec. **87**, we see that the left-hand sides of equations (89-2) take values on S that depend upon only the Cauchy data and their derivatives with respect to the x^k, and, as a result, these equations will provide conditions that must be satisfied by the Cauchy data on S.

Likewise, the system (85-3) may be divided into the equations:

(89-6) $$\partial_\rho \underline{g}^{[\rho i]} \equiv \partial_0 \underline{g}^{[0i]} + \partial_k \underline{g}^{[ki]} = 0$$

and:

(89-7) $$\partial_\rho \underline{g}^{[\rho 0]} \equiv \partial_k \underline{g}^{[k0]} = 0.$$

Equations (89-6) provide the values of the $\partial_0 \underline{g}^{[0i]}$ as functions of the fundamental tensor with respect to the x^k. The last equation provides a condition that must be satisfied by the $\underline{g}^{[k0]}$ on S.

We are thus led to decompose the system of field equations (85-3) and (85-4) into two systems of equations that are defined in the following manner: The first system is composed of equations (89-1) and (89-6), and the second one is composed of equations (89-2) and (89-7).

On the subject of this decomposition, we propose to establish the following theorem:

THEOREM – *Given a solution* ($g_{\lambda\mu}$, S_α) *of the system* (89-1), (89-6) *on S that satisfies equations* (89-2), (89-7), *the set* ($g_{\lambda\mu}$, S_α) *satisfies* (89-2) *and* (89-7) *outside of S.*

Indeed, first of all, the left-hand sides $\partial_\rho g^{[\rho\sigma]}$ of equations (85-3) satisfy the relation:

$$\partial_\sigma [\partial_\rho g^{[\rho\sigma]}] = 0$$

identically. It results from this identity that for a solution of (89-6) one will have:

$$\partial_0 [\partial_\rho g^{[\rho 0]}] = 0,$$

and since $\partial_\rho g^{[\rho 0]}$ is zero on S, it is also zero outside of S. The system (85-3) is thus satisfied. It results from this that one can deduce a set $(g_{\lambda\mu}, \Gamma^\alpha_{\beta\gamma})$ from the set envisioned ($g_{\lambda\mu}$, S_α) that satisfies the conservation identities (83-9), namely:

(89-8) $$\partial_\lambda \underline{M}_\rho{}^\lambda + \tfrac{1}{2} R_{\alpha\beta}\, \partial_\rho \underline{g}^{\alpha\beta} = 0,$$

or:

$$\partial_0 \underline{M}_\rho{}^0 + \partial_i \underline{M}_\rho{}^i + \tfrac{1}{2} R_{\alpha\beta}\, \partial_\rho \underline{g}^{\alpha\beta} = 0.$$

Now:

$$M_\rho^i = R_{[\rho\sigma]}\, g^{(i\sigma)} + R_{[\rho\sigma]}\, g^{[i\sigma]} - \tfrac{1}{2} \delta^i_\rho [R_{(\alpha\beta)}\, g^{(\alpha\beta)} + R_{[\alpha\beta]}\, g^{[\alpha\beta]}].$$

For a solution of (89-1):

$$M_j^i = R_{(j0)}\, g^{(i0)} - \tfrac{1}{2}\,\delta_\rho^i\, R_{(\lambda 0)}\, g^{(\lambda 0)}, \qquad M_j^i = R_{(0\lambda)}\, g^{(i\lambda)}.$$

It results from the calculations of the preceding theorem that the conservation identities (89-8) imply that one will have relations of the form:

$$\partial_0 M_\rho{}^0 = A^i{}_\rho{}^\lambda\, \partial_i M_\lambda{}^0 + B_\rho^\lambda\, M_\lambda{}^0 \tag{89-9}$$

for a solution to (89-1), in which the A and B are regular functions. For given $(M_\rho^0)_S$ that are zero on S, the system (89-9) will admit no other solution than the zero solution. It will result from this that $M_\rho^0 = 0$ outside of S.

The problem local integration of the field equations is thus found to come down to:

a) The search for Cauchy data that satisfy equations (89-2) and (89-7) on S;

b) The study of the system (89-1) and (89-6) for such Cauchy data.

This situation is completely analogous to the one in general relativity that we have encountered in various forms already.

90. – Remarks on the search for Cauchy data. – It is possible to make several remarks concerning the search for Cauchy data that suggest possibility conditions and the order of difficulty for the problem. Suppose that we have chosen Cauchy data relative to the fundamental tensor such that:

$$\partial_k \underline{g}^{[k0]} = 0.$$

We discard the case in which the $g^{[0i]}$ are all zero. The quantities K_ρ^0 are known on S, and in order to determine the components of S_α, one has the relations:

$$g^{[0i]}\, S_{ik} = \tfrac{3}{2}\, K_i^0, \tag{90-1}$$

$$g^{[ij]}\, S_{ij} = -\,3K_0^0, \tag{90-2}$$

in which one has set:

$$S_{ij} = \partial_i S_j - \partial_j S_i .$$

One obviously has the following possibility condition for equations (90-1):

$$K_i^0\, g^{[0i]} = 0. \tag{90-3}$$

If this condition is satisfied then equations (90-1) will provide a solution for the S_{ij} that depends on one scalar parameter λ. If u_i denotes a covariant vector such that $u_i\, g^{[0i]} = 1$ then one will have the explicit solution:

$$S_{ik} = \tfrac{3}{2}\,(K_i^0\, u_k - K_k^0\, u_i) + \lambda\, \varepsilon_{ikl}\, \underline{g}^{[0i]},$$

in which ε_{ikl} is the indicator of the permutation. Equation (90-2) fixes the value of λ uniquely in the case for which:

$$g^{[23]}g^{[01]} + g^{[31]}g^{[02]} + g^{[12]}g^{[03]} \neq 0,$$

i.e., the one for which:

$$m = \det (m^{\alpha\beta}) \neq 0.$$

If this is not the case then the problem of determining the S_{ik} by means of (90-1) and (90-2) might be impossible or undetermined. In any case, it will be convenient to impose the requirement upon the S_{ik} that are obtained that they must define a quadratic exterior form with zero exterior differential. If this is the case then the S_i will be found to be defined locally up to a gradient, while S_0 naturally remains arbitrary.

91. – Relations between the derivatives of index 2 **of the fundamental tensor and the derivatives of the coefficients of the connection. –** In order for us to proceed with the study of the system of equations (89-1), (89-6), we shall first analyze certain relations that exist between the derivatives of order 2 of the fundamental tensor and the derivative $\partial_0 L^\alpha_{\beta\gamma}$ of the coefficients of the connection.

In the present section, we suppose that the fundamental tensor and the connection L satisfy:

1. The connecting relations (85-1):

2. Equations (89-6) and (on S) equation (89-7).

It will result from the argument that was given in sec. **89** that the system (85-3) is then satisfied; as a result, the connection $L^\alpha_{\beta\gamma}$ will admit a zero torsion vector. Equations (89-6) will provide the values of the $\partial_0 \underline{g}^{[0i]}$ and, by derivation, those of the $\partial_{00} \underline{g}^{[0i]}$, as a function of the Cauchy data and their derivatives with respect to the (x^k). We will always suppose in what follows that one has replace the quantities $\partial_0 \underline{g}^{[0i]}$ and $\partial_{00} \underline{g}^{[0i]}$ with their expressions as provided by (89-6).

By solving the connecting relations (85-1), one will see that the $L^\alpha_{\beta\gamma}$ are expressed by linear functions of the first derivatives of the components of the fundamental tensor whose coefficients are functions of these components.

In order to simplify the notation, we shall use a congruence symbol (~); this congruence is intended to mean *modulo* functions of the Cauchy data relative to the fundamental tensor and their first derivatives with respect to the (x^k). One deduces from the expression for the components of the Ricci tensor that:

(91-1) $$P_{ij} \sim \partial_0 L^0_{ij}.$$

From the theorem in sec. **88**, and on account of equations (89-6), the only derivatives of index 2 that will appear in the P_{ij} are the $\partial_{00} g_{kl}$. One will then have:

(91-2) $$\partial_0 L^0_{ij} \sim A^{kl}_{ij} \partial_{00} g_{kl}\,.$$

We propose to establish that *equations* (91-2) *are, in general, soluble with respect to the* $\partial_{00} g_{kl}$; in other words, the matrix A that figures in (91-2) is generally invertible.

To that end, we use the following relations that are easily deduced by deriving relations (85-1) – or, equivalently, relations (78-4) – with respect to x^0:

(91-3) $$\partial_{0k} g_{ij} \sim \quad \partial_0 L^h_{ik}\, g_{hj} + \partial_0 L^h_{kj}\, g_{ih} + \partial_0 L^0_{ik}\, g_{0j} + \partial_0 L^0_{kj}\, g_{i0}\,,$$

(91-4) $$\partial_{0k} g^{00} \sim -\partial_0 L^0_{hk}\, g^{h0} - \partial_0 L^0_{kh}\, g^{0h} - \partial_0 L^0_{0k}\, g^{00} - \partial_0 L^0_{k0}\, g^{00}\,,$$

(91-5) $$\partial_{0k} g^{0i} \sim -\partial_0 L^0_{hk}\, g^{hi} - \partial_0 L^i_{kh}\, g^{0h} - \partial_0 L^0_{0k}\, g^{0i} - \partial_0 L^i_{k0}\, g^{00}\,,$$

(91-6) $$\partial_{0k} g^{i0} \sim -\partial_0 L^0_{hk}\, g^{ih} - \partial_0 L^i_{kh}\, g^{h0} - \partial_0 L^0_{0k}\, g^{i0} - \partial_0 L^i_{k0}\, g^{00}.$$

On the other hand, by the same process, one will get:

$$\partial_{00} g^{0i} \sim -\partial_0 L^0_{h0}\, g^{hi} - \partial_0 L^i_{0h}\, g^{0h} - \partial_0 L^0_{00}\, g^{0i} - \partial_0 L^i_{00}\, g^{00}\,,$$

$$\partial_{00} g^{i0} \sim -\partial_0 L^0_{0h}\, g^{ih} - \partial_0 L^i_{h0}\, g^{h0} - \partial_0 L^0_{00}\, g^{i0} - \partial_0 L^i_{00}\, g^{00}.$$

Upon subtracting term-by-term, it will follow that:

(91-7) $$2\partial_{00} g^{[0i]} \sim -\partial_0 L^0_{h0}\, g^{hi} + \partial_0 L^0_{0h}\, g^{ih} - \partial_0 L^i_{0h}\, g^{0h} - \partial_0 L^i_{h0}\, g^{h0} - 2\,\partial_0 L^0_{00}\, g^{[0i]}.$$

On the other hand, from (78-6):

$$\gamma_0 = \frac{\partial_0 \sqrt{|\,g\,|}}{\sqrt{|\,g\,|}} = L^\rho_{0\rho}\,.$$

One deduces from this that:

$$\partial_{00} \underline{g}^{[0i]} \sim \partial_{00} g^{[0i]} \sqrt{|\,g\,|} + \partial_0 L^\rho_{0\rho}\, \underline{g}^{[0i]}\,.$$

It will then follow from (91-7) that:

(91-8) $$2\partial_{00} g^{[0i]} \sim -\partial_0 L^0_{h0}\, \underline{g}^{hi} + \partial_0 L^0_{0h}\, \underline{g}^{0h} - \partial_0 L^i_{0h}\, \underline{g}^{0h} - \partial_0 L^i_{h0}\, \underline{g}^{h0} - 2\,\partial_0 L^j_{0j}\, \underline{g}^{[0i]}\,.$$

Finally, one has, from (85-1):

(91-9) $$\partial_{00} g_{ij} \sim \partial_0 L^h_{i0}\; g_{hj} + \partial_0 L^h_{0j}\; g_{ih} + \partial_0 L^0_{i0}\; g_{0j} + \partial_0 L^0_{0j}\; g_{i0}\,.$$

Having said that, one will note that equations (91-3), when they are considered to be equations in the unknowns $\partial_0 L^h_{ik}$, have a form that is identical to the connecting relations (85-1), since the coefficients of the unknowns are the g_{ij} for which:

$$\det(g_{ij}) = g\, g^{00} \neq 0.$$

From the cited results of Hlavaty, one deduces that this system is invertible, except for the exceptional case that we discarded ([1]). One will thus obtain:

(91-10) $$\partial_0 L^h_{ik} \sim T^h_{ik}\,,$$

in which the letter T will henceforth denote terms that depend linearly upon the $\partial_0 L^0_{rs}$.

Since $g^{00} \neq 0$, one will then have, from (91-4), that:

(91-11) $$\partial_0 L^0_{0k} + \partial_0 L^0_{k0} \sim T_k\,.$$

Now take equations (91-5) and (91-6). They may be written:

(91-12) $$\partial_0 L^i_{k0} \sim \partial_0 L^0_{0k} \frac{g^{0i}}{g^{00}} + T^i_k\,,$$

(91-13) $$\partial_0 L^i_{0k} \sim \partial_0 L^0_{k0} \frac{g^{i0}}{g^{00}} + T'^i_k\,.$$

One deduces from this, by contraction, that:

$$\partial_0 L^j_{j0} = \partial_0 L^j_{0j} \sim -\partial_0 L^0_{0j} \frac{g^{0j}}{g^{00}} + T \sim -\partial_0 L^0_{j0} \frac{g^{j0}}{g^{00}} + T'\,.$$

By replacing $\partial_0 L^0_{j0}$ with its value in (91-11), it will follow that:

(91-14) $$\partial_0 L^j_{j0} = \partial_0 L^j_{0j} \sim -\,\partial_0 L^0_{0j} \frac{g^{[0j]}}{g^{00}} + T''\,.$$

Upon substituting the values of $\partial_0 L^i_{h0}$, $\partial_0 L^i_{0h}$, $\partial_0 L^j_{0j}$ that one finds in (91-12), (91-13), (91-14) into (91-8), one will obtain a system in the unknowns $\partial_0 L^0_{0h}$ of the form:

([1]) That case has been studied by M[lle] TISON.

$$\partial_0 L^0_{0h}\left(g^{ih}+g^{hi}-\frac{g^{0i}g^{h0}}{g^{00}}-\frac{g^{i0}g^{0h}}{g^{00}}\right)-2\partial_0 L^0_{0h}\,\frac{g^{[0h]}g^{[0i]}}{g^{00}}\sim 2T^i,$$

which, by a simple calculation, can be put into the form:

$$\partial_0 L^0_{0h}\left[g^{(ih)}-\frac{g^{(0h)}g^{(0i)}}{g^{00}}\right]\sim T^i\,; \tag{91-15}$$

namely:

$$\partial_0 L^0_{0h}\,\dot{l}^{ih}\sim T^i,$$

in which we have set:

$$\dot{l}^{ih}=l^{ih}-\frac{l^{0h}l^{0i}}{l^{00}}\,.$$

One obviously has:

$$\det(\dot{l}^{ih})\neq 0,$$

since the quadratic form $\dot{l}^{ih}\,Y_i\,Y_h$ is deduced from the form $l^{\alpha\beta}Y_\alpha Y_\beta$ by suppressing the square that is associated with the direction variable Y_0. One deduces from this that one may solve equations (91-15) and obtain relations of the form:

$$\partial_0 L^0_{0h}\sim T_h\,,\qquad \partial_0 L^0_{h0}\sim T'_h\,. \tag{91-16}$$

By substituting the values of $\partial_0 L^0_{0h}$, $\partial_0 L^0_{h0}$ into (91-9), as well as those of $\partial_0 L^h_{i0}$ and $\partial_0 L^h_{0i}$ that are deduced from (91-12), (91-13), the following relations ensue:

$$\partial_{00}g_{ij}\sim B^{kl}_{ij}\,\partial_0 L^0_{kl}\,, \tag{91-17}$$

which realize the inversion of relations (91-2).

Therefore, provided that equations (91-3) in the unknowns $\partial_0 L^k_{ih}$ are invertible, *one may express the* $\partial_{00}g_{ij}$ *as linear combinations of the* $\partial_0 L^0_{kl}$*, up to the addition of a function of the Cauchy data and their derivatives with respect to the* (x^k).

92.– The integration of the field equations. – Now consider a solution of equations (89-1), (89-6) that corresponds to the Cauchy data on S and satisfies (89-2) and (89-7). We propose to evaluate the values on S of the successive derivatives of the fundamental tensor and the vector S_α.

First of all, as we have already observed, equations (89-6) provide the values on S of the derivatives $\partial_0\underline{g}^{[0i]}$ and $\partial_{00}\underline{g}^{[0i]}$. Equations $(89\text{-}1)_a$ and $(89\text{-}1)_b$ then provide the values on S of the P_{ij}, and, as a result, those of the $\partial_0 \mathrm{L}^0_{ij}$. One will then deduce the values of the $\partial_{00}g_{ij}$ from the preceding analysis and (91-17) for these values of $\partial_{00}g_{ij}$.

Conversely, it results from our analysis that the left-hand sides of $(89\text{-}1)_a$ and $(89\text{-}1)_b$ have the values that were imposed for these values of $\partial_{00} g_{ij}$.

Equations $(89\text{-}1)_c$ then provide the values of $\partial_0 S_i$. As for the $\partial_0 S_i$, they are provided by the auxiliary condition (86-7).

Conforming to the theorem of sec. **88**, no equation will contain the $\partial_{00} \underline{g}^{(0\lambda)}$, which corresponds precisely to the possibility of the coordinate changes that we studied. The $\partial_{00} \underline{g}^{(0\lambda)}$ may admit discontinuities upon crossing S, but, from the differential structure of V_4, these discontinuities will be devoid of any intrinsic significance, and may be annulled by an admissible coordinate change.

Therefore, *the second derivatives* $\partial_{00} g_{ij}$ *and* $\partial_{00} \underline{g}^{[0i]}$ *are continuous upon crossing a hypersurface* S ($x^0 = 0$) *for which* $g^{00} \neq 0$, *and the same thing will be true for the derivatives* $\partial_0 S_i$, $\partial_0 S_0$. These results may be extended to derivatives of higher order by differentiating equations (89-1), (89-6), and the auxiliary condition with respect to x^0.

One sees that, except for a singular case that we pointed out, *the Cauchy problem that relates to the field equations* (85-3), (85-4) *and the Cauchy data* (g_{ij}, $\underline{g}^{[0i]}$, $\underline{g}^{(0\lambda)}$, $\partial_0 g_{ij}$, $\partial_0 \underline{g}^{(0\lambda)}$, S_i, S_0), *which are defined on the hypersurface* S ($x^0 = 0$, $g^{00} \neq 0$) *and satisfy equations* (89-2) *and* (89-7) *on* S *admit a unique solution*(*at least for the analytic case*), *up to an admissible coordinate change.* It seems, moreover, that the method of Mme. Fourés may be extended to this case, as well.

The results will be totally different when S is tangent to the cone C_x that was defined in sec. **86**. The hypersurfaces that are tangent to this cone will appear to be the *wave surfaces of the unitary field envisioned.* The associated rays will be the characteristics of the equation:

$$\Delta_1 f \equiv \nabla l^{\alpha\beta} \partial_\alpha f \partial_\beta f = 0 \; ; \tag{92-1}$$

i.e., they will be the *null-length geodesics* of the Riemannian metric of hyperbolic normal type:

$$ds^2 = l_{\alpha\beta}\, dx^\alpha dx^\beta . \tag{92-2}$$

We are thus led to consider that, in the present theory, *it is the tensor* $l_{\alpha\beta}$ (*or a tensor that is proportional to it*) *that defines the gravitational part of the unitary field,* and that it is the behavior of such a tensor that must be compared to that of the gravitational tensor $g_{\alpha\beta}$ of general relativity.

BIBLIOGRAPHY FOR PART TWO

S. S. CHERN. – *Topics in Differential Geometry,* Mimeographed course notes from the Institute for Advanced Study. Princeton, (1951).

A. EINSTEIN. – *The Meaning of Relativity* (Appendix *H*).

L. P. EISENHART. – *Non-Riemannian Geometry,* Amer. Math. Colloq., (1927).

V. HLAVATY. – Journ. Rat. Mech., **1** (1952), 539-562; **2** (1953), 1-52; **3** (1954), 103-146.

V. HLAVATY and A. W. SAENZ. – Journ. Rat. Mech., **2** (1953), 523-536.

A. LICHNEROWICZ. – Comptes rendus Acad. Sc., **237** (1953), 1383-1386, and Journ. Rat. Mech., **3** (1954), 487-522.

E. SCHRÖDINGER. – *Spacetime Structure.* Cambridge, (1950).

M. A. TONNELAT. – Journ. Phys. et Rad., **12** (1951), 81-88, and **13** (1952), 177-185.

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