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The foundations of the theory of infinite continuous transformation groups – II.

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Now that we know that any infinite continuous group contains infinitely many independent linear transformations, we would now like to show that the entire theory of infinite continuous groups comes down to the examination of the infinitesimal transformations of such groups.

§ 9. The defining equations of the infinitesimal transformations of an infinite group.

1. As before, let:

$$W_k\left(x_1,\cdots,x_n,\mathfrak{x}_1,\cdots,\mathfrak{x}_n,\frac{\partial\mathfrak{x}_1}{\partial x_1},\cdots,\frac{\partial\mathfrak{x}_n}{\partial x_n},\frac{\partial^2\mathfrak{x}_1}{\partial x_1^2},\cdots\right)=0 \qquad (k = 1, 2, \ldots) \tag{1}$$

be the defining equations of the finite transformations of an infinite continuous group with pair-wise inverse transformations. However, from now on, we would like to write:

$$\frac{\partial\mathfrak{x}_i}{\partial x_\nu}=\mathfrak{x}_{i,\nu}, \qquad \frac{\partial^2\mathfrak{x}_i}{\partial x_\mu\,\partial x_\nu}=\mathfrak{x}_{i,\mu\nu},\ \ldots,$$

for the differential quotients of $\mathfrak{x}$ with respect to x, such that our defining equations take the form:

$$W_k(x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \mathfrak{x}_{1,1}, \ldots, \mathfrak{x}_{n,n}, \mathfrak{x}_{1,11}, \ldots) = 0 \qquad (k = 1, 2, \ldots). \tag{2}$$

Should the infinitesimal transformation:

$$Xf=\sum_{i=1}^{n}\xi_i(\mathfrak{x}_1,\cdots,\mathfrak{x}_n)\frac{\partial f}{\partial\mathfrak{x}_i}$$

belong to our group, then, from Theorem I, pp. 336 [here, pp. 317], it is necessary and sufficient that the system of differential equations (1) admit the infinitesimal transformation Xf, or, what amounts to the same thing, that the system of equations (2) in the variables x, $\mathfrak{x}$, $\mathfrak{x}_i$, …, $\mathfrak{x}_{i,\mu\nu}$, … admit the extended infinitesimal transformation:

(3) $$X^{(n)}f = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \frac{\partial f}{\partial \mathfrak{x}_i} + \sum_{i,\mu,\nu=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\mu} \mathfrak{x}_{\mu,\nu} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu}} + \dots$$

Analytically, this condition may be expressed as follows: All expressions of the form:

(4) $$X^{(m)}f = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \frac{\partial W_k}{\partial \mathfrak{x}_i} + \sum_{i,\mu,\nu=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\mu} \mathfrak{x}_{\mu,\nu} \frac{\partial W_k}{\partial \mathfrak{x}_{i,\nu}} + \dots$$

vanish, due to (2).

2. Now, if:

(5) $$\mathfrak{x}_i = F_i(x_1, \dots, x_n) \qquad (i = 1, \dots, n)$$

is any finite transformation of our group then equations (2) will be fulfilled identically under the substitutions:

(6) $$\mathfrak{x}_i = F_i(x), \qquad \mathfrak{x}_{i,\nu} = \frac{\partial F_i}{\partial x_\nu}, \qquad \mathfrak{x}_{i,\mu\nu} = \frac{\partial^2 F_i}{\partial x_\mu \, \partial x_\nu}, \dots$$

If we then think of equations (5) as having been solved for $x_1, \dots, x_n$:

(7) $$x_i = \Phi_i(\mathfrak{x}_1, \dots, \mathfrak{x}_n) \qquad (i = 1, \dots, n),$$

and further imagine these values of $x_1, \dots, x_n$ as having been substituted in the expressions for $\mathfrak{x}_{i,\nu}$, $\mathfrak{x}_{i,\mu\nu}$, … that follow from (6):

(8) $$\mathfrak{x}_{i,\nu} = \Phi_{i\nu}(\mathfrak{x}), \qquad \mathfrak{x}_{i,\mu\nu} = \Phi_{i\mu\nu}(\mathfrak{x}), \dots$$

then equations (2) must also go to mere identities under the substitution that is defined by (7) and (8). However, we saw before that Xf is an infinitesimal transformation of our group when and only when the expression (4) vanishes due to (2). Thus, we can also say: Xf is an infinitesimal transformation of our group when and only when the expression (4) always vanishes identically under the substitution (7), (8), which might also make the transformation (5) belong to our group. If we imply the substitution (7), (8) by including the variables in square brackets then it emerges from this requirement that $\xi_1(\mathfrak{x}), \dots, \xi_n(\mathfrak{x})$ must satisfy the differential equations:

(9) $$[X^{(m)} W_k] = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \left[\frac{\partial W_k}{\partial \mathfrak{x}_i}\right] + \sum_{i,\mu,\nu=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\mu} \left[\sum_{\nu=1}^{n} \mathfrak{x}_{\mu,\nu} \frac{\partial W_k}{\partial \mathfrak{x}_{i,\nu}}\right] + \dots = 0 \qquad (k = 1, 2, \dots).$$

3. Obviously, we obtain infinitely many different differential equations for $\xi_1, \dots, \xi_n$ in this way, since our group indeed contains infinitely many different transformations (5). However, it may be shown that equations (9) are completely independent of the special choice of the transformation (5). If one employs two different transformations (5) of our group for the definition of equations (9) then one obtains the same system of differential equations for $\xi_1, \dots, \xi_n$ in both cases.

In order to prove this assertion, we first remark that the form in which we have employed the system of equations (2) has no influence on the system of differential equations (9). We then replace the system of equations (2) with the equivalent one:

$$U_k(x_1, \dots, x_n, \mathfrak{x}_1, \dots, \mathfrak{x}_n, \mathfrak{x}_{1,1}, \dots, \mathfrak{x}_{n,n}, \mathfrak{x}_{1,11}, \dots) = 0 \qquad (k = 1, 2, \dots)$$

then the system of differential equations:

$$[X^{(m)}\, U_1] = 0 \qquad\qquad (k = 1, 2, \dots)$$

differs from the system (9) only in form, when we naturally assume that the same transformation (5) was employed both times for the substitution [ ]. In order to convince oneself of this, confer *Theorie der Transformationsgruppen*, Abschnitt I, pp. 109-111 [Leipzig 1888].

4. Now, if:

(10) $$\mathfrak{x}_i = \mathfrak{F}_i(\bar{x}_1, \dots, \bar{x}_n) \qquad (i = 1, \dots, n)$$

is any other transformation of our group, and if one might obtain :

(11) $$x_i = \Psi_i(\bar{x}_1, \dots, \bar{x}_n) \qquad (i = 1, \dots, n)$$

from (5) and (10), by dropping the $\mathfrak{x}$ then the transformation (11) would also belong to our group, and the transformation (10) can obviously be obtained when one first performs (11) and then (5).

We now recall Theorem II, pp. 338 (here, pp. 319). From this theorem, the system of differential equations (1) preserves its form when we introduce the new variables $\bar{x}_1, \dots, \bar{x}_n$ in place of $x_1, \dots, x_n$ by means of the transformation (11) of our group. If we then set:

$$\frac{\partial \mathfrak{x}_i}{\partial \bar{x}_\nu} = \bar{\mathfrak{x}}_{i,\nu}, \qquad \frac{\partial^2 \mathfrak{x}_i}{\partial \bar{x}_\mu\, \partial \bar{x}_\nu} = \bar{\mathfrak{x}}_{i,\mu\nu}, \dots$$

and define the equations:

(12) $$x_i = \Psi_i(\bar{x}), \qquad \mathfrak{x}_{i,\nu} = \sum_{\tau=1}^{n} \bar{\mathfrak{x}}_{i,\tau} \frac{\partial \bar{x}_\nu}{\partial x_\nu}, \qquad \mathfrak{x}_{i,\mu\nu} = \sum_{\tau,\pi=1}^{n} \bar{\mathfrak{x}}_{i,\tau\pi} \frac{\partial \bar{x}_\nu}{\partial x_\nu} \frac{\partial \bar{x}_\pi}{\partial x_\mu} + \sum_{\tau=1}^{n} \bar{\mathfrak{x}}_{i,\tau} \frac{\partial^2 \bar{x}_\tau}{\partial x_\nu\, \partial x_\mu},$$

in which we think of all of the differential quotients:

$$\frac{\partial \overline{x}_\nu}{\partial x_\nu}, \frac{\partial^2 \overline{x}_\tau}{\partial x_\nu \, \partial x_\mu}, \ldots$$

by means of (11) then under the substitution (12), the system of equations (2) goes to a system of equations:

$$(13) \qquad U_k(\overline{x}_1, \ldots, \overline{x}_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \overline{\mathfrak{x}}_{1,1}, \ldots, \overline{\mathfrak{x}}_{n,n}, \overline{\mathfrak{x}}_{1,11}, \ldots) = 0 \qquad (k = 1, 2, \ldots)$$

that is equivalent to the system of equations:

$$(14) \qquad W_k(x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \overline{\mathfrak{x}}_{1,1}, \ldots, \overline{\mathfrak{x}}_{n,n}, \overline{\mathfrak{x}}_{1,11}, \ldots) = 0 \qquad (k = 1, 2, \ldots).$$

If we would then like to form the system of differential equations that we get for ξ_1, …, ξ_n by the use of the transformation (10) then, from the previous statements, we can use the system (13) in place of system (14) with no further assumptions.

5. In order to obtain the differential equations for the $\xi_i(\mathfrak{x})$ that follow from (10), we extend the infinitesimal transformation Xf, when we consider the $\mathfrak{x}_1$, …, $\mathfrak{x}_n$ to be functions of $\overline{x}_1$, …, $\overline{x}_n$:

$$(15) \qquad \overline{X}^{(m)} f = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \frac{\partial f}{\partial \mathfrak{x}_i} + \sum_{i,\mu,\nu=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\mu} \overline{\mathfrak{x}}_{\mu,\nu} \frac{\partial f}{\partial \overline{\mathfrak{x}}_{i,\nu}} + \ldots,$$

and then define the expressions:

$$(16) \qquad \overline{X}^{(m)} U_k = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \frac{\partial U_k}{\partial \mathfrak{x}_i} + \sum_{i,\mu,\nu=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\mu} \overline{\mathfrak{x}}_{\mu,\nu} \frac{\partial U_k}{\partial \overline{\mathfrak{x}}_{i,\nu}} + \ldots,$$

and we finally make the substitution (17) into the expressions (16), which may be implied by curly brackets. Then:

$$(18) \qquad \{\overline{X}^{(m)} U_k\} = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \left\{ \frac{\partial U_k}{\partial \mathfrak{x}_i} \right\} + \sum_{i,\mu,\nu=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\mu} \left\{ \sum_{\nu=1}^{n} \overline{\mathfrak{x}}_{\mu,\nu} \frac{\partial U_k}{\partial \overline{\mathfrak{x}}_{i,\nu}} \right\} + \ldots = 0$$

are the differential equations for ξ_1, …, ξ_n that one obtains by means of the transformation (10).

6. We will verify that the differential equations (18) and (9) are identical with each other.

Under the assumptions that were made, the functions W_1, W_2, …, go to U_1, U_2, … when the $\overline{x}_i$, $\overline{\mathfrak{x}}_i$, $\overline{\mathfrak{x}}_{i,\mu}$, … are introduced in place of the x_i, $\mathfrak{x}_i$, $\mathfrak{x}_{i,\mu\nu}$, … by means of (12). However, at the same time, the infinitesimal transformation $X^{(m)}f$ is converted into $\overline{X}^{(m)} f$

under the transformation (12). $X^{(m)}f$ is then defined by the fact that it must leave invariant the system of Pfaffian equations:

$$(19)\qquad d\mathfrak{x}_i - \sum_{\nu=1}^{n} \mathfrak{x}_{i,\nu} dx_\nu = 0, \qquad d\mathfrak{x}_{i,\nu} - \sum_{\mu=1}^{n} \mathfrak{x}_{i,\nu\mu} dx_\mu = 0, \ \ldots;$$

on the other hand, $\bar{X}^{(m)}f$ is defined by the fact that it leaves invariant the system:

$$(20)\qquad d\mathfrak{x}_i - \sum_{\nu=1}^{n} \bar{\mathfrak{x}}_{i,\nu} d\bar{x}_\nu = 0, \qquad d\bar{\mathfrak{x}}_{i,\nu} - \sum_{\mu=1}^{n} \bar{\mathfrak{x}}_{i,\nu\mu} d\bar{x}_\mu = 0, \ \ldots$$

However, the system (19) takes on the form (20) by means of the transformation (12), so $X^{(m)}f$ must take on the form $\bar{X}^{(m)}f$ under the transformation (12).

This illuminates the fact that the expression (4) goes to the expression (16) under the substitution (12). However, the left-hand sides of equations (18) arise from the expressions (16) by way of the substitution (17), so we can also say: The left-hand sides of equations (18) will be obtained from the expressions (4) when one first performs the substitution (12) and the substitution (17). If we ultimately recall that the two transformations (11) and (5), when carried out one after the other, deliver the transformation (11), then we recognize that the two substitutions (12) and (17), when performed in succession, yield precisely the same thing as if we had performed just the substitution (7), (8). The left-hand sides of equations (18) will then be obtained from the expressions (4) by the substitution (7), (8). In other words: Equations (18) are identical with equations (9).

With that, we have proved that the particular choice of transformation (5) has no influence on the differential equations (9), just as the transformation (5) of our group may be chosen so that one still always obtains the same system of differential equations for $\xi_1(\mathfrak{x}), \ldots, \xi_n(\mathfrak{x})$.

7. We are therefore completely free to choose which transformation (5) of our group to use for the definition of the differential equations (9). Naturally, everything becomes simplest when we let the transformation (5) coincide with the identity transformation, so that one then has $\mathfrak{x}_{i,\nu} = \varepsilon_{i\nu}$, where $\varepsilon_{i\nu}$ equals 1 whenever $i = \nu$ and vanishes whenever $i \neq \nu$, although the $\mathfrak{x}_{i,\mu\nu}$, and likewise all differential quotients of higher order in the $\mathfrak{x}_i$, are equal to zero; the differential equations (9) then assume the simple form:

$$(21)\qquad \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \left[\frac{\partial W_k}{\partial \mathfrak{x}_i}\right]_0 + \sum_{i,\nu=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} \left[\frac{\partial W_k}{\partial \bar{\mathfrak{x}}_{i,\nu}}\right]_0 + \ldots = 0 \quad (k = 1, 2, \ldots),$$

where the symbol 0 on the square bracket means that one must set all $x_i = \mathfrak{x}_i$, all $\mathfrak{x}_{i,\nu} = \varepsilon_{i\nu}$, and all $\mathfrak{x}_{i,\mu\nu}, \ldots$ equal to zero.

If one thinks of the system of equations (2) as being solved before one uses them for the definition of the differential equations (21) then one sees that the system of differential equations (21) is of the same order as the system (1) and that it contains just as many independent equations as that one. It is likewise clear that all of the differential equations that are derivable from (21) by differentiation and elimination whose order does not exceed the order of (21) already follow from (21) without differentiations.

8. The differential equations (21), whose most general solutions $\xi_1(\mathfrak{x}), \ldots, \xi_n(\mathfrak{x})$ determine the most general infinitesimal transformation:

$$Xf = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \frac{\partial f}{\partial \mathfrak{x}_i}$$

of our group, are what we shall call the defining equations of the infinitesimal transformations of this group.

These defining equations (21) possess a characteristic property. Namely, if Xf and:

$$Yf = \sum_{i=1}^{n} \eta_i(\mathfrak{x}) \frac{\partial f}{\partial \mathfrak{x}_i}$$

are two infinitesimal transformations of our group then, from Theorem IV, pp. 348 [here, pp. 327]:

$$(X\ Y) = \sum_{i,\nu=1}^{n} \left\{ \xi_\nu \frac{\partial \eta_i}{\partial \mathfrak{x}_\nu} - \eta_i \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} \right\} \frac{\partial f}{\partial \mathfrak{x}_i}$$

is also always an infinitesimal transformation of the group. In other words: when $\xi_1, \ldots, \xi_n$ and $\eta_1, \ldots, \eta_n$ are any two systems of solutions of the differential equations (21) then:

$$\sum_{i,\nu=1}^{n} \left(\xi_\nu \frac{\partial \eta_i}{\partial \mathfrak{x}_\nu} - \eta_i \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} \right) \qquad (i = 1, \ldots, n)$$

is likewise a system of solutions of (21).

We summarize the results obtained in:

Theorem V. *If the finite transformations of an infinite continuous group can be defined by a finite number of partial differential equations then the infinitesimal transformations contained in this group may also be defined by a finite number of differential equations, where the latter differential equations have the form:*

$$\sum_{i=1}^{n} \alpha_{ki}(\mathfrak{x}) \xi_1 + \sum_{i,\nu=1}^{n} \alpha_{ki\nu}(\mathfrak{x}) \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} + \sum_{i,\mu,\nu=1}^{n} \alpha_{ki\mu\nu}(\mathfrak{x}) \frac{\partial^2 \xi_i}{\partial \mathfrak{x}_\mu \partial \mathfrak{x}_\nu} + \ldots = 0, \tag{22}$$

so they are linear and homogeneous in $\xi_1(\mathfrak{x}), \ldots, \xi_n(\mathfrak{x})$, *and their differential quotients possess the following property, in addition: If* $\xi_1, \ldots, \xi_n$ *and* $\eta_1, \ldots, \eta_n$ *are any two systems of solutions of the differential equations* (22) *then:*

$$\sum_{i,\nu=1}^{n}\left(\xi_\nu \frac{\partial \eta_i}{\partial \mathfrak{x}_\nu} - \eta_i \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu}\right) \qquad (i = 1, \ldots, n)$$

is a system of solutions of (22).

9. The foregoing theorem shall be referred to as the *First Fundamental Theorem* of the theorem of infinite continuous groups; it corresponds to the theorem that r independent infinitesimal transformations:

$$X_k f = \sum_{i=1}^{n} \xi_{ki}(x_1, \ldots, x_n) \frac{\partial f}{\partial x_i}$$

of an r-parameter group satisfy the pair-wise relationships:

$$(X_i X_k) = \sum_{s=1}^{r} c_{iks} X_s f \qquad (i, k = 1, \ldots, r).$$

The theorem that was proved above – viz., that the differential equations (9) are independent of the choice of the transformation (5) – has its analogue in the theory of finite groups: As might be remarked in passing, it corresponds to the theorem that the finite equations:

$$x_i' = f_i(x_1, \ldots, x_n, a_1, \ldots, a_r) \qquad (i = 1, \ldots, n)$$

satisfy the differential equations of an r-parameter group of the form:

$$\frac{\partial x_i'}{\partial a_k} = \sum_{j=1}^{r} \Psi_{jk}(a_1, \ldots, a_r)\, \xi_{ji}(x_1', \ldots, x_n') \quad (i = 1, \ldots, n;\ k = 1, \ldots, r).$$

§ 10. Infinite groups of infinitesimal transformations. Differential invariants of such groups.

10. In the previous paragraphs, we have seen that the infinitesimal transformations of an infinitely continuous group can be defined by a system of partial differential equations that possesses a certain special property: It is linear and homogeneous in the ξ_i and their differential quotients. From any two of its systems of solutions, one can derive a third system of solutions by a certain operation. Finally, its most general system of solutions does not depend upon merely a finite number of arbitrary constants – the latter property follows from the fact that any infinite group contains an infinite number of independent infinitesimal transformations.

A closely-related problem is to confirm that, conversely, any system of partial differential equations that possesses the aforementioned property defines the infinitesimal transformations of an infinite group. It is therefore natural for us to now consider an arbitrary system of partial differential equations that fulfills the given requirements.

11. We thus imagine that we are now given an arbitrary system of linear, homogeneous partial differential equations:

$$\sum_{i=1}^{n} \alpha_{ki}(\mathfrak{x})\xi_i + \sum_{i=1}^{n} \alpha_{ki\nu}(\mathfrak{x}) \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} + \ldots = 0 \qquad (k = 1, 2, \ldots) \tag{22}$$

that possesses the following two properties:

1. *The most general system of solutions of* (22) *shall not depend upon just a finite number of arbitrary constants.*

2. *Whenever* $\xi_1(\mathfrak{x}), \ldots, \xi_n(\mathfrak{x})$ *and* $\eta_1(\mathfrak{x}), \ldots, \eta_n(\mathfrak{x})$ *are any two systems of solutions of* (22):

$$\sum_{i,\nu=1}^{n} \left(\xi_\nu \frac{\partial \eta_i}{\partial \mathfrak{x}_\nu} - \eta_i \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} \right) \qquad (i = 1, \ldots, n)$$

is likewise a system of solutions of (22).

In addition, corresponding to the assumptions on pp. 318 [here, pp. 302], we make the following assumption: If q is the order of the system (22) then all differential equations of order r and less that can be derived from (22) by differentiations and eliminations already follow from (22) without differentiation.

Obviously, our system (22) defines a family of infinitely many independent infinitesimal transformations, and in fact *only one* family, that always contains, along with the two infinitesimal transformations Xf and Yf, likewise the infinitesimal transformations $a\,Xf + b\,Yf$. This suggests that one refer to such a family as an *infinite group of infinitesimal transformations.* We pose the following definition:

A family of infinitely many independent infinitesimal transformations shall be called an infinite group of infinitesimal transformations when it is defined by a system of differential equations of the form (22) that possesses the aforementioned property.

12. From Theorem V, it follows that the infinitesimal transformations of an infinite group always define an infinite group of infinitesimal transformations. Later, we will see that conversely, the infinitesimal transformations of an infinite group of infinitesimal transformations are also always the infinitesimal transformations of a certain infinite group. If that were true to begin with then naturally there would be no point in speaking of infinite groups of infinitesimal transformations; however, although this manner of expression is superfluous, it is nevertheless very convenient and advantageous.

We will next prove that any infinite group of infinitesimal transformations determines infinitely many differential invariants. From this, one can show without any difficulty that also any infinite group of finite transformations determines such differential invariants. By considering certain differential invariants of a particular nature we will then prove in the next paragraph that any infinite group of infinitesimal transformations consists of the infinitesimal transformations of a certain infinite group.

13. The infinite group of infinitesimal transformations transforms the variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$. To these variables, we add certain auxiliary variables $\mathfrak{y}_1, \ldots, \mathfrak{y}_l$ that are not transformed by our group at all, and we reserve the right to choose the number of these auxiliary variables as needed. The $n + l$ variables: $\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \mathfrak{h}_1, \ldots, \mathfrak{h}_n$ will then be transformed under our group by an infinitesimal transformation of the form:

$$\sum_{\nu=1}^{n} \xi_\nu \frac{\partial f}{\partial \mathfrak{x}_\nu} + \sum_{\mu=1}^{l} \eta_\mu \frac{\partial f}{\partial \mathfrak{y}_\mu},$$

where the ξ_ν and η_ν are defined by (22) and the equations:

$$(23)\qquad \begin{cases} \dfrac{\partial \xi_\nu}{\partial \mathfrak{y}_\mu} = 0, \ \dfrac{\partial^2 \xi_\nu}{\partial \mathfrak{y}_\mu \partial \mathfrak{y}_\pi} = 0, \ \dfrac{\partial^2 \xi_\nu}{\partial \mathfrak{y}_\mu \partial \mathfrak{x}_\pi} = 0, \ \cdots, \\ \eta_\mu = 0, \ \dfrac{\partial \eta_\mu}{\partial \mathfrak{y}_\pi} = 0, \ \dfrac{\partial \eta_\mu}{\partial \mathfrak{x}_\nu} = 0, \ \dfrac{\partial^2 \eta_\mu}{\partial \mathfrak{y}_\pi \partial \mathfrak{y}_\rho} = 0, \ \ldots \end{cases}$$

If q is the order of the system (22) then we imagine that all differential quotients up to order q of the ξ and η with respect to the $\mathfrak{x}$ and $\mathfrak{y}$ have been included.

Among the $n + l$ variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \mathfrak{y}_1, \ldots, \mathfrak{y}_l$, we consider a certain number of them – say, $\mathfrak{n}$ – to be independent, and write them as $z_1, \ldots, z_\mathfrak{n}$, while we regard the remaining $n + l - \mathfrak{n} = \mathfrak{l}$ of them to be functions of the $z_1, \ldots, z_\mathfrak{n}$, and call them $u_1, \ldots, u_\mathfrak{n}$. Our infinite group of infnitesimal transformations then transforms the $n + l = \mathfrak{n} + \mathfrak{l}$ variables $z_1, \ldots, z_\mathfrak{n}$, $u_1, \ldots, u_\mathfrak{n}$ by an infinitesimal transformation:

$$Zf = \sum_{\nu=1}^{\mathfrak{n}} \zeta_\nu (z_1, \ldots, z_\mathfrak{n}, u_1, \ldots, u_\mathfrak{l}) \frac{\partial f}{\partial z_\nu} + \sum_{\mu=1}^{\mathfrak{l}} \omega_\mu (z_1, \ldots, z_\mathfrak{n}, u_1, \ldots, u_\mathfrak{l}) \frac{\partial f}{\partial u_\mu}$$

that is, in turn, defined by certain linear, homogeneous, partial differential equations:

$$(24)\qquad \begin{cases} \sum_{\nu=1}^{\mathfrak{n}} \beta_{k\nu}(z,u)\zeta_\nu + \sum_{\mu=1}^{\mathfrak{l}} \gamma_{k\mu}(z,u)\omega_\mu \\ + \sum_{\nu,\tau=1}^{\mathfrak{n}} \beta_{k\nu\tau}(z,u)\dfrac{\partial \zeta_\nu}{\partial z_\tau} + \sum_{\mu,\pi=1}^{\mathfrak{l}} \gamma_{k\mu\pi}(z,u)\dfrac{\partial \omega_\mu}{\partial u_\pi} \\ \sum_{\nu=1}^{\mathfrak{n}}\sum_{\mu=1}^{\mathfrak{l}} \left\{ \beta'_{k\nu\mu}(z,u)\dfrac{\partial \zeta_\nu}{\partial u_\mu} + \gamma'_{k\mu\nu}(z,u)\dfrac{\partial \omega_\mu}{\partial z_\nu} \right\} + \cdots = 0 \end{cases} \qquad (k = 1, 2, \ldots).$$

One clearly obtains these differential equations (24) when one replaces the $\mathfrak{x}$, $\mathfrak{y}$, ξ, and η in (22) and (23) with the z, u, ζ, and ω. Among equations (24), one finds some, in particular, that express the idea that l of the functions $\zeta_1, \ldots, \zeta_{\mathfrak{n}}, \omega_1, \ldots, \omega_{\mathfrak{l}}$ vanish, along with all of their differential quotients up to order q. Furthermore, some of the equations express the idea that for n of the functions ζ_ν, ω_μ, all of the differential quotients with respect to $\mathfrak{y}_1, \ldots, \mathfrak{y}_{\mathfrak{l}}$ vanish.

It is self-explanatory that equations (24) define an infinite group of infinitesimal transformations and that this group transforms the variables in precisely the same way as the infinite group of infinitesimal transformations that is defined by (22).

14. We now extend the infinitesimal transformation Zf by including all differential quotients:

$$\frac{\partial u_\mu}{\partial z_\nu} = u_{\mu,\nu}\,, \qquad \frac{\partial^2 u_\mu}{\partial z_\nu\,\partial z_\tau} = u_{\mu,\nu\tau}\,, \ldots$$

perhaps up to order N, inclusive. Thus, we nonetheless do not yet consider that the ξ_ν and ω_μ satisfy the differential equations (21). The extended transformation in question, which we call $Z^{(N)}f$, is defined by the fact that it leaves invariant the system of Pfaffian equations:

$$(25)\qquad du_\mu - \sum_{\nu=1}^{\mathfrak{n}} u_{\mu,\nu}dz_\nu = 0, \qquad du_{\mu,\nu} - \sum_{\tau=1}^{\mathfrak{n}} u_{\mu,\nu\tau}dz_\tau = 0, \ldots$$

The expressions:

$$Z^{(N)}\,u_{\mu,\nu}\,, \qquad Z^{(N)}\,u_{\mu,\nu\tau}\,, \ldots$$

then become linear, homogeneous functions of $\zeta_1, \ldots, \zeta_{\mathfrak{n}}, \omega_1, \ldots, \omega_{\mathfrak{l}}$ and their differential quotients. Therefore, we can write $Z^{(N)} f$ in the following way:

$$Z^{(N)} f = \sum_{\nu=1}^{\mathfrak{n}} \zeta_\nu A_\nu f + \sum_{\mu=1}^{\mathfrak{l}} \omega_\mu B_\mu f + \sum_{\nu,\tau=1}^{\mathfrak{n}} \frac{\partial \zeta_\nu}{\partial z_\tau} A_{\nu\tau} f + \sum_{\mu,\pi=1}^{\mathfrak{l}} \frac{\partial \omega_\mu}{\partial u_\pi} B_{\mu\pi} f$$
$$+ \sum_{\nu=1}^{\mathfrak{n}}\sum_{\mu=1}^{\mathfrak{l}} \left\{ \frac{\partial \zeta_\nu}{\partial u_\mu}\mathfrak{A}_{\nu\mu} f + \frac{\partial \omega_\mu}{\partial z_\nu}\mathfrak{B}_{\mu\nu} f \right\} + \ldots,$$

where the $A_\nu f$, $B_\nu f$, $A_{\nu\tau} f$, … are completely well-defined infinitesimal transformations in the variables:

$$z_\nu,\ u_\mu,\ u_{\mu,\nu},\ u_{\mu,\nu\tau},\ \ldots$$

that do not include the ζ_ν, ω_μ, or any of their differential quotients, at all.

15. Now, the differential equations (24) might also be considered.

We would like to assume that precisely m_0 independent equations of zero order can be defined from (24):

$$(24^0)\qquad \sum_{\nu=1}^{\mathfrak{n}} \alpha_{j\nu}^{(0)}(z,u)\zeta_\nu + \sum_{\mu=1}^{\mathfrak{l}} \beta_{j\mu}^{(0)}(z,u)\omega_\mu = 0 \qquad (j = 1, \ldots, m_0),$$

as well as precisely m_1 independent equations of first order:

$$(24^1)\qquad \begin{cases} \displaystyle\sum_{\nu=1}^{\mathfrak{n}} \alpha_{k\nu}^{(1)}\,\zeta_\nu + \sum_{\mu=1}^{\mathfrak{l}} \beta_{k\mu}^{(1)}\,\omega_\mu + \sum_{\nu,\tau=1}^{\mathfrak{n}} \alpha_{k\nu\tau}^{(1)} \frac{\partial \zeta_\nu}{\partial z_\tau} + \sum_{\mu,\pi=1}^{\mathfrak{l}} \beta_{k\mu\pi}^{(1)} \frac{\partial \omega_\mu}{\partial u_\pi} \\ \displaystyle\qquad + \sum_{\nu=1}^{\mathfrak{n}} \sum_{\mu=1}^{\mathfrak{l}} \left\{ \gamma_{k\nu\mu}^{(1)} \frac{\partial \zeta_\nu}{\partial u_\mu} + \vartheta_{k\mu\nu}^{(1)} \frac{\partial \omega_\mu}{\partial z_\nu} \right\} = 0 \end{cases} \qquad (k = 1, \ldots, m_1),$$

from which, not all of the differential quotients of first order can be removed, and so on, such that finally there are precisely m_q independent equations of order q (24^q), from which not all of the differential quotients of order q can be removed.

The system of $m_0 + m_1 + \ldots + m_q$ mutually independent equations:

$$(26)\qquad (24^0),\ (24^1),\ \ldots,\ (24^q)$$

is then equivalent to the system (24), and one can be sure that only such equations of order q and less can be derived by differentiation and elimination starting with (24) that already follow from (26) without differentiation. In addition, it must be remarked that among the assumptions that were made, the number m_0 is equal to at least l, the number m_1 is equal to at least $ll + 2nl$, and so on.

16. If the number N that appeared above is greater than q then we must also include the differential quotients of order $(q + 1)$ up to N in the ζ_ν and ω_μ. We thus differentiate equations (24^q) $N - q$ more times with respect to $z_1, \ldots, z_{\mathfrak{n}}$ and $u_1, \ldots, u_{\mathfrak{l}}$, and obtain, in this way, m_{q+1} independent equations of order $q + 1$, from which not all differential quotients of order $q + 1$ can be removed, until finally we obtain m_N independent equations of order N, from which not all differential quotients of order N can be removed. The system of equations:

$$(27)\qquad (24^0),\ (24^1),\ \ldots,\ (24^N)$$

then subsumes all linear, homogeneous relations that exist between the general solutions $\zeta_1, \ldots, \zeta_{\mathfrak{n}}, \omega_1, \ldots, \omega_{\mathfrak{l}}$ of the differential equations (24) and the differential quotients of these solutions of first up to N^{th} order.

17. We now think of the system of equations (27) as having been solved for m_0 of the functions ζ_ν, ω_μ, and then for m_1 of the differential quotients of first order:

$$\frac{\partial \zeta_\nu}{\partial z_\tau}, \frac{\partial \zeta_\nu}{\partial u_\mu}, \frac{\partial \omega_\mu}{\partial z_\nu}, \frac{\partial \omega_\mu}{\partial u_\pi},$$

and so on, until finally, for m_N of the differential quotients of N^{th} order of the ζ_ν, ω_μ. We substitute the expressions thus found in the infinitesimal transformation $Z^{(N)}f$ that was defined above, and obtain an abbreviated infinitesimal transformation:

$$\bar{Z}^{(N)} f = \sum \zeta_\nu \bar{A}_\nu f + \sum \omega_\mu \bar{B}_\mu f + \sum_{\nu,\tau} \frac{\partial \zeta_\nu}{\partial z_\tau} \bar{A}_{\nu\tau} f + \sum_{\mu,\pi} \frac{\partial \omega_\mu}{\partial u_\pi} \bar{B}_{\mu\pi} f$$
$$+ \sum_{\mu,\nu} \left\{ \frac{\partial \zeta_\nu}{\partial u_\mu} \bar{\mathfrak{A}}_{\nu\mu} f + \frac{\partial \omega_\mu}{\partial z_\nu} \bar{\mathfrak{B}}_{\mu\pi} f \right\} + \ldots$$

Here, the $\bar{A}_\nu f$, $\bar{B}_\mu f$, …, like the $A_\nu f$, $B_\mu f$, … before, are completely well-defined infinitesimal transformations in the variables:

$$z_\nu, \quad u_\mu, \quad u_{\mu,\nu}, \quad u_{\mu,\nu\tau}, \ldots$$

except that now not all of the $\mathfrak{n} + \mathfrak{l} = n + l$ functions ζ_ν, ω_μ appear, but only $\mathfrak{n} + \mathfrak{l} - m_0 = \varepsilon_0$ of them, and furthermore, of the $(n + l)^2$ differential quotients of first order of the ζ_ν, ω_μ, only:

$$(n + l)^2 - m_1 = \varepsilon_1,$$

and in general, only ε_k of the differential quotients of order k of the ζ_ν, ω_μ. The numbers in question ε_0, ε_1, ε_2, … are thus clearly independent of the number l, and are already completely well-defined by equations (22), a situation that will likewise be of use to us.

18. With these preparations, we can finally prove that the infinite group of infinitesimal transformations that are defined by equations (24) possess differential invariants.

A differential invariant of the group in question is any function of $z_1, \ldots, z_{\mathfrak{n}}, u_1, \ldots, u_{\mathfrak{l}}$ and the differential quotients of the u_μ with respect to the z_ν that remains invariant under any infinitesimal transformation of the group:

$$Zf = \sum_{\nu=1}^{\mathfrak{n}} \zeta_\nu \frac{\partial f}{\partial z_\nu} + \sum_{\mu=1}^{\mathfrak{l}} \omega_\mu \frac{\partial f}{\partial u_\mu}.$$

The determination of all differential invariants of order N of the group (24) then comes down to the determination of all functions of:

(28) $$z_\nu\,,\quad u_\mu\,,\quad u_{\mu,\nu}\,,\quad u_{\mu,\nu\tau},\ \ldots$$

that admit the infinitesimal transformation, which then assumes that $\zeta_1, \ldots, \zeta_{\mathfrak{n}}, \omega_1, \ldots, \omega_{\mathfrak{l}}$ are understood to be the most general system of solutions of the differential equations (24).

Now, only $\varepsilon_0 + \varepsilon_1 + \ldots + \varepsilon_N$ of the quantities:

$$\zeta_\nu\,,\quad \omega_\mu\,,\quad \frac{\partial\zeta_\nu}{\partial z_\tau}\,,\quad \frac{\partial\zeta_\nu}{\partial u_\mu}\,,\quad \frac{\partial\omega_\mu}{\partial z_\nu}\,,\quad \frac{\partial\omega_\mu}{\partial u_\pi}\,,\ \ldots$$

enter into $\bar{Z}^{(N)}f$, and these $\varepsilon_0 + \ldots + \varepsilon_N$ quantities are not coupled by any linear, homogeneous relation, if $\zeta_1, \ldots, \zeta_{\mathfrak{n}}, \omega_1, \ldots, \omega_{\mathfrak{l}}$ refers to the most general system of solutions (24). Thus, a function of the variables (28) can admit the infinitesimal transformation $\bar{Z}^{(N)}f$ when and only when it is a common solution of the $\varepsilon_0 + \ldots + \varepsilon_N$ linear partial differential equations:

(29) $$\begin{cases} \bar{A}_\nu f = 0,\ \bar{B}_\mu f = 0,\ \bar{A}_{\nu\tau} f = 0,\ \bar{B}_{\mu\pi} f = 0, \\ \qquad \bar{\mathfrak{A}}_{\nu\mu} f = 0,\ \bar{\mathfrak{B}}_{\mu\nu} f = 0, \cdots \end{cases}$$

We will prove that equations (29) determine a complete system that has at most $\varepsilon_0 + \ldots + \varepsilon_N$ parameters.

19. If $\zeta_1, \ldots, \zeta_{\mathfrak{n}}, \omega_1, \ldots, \omega_{\mathfrak{l}}$ means the most general system of solutions of (24) then the expression $\bar{Z}^{(N)}f$ obviously represents the most general infinitesimal transformation that includes a certain infinite group of infinitesimal transformations in the variables (28). Therefore, if:

$$Z_k f = \sum_{\nu=1}^{\mathfrak{n}} \zeta_{k\nu}\frac{\partial f}{\partial z_\nu} + \sum_{\mu=1}^{\mathfrak{l}} \omega_{k\mu}\frac{\partial f}{\partial u_\mu} \qquad (k = 1, 2)$$

are any two infinitesimal transformations of the group that is defined by (24), and we set:

$$(Z_1\, Z_2) = \sum_{\nu=1}^{\mathfrak{n}} \varphi_\nu\frac{\partial f}{\partial z_\nu} + \sum_{\mu=1}^{\mathfrak{l}} \vartheta_\mu\frac{\partial f}{\partial u_\mu} = \mathfrak{Z} f,$$

then the three infinitesimal transformations:

$$\bar{Z}_1^{(N)}f\,,\quad \bar{Z}_2^{(N)}f\,,\quad \bar{\mathfrak{Z}}^{(N)}f$$

are related by:

$$(\overline{Z}_1^{(N)} f\,,\quad \overline{Z}_2^{(N)} f\,) = \overline{\mathfrak{Z}}^{(N)} f\,.$$

In other words: The totality of all linear, partial differential equations that one obtains when one thinks of all solutions of (24) as having been substituted for $\zeta_1, \ldots, \zeta_n$, $\omega_1, \ldots, \omega_l$ in:

$$\overline{Z}^{(N)} f = 0$$

defines a complete system in the variables (28). However, since the $\varepsilon_0 + \ldots + \varepsilon_N$ quantities:

$$\zeta_\nu,\ \omega_\mu,\ \frac{\partial \zeta_\nu}{\partial z_\tau},\ \frac{\partial \omega_\mu}{\partial u_\pi},\ \ldots$$

that enter into $\overline{Z}^{(N)} f$ cannot be coupled by a linear, homogeneous relation as long as $\zeta_1, \ldots, \zeta_n$, $\omega_1, \ldots, \omega_n$ is an arbitrary system of solutions of (24), the expressions:

$$\begin{array}{llll}
(\overline{A}_\nu \overline{A}_\tau)\,, & (\overline{A}_\nu \overline{B}_\mu)\,, & (\overline{A}_\nu \overline{A}_{\tau\rho})\,, & \ldots \\
 & (\overline{B}_\mu \overline{B}_\pi)\,, & (\overline{B}_\mu \overline{A}_{\nu\tau})\,, & \ldots \\
 & & (\overline{A}_{\nu\tau} \overline{A}_{\sigma\rho})\,, & \ldots
\end{array}$$

can all be expressed linearly and homogeneously in terms of the:

$$\overline{A}_\nu f\,,\quad \overline{B}_\mu f\,,\quad \overline{A}_{\nu\tau} f\,,\quad \ldots$$

with coefficients that are functions of the variables (28). However, that means nothing more than the fact that equations (29) define a complete system that has at most $\varepsilon_0 + \ldots + \varepsilon_N$ parameters.

20. The question of whether the infinite group that is defined by (24) possesses N^{th} order differential invariants is now resolved when we succeed in proving that the complete system (29) possesses solutions.

Such solutions will always exist when the number of variables (28) that appear in (29) is larger than the number of mutually independent ones in equations (29). Now, we know that among equations (29) at most $\varepsilon_0 + \ldots + \varepsilon_N$ of them are mutually independent. On the other hand, we know that the numbers $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_N$ are independent of *l*, while we can, on the contrary, make the number of variables (28) as large as we want by a suitable choice of *l*. Thus, the complete system (29) always has a solution when we choose *l* in a suitable way.

If we now turn from the group (24) to the group that is defined by (22) then we obtain:

Theorem 9. *Any infinite group of infinitesimal transformations in the variables* $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ *possesses differential invariants in any event when one appends a certain number of variables* $\mathfrak{y}_1, \ldots, \mathfrak{y}_l$ *to the variables* $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ *that are not transformed by the group. If one has chosen l in a suitable way, and one considers that among the variables* $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$, $\mathfrak{y}_1, \ldots, \mathfrak{y}_l$, *some of them are functions of the other ones then all differential invariants of* N^{th} *order of the group can be found by integration of a complete system, in which the* $\mathfrak{x}$, *the* $\mathfrak{y}$, *and the differential quotients of the dependent ones with respect to the independent ones appear as variables.*

21. It is indeed self-explanatory, although it must still be mentioned, that the differential invariants of an infinite group of infinitesimal transformations also remain invariant under all one-parameter groups that are generated by the infinitesimal transformations of the group.

Moreover, one can also ask whether there are systems of differential equations of N^{th} order that remain invariant under the group that is defined by (22). Obviously, one finds systems of this type when one determines all systems of equations in the variables (28) that admit the infinitesimal transformations:

$$\bar{A}_\nu f\,, \quad \bar{B}_\mu f\,, \quad \bar{A}_{\nu\tau} f\,, \quad \bar{B}_{\mu\pi} f\,, \ldots$$

Admittedly, one must still examine whether the system of differential equations is integrable in the individual cases.

§ 11. Any infinite continuous group of finite transformation possess differential invariants.

22. We again think of an infinite group of finite transformations as being defined by a system of partial differential equations:

$$W_k\left(x_1, \cdots, x_n, \mathfrak{x}_1, \cdots, \mathfrak{x}_n, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \cdots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \cdots, \frac{\partial^2 \mathfrak{x}_1}{\partial x_1^2}, \cdots\right) = 0 \qquad (k = 1, 2, \ldots). \tag{30}$$

Let the defining equations of the infinitesimal transformations of these group be:

$$\sum_{\nu=1}^{n} \alpha_{k\nu}(x)\xi_\nu + \sum_{i,\nu=1}^{n} \alpha_{ki\nu}(x)\frac{\partial \xi_i}{\partial x_\nu} + \ldots = 0 \qquad (k = 1, 2, \ldots). \tag{31}$$

Among the assumptions that were made, the infinitesimal transformations that were defined by (31) define an infinite group of infinitesimal transformations, and naturally any differential invariant of the group that is defined by (30) is likewise a differential invariant of the group of infinitesimal transformations defined by (31). We assert that, conversely, any differential invariant of the latter group is also one of the former.

23. Let:

(32) $$\mathfrak{x}_i = F_i(x_1, \dots, x_n; \varepsilon) \qquad (i = 1, \dots, n)$$

be any family of ∞^1 transformations of group (30). This family yields the identity transformation for $\varepsilon = 0$, and the F_i might still be regular for $\varepsilon = 0$.

With the guidance of the previous paragraphs, we next define differential invariants of the infinite group of infinitesimal transformations (31). We then append certain variables $y_1, \dots, y_l$ to the $x_1, \dots, x_n$ that are not transformed under the group. Among the variables $x_1, \dots, x_n, y_1, \dots, y_l$, we consider $\mathfrak{n}_\nu$ of them, which might be called $z_1, \dots, z_n$, to be independent, while the remaining $n + l - \mathfrak{n} = \mathfrak{l}$ of them $u_1, \dots, u_\mathfrak{l}$ are functions of $z_1, \dots, z_n$. Let:

$$J(z_1, \dots, z_n, u_1, \dots, u_\mathfrak{l}, u_{1,1}, \dots, u_{\mathfrak{l},\mathfrak{n}}, u_{1,11}, \dots)$$

be any one of the differential invariants of N^{th} order that arise in this way.

Under the transformations (32), the variables $x_1, \dots, x_n, y_1, \dots, y_l$ will be transformed as follows:

$$\mathfrak{x}_i = F_i(x_1, \dots, x_n, \varepsilon), \quad \mathfrak{y}_\mu = y_\mu \qquad (i = 1, \dots, n;\ \mu = 1, \dots, l).$$

If we write these transformations in $z_1, \dots, z_n, u_1, \dots, u_\mathfrak{l}$ as:

(33) $$\begin{cases} \mathfrak{z}_i = \Psi_i(z_1, \cdots, z_\mathfrak{n}, u_1, \cdots, u_\mathfrak{l}, \varepsilon) \\ \mathfrak{u}_\mu = \mathrm{X}_\mu(z_1, \cdots, z_\mathfrak{n}, u_1, \cdots, u_\mathfrak{l}, \varepsilon) \end{cases} \qquad (i = 1, \dots, \mathfrak{n};\ \mu = 1, \dots, \mathfrak{l})$$

then it emerges from our assertion above that J should remain invariant under the ∞^1 transformations (33).

24. In order to prove that this is the case, we proceed as we did in the proof of Theorem 2 on pp. 330 [here, pp. 312].

We imagine that equations (33) have been solved for $z_1, \dots, z_n, u_1, \dots, u_\mathfrak{l}$, which might yield:

(34) $$\begin{cases} z_i = \bar{\Psi}_i(\mathfrak{z}_1, \cdots, \mathfrak{z}_\mathfrak{n}, \mathfrak{u}_1, \cdots, \mathfrak{u}_\mathfrak{l}, \varepsilon) \\ u_\mu = \bar{\mathrm{X}}_\mu(\mathfrak{z}_1, \cdots, \mathfrak{z}_\mathfrak{n}, \mathfrak{u}_1, \cdots, \mathfrak{u}_\mathfrak{l}, \varepsilon) \end{cases} \qquad (i = 1, \dots, \mathfrak{n};\ \mu = 1, \dots, \mathfrak{l}).$$

Furthermore, we define the equations:

$$\frac{d\mathfrak{z}_i}{d\varepsilon} = \left[\frac{\partial \Psi_i(z, u, \varepsilon)}{\partial \varepsilon}\right]_{z=\bar{\Psi}, u=\bar{X}} = \zeta_i(\mathfrak{z}, \mathfrak{u}, \varepsilon),$$

$$\frac{d\mathfrak{u}_\mu}{d\varepsilon} = \left[\frac{\partial X_\mu(z,u,\varepsilon)}{\partial\varepsilon}\right]_{z=\bar{\Psi},u=\bar{X}} = \omega_\mu(\mathfrak{z}, \mathfrak{u}, \varepsilon).$$

The expression:

$$Zf = \sum_{i=1}^{n} \zeta_i(z,u,\varepsilon)\frac{\partial f}{\partial z_i} + \sum_{\mu=1}^{\mathfrak{l}} \omega_i(z,u,\varepsilon)\frac{\partial f}{\partial u_\mu}$$

then represents an infinitesimal transformation for each ε, under which $z_1, \ldots, z_n, u_1, \ldots, u_{\mathfrak{l}}$ will be transformed by the infinite group (30) (cf., pp. 324-326 and Theorem III, pp. 342 [here, pp. 306-308 and pp. 322]. One then has:

$$Z^{(N)} J(z_1, \ldots, z_n, u_1, \ldots, u_{\mathfrak{l}}) \equiv 0.$$

If we now make the substitution (33) in the function:

$$J(\mathfrak{z}_1, \ldots, \mathfrak{z}_n, \mathfrak{u}_1, \ldots, \mathfrak{u}_{\mathfrak{l}}, \mathfrak{u}_{1,1}, \ldots)$$

then we obtain an equation of the form:

(35) $$J(\mathfrak{z}_1, \ldots, \mathfrak{z}_n, \mathfrak{u}_1, \ldots, \mathfrak{u}_{\mathfrak{l}}, \mathfrak{u}_{1,1}, \ldots) = X(\varepsilon; \mathfrak{z}_1, \ldots, \mathfrak{z}_n, \mathfrak{u}_1, \ldots, \mathfrak{u}_{\mathfrak{l}}, \mathfrak{u}_{1,1}, \ldots),$$

and it obviously becomes:

$$\frac{\partial X}{\partial\varepsilon} = \mathfrak{Z}^{(N)} J(\mathfrak{z}_1, \ldots, \mathfrak{z}_n, \mathfrak{u}_1, \ldots, \mathfrak{u}_{\mathfrak{l}}, \mathfrak{u}_{1,1}, \ldots) \equiv 0,$$

where $\mathfrak{Z}^{(N)}f$ is obtained from $Z^{(N)}$ when one replaces $z_1, \ldots, z_n, u_1, \ldots, u_{\mathfrak{l}}$ with the corresponding German symbols. From this, it follows that X is free of ε, and that the value of X will be found when one sets $\varepsilon = 0$ in equation (35).

For $\varepsilon = 0$, however, (33) goes to the identity transformation, so equation (35), which comes about by means of (33), has the simple form:

(36) $$J(\mathfrak{z}_1, \ldots, \mathfrak{z}_n, \mathfrak{u}_1, \ldots, \mathfrak{u}_{\mathfrak{l}}, \mathfrak{u}_{1,1}, \ldots) = J(z_1, \ldots, z_n, u_1, \ldots, u_{\mathfrak{l}}, u_{1,1}, \ldots).$$

In other words: The function $J(z, u, u_{1,1}, \ldots)$ remains invariant under each of the ∞^1 transformations (33).

25. With that, the aforementioned assertion is proved and we have that:

Theorem VI. *Any infinite continuous group whose finite transformations can be defined by a system of partial differential equations:*

$$W_k\left(x_1,\dots,x_n,\mathfrak{x}_1,\dots,\mathfrak{x}_n,\frac{\partial\mathfrak{x}_1}{\partial x_1},\dots\right)=0 \qquad (k = 1, 2, \dots) \tag{30}$$

simultaneously possesses differential invariants when one appends a certain number of variables $y_1, \dots, y_l$ *that are not transformed under the group to the* $x_1, \dots, x_n$.

In order to define such differential invariants, one must, for a suitable choice of l, consider any of the variables $x_1, \dots, x_n, y_1, \dots, y_l$ *to be functions of the remaining ones. The differential invariants are then functions of* $x_1, \dots, x_n, y_1, \dots, y_l$ *and the differential quotients of the independent variables with respect to the dependent ones. As differential invariants, they are characterized by the fact that they preserve their form under any transformation of the infinite group.*

The differential invariants in question may be defined, moreover, to be the differential invariants of the infinite group of infinitesimal transformations that are defined by the infinitesimal transformations of the group that is determined by (30). *Their number is unbounded, but all differential invariants of given order can be found by the integration of a certain complete system.*

§ 12. The infinite group of all point transformations in *n* variables.

26. From now on, our problem is to prove the converse of Theorem V, and thus to show that any infinite group of infinitesimal transformations consists of infinitesimal transformations of a certain infinite group of finite transformations. We will thus employ the fact that the defining equations of the finite transformations of an infinite group remain invariant, *in a certain sense*, under the infinitesimal transformations of this group (Theorem I, pp. 336 [here, pp. 317]).

Now, if an infinite group of infinitesimal transformations is given then we will show that a system of differential equations remains invariant under this group, which defines an infinite group of finite transformations, and indeed an infinite group whose infinitesimal transformations define precisely the given infinite group of infinitesimal transformations.

Before we commence the implied investigation in full generality, we would like to treat a particularly simple case. We will arrive at the result that will later essentially simplify the definition of the aforementioned invariant system of differential equations in the case of an arbitrary infinite group of infinitesimal transformations.

27. We consider the infinite group of all infinitesimal transformations:

$$Xf=\sum_{i=1}^{n}\xi_i(\mathfrak{x}_1,\dots,\mathfrak{x}_n)\frac{\partial f}{\partial\mathfrak{x}_i}$$

in the *n* variables $\mathfrak{x}_1, \dots, \mathfrak{x}_n$. This case is especially simple for the fact that we know from the outset that the group of infinitesimal transformations in question consists of all of the infinitesimal transformations of the infinite group of all finite transformations in $\mathfrak{x}_1, \dots, \mathfrak{x}_n$. From Theorem VI, the differential invariants of our infinite group of infinitesimal transformations are likewise the differential invariants of the infinite group of all finite transformations in $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$.

To the variables $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$, we append n more variables $x_1, \ldots, x_n$ that are not transformed by Xf at all. We regard $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ as functions of $x_1, \ldots, x_n$, and look for all functions of:

$$\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \qquad \frac{\partial \mathfrak{x}_i}{\partial x_\nu} = \mathfrak{x}_{i,\nu}, \qquad \frac{\partial^2 \mathfrak{x}_i}{\partial x_\nu \partial x_\tau} = \mathfrak{x}_{i,\nu\tau}, \ldots$$

that remain invariant under all infinitesimal transformations $X f$. These functions are then naturally certain differential invariants of the infinite group of all transformations in $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$.

28. It is now clear that there are no functions of only $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ that remain invariant under all infinitesimal transformations $X f$. The n infinitesimal transformations:

$$\frac{\partial f}{\partial \mathfrak{x}_1}, \ldots, \frac{\partial f}{\partial \mathfrak{x}_n}$$

then already possess no common invariant.

If we now extend $X f$ by the inclusion of the $\mathfrak{x}_{i,\nu}$ then we get:

$$X^{(1)} f = Xf + \sum_{i,\nu,\tau=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} \mathfrak{x}_{\nu,\tau} \frac{\partial f}{\partial \mathfrak{x}_{i,\tau}} .$$

Any function of $\mathfrak{x}_i$ and $\mathfrak{x}_{i,\nu}$ that remains invariant under all infinitesimal transformations $X f$ must therefore fulfill the equations:

$$\text{(37)} \qquad \frac{\partial f}{\partial \mathfrak{x}_1} = 0, \ldots, \frac{\partial f}{\partial \mathfrak{x}_n} = 0, \quad \sum_{\tau=1}^{n} \mathfrak{x}_{\nu,\tau} \frac{\partial f}{\partial \mathfrak{x}_{i,\tau}} = 0 \qquad (\nu, i = 1, \ldots, n).$$

The first n of them say that the function f is free of $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$; the last nn likewise show that it is free of the $\mathfrak{x}_{i,\nu}$, so the determinant:

$$\Delta = \sum \pm \mathfrak{x}_{1,1} \cdots \mathfrak{x}_{n,n}$$

does not vanish identically.

There is therefore also no differential invariant of first order with the desired behavior; on the contrary, the equation $\Delta = 0$ obviously represents a differential equation of first order that is invariant under our infinite group.

29. If we also include the $\mathfrak{x}_{i,\nu\tau}$ then we obtain the extended infinitesimal transformation:

$$X^{(2)}f = X^{(1)}f + \sum_{i,\nu,\tau=1}^{n} \frac{1+\varepsilon_{\nu\tau}}{2} \left\{ \sum_{\rho=1}^{n} \frac{\partial \xi_i}{\partial \mathfrak{x}_\rho} + \sum_{\rho,\pi=1}^{n} \frac{\partial^2 \xi_i}{\partial \mathfrak{x}_\rho \partial \mathfrak{x}_\pi} \mathfrak{x}_{\rho,\nu}\, \mathfrak{x}_{\pi,\tau} \right\} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu,\tau}},$$

from Xf, where, as usual, $\varepsilon_{\nu\tau}$ vanishes when $\nu \neq \tau$ and has the value 1 when $\nu = \tau$. This $\varepsilon_{\nu\tau}$ is introduced because one has $\mathfrak{x}_{i,\nu\tau} = \mathfrak{x}_{i,\tau\nu}$, and the differential quotient of f with respect to $\mathfrak{x}_{i,\nu\tau}$ may then be considered only once. Thus, should a function of the $\mathfrak{x}_i$, $\mathfrak{x}_{i,\nu}$, $\mathfrak{x}_{i,\nu\tau}$ remain invariant under all infinitesimal transformations Xf then it must satisfy the equations:

$$(38)\qquad \begin{cases} \dfrac{\partial f}{\partial \mathfrak{x}_1} = 0, \ \cdots \ \dfrac{\partial f}{\partial \mathfrak{x}_n} = 0, \\ \displaystyle\sum_{\tau=1}^{n} \mathfrak{x}_{\rho,\tau} \frac{\partial f}{\partial \mathfrak{x}_{i,\tau}} + \sum_{\nu,\tau=1}^{n} \frac{1+\varepsilon_{\nu\tau}}{2} \mathfrak{x}_{\rho,\nu\tau} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu\tau}} = 0, \\ \displaystyle\sum_{\nu,\tau=1}^{n} (1+\varepsilon_{\nu\tau})\, \mathfrak{x}_{\rho,\nu}\, \mathfrak{x}_{\pi,\tau} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu\tau}} = 0. \end{cases}$$

The equations in the last row of (38) may be written as follows:

$$\sum_{\nu=1}^{n} \mathfrak{x}_{\rho,\nu} \sum_{\tau=1}^{n} (1+\varepsilon_{\nu\tau})\, \mathfrak{x}_{\pi,\tau} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu\tau}} = 0,$$

so, since Δ does not vanish, they imply the equations:

$$\sum_{\tau=1}^{n} (1+\varepsilon_{\nu\tau})\, \mathfrak{x}_{\pi,\tau} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu\tau}} = 0.$$

On the same grounds, this, in turn, implies the equations:

$$\frac{\partial f}{\partial \mathfrak{x}_{i,\nu\tau}} = 0 \qquad (i,\ \nu,\ \tau = 1, \ldots, n),$$

which shows that f is free of $\mathfrak{x}_{i,\nu\tau}$. However, since we have already seen that there are no differential invariants of first order with the desired behavior, it is thus also proved that there are none of second order.

30. In order to prove in general that for any arbitrary N no differential invariants of N^{th} order exist with the desired behavior, we imagine that Xf has been extended by the inclusion of the differential quotients $\mathfrak{x}_i$, $\mathfrak{x}_{i,\nu}$, $\mathfrak{x}_{i,\nu\tau}$ up to N^{th} order. We can briefly write the infinitesimal transformation that comes about as:

(39) $$X^{(N)} f = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) \frac{\partial f}{\partial \mathfrak{x}_i} + \sum_{i,\nu=1}^{n} \frac{\partial \xi_i(\mathfrak{x})}{\partial x_\nu} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu}} + \cdots + \sum_{i,\nu=1}^{n} \frac{\partial^2 \xi_i(\mathfrak{x})}{\partial x_{\nu_1} \cdots \partial x_{\nu_N}} \frac{\partial f}{\partial \mathfrak{x}_{i,\nu}} \quad ,$$

where the summation symbols $\nu_1, \nu_2, \ldots, \nu_N$ must satisfy the conditions:

$$\nu_1 \leq \nu_2 \leq \nu_3 \leq \ldots \leq \nu_N .$$

If we imagine that the differentiations that were implied in (39) have been carried out then we get:

(40) $$X^{(N)} f = \sum_{i=1}^{n} \xi_i(\mathfrak{x}) A_i f + \sum_{i,\tau=1}^{n} \frac{\partial \xi_i(\mathfrak{x})}{\partial \mathfrak{x}_\tau} A_{i,\tau} f + \cdots + \sum_{i,\tau_1 \cdots \tau_N = 1}^{n} \frac{\partial^N \xi_i(\mathfrak{x})}{\partial \mathfrak{x}_1 \cdots \partial \mathfrak{x}_N} A_{i,\tau_1 \cdots \tau_N} f \ ,$$

where the $A_i f$, $A_{i,\tau} f$, … mean certain infinitesimal transformations in the variables $\mathfrak{x}_1$, $\mathfrak{x}_{i,\nu}$, …, and where $\tau_1, \ldots, \tau_N$ must satisfy the same conditions as $\nu_1, \ldots, \nu_N$ did above.

Now, should there be a function of the $\mathfrak{x}$ and its differential quotients up to N^{th} order that remains invariant under all infinitesimal transformations $X f$ then it must satisfy all of the equations:

(41) $$A_i f = 0, \quad A_{i,\tau} f = 0, \ \ldots, \ A_{i,\tau_1 \cdots \tau_N} f = 0.$$

However, it is easy to see that these equations are independent of each other and can only be satisfied when f is free of the $\mathfrak{x}$ and their differential quotients.

In fact, the $\mathfrak{x}_i$ are undetermined functions of $x_1, \ldots, x_n$. It thus suffices to verify that the equations (41) are then also independent of each other when one substitutes any well-defined functions of the x for the $\mathfrak{x}_i$ in their coefficients – say, $\mathfrak{x}_i = x_i$. However, if we make the substitution $\mathfrak{x}_i = x_i$ in the coefficients of (41), or, more precisely, the substitution:

(42) $$\mathfrak{x}_i = x_i , \qquad \mathfrak{x}_{i,\nu} = \varepsilon_{i\nu} , \qquad \mathfrak{x}_{i,\nu\tau} = \ldots = \mathfrak{x}_{i,\nu_1 \cdots \nu_N} = 0,$$

then this easily gives:

$$A_i f = \frac{\partial f}{\partial \mathfrak{x}_i} , \qquad A_{i,\tau} f = \frac{\partial f}{\partial \mathfrak{x}_{i,\tau}} , \ \ldots, \ A_{i,\tau_1 \cdots \tau_N} f = \frac{\partial f}{\partial \mathfrak{x}_{i,\tau_1 \cdots \tau_N}} .$$

One sees this immediately, when one carries out the substitution (42) on the coefficients of the two forms (39) and (40) of $X^{(N)} f$ and compares the two expressions obtained with each other, in which one must observe that $\xi_1, \ldots, \xi_n$ are arbitrary functions of their arguments.

With that, it is proved that equations (41) are independent of each other, and the vanishing of all differential quotients:

$$\frac{\partial f}{\partial \mathfrak{x}_i}, \quad \frac{\partial f}{\partial \mathfrak{x}_{i,\tau}}, \ldots, \frac{\partial f}{\partial \mathfrak{x}_{i,\tau_1 \cdots \tau_N}}$$

is implied by this; there are therefore also no differential invariants of N^{th} order with the desired behavior.

§ 13. The second fundamental theorem.

31. Now, let an arbitrary infinite group of infinitesimal transformations be given once more, and, in fact, by a system of partial differential equations:

$$\sum_{i=1}^{n} \alpha_{ki}(\mathfrak{x})\,\xi_i + \sum_{i,\nu=1}^{n} \alpha_{ki\nu}(\mathfrak{x})\frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} + \ldots = 0 \qquad (k = 1, 2, \ldots). \tag{43}$$

We assume that this system is of order q, and that no new equations of order q and less can be derived by differentiation and elimination. The most general system of solutions of (43) shall naturally not depend upon merely a finite number of arbitrary constants.

We understand N to mean any positive whole number, which can also be larger than q. If $N > q$ then we append to (43) all equations of order $(q + 1), \ldots, N$ that come about by differentiation. We imagine that the system thus obtained is then solved in a way that is similar to the one on pp. 367 [here, pp. 342], namely, for $m_0, m_1, \ldots, m_N$ of the differential quotients of order zero, one, …, N, resp., and indeed such that one always has that m_ν of the p_ν differential quotients of order n are expressible in terms of the remaining $p_\nu - m_\nu$ and ones of lower order.

From now on, we consider $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ to be functions of certain variables $x_1, \ldots, x_n$ that are not transformed at all, and look for all functions of:

$$\mathfrak{x}_i, \quad \frac{\partial \mathfrak{x}_i}{\partial x_\nu} = \mathfrak{x}_{i,\nu}, \quad \frac{\partial^2 \mathfrak{x}_i}{\partial x_\nu \partial x_\tau} = \mathfrak{x}_{i,\nu\tau}, \ldots$$

that remain invariant under all of the infinitesimal transformations that are defined by (43).

32. We extend the infinitesimal transformation:

$$X f = \sum_{i=1}^{n} \xi_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n)\frac{\partial f}{\partial \mathfrak{x}_i}$$

by the inclusion of all differential quotients $\mathfrak{x}_{i,\nu}, \ldots$ up to N^{th} order:

$$X^{(N)} f = \sum_{i=1}^{n} \xi_i(\mathfrak{x})\, A_i\, f + \sum_{i,\tau=1}^{n} \frac{\partial \xi_i(\mathfrak{x})}{\partial \mathfrak{x}_\tau} A_{i,\tau}\, f + \cdots + \sum_{i,\tau_1 \cdots \tau_N=1}^{n} \frac{\partial^N \xi_i(\mathfrak{x})}{\partial \mathfrak{x}_1 \cdots \partial \mathfrak{x}_N} A_{i,\tau_1 \cdots \tau_N}\, f\,. \tag{40}$$

In this infinitesimal transformation, we now express, by means of (43), all of the m_ν differential quotients of order ν of the ξ_i in terms of the $p_\nu - m_\nu$ remaining ones and the ones of lower order, and obtain an abbreviated infinitesimal transformation:

$$\bar{X}^{(N)} f = \sum_{i=1}^{n} \xi_i(\mathfrak{x})\,\bar{A}_i f + \sum_{i,\tau=1}^{n} \frac{\partial \xi_i(\mathfrak{x})}{\partial \mathfrak{x}_\tau}\,\bar{A}_{i,\tau} f + \cdots + \sum_{i,\tau_1\cdots\tau_N=1}^{n} \frac{\partial^N \xi_i(\mathfrak{x})}{\partial \mathfrak{x}_1 \cdots \partial \mathfrak{x}_N}\,\bar{A}_{i,\tau_1\cdots\tau_N} f ,$$

in which, of the ξ_i, only $p_0 - m_0$ of them enter in, while ultimately, of the differential quotients of order ν of the ξ_i, only $p_\nu - m_\nu$ of them enter in. The $\bar{A}_i f$, $\bar{A}_{i,\tau} f$, … are $A_i f$, $A_{i,\tau} f$, …, and in fact, one has:

$$(44) \qquad \begin{cases} \bar{A}_i f = A_i f + \sum\limits_{\mu} \varphi_{i\mu}(\mathfrak{x}) A_\mu f + \sum\limits_{\mu} \varphi_{i\mu\nu}(\mathfrak{x}) A_{\mu,\nu} f + \cdots \\ \bar{A}_{i,\nu} f = A_{i,\nu} f + \sum\limits_{\mu,\nu} \psi_{i\mu\nu}(\mathfrak{x}) A_{\mu,\nu} f + \sum\limits_{\mu,\nu,\pi} \psi_{i\mu\nu\pi}(\mathfrak{x}) A_{\mu,\nu\pi} f + \cdots \\ \qquad \cdots \end{cases}$$

Under the substitution (42), one will then have:

$$(45) \qquad \begin{cases} \bar{A}_i f = \dfrac{\partial f}{\partial \mathfrak{x}_i} + \sum\limits_{\mu} \varphi_{i\mu}(\mathfrak{x}) \dfrac{\partial f}{\partial \mathfrak{x}_\mu} + \sum\limits_{\mu} \varphi_{i\mu\nu}(\mathfrak{x}) \dfrac{\partial f}{\partial \mathfrak{x}_{\mu,\nu}} + \cdots \\ \bar{A}_{i,\nu} f = \dfrac{\partial f}{\partial \mathfrak{x}_{i,\nu}} + \sum\limits_{\mu,\nu} \psi_{i\mu\nu}(\mathfrak{x}) \dfrac{\partial f}{\partial \mathfrak{x}_{\mu,\nu}} + \sum\limits_{\mu,\nu,\pi} \psi_{i\mu\nu\pi}(\mathfrak{x}) \dfrac{\partial f}{\partial \mathfrak{x}_{\mu,\nu\pi}} + \cdots \\ \qquad \cdots \end{cases}$$

in which obviously no other differential quotients:

$$\frac{\partial f}{\partial \mathfrak{x}_{\mu,\nu}}$$

enter into the $\bar{A}_i f$ than the ones for which the $\bar{A}_{i,\tau} f$ are solved, and so on.

33. The differential invariants that we seek are obviously the common solutions to the $p_0 - m_0 + \ldots + p_N - m_N$ linear, partial differential equations:

$$(46) \qquad \bar{A}_i f = 0, \qquad \bar{A}_{i,\tau} f = 0, \; \ldots, \; \bar{A}_{i,\tau_1\cdots\tau_N} f = 0.$$

Of these equations, we already know that they determine a complete system (see pp. 369, *et seq.* [here, pp. 344, *et seq.*]). In our case, they are, moreover, obviously independent of each other, so they define a complete system of $p_0 - m_0 + \ldots + p_N - m_N$ parameters, and since they contain $p_0 + \dots + p_N$ independent variables, they have precisely $m_0 + \dots + m_N$ common independent solutions.

The statement is true for any value of N, so our infinite group of infinitesimal transformations possesses the following differential invariants with the desired behavior: Precisely m_0 independent differential invariants of order zero:

$$J_\mu^0(\mathfrak{x}_1, \dots, \mathfrak{x}_n) \qquad (\mu = 1, \dots, m_0),$$

precisely m_1 differential invariants of first order:

$$J_\mu^1(\mathfrak{x}_1, \dots, \mathfrak{x}_n, \mathfrak{x}_{1,1}, \dots, \mathfrak{x}_{n,n}) \qquad (\mu = 1, \dots, m_1)$$

that are independent of each other the J_μ^0, and ultimately, m_N differential invariants of N^{th} order that are independent of each other and the ones of lower order.

The differential invariants that we spoke of possess some important properties. Namely, first of all, one can derive arbitrarily many new differential invariants of higher order from any of them by simply differentiating. Secondly, as long as all differential quotients of order q are given, one can arrive at all differential invariants of $(q + 1)^{\text{th}}$ and higher order by differentiation. Finally, the totality of all differential invariants of order zero up to q (and higher) is only invariant under the infinitesimal transformations that are defined by (43), but none of them in the variables $\mathfrak{x}_1, \dots, \mathfrak{x}_n$.

34. The first property can be seen quite simply. Namely, if:

$$J(\mathfrak{x}_1, \dots, \mathfrak{x}_n, \mathfrak{x}_{1,1}, \dots)$$

is any differential invariant of order $N - 1$ with the given property then, as a function of the $x_1, \dots, x_n$, J satisfies the equation:

$$dJ - \sum_{\nu=1}^{n} \frac{\partial J}{\partial x_\nu} dx_\nu = 0, \tag{47}$$

where $\partial J : \partial x_\nu$ has the form:

$$\frac{\partial J}{\partial x_\nu} = \sum_{i=1}^{n} \frac{\partial J}{\partial \mathfrak{x}_i} \mathfrak{x}_{i,\nu} + \sum_{i,\tau=1}^{n} \frac{\partial J}{\partial \mathfrak{x}_{i,\tau}} \mathfrak{x}_{i,\tau\nu} + \dots$$

Now, if $X f$ is an arbitrary infinitesimal transformation of the group that is defined by (43) then $X^{(N)} J \equiv 0$, and likewise, one has:

$$X^{(N)}\left(\frac{\partial J}{\partial x_\nu}\right) \equiv 0,$$

so $X^{(N)}f$ must leave invariant the system of Pfaffian forms that is defined by equation (47) and the equations:

$$d\mathfrak{x}_i - \sum_{\nu=1}^{n} \mathfrak{x}_{i,\nu} dx_\nu = 0, \qquad d\mathfrak{x}_{i,\tau_1\cdots\tau_{N-1}} - \sum_{\nu=1}^{n} \mathfrak{x}_{i,\tau_1\cdots\tau_{N-1}\nu} dx_\nu = 0.$$

Thus, $\partial J / \partial x_\nu$ is actually a differential invariant of order N.

35. In order to prove the other two properties of our differential invariants, we must elaborate somewhat.

Equations (43) were soluble for $m_0 + \ldots + m_q$ of the ξ_i and their differential quotients of order one up to q; we would like to denote the quantities in question by:

$$(48) \qquad \xi_i, \qquad \frac{\partial \xi_i}{\partial \mathfrak{x}_{\mathfrak{n}}}, \ldots, \frac{\partial^q \xi_i}{\partial \mathfrak{x}_{\mathfrak{n}_1} \cdots \partial \mathfrak{x}_{\mathfrak{n}_q}}.$$

In addition, we would like to assume that the coefficients in the solutions of (43) are regular for $\mathfrak{x}_1 = 0, \ldots, \mathfrak{x}_q = 0$.

If we now choose $N = q$ then we can write equations (46) more precisely:

$$(50) \qquad \bar{A}_{\mathfrak{i}} f = 0, \qquad \bar{A}_{\mathfrak{i},\mathfrak{n}} f = 0, \ldots, \qquad \bar{A}_{\mathfrak{i},\mathfrak{n}_1\cdots\mathfrak{n}_q} f = 0,$$

where the indices $\mathfrak{i}$, $\mathfrak{n}$ must range through the same values as in (49). If we further recall the form (45) that the $\bar{A}_{\mathfrak{i}} f, \ldots$ assume under the substitution (42) then we recognize that the equations (50) can be solved for the differential quotients:

$$\frac{\partial f}{\partial \mathfrak{x}_{\mathfrak{i}}}, \quad \frac{\partial f}{\partial \mathfrak{x}_{\mathfrak{i},\mathfrak{n}}}, \ldots, \quad \frac{\partial f}{\partial \mathfrak{x}_{\mathfrak{i},\mathfrak{n}_1\cdots\mathfrak{n}_q}},$$

and that the coefficients in the equations thus solved are regular for the system of values:

$$\mathfrak{x}_\tau = 0, \quad \mathfrak{x}_{\tau,\pi} = \varepsilon_{\tau\pi}, \quad \mathfrak{x}_{\tau,\pi} = 0, \ldots \qquad (\tau, \pi, \rho, \ldots = 1, \ldots, n).$$

As solutions of the complete system (50), we can thus employ its $m_0 + \ldots + m_q$ principal solutions for:

$$(51) \qquad \mathfrak{x}_\tau = 0, \qquad \mathfrak{x}_{\tau,\pi} = \varepsilon_{\tau\pi}, \qquad \mathfrak{x}_{\tau,\pi} = 0, \ldots, \qquad \mathfrak{x}_{\mathfrak{i},\mathfrak{n}_1\cdots\mathfrak{n}_q} = 0.$$

This principal solutions are ordinary power series in the $n + p_1 + \ldots + p_q$ quantities:

$$\mathfrak{x}_1, \ldots, \mathfrak{x}_n, \quad \mathfrak{x}_{\tau,\pi} - \varepsilon_{\tau\pi}, \quad \mathfrak{x}_{\tau,\pi_1\pi_2}, \ldots, \quad \mathfrak{x}_{\tau,\pi_1\cdots\pi_q} \qquad (\tau, \pi, \pi_1, \ldots = 1, \ldots, n);$$

under the substitution (51), the series reduces to the quantities:

$$\mathfrak{x}_i\,,\ \mathfrak{x}_{i,\nu}\,,\ \ldots,\ \mathfrak{x}_{i,\nu_1\cdots\nu_q}\,,$$

where the indices i, n, … must range through the same values as in (48); for that reason, we would like to call them:

(52) $$J_i,\qquad J_{i,\nu}\,,\ \ldots,\ J_{i,\nu_1\cdots\nu_q}\,.$$

Thus, it is clear from the outset that the J_i are of order zero, the $J_{i,\nu}$ are of first order, and so on. Since, in fact, equations (46) define a complete system for any value of N then the principal solutions of the system (46) are likewise principal solutions of the system (50) in each of the cases $N = 0, 1, \ldots, q$. It is self-explanatory that all differential invariants of order zero up to q may also be expressed by the $m_0 + \ldots + m_q$ functions (52).

36. It is, moreover, easy to prove that the infinitesimal transformations that are defined by (43) are the only ones that leave each of the functions (52) invariant.

In fact, should the infinitesimal transformation:

$$Xf = \sum_{i=1}^{n} \xi_i(\mathfrak{x})\frac{\partial f}{\partial \mathfrak{x}_i}$$

leave any of the functions (52) invariant then it is necessary and sufficient that the expressions:

(53) $$X^{(q)}J_i\,,\quad X^{(q)}J_{i,\nu}\,,\ \ldots,\ X^{(q)}J_{i,\nu_1\cdots\nu_q}$$

all vanish identically for all values of the $n + p_1 + \ldots + p_q$ quantities:

$$\mathfrak{x}_1\,,\ \ldots,\ \mathfrak{x}_n,\ \mathfrak{x}_{\tau,\nu}\,,\ \ldots,\ \mathfrak{x}_{\tau,\pi_1\cdots\pi_q}\,,\qquad\qquad (\tau,\ \pi,\ \ldots = 1,\ \ldots,\ n).$$

A necessary condition for this is therefore that the expressions (53) vanish identically under the substitution:

(54) $$\mathfrak{x}_{\tau,\pi} = \varepsilon_{\tau\pi},\qquad \mathfrak{x}_{\tau,\pi_1\pi_2} = 0,\ \ldots,\ \mathfrak{x}_{\tau,\pi_1\cdots\pi_q} = 0\qquad (\tau,\ \pi,\ \pi_1,\ \ldots = 1,\ \ldots,\ n)$$

for all values of the $\mathfrak{x}_1\,,\ \ldots,\ \mathfrak{x}_n$. In other words: Xf leaves the functions (52) invariant only when $\xi_1, \ldots, \xi_n$ satisfy the $m_0 + \ldots + m_q$ equations:

(55) $$(X^{(q)}J_i) = 0,\ \ldots,\ (X^{(q)}J_{i,\nu_1\cdots\nu_q}) = 0,$$

where the parentheses imply that the substitution (54) has been carried out.

As would follow easily from the properties of the principal solutions (52), equations (55) now represent precisely $m_0 + \ldots + m_q$ independent linear, homogeneous differential equations for $\xi_1, \ldots, \xi_n$, and indeed these differential equations are soluble for the $m_0 + \ldots + m_q$ quantities:

(48) $$\xi_i, \qquad \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu}, \ldots, \frac{\partial^q \xi_i}{\partial \mathfrak{x}_{\nu_1} \cdots \partial \mathfrak{x}_{\nu_q}},$$

in precisely the same way as we imagined equations (43) being solved on pp. 383 [here, pp. 355]. On the other hand, we known that Xf always leaves the functions (52) invariant whenever $\xi_1, \ldots, \xi_n$ fulfill equations (43). We can also conclude that equations (55) are nothing but equations (43) in another form.

With that, we have proved that equations (43) define the most general infinitesimal transformation under which all of the functions (52) remain invariant.

37. Finally, we still have to prove that, of the m_q differential invariants of order q:

(56) $$J_{i,\nu_1 \cdots \nu_q},$$

one obtains precisely the required number by differentiation with respect to $x_1, \ldots, x_n$, namely, m_{q+1} such differential invariants of order $(q + 1)$ that are independent of each other and the differential invariants of lower order. This is also not difficult to do.

We know that differentiating (43) once with respect to $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ produces just enough equations of order $(q+ 1)$ for one to be able to solve for precisely m_{q+1} of the differential quotients of the ξ of order $(q + 1)$. Now, since the two systems (43) and (55) are equivalent, this implies that the $n \cdot m_q$ expressions:

(57) $$\frac{\partial}{\partial \mathfrak{x}_\tau}\left(X^{(q)} J_{i,\nu_1 \cdots \nu_q}\right) \qquad (\tau = 1, \ldots, n)$$

are independent of each other relative to m_{q+1} of the differential quotients:

(58) $$\frac{\partial^{q+1} \xi_i}{\partial \mathfrak{x}_{\nu_1} \cdots \partial \mathfrak{x}_{\nu_{q+1}}}.$$

On the other hand, it is clear that in the $n \cdot m_q$ expressions:

(59) $$\left(\frac{\partial}{\partial x_\tau} J_{i,\nu_1 \cdots \nu_q}\right) \qquad (\tau = 1, \ldots, n),$$

the coefficient of:

(60) $$\mathfrak{x}_{i,\tau_1 \cdots \tau_{q+1}}$$

is, in each case, the same as the coefficient of the differential quotients (58) in the expression that corresponds to (57). With that, the expressions (59) are independent of each other relative to m_{q+1} of the quantities (60). However, from this it follows that among the $n \cdot m_q$ differential invariants of order $(q+ 1)$:

$$\frac{\partial}{\partial x_\tau} J_{i,\nu_1\cdots\nu_q},$$

one actually finds precisely m_{q+1} of them that are independent of each other and the ones of lower order.

38. With these preparations, we are finally in a position to achieve the goal that set in the beginning of § 12.

We now directly seek a system of partial differential equations:

$$W_k(x_1, \ldots, x_n, \mathfrak{x}_1, \ldots, \mathfrak{x}_n, \mathfrak{x}_{1,1}, \ldots) = 0 \qquad (k = 1, 2, \ldots) \tag{61}$$

that defines the finite transformations of a certain infinite group, and indeed this group shall include all of the infinitesimal transformations that are defined by (43), but no other ones.

It is clear from the outset that the differential equations (61), when they exist, must be satisfied by the identity transformation $\mathfrak{x}_i = x_i$, so the substitution:

$$\mathfrak{x}_i = x_i, \qquad \mathfrak{x}_{i,\tau} = \varepsilon_{i\tau}, \qquad \mathfrak{x}_{i,\tau\nu} = 0, \ldots, \tag{62}$$

when applied to (61), must yield nothing but identities. From the developments on pp. 353-359 [here, pp. 331-336], it emerges, moreover, that equations (61) must be of order q and that there must be precisely $m_0 + \ldots + m_q = s$ of them that are independent of each other.

39. From Theorem I, pp. 336 [here, pp. 317], the system of differential equations (61) must admit each of the infinitesimal transformations that are defined by (43). The system of equations (61) in the $n + p_0 + \ldots + p_q$ variables x_i, $\mathfrak{x}_i$, $\mathfrak{x}_{i,\tau}$, …, assumes that one has substituted the most general system of solutions of (43) in them for the $\xi_1, \ldots, \xi_n$. Thus, it is, in turn, necessary and sufficient that the system of equations (61) admit the:

$$p_0 - m_0 + \ldots + p_q - m_q$$

infinitesimal transformations:

$$\bar{A}_i f, \quad \bar{A}_{i,\tau} f, \ldots, \bar{A}_{i,\tau_1\cdots\tau_q} f$$

that are defined on pp. 380 [here, pp. 353].

We now recall the properties of these infinitesimal transformations, and imagine that the system of equations (61) must be fulfilled identically under the substitution (62), so we see that the system of equations (61) must be expressed by relations among the solutions of the complete system:

$$\bar{A}_i f = 0, \qquad \bar{A}_{i,\tau} f = 0, \ldots, \bar{A}_{i,\tau_1\cdots\tau_q} f = 0.$$

The complete system [certainly] possesses precisely $n + s$ independent solutions, namely, first of all, $x_1, \ldots, x_n$, and secondly, the s differential invariants (52), which we would like to briefly call $J_1, \ldots, J_s$. Now, since the system (61) must contain precisely s independent equations, and since it certainly yields no relation between just $x_1, \ldots, x_n$, it must necessarily be soluble for $J_1, \ldots, J_s$, and thus have the form:

$$J_k\left(\mathfrak{x}_1,\ldots,\mathfrak{x}_n,\frac{\partial\mathfrak{x}_1}{\partial x_1},\ldots,\frac{\partial\mathfrak{x}_n}{\partial x_n},\ldots\right) = \alpha_k(x_1, \ldots, x_n) \qquad (k = 1, \ldots, s), \tag{63}$$

where the α_k refer to the functions of $x_1, \ldots, x_n$ that arise from J_k by the substitution (62).

40. The equations (63) represent a system of differential equations order q for $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$. Due to the properties of the differential invariants $J_1, \ldots, J_s$, it is certain that no equation of order q or less can be derived from (63) by differentiation and elimination that does not already follow from (63) with no differentiation. Therefore, equations (63) define a family of finite transformations that includes the identity transformation. We assert that this family defines a group, and that the group in question nothing other than the infinitesimal transformations that are defined by (43).

Let:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{64}$$

and:

$$\mathfrak{x}'_i = \mathfrak{F}_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad (i = 1, \ldots, n) \tag{65}$$

be two arbitrary transformations of the family that is defined by (63). One then proves that the transformation:

$$\mathfrak{x}'_i = \mathfrak{F}_i(F_1(x), \ldots, F_n(x)) \qquad (i = 1, \ldots, n) \tag{66}$$

always belongs to this family, as well.

41. By the assumptions that were made, the $\mathfrak{x}_i$ satisfy the differential equations (63) as functions of $x_1, \ldots, x_n$, and the $\mathfrak{x}'_i$ satisfy the differential equations:

$$J_k\left(\mathfrak{x}'_1,\ldots,\mathfrak{x}'_1,\frac{\partial\mathfrak{x}'_1}{\partial\mathfrak{x}_1},\ldots,\frac{\partial\mathfrak{x}'_n}{\partial\mathfrak{x}_n},\ldots\right) = \alpha_k(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad (k = 1, \ldots, s) \tag{63$'$}$$

as functions of $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$. Thus, as functions of $x_1, \ldots, x_n$, the $\mathfrak{x}'_i$ satisfy certain differential equations:

$$U_k\left(x_1,\ldots,x_n,\mathfrak{x}'_1,\ldots,\mathfrak{x}'_1,\frac{\partial\mathfrak{x}'_1}{\partial\mathfrak{x}_1},\ldots,\frac{\partial\mathfrak{x}'_n}{\partial\mathfrak{x}_n},\ldots\right) = \alpha_k(F_1(x), \ldots, F_n(x)) \qquad (k = 1, \ldots, s) \tag{67}$$

that one obtains from (63′) from the substitution:

(68) $$\mathfrak{x}_i = F_i(x), \qquad \frac{\partial \mathfrak{x}_i'}{\partial \mathfrak{x}_\nu} = \sum_{\tau=1}^{n} \frac{\partial \mathfrak{x}_i'}{\partial x_\tau}\frac{\partial x_\tau}{\partial \mathfrak{x}_\nu}, \ldots,$$

in which one must, however, imagine that the differential quotients of the x_τ with respect to the $\mathfrak{x}_\nu$ are expressed in terms of $x_1, \ldots, x_n$ by means of (64).

The system (67) naturally consists of s independent equations and has the same order as the system (63). Ostensibly, its form depends upon the choice of transformation (64). However, we will see that in reality it does not depend upon it.

42. The system (63′) obviously admits all infinitesimal transformations:

$$X'f = \sum_{i=1}^{n} \xi_i(\mathfrak{x}_1', \ldots, \mathfrak{x}_n')\frac{\partial f}{\partial \mathfrak{x}_i'}$$

that belong to the family that was defined by (43). Moreover, it will be satisfied by the substitution $\mathfrak{x}_i' = x_i$ identically. From this, it follows that the system (67) likewise admits all of the infinitesimal transformations $X'f$ in question, and that under the substitution:

(69) $$\mathfrak{x}_i' = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

it is fulfilled identically. The system (67) is now completely determined by these two properties, so it must, in fact, possess the earlier form:

$$J_k\left(\mathfrak{x}_1', \ldots, \mathfrak{x}_1', \frac{\partial \mathfrak{x}_1'}{\partial \mathfrak{x}_1}, \ldots, \frac{\partial \mathfrak{x}_n'}{\partial \mathfrak{x}_n}, \ldots\right) = \beta_k(x_1, \ldots, x_n) \qquad (k = 1, \ldots, s).$$

The functions β_k are thus to be chosen such that one obtains nothing but identities under the substitution (69); in other words: $\beta_k = \alpha_k$.

Thus, the system (67) may be brought into the form:

(70) $$J_k\left(\mathfrak{x}_1', \ldots, \mathfrak{x}_1', \frac{\partial \mathfrak{x}_1'}{\partial \mathfrak{x}_1}, \ldots, \frac{\partial \mathfrak{x}_n'}{\partial \mathfrak{x}_n}, \ldots\right) = \alpha_k(x_1, \ldots, x_n) \qquad (k = 1, \ldots, s),$$

which is independent of the special choice of the transformation (64). However, with this, it is proved that the transformation (66) always belongs to the family that is defined by (63), so this family defines a group, as we asserted.

43. Finally, it is not difficult to show that the group that is defined by (63) includes all of the infinitesimal transformations that are defined by (43), but no other ones.

In fact, from Theorem I, pp. 336 [here, pp. 317], the transformation:

$$\mathfrak{x}_i = F_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n)$$

belongs to the group that is defined by (63) when and only when the system of partial differential equations (63) remains invariant under the transformation:

(71) $$\mathfrak{x}'_i = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n), \qquad x'_i = x_i \qquad (i = 1, \ldots, n).$$

For this to be true, it is, in turn, obviously necessary and sufficient that the transformation (71) leaves invariant each of the *s* functions:

$$J_k\left(\mathfrak{x}_1, \ldots, \mathfrak{x}_1, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \ldots\right) \qquad (k = 1, \ldots, s).$$

Now, we have seen on pp. 385 [here, pp. 357] that the infinitesimal transformations that are defined by (43) are the only ones that leave all functions $J_1, \ldots, J_s$ invariant. Thus, we can conclude that these infinitesimal transformations are the only ones that belong to the group that was defined by (63).

44. We can now express the second fundamental theorem in the theory of infinite groups. It reads as follows:

Theorem VII. *Let a system of s independent linear, homogeneous, partial differential equations be given:*

(72) $$\sum_{i=1}^{n} \alpha_{ki}(\mathfrak{x})\, \xi_i + \sum_{i,\nu=1}^{n} \alpha_{ki\nu}(\mathfrak{x}) \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu} + \ldots = 0 \qquad (k = 1, \ldots, s)$$

that possesses the following properties: It shall be of order q and shall yield no equations of order q or less by differentiation and elimination that are independent of (72). *Furthermore, its most general system of solutions shall not merely depend upon a finite number of arbitrary constants. Finally, if* $\xi_1, \ldots, \xi_n$ *and* $\eta_1, \ldots, \eta_n$ *are two solutions of* (72) *then:*

$$\sum_{\nu=1}^{n}\left(\xi_\nu \frac{\partial \eta_i}{\partial \mathfrak{x}_\nu} - \eta_\nu \frac{\partial \xi_i}{\partial \mathfrak{x}_\nu}\right) \qquad (i = 1, \ldots, n)$$

is a system of solutions.

By these assumptions, the system (72) *defines the most general infinitesimal transformation:*

$$Xf = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial \mathfrak{x}_i}$$

of a certain infinite group. The finite transformations of this group are determined by s independent partial differential equations of order q of the form:

(73) $$J_k\left(\mathfrak{x}_1, \ldots, \mathfrak{x}_1, \frac{\partial \mathfrak{x}_1}{\partial x_1}, \ldots, \frac{\partial \mathfrak{x}_n}{\partial x_n}, \ldots\right) = \alpha_k(x_1, \ldots, x_n) \qquad (k = 1, \ldots, s),$$

which go to just identities under the substitution $\mathfrak{x}_i = x_i$; *moreover, the functions* $J_1, \ldots, J_s$ *have the property that they remain invariant under any transformation:*

$$\mathfrak{x}'_i = F_i(\mathfrak{x}_1, \ldots, \mathfrak{x}_n) \qquad\qquad (i = 1, \ldots, n)$$

that belongs to the infinite group. If the equations (72) *are given then one can find equations* (73) *by integrating a complete system.*

45. It is self-explanatory that the infinite group contains all one-parameter groups that are generated by the infinitesimal transformations that are defined by (72), although it contains no one-parameter group. Whether, conversely, any of its transformations belong to one of the one-parameter groups that were mentioned, and whether it therefore consists of nothing but one-parameter groups is still undecided (cf. pp. 344 [here, pp. 323, *et seq.*, no. 31]).

A similar theorem for infinite groups of contact transformations of an $(n + 1)$-fold extended space $z, x_1, \ldots, x_n, p_1, \ldots, p_n$ follows from Theorem VII with no further assumptions. Certain linear, homogeneous, partial differential equations appear instead of equations (72) that preserve an unknown function W and $2n + 1$ independent variables $z, x_1, \ldots, x_n, p_1, \ldots, p_n$. The most general solution W of these equations is then the characteristic function of the most general infinitesimal contact transformation that belongs to the group of contact transformations in question.

Finally, let it be emphasized that the foregoing developments deliver a new basis for the theory of finite continuous groups (cf., pp. 320, *et seq.* [here, pp. 304, no. 6]).

§ 14. Concluding remarks.

46. The foundations of the theory of infinite groups is thus laid. I will now say a few words about the special problems of this theory that I have already resolved.

The concepts of transitive and intransitive may be carried over to infinite groups immediately. If the defining equations of the infinitesimal transformations of an infinite group are given then one can always decide whether the group is intransitive or not, and, in addition, one can also determine all systems of equations that are invariant under the group, and indeed, by forming the determinants and integrating complete systems.

In the same way, one can systematically and asystatically (*asystatisch*) carry over the concepts of primitive and imprimitive to infinite groups. The concepts of connection, isomorphism, simple group, invariant subgroup, and derived group may be applied to infinite groups.

Already in the year 1883, I published a determination of all infinite groups of point transformations in the plane (Ges. d. Wiss. zu Christiania [this collection, v. V, Abh. XIII]). In recent times, I have further determined all infinite, irreducible groups of contact transformations of the plane, and an important class of groups of this type in the space of $n + 1$ dimensions. Likewise, I have arrived at the description of all primitive, infinite groups of point transformations of the ordinary space and at least a certain class of groups of this type in the space of n dimensions.

47. The investigations that were mentioned here shall be published in the third part of my theory of transformation groups. The general theory that was detailed in the present work shall appear there in a revamped form. For that reason, the present treatise, like the first treatise of Herrn Professor Engel, was developed by taking a manuscript of mine as its basis.