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Contributions to the general theory of transformations

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Translated by D. H. Delphenich

In what follows, permit me to communicate several theories of transformations to the Kgl. Gesellschaft der Wissenschaften that I have developed thoroughly in my lectures at the University of Leipzig.

I.

1. All rotations that leave the coordinate origin $x = 0$, $y = 0$, $z = 0$ fixed define a three-parameter transformation group whose infinitesimal transformations:

$$yp - xq = X_1 f, \qquad zq - yr = X_2 f, \qquad xr - zp = X_3 f$$

fulfill my known relations:

$$(X_1\, X_2) = X_3 f, \qquad (X_2\, X_3) = X_1 f, \qquad (X_3\, X_1) = X_2 f.$$

The three expressions X_1, X_2, X_3 simultaneously define a homogeneous function group. Among the functions of that group, there are infinitely many of them that are *homogeneous of degree one* in p, q, r., namely, all of them that have the form:

$$X_3\, \Phi\left(\frac{X_1}{X_3},\ \frac{X_2}{X_3}\right).$$

All of these first-degree functions can be regarded as characteristic functions of infinitesimal contact transformations, and indeed – as I have often stressed – all of the transformations:

$$X = X_3\, \Phi\left(\frac{X_1}{X_3},\ \frac{X_2}{X_3}\right)$$

define an *infinite* group of contact transformations that includes the group of rotations X_1, X_2, X_3 as a finite subgroup.

2. Now, on the other hand, there are infinitely many contact transformations:

$$H = r\,W\left(x, y, z, \frac{p}{r}, \frac{q}{r}\right),$$

which commute with the rotations X_1, X_2, X_3, and thus fulfill the relations:

$$(X_1\,H) = 0, \qquad (X_2\,H) = 0, \qquad (X_3\,H) = 0,$$

as well as the homogeneity condition:

$$p\frac{\partial H}{\partial p} + q\frac{\partial H}{\partial q} + r\frac{\partial H}{\partial r} - H = 0.$$

Among these quantities H, we point out the three quantities:

$$\begin{aligned} xp + yq + zr &= H_1, \\ \sqrt{p^2 + q^2 + r^2} &= H_2, \\ \sqrt{(xq - yp)^2 + (yr - zq)^2 + (zp - xr)^2} &= H_3, \end{aligned}$$

in particular. The most general quantity H possesses the form:

$$H = H_3\,\Phi\left(\frac{H_1}{H_3}, \frac{H_2}{H_3}\right).$$

Here, H is the most general symbol of all infinitesimal contact transformations that commute with all rotations X_1, X_2, X_3, and as a result, likewise commute with all contact transformations X.

3. As a matter of fact, all H generate an infinite transformation group, while all X determine another such group. These two groups of contact transformations relate to each other in such a way that any transformation of the one group commutes with any transformation of the second group.

On the other hand, as was already pointed out, the functions X_1, X_2, X_3 determine another function group, and likewise the functions H_1, H_2, H_3 determine another function group. With my usual terminology, these two function groups are *reciprocal* function groups. Every function in the one group is in involution with every function of the second group.

However, when *two* reciprocal *three*-parameter function groups are present in the variables x, y, z, p, q, r, from my older studies, there is always a function that belongs to the two groups and appears in both groups as a *distinguished* function. In the present case, that quantity will have the form:

$$\sqrt{(xq-yp)^2+(yr-zq)^2+(zp-xr)^2} = H_3 ,$$

and since it is homogeneous of degree one, it can then be regarded as the characteristic function of an infinitesimal contact transformation.

4. The infinitesimal contact transformation that is thus found, as well as the finite transformation of the associated one-parameter group, possesses some remarkable properties. Indeed, one has the equations:

$$(X_1\, H_3) = 0, \quad (X_2\, H_3) = 0, \quad (X_3\, H_3) = 0,$$

as well as the analogous ones:

$$(H_1\, H_3) = 0, \quad (H_2\, H_3) = 0, \quad (H_3\, H_3) = 0.$$

As a result, H_3 is in involution with every quantity Ω that has the form:

$$\Omega = W\,(X_1, X_2, X_3, H_1, H_2).$$

The infinitesimal, like the finite, transformations of the one-parameter group of H_3 then commute:

1. With all rotations around the coordinate origin.
2. With all spiral transformations that fix the coordinate origin.
3. With all dilatations.
4. With all finite and infinitesimal base-point transformations.
5. With all finite and infinitesimal contact and point transformations that commute with all rotations around the coordinate origin.
6. Absolutely all finite and infinitesimal transformations of that *infinite* group of contact transformations whose characteristic function possesses the general form:

$$H_3\, W\left(\frac{X_1}{H_3}, \frac{X_2}{H_3}, \frac{X_3}{H_3}, \frac{H_1}{H_3}, \frac{H_2}{H_3}\right).$$

One must observe in this that only four of the five arguments of the function W are independent.

A set of quantities remains invariant under the infinitesimal and finite transformations of the one-parameter group H_3; in particular, the quantities:

$$\sqrt{x^2+y^2+z^2}\,,$$
$$\sqrt{p^2+q^2+r^2}\,,$$

$$\frac{xp + yq + zr}{\sqrt{x^2 + y^2 + z^2} \cdot \sqrt{p^2 + q^2 + r^2}},$$

as well as any quantity at all that can be expressed as a function of X_1, X_2, X_3, H_1, H_2, H_3.

5. It seems superfluous to go further into the various consequences of these theorems. Then again, I shall remark that it possible to write down the *aequationes directrices* that will yield all finite transformations of our group.

Two *aequationes directrices* belong to each such transformation, one of which will always have the form:

$$\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2 = x^2 + y^2 + z^2.$$

The other equation will have the general form:

$$(x\,\mathfrak{x} + y\,\mathfrak{y} + z\,\mathfrak{z})^2 - m^2\,(x^2 + y^2 + z^2)\,(\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2) = 0,$$

in which one assumes that m denotes a constant.

It is easy to interpret these equations geometrically. It shows that every point (x, y, z) goes around *a circle* whose point $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ has the same distance from the coordinate origin as the point (x, y, z) itself. On the other hand, the *radii vectores* to (x, y, z) and $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z})$ define an angle that has a constant value for every individual transformation of our one-parameter group, namely:

$$\text{arc cos } m.$$

The concentric spheres:

$$x^2 + y^2 + z^2 = \text{const.}$$

remain invariant under the transformations of our group; the *line element* of each individual sphere will be transformed by a *dilatation* under it.

6. The one-parameter group of contact transformations that we speak of here includes a transformation that has been known for some time, but has generally not been regarded as a contact transformation before. Namely, if one sets $m = 0$ then the corresponding *aequationes directrices*:

$$\mathfrak{x}^2 + \mathfrak{y}^2 + \mathfrak{z}^2 - x^2 - y^2 - z^2 = 0, \; = 0, \qquad x\,\mathfrak{x} + y\,\mathfrak{y} + z\,\mathfrak{z}$$

will determine a a contact transformation that takes every surface into a so-called *apsidal surface.*

I therefore refer to this contact transformation as the *apsidal transformation;* correspondingly, I shall refer to the infinitesimal contact transformation:

$$\sqrt{(xq-yp)^2+(yr-zq)^2+(zp-xr)^2}$$

as the *infinitesimal apsidal transformation.*

I have no doubt that the developments that were given here will permit one to simplify and complete the investigations into apsidal transformations that were made up to now in an essential way.

5. The foregoing developments have a close connection with a theory whose origin goes back to **Monge**.

Any first-order partial differential equation of the form:

$$\frac{X_2}{X_3} = \varphi\left(\frac{X_1}{X_3}\right)$$

is, in fact, *in involution* with any equation of the form:

$$\frac{H_2}{H_3} = \psi\left(\frac{H_1}{H_3}\right).$$

These two equations thus represent intermediate integrals, and indeed, general intermediate integrals of a certain second-order partial differential equation whose geometric interpretation can be found rather easily. The last intermediate integral can, in fact, be brought into the form:

$$\frac{xp+yq+zr}{\sqrt{x^2+y^2+z^2}\cdot\sqrt{p^2+q^2+r^2}} = \Omega\,(x^2+y^2+z^2)\ ;$$

one can thus define the associated integral surface by the fact that its lines of curvature of the family lie on the concentric spheres $x^2 + y^2 + z^2 = c$. Therefore, according to **Monge**, the lines of curvature of the second family are straight and lie in planes that go through the coordinate origin.

8. If one now endows the arbitrary functions φ and ψ with forms that are determined by the intermediate integral equations above then, as was remarked already, the first-order equations that emerge:

$$\frac{X_2}{X_3} - \varphi_0\left(\frac{X_1}{X_3}\right) = 0, \qquad \frac{H_2}{H_3} - \psi_0\left(\frac{H_1}{H_3}\right) = 0$$

will be in involution. If one would wish to find ∞^1 common integral surfaces then one would have to exhibit an integrability factor. This leads one, in the simplest way, to the remark that, from the previous developments, the *infinitesimal apsidal transformation* will transform *the desired* ∞^1 *surfaces between each other.* That remark is noteworthy due to the fact that it can also be regarded as a direct consequence of my general theory of the integrable **Monge-Ampère** equation ([1]).

It can be further added that the developments above show that many interesting contact transformation will take any surface whose lines of curvature lie on concentric spheres to other such spheres ([2]).

II.

9. Any three-parameter transformation group X_1, X_2, X_3 of space (x, y, z) can serve as the starting point for theories that are completely analogous to the ones that were just developed. That is the case, in particular, when the three characteristic functions X_1, X_2, X_3 in question are independent and are not pair-wise in involution.

The three infinitesimal point transformations:

$$\begin{aligned} X_1 f &= q + xr, \\ X_2 f &= yq + zr, \\ X_3 f &= y\,(xp + yq + zr) - zp \end{aligned}$$

define a three-parameter transformation group; these transformations commute with the three transformations:

$$\begin{aligned} H_1 f &= p + yr, \\ H_2 f &= xp + zr, \\ H_3 f &= x\,(xp + yq + zr) - zp, \end{aligned}$$

which generate a simply-transitive three-parameter group in their own right. I refer to two three-parameter transformation groups of space x, y, z that have the given relationship as *reciprocal transformation groups.*

However, on the other hand, the X_1, X_2, X_3 and H_1, H_2, H_3 define two three-parameter and homogeneous function groups in the variables:

$$x, y, z, p, q, r.$$

I refer to these function groups, in turn, as reciprocal groups.

From my general theory, these function groups contain a common *distinguished* function, namely:

([1]) Cf., Nor. Arch., Bd. II, (1877). [this coll., v. III, art. XIX, pp. 287, *et seq.*]

([2]) On several occasions, I have already referred to the fact that geometrical optics takes on simplicity and clarity by the explicit introduction of the concepts of infinitesimal contact transformation and one-parameter group of contact transformations.

In this way of looking at things, reflections are contact transformations that leave the infinitesimal contact transformation in question invariant. One can make an analogous statement for refractions when the wave surfaces in both media are similar and lie similarly to each other. If the wave surfaces in the two media are different then one must regard refraction as a contact transformation that takes a certain one-parameter group into a certain other one.

$$\begin{aligned} &\sqrt{(q+xr)(y(xp+yq+zr)-zp)-(yq+zr)^2} \\ = &\sqrt{(p+yr)(x(xp+yq+zr)-zq)-(xp+zr)^2} \\ = &\sqrt{(xy-z)(pq+r(xp+yq+zr))} \ = \Omega. \end{aligned}$$

10. Now, the first-order partial differential equation:

$$\frac{X_2}{X_3} - \varphi\left(\frac{X_1}{X_3}\right) = 0, \qquad \frac{H_2}{H_3} - \psi\left(\frac{H_1}{H_3}\right) = 0,$$

with the arbitrary functions φ and ψ, in turn, determines an important second-order partial differential equation:

$$\frac{rt-s^2}{(1+p^2+q^2)^2} = Q\,(x, y, z, p, q)$$

that I have considered on various occasions. The integral surfaces of this second-order equations can be characterized by the fact that all of their principal tangent curves belong to linear complexes; hence, all of these linear complexes define two bundles in involution.

If we now endow the arbitrary functions φ and ψ with specific forms – say, φ_0 and ψ_0 – then the corresponding first-order equations:

$$\frac{X_2}{X_3} - \varphi_0\left(\frac{X_1}{X_3}\right) = 0, \quad \frac{H_2}{H_3} - \psi_0\left(\frac{H_1}{H_3}\right) = 0$$

will always admit the infinitesimal contact transformation Ω, and we thus find the associated ∞^1 common integral surfaces by a quadrature in all cases.

11. The infinitesimal transformation Ω generate an important one-parameter group of contact transformations whose finite equations will be found in such a way that one can consider the equation:

$$0 = (z - xy)\,(z_1 - x_1\,y_1) - m\,(z + z_1 - x\,y_1 - x_1\,y)^2,$$

which is known from the theory of polars, to be the *aequatio directrix*. If I am not mistaken, the theorem that the ∞^1 contact transformations that emerge in that way – *among which, one finds the transformation by reciprocal polars – define a one-parameter group* has not come to the attention of mathematicians up to now.

12. By geometric considerations, one easily recognizes (cf., for example, my paper in the Math. Ann., Bd. V [this coll., v. II, art. I]) that any partial differential equation:

$$\frac{X_2}{X_3} - \varphi\left(\frac{X_1}{X_3}\right) = 0$$

will determine all non-rectilinear surfaces whose principal tangents belong to the one family of a line complex that can be regarded as the enveloping complex of ∞^1 linear line complexes.

We shall not go further into these considerations at this point, although they are interesting in their own right.

13. However, the following remark might find its place here: The question of whether every finite contact transformation belongs to a one-parameter group of contact transformations has not been taken up in earnest in my publications up to now. Under the circumstances, it does not seem uninteresting to me that two one-parameter groups were presented in the foregoing, among which, one of them subsumed the transformation by reciprocal polars, while the other one subsumed the apsidal transformation.

By considering the projective group of a twisted third-degree curve, one finds, in an entirely similar way, a one-parameter group that subsumes the duality with respect to a linear line complex.

III.

14. I consider r independent functions $u_1, u_2, \ldots, u_r$ of the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$, and thus refer to the totality of all quantities of the form:

$$\varphi(u_1, \ldots, u_r)$$

as an r-parameter *family of functions.*

Amongst all possible r-parameter families of functions $u_1, \ldots, u_r$, I concerned myself exclusively with certain families that I referred to as *function groups* in my own studies of partial differential equations and contact transformations.

If one is given an r-parameter function group $v_1, \ldots, v_r$ then the linear partial differential equation:

$$(V(v_1, v_2, \ldots, v_r), f) = 0$$

will always possess $r - 1$ (or even r) independent solutions that themselves belong to the function group.

Now, the function groups are the only families of functions that possess the aforementioned property. Namely, if we take any *homogeneous* function group $\omega_1, \omega_2, \ldots, \omega_s$ whose brackets $(\omega_i\ \omega_k)$ do not all vanish identically then the group will contain $s - 1$ independent functions of degree zero that, from my older studies, will define an $(s - 1)$-parameter family of functions that likewise possesses the stated property.

It is easy to construct many more general families of functions that likewise possess that property. Namely, if one performs an entirely arbitrary contact transformation on a homogeneous function group that has the form:

$$\begin{aligned} x'_k &= X_k(x_1, \ldots, x_n, p_1, \ldots, p_s), \\ p'_i &= P_i(x_1, \ldots, x_n, p_1, \ldots, p_s), \\ z' &= \text{const. } z + W(x_1, \ldots, p_s) \end{aligned}$$

then the transforms of the functions of degree zero in our group will always yield a family of functions that possess the desired property.

15. We will now pose the problem of finding all families of functions $u_1, \ldots, u_r$ in the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$ that have the property that every linear partial differential equation:

$$(U(u_1, \ldots, u_r), f) = 0$$

possess $r - 1$ (or even r) independent solutions that belong to the same family of functions as the u.

As one recognizes almost immediately, this general problem coincides with the following one:

Problem. *Which families of functions $u_1, \ldots, u_r$ possess the property that all bracket expressions $(u_i\, u_k)$ have the form:*

$$(u_i\, u_k) - \rho \cdot \varphi_{ik}(u_1, \ldots, u_r),$$

in which ρ denotes a quantity that is independent of the u?

We make the assumption that ρ should be independent of the u, since otherwise the quantities u would define a function group; however, all function groups were determined by me in my older papers and were brought into simple canonical forms.

16. If we now address our problem then we will find it convenient to first deal with the special case of $r = 4$.

There now exist six relations of the form:

$$(u_i\, u_i) = \rho\, \varphi_{ik}(u_1, \ldots, u_r) \qquad (i \neq 1, 2, 3, 4;\ k = 1, 2, 3, 4;\ i \neq k).$$

The identity:

$$((u_1\, u_2)\, u_3) + ((u_2\, u_3)\, u_1) + ((u_3\, u_1)\, u_2) = 0$$

will then give the relation:

$$0 = \varphi_{12}\cdot(\rho\, u_3) + \varphi_{23}\cdot(\rho\, u_1) + \varphi_{31}\cdot(\rho\, u_2) + \rho^2 \sum_s \left(\frac{\partial\varphi_{12}}{\partial u_s}\varphi_{s3} + \frac{\partial\varphi_{23}}{\partial u_s}\varphi_{s1} + \frac{\partial\varphi_{31}}{\partial u_s}\varphi_{s2}\right),$$

which is linear in the $(\rho\, u_k)$, while the factor of ρ^2 depends upon only the u.

If we then cyclically permute the indices 1, 2, 3, 4 then we will obtain four equations for the determination of the four quantities $(\rho\, u_1)$, $(\rho\, u_2)$, $(\rho\, u_3)$, $(\rho\, u_4)$ that have the form:

$$\begin{array}{llllcl}
\varphi_{23}(\rho\, u_1) + & \varphi_{31}(\rho\, u_2) + & \varphi_{12}(\rho\, u_3) + & & = & \psi_{123}(u)\,\rho^2, \\
& \varphi_{34}(\rho\, u_2) + & \varphi_{42}(\rho\, u_3) + & \varphi_{23}(\rho\, u_4) & = & \psi_{234}(u)\,\rho^2, \\
\varphi_{34}(\rho\, u_1) & + & \varphi_{41}(\rho\, u_3) + & \varphi_{31}(\rho\, u_4) & = & \psi_{341}(u)\,\rho^2, \\
\varphi_{24}(\rho\, u_1) + & \varphi_{41}(\rho\, u_2) & + & \varphi_{12}(\rho\, u_4) & = & \psi_{412}(u)\,\rho^2.
\end{array}$$

The [answer to the] question of whether these equations can be solved for the $(\rho\, u_k)$ comes down to the determinant:

$$\begin{vmatrix} \varphi_{23} & \varphi_{31} & \varphi_{12} & 0 \\ 0 & \varphi_{34} & \varphi_{42} & \varphi_{23} \\ \varphi_{34} & 0 & \varphi_{41} & \varphi_{13} \\ \varphi_{24} & \varphi_{41} & 0 & \varphi_{12} \end{vmatrix} = - \begin{vmatrix} 0 & \varphi_{12} & \varphi_{13} & \varphi_{14} \\ \varphi_{21} & 0 & \varphi_{23} & \varphi_{24} \\ \varphi_{31} & \varphi_{32} & 0 & \varphi_{34} \\ \varphi_{41} & \varphi_{42} & \varphi_{43} & 0 \end{vmatrix}.$$

If the skew four-rowed determinant $|\,\varphi_{ik}\,|$ is non-zero then one will find expressions of the form:

$$(\rho\, u_i) = \rho^2\,\psi_i\,(u_1, \ldots, u_4)$$

by solving for the $(\rho\, u_i)$, and correspondingly:

$$\left(\frac{1}{\rho}, u_i\right) = -\,\psi_i\,(u_1, \ldots, u_4).$$

The formulas show that the five quantities u_1, u_2, u_3, u_4, ρ generate a five-parameter function group and that there is a contact transformation in the x, p that takes that function group into a *homogeneous* group whose functions of degree zero are the transforms of the u.

17. We now turn to the general case, but assume that it is possible to choose *four* of the r quantities u – say, u_α, u_β, u_γ, u_δ – such that the determinant of the φ, which is the analogue of the skew determinant above, but with the indices α, β, γ, δ, instead of 1, 2, 3, 4, does not vanish identically.

We then recognize just as before that the four quantities $(\rho\, u_\alpha)$, $(\rho\, u_\beta)$, $(\rho\, u_\gamma)$, $(\rho\, u_\delta)$ are expressed as products of ρ^2 and some function of the u. Therefore, if say $\varphi_{\alpha\beta} \neq 0$, as we can assume, then we will recognize upon the forming the identity:

$$((u_\alpha\, u_\beta)\, u_i) + ((u_\beta\, u_i)\, u_\alpha) + ((u_i\, u_\alpha)\, u_\beta) = 0,$$

which assumes the form:

$$\varphi_{\alpha\beta}\,(\rho\, u_i) + \varphi_{\beta i}\,(\rho\, u_\alpha) + \varphi_{i\,\alpha}\,(\rho\, u_\beta) + \rho^2\,\Theta(u) = 0,$$

that all $(\rho\, u_i)$ can be expressed as follows:

$$(\rho\, u_i) = \rho^2\,\psi_i\,(u_1, \ldots, u_r)\,.$$

From this, we conclude, as before, that the quantities $u_1, \ldots, u_r$, r define an $(r + 1)$-term group that goes to a homogeneous group whose functions of degree zero are the transforms of the u under a contact transformation.

18. By contrast, if *all* four-rowed determinants that can be defined analogously to the above determinant in the quantities φ then the considerations above will no longer be valid. In this case, the quantities $u_1, \ldots, u_r$, ρ do not need to define a group.

At this point, we confine ourselves to the remark that the discussion of the case that was excluded here, but is very interesting in its own right, is based upon the theory of Pfaffian systems. I will take up the questions that were left unresolved here on another occasion.

IV.

19. Since I have still not found the time to thoroughly edit the discussion of my general theory of the integration of a complete system with known infinitesimal transformations, I will show here how I have treated the most interesting (because it is simplest) of these general problems in my lectures. Those who know of my general theories will see with no difficulty that all cases can be dealt with by entirely analogous considerations.

20. I assume that an equation $Af = 0$ admits certain known infinitesimal transformations $X_1 f$, $X_2 f$, …, as well as assuming that certain solutions φ_1, φ_2, … of $Af = 0$ are given. I further assume that relations of the form:

$$(X_i\, X_k) = \psi_{ik1}\,(\varphi)\, X_1\, f) + \psi_{ik2}\,(\varphi)\, X_2\, f) + \ldots,$$
$$X_i\,\varphi_k = \omega_{ik}\,(\varphi_1, \varphi_2, \ldots)$$

exist such that no further transformations and solutions can be derived.

I can now assume, with no *essential* restriction, that $Af = 0$ is given in the form:

$$Af = \frac{\partial f}{\partial \varphi_k} + \sum \alpha_i\,(z, \varphi_1, \ldots, \varphi_q, y_1, y_2, \ldots)\frac{\partial f}{\partial y_i} = 0,$$

and that the $X_k f$ are given in the form:

$$\Phi_k f = \frac{\partial f}{\partial \varphi_k} + \sum \psi_{ki}(z, \varphi_1, \ldots, \varphi_q, y_1, y_2, \ldots) \frac{\partial f}{\partial y_i},$$

$$Y_\nu f = \qquad \sum \eta_{\nu i}(z, \varphi_1, \ldots, \varphi_q, y_1, y_2, \ldots) \frac{\partial f}{\partial y_i}.$$

21. We now restrict ourselves to the case in which the number of y, as well as the number of Y, is equal to 3.

Here, there surely exist relations of the form:

$$(A\ \Phi_k) = 0, \qquad (A\ Y_\nu) = 0,$$
$$(\Phi_i\ \Phi_k) - \sum \omega_{iks}(\varphi)\ Y_s f, \qquad (\Phi_i\ Y_k) - \sum v_{iks}(\varphi)\ Y_s f,$$
$$(Y_i\ Y_\nu) = \sum u_{iks}(\varphi)\ Y_s f.$$

We can then set:

$$(Y_1\ Y_2) = Y_1, \qquad (Y_1\ Y_3) = 2Y_2, \qquad (Y_2\ Y_3) = Y_3,$$

with no restriction, since the cases that can be solved by quadratures are of no interest to us here.

When we introduce suitable expressions:

$$\Phi_k f + \sum v_{ki}(\varphi)\ Y_i f$$

in place of $\Phi_k f$ as new $\Phi_k f$, we will find that all $(\Phi_i\ Y_k)$ vanish:

$$(\Phi_i\ Y_k) = 0.$$

If we then form the identity:

$$((\Phi_i\,\Phi_k)\ Y_j) + ((\Phi_k\ Y_j)\ \Phi_i) + ((Y_j\ \Phi_i)\ \Phi_k) = 0$$

then we will recognize that all of the $((\Phi_i\,\Phi_k)\ Y_j)$, and at the same time, all of the $(\Phi_i\,\Phi_k)$, are equal to zero.

22. There then now exist the relations:

$$(A\ \ \Phi_k) = 0, \qquad (A\ \ Y_\nu) = 0,$$
$$(\Phi_i\,\Phi_k) = 0, \qquad (\Phi_i\,Y_k) = 0,$$
$$(Y_1\,Y_2) = Y_1, \quad (Y_1\,Y_3) = 2Y_2, \quad (Y_2\,Y_3) = Y_3.$$

If we then denote the generally unknown solutions of the complete system:

$$Af = 0, \qquad \Phi_1 f = 0, \ldots, \quad \Phi_q f = 0$$

by:

$$y_1', \; y_2', \; y_3',$$

and then introduce the quantities:

$$z, \; \varphi_1, \ldots, \varphi_q, \; y_1', \; y_2', \; y_3'$$

as new independent variables then one will get:

$$\begin{aligned} Af &= \frac{\partial f}{\partial z} + \sum \alpha_i'(y_1', y_2', y_3') \frac{\partial f}{\partial y_i'} = 0, \\ \Phi_k f &= \frac{\partial f}{\partial \varphi_k}, \\ Y_k f &= \sum \eta_{vi}'(y_1', y_2', y_3') \frac{\partial f}{\partial y_i'}, \end{aligned}$$

and it will become immediately obvious here that a further reduction would be impossible.

Therefore, my old theory of integration proves to be the best possible one in the present case.