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On complexes – in particular, line and sphere complexes – with applications to the theory of partial differential equations

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I.

The rapid development of geometry in our century is, as is well-known, intimately linked with philosophical arguments about the essence of Cartesian geometry, arguments that were set down in their most general form by Plücker in his early papers. For anyone who proceeds in the spirit of Plücker's work, the thought that one can employ every curve that depends on three parameters as a space element will convey nothing that is essentially new, so if no one, as far as I know, has pursued this idea then I believe that the reason for this is that no one has ascribed any practical utility to that fact. I was led to study the aforementioned theory when I discovered a remarkable transformation that represented a precise connection between lines of curvature and principal tangent curves, and it is my intention to summarize the results that I obtained in this way in the following treatise.

In the first section, I concern myself with curve complexes – that is, manifolds that are composed of a three-fold infinitude of curves. All surfaces that are composed of a single infinitude of curves from a given complex satisfy a partial differential equation of second order that admits a partial differential equation of first order as its singular first integral. In this way, I obtain a new geometric interpretation for partial differential equations of first order that presents a certain interest, especially when the complex in question is a Plücker line complex. Just as the general equation $F(x, y, z, X, Y, Z) = 0$, as the *aequatio directrix*, determines a reciprocity in space, I show that the simultaneous system of equations:

$$F_1(x, y, z, X, Y, Z) = 0, \qquad F_2(x, y, z, X, Y, Z) = 0$$

also establishes a correspondence between the surface elements of two spaces. These two types of transformations, together with all point transformations, are the only spatial deformations for which contact is an invariant relation, and thus all such transformations consist of either an exchange of space elements or the introduction of a new coordinate system. Finally, I consider the application of such transformations to partial differential equations.

In the second section, I assume that the equations $F_1 = 0$, $F_2 = 0$ are linear with respect to both systems of variables, and indeed I examine, in particular, the system:

$$- Zz = x - (X + iY), \qquad (X - iY)\, z = y - Z$$

and the spatial structures that are determined by it. It thus transforms the lines in the space of (x, y, z) into spheres in the second space – i.e., all surface elements that contain two consecutive points of a line go over to the elements of a sphere. Lines that intersect then map to spheres that contact. *On this, I base an exact and – in my opinion – fundamental connection between line geometry and sphere geometry, and, as a consequence, between several projective and metric theories, a connection whose main interest resides in the fact that the principal tangent curves occupy the same position in the former geometry as the lines of curvature do in the latter.* This theory gives the determination of the principal tangent curves for particular surfaces, especially, the Kummer surfaces of fourth order with sixteen nodes. Finally, I establish that the most general transformation of the space of (X, Y, Z) for which, on the one hand, contact is an invariant relation, and on the other, lines of curvature are covariant curves, corresponds to my map of the general linear or dualistic transformation of the space of (x, y, z). All of these transformations can be composed from reciprocal radii and parallel transformations (dilatations).

In the third section I determine, with applications, the following concepts: Line and sphere complex, line and sphere congruence, all partial differential equations of first and second order whose characteristics are principal tangent curves or lines of curvature on the integral surfaces. Among the aforementioned equations of second order, I arrive at all of them that possess a general first integral relative to two given general first integrals, and I then show that when a general first integral exists, it can always be constructed, as well as the associated singular first integral. Here, I come to the well-known investigations of surfaces whose lines of curvature are planar or spherical, surfaces that possess a given spherical map, etc.

In the fourth section, I finally give some theorems that are connected with the foregoing discussion.

While I was concerned with these developments, I was in active communication with F. Klein, whom I must thank for many of the ideas that emerged, and more than is possible for me to cite.

I also remark that the theory presented here has some points of contact with my earlier investigations into the representation of the imaginaries. If I do not present my current understanding of this connection here then this is because, on the one hand, I find it, to some degree, casual, and on the other hand, because I do not wish to deviate from the customary language of mathematics. [*])

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[*]) The most essential viewpoints and results of this treatise can be found in a brief note (October, 1870) and in two larger works (1871) that were recorded in the Verhandlungen der Akademie zu Christiana. I communicated the connection between principal tangent curves and lines of curvature as being like the one between line geometry and sphere geometry to that society in July, 1870. Cf., also the Comptes rendus of October, 1870, as well as a note that Klein and myself published in the Monatsberichten der Berliner Akademie, 15 December 1870.

First section

On a new reciprocity in space

In the first two paragraphs of this section, I give a brief overview of some well-known theories, in order to simplify the understanding of paragraph 4. This latter paragraph gives all of the assumptions that are necessary in order to be able to understand the second section; I will employ the following paragraphs in the third and fourth sections, as well.

§ 1.

Reciprocity between two planes or two spaces

1. The Poncelet-Gergonne theory of reciprocity in the plane can, as is known, be developed from the equation:

$$X(a_1x + b_1y + c_1) + Y(a_2x + b_2y + c_2) + (a_3x + b_3y + c_3) = 0, \tag{1}$$

or, what amounts to the same thing:

$$x(a_1X + a_2Y + aZ) + y(b_1X + b_2Y + b_3) + (c_1X+ c_2Y + c_3) = 0,$$

assuming that one interprets x, y and X, Y as the Cartesian point-coordinates of two planes.

Namely, if one refers to two points whose coordinate values (x, y) and (X, Y) satisfy equation (1) as *conjugate* then one can say that the points that are conjugate to a point define a line in the second plane, and we regard the latter as corresponding to the given point. All points of a line have a common conjugate point in the second plane, and thus their corresponding lines go through this common point.

The two planes will thus be related to each other by (1)*, in such a way that the lines of each plane will be mapped to the points of the other one. The points of a line λ thus correspond to the lines that go through the image point of λ.* This reciprocal relationship is the fundamental principle of the aforementioned reciprocity.

Now, let a polygon be given in the one plane and in the other one, a polygon whose sides map to the vertices of the other one; from the foregoing, it is then clear that the vertices of the latter polygon also correspond to the sides of the former one, so the two polygons have a reciprocal relationship. From these polygons, one obtains two curves upon passing to the limit, which, as one discovers, are reciprocal to each other relative to the equation (1). It is obvious that the tangents of each curve are mapped to the reciprocal points of the other one.

2. Plücker *) based a generalization of the theory that we just developed on the interpretation of the general equation:

(2) $$F(x, y, X, Y) = 0.$$

The points (X, Y) that are conjugate to points (x, y) now define a curve C that is represented by (2), if one regards the (x, y) as a parameter and (X, Y) as the running coordinates; conversely, a point (X, Y) defines conjugate points (x, y) on a curve c, which will likewise be represented by (2). The two planes will thus be related to each other by (2) in such a way that the points of the one plane correspond to the curves of a net of curves (C or c) in the second one. Exactly as before, one sees that the points of one curve C correspond to those curves c that go through the image points of C.

A curve polygon $(C_1, C_2, \ldots, C_n)$ corresponds to a point-system $(p_1, p_2, \ldots, p_n)$, and obviously these points lie pairwise (p_1, p_2), (p_3, p_4), … on those curves c whose image points are vertices of the given curvilinear polygon. By passing to the limit, one also obtains curves here that correspond to such curves c or C that envelope the other. Thus, the reciprocity relationship is not complete in general. Namely, if Σ is the enveloping curve of all C that are mapped to as the points of a curve σ then certainly the given curve σ envelops the c that correspond to the points of Σ, but in general it is a second curve.

3. Plücker *) based the general reciprocity between two spaces r and R on the general equation:

$$F(x, y, z, X, Y, Z) = 0.$$

In particular, when F is linear with respect to both systems of variables, one obtains the Poncelet-Gergonne reciprocity between the points and planes of two spaces.

It is now my purpose in this treatise, and in the first section, in particular, to consider a new reciprocity that relates to Plücker's, which is determined by the system of equations:

$$F_1(x, y, z, X, Y, Z) = 0, \qquad\qquad F_2(x, y, z, X, Y, Z) = 0,$$

assuming that one regards (x, y, z) *and* (X, Y, Z) *as point-coordinates of two spaces* r *and* R.

§ 2.

Curves that depend upon three parameters can be introduced as space elements

4. The transformation of geometric theorems that are based on the Poncelet-Gergonne or Plücker reciprocity theory can, as Gergonne and Plücker have suggested, be

*) Analytisch geometrische Entwicklungen. v. 1, second edition.

*) I believe it is correct to attribute this reciprocity to Plücker, although I can point to no place in Plücker's works where he explicitly laid out the reciprocity of spaces in this way.

considered from a higher viewpoint, which we will likewise give. The same is true for our new reciprocity.

The Cartesian analytic geometry translates arbitrary geometric theorems into algebraic ones, and thus makes of the geometry of the plane, a tangible representation of the algebra of two variables, and likewise, of the geometry of space, a representation of an algebra that pertains to three variable quantities. Now, Plücker especially directed his remarks to the fact that the Cartesian geometry includes a two-fold arbitrariness.

Descartes represented a value system of two variables x and y by a *point* in the plane; he has, as one cares to say, chosen the point to be the element of plane geometry, while one can just as well use the line or even an arbitrary two-parameter curve as the element. *However, one can, as is known, regard the transformation that mediates the Poncelet-Gergonne reciprocity as based on the transition from points as elements to lines as elements, and likewise the Plücker reciprocity in the plane consists, to some extent, in the transition from the point as element to a curve that depends upon two parameters as element.*

Furthermore, Descartes represented the value system (x, y) by those points in the plane whose distance from two given lines – viz., the coordinate axes – were equal to x and y, respectively; among the unbounded number of possible coordinate systems, he chose a particular one.

It is known that the advance of geometry in our century was essentially based on the fact that the two aforementioned sources of arbitrariness in the Cartesian representation were clearly regarded as such, and it is therefore a closely related task for us to attempt to take still more from this bountiful well.

5. The new theory that is presented in the following relates to the fact that *one can employ any space curve that depends upon three parameters as a space element.* If one recalls, e.g., that the equations of a space line include four essential constants then one sees that one can choose the straight lines that satisfy one condition to be the elements of a space geometry that gives a tangible representation of an algebra with three variables. However, since a system of lines – viz., the Plücker line complex – will be distinguished by this, it is obvious that a definite representation of the given type can find only limited application. Meanwhile, when one is engaged in the study of space relative to a line complex, it can therefore be very profitable to employ the lines of this complex as space elements.

In *metric geometry*, one distinguishes, for example, the infinitely distant imaginary circle and, as a result of this, the complex of imaginary lines that intersect it, *and it can thus seem obvious, a priori, that if one is treating certain metric problems then it is worthwhile to introduce these imaginary lines as elements.*

Although we have just said, by way of example, that it is possible to choose the lines of a line complex to be space elements, we must nonetheless remark that this is something quite different – something more particular, if one will – from the ideas that Plücker developed in his last work: "Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement." Plücker had already *) directed one's attention to the fact that it is possible to create a representation of an algebra with

*) Geometrie des Raumes, no. 258 (1816).

arbitrarily many variables if one introduces a structure that includes just as many parameters as a space element. In particular, he suggested that the line in space possesses four coordinates, so one obtains a geometry of a manifold with *four* dimensions when one considers the line to be the element of space.

§ 3.

Curve complexes. New geometric interpretation of partial differential equations of first order. Principal tangent curves of a line complex.

6. Following Plücker, one calls the totality of all lines that are subject to *one* condition, and which then depend upon *three* essential constants, a *line complex*. By analogy to this, in the following I will refer to any system of curves c whose equations include three independent parameters as a *curve complex*. The equations:

(1) $$f_1(x, y, z, a, b, c) = 0, \qquad f_2(x, y, z, a, b, c) = 0,$$

in which a, b, c are constants, thus represent a curve complex. By differentiation of these two equations with respect to x, y, z and eliminating the quantities a, b, c one obtains a resultant:

(2) $$f(x, y, z, dx, dy, dz) = 0,$$

which, if dx, dy, dz are regarded as the determining elements of a direction, associates each point of space with a cone, namely, the totality of all directions of all curves of the complex c that go through the point in question. I call this cone the *elementary cone of the complex;* likewise, I employ the expression *elementary direction of the complex* in order to indicate an arbitrary line element (dx, dy, dz) of a curve of the complex c. *The totality of all elementary directions of the complex that correspond to a point defines the elementary cone of the complex associated with the point in question.*

A given system (1) – or, as one can also say, a given curve complex – corresponds to a certain differential equation ($f = 0$); on the other hand, there are unboundedly many systems (1) that lead to the same differential equation ($f = 0$). Namely, if one chooses an arbitrary relation:

$$y(x, y, z, dx, dy, dz) = 0,$$

in which α denotes a constant, and if:

(3) $$\varphi_1(x, y, z, \alpha, \beta, \gamma) = 0, \qquad \varphi_2(x, y, z, \alpha, \beta, \gamma) = 0,$$

are the integrals of simultaneous systems:

$$f = 0, \quad \psi = 0$$

then it is clear that equations (3) also give, by differentiation with respect to x, y, z and elimination of the quantities α, β, γ, the result:

$$f = 0.$$

There thus exist unboundedly many curve complexes whose equations satisfy a given relation $f(x, y, z, dx, dy, dz) = 0$. Any curve of such a complex will envelop curves c of the original complex, for which all their elements are directions of the complex.

7. According to Monge, a partial differential equation of first order between the variables x, y, z is equivalent to the problem: Find the general surface that contacts a cone at each of its points according to an arbitrary rule. Thus, when one regards x, y, z as parameters in the given partial differential equation:

$$F(x, y, z, p, q) = 0,$$

while p and q are the plane coordinates, one can say that $F = 0$ represents the general equation of the aforementioned cone system in plane coordinates. Lagrange and Monge have reduced the integration of a partial differential equation of first order to the determination of a certain curve complex – the so-called *characteristics* – by showing that when a single infinitude of characteristics envelope a curve they always generate an integral surface. One must remark that the differential equation of the characteristics:

$$f(x, y, z, dx, dy, dz) = 0,$$

can be regarded as equivalent to the partial differential equation; namely, both equations give the analytical definition of the aforementioned cone system.

8. Now, let a partial differential equation of first order D_1 be given, along with the associated differential equation of the characteristics $f = 0$, and finally, a triply infinite number of curves c that likewise satisfy $f = 0$. Here, if we consider an arbitrary integral surface U_0, a characteristic k_0 that lies on it, and furthermore, a curve of the complex c_0 that contacts k_0, then I assert that c_0 *contacts the surface* U_0 *at three points.*

Namely, if we call the two common, infinitely close points of the curves c_0 and k_0, p_1 and p_2, and furthermore, the subsequent points of our curves π and p_3, resp., then it is clear that the elementary cone of the complex whose vertex lies at p_2 includes the surface element (p_2, p_3, π). Now, the cone contacts the integral surface U_0 along the generators p_2 , p_3 and thus the element (p_2, p_3, π) likewise belongs to the surface U_0, which thus includes the point π. Our assertion is thus proved.

This theorem may be generalized by saying that when c_0 has n consecutive points points in common with the characteristic k_0, $(n + 1)$ intersection points of c_0 and the integral surface U_0 coincide. [*])

9. The requirement that a surface $z = F(x, y)$ should have three consecutive points in common with a curve of the complex c at each point can be expressed by a differential equation of second order D_2 [**]), and indeed one easily sees that a single infinitude of c

[*]) Corresponding theorems are true for a space with an arbitrary number of dimensions.
[**]) This differential equation has the following form:

always defines an integral surface. One thus knows the general second integral of the equation D_2 with two arbitrary functions. Now, the theorem of last paragraph says that the integral surfaces of the equations D_1 – which, in general, will not be generated by curves of the complex c – will contact three points of a c at each of its points, and thus D_1 is a singular first integral of our differential equation of second order. *I assert that there can be no other singular integral.*

Namely, let ϑ be an integral surface of the equation D_2 that is not composed of curves c. Through each point of the surface there thus go two coincident curves c that contact it at these points. From this, it follows that the associated elementary cone of the complex contacts the surface, which thus satisfies the same equation D_1. The last two paragraphs give the following noteworthy geometric interpretation for partial differential equations of first order between three variables.

The problem: Find all surfaces that have three consecutive points in common with a two-fold infinitude of curves of a given complex and express it by means of a partial differential equation of first order. All complexes whose equations satisfy a given relation:

$$f(x, y, z, dx, dy, dz) = 0$$

lead to the same partial differential equation, and indeed the associated characteristics satisfy the equation $f = 0$.

10. *Corollary. The problem: Find all surfaces whose two-fold infinitude of principal tangents of a system belong to a given line complex and lead to a partial differential equation of first order whose characteristics will be enveloped by lines of the complex. In this case, the characteristics are principal tangent curves of the one system on the integral surfaces* *).

I will suggest how one can prove this theorem independently.

The partial differential equation of first order whose characteristics are enveloped by the lines of a line complex is, as is known, an expression of the following problem: Find the general surface that contacts the cone of the complex in question at all of its points. Now, the osculating plane of a curve whose tangents belong to a line complex is a tangent plane to the corresponding cone of the complex, and thus, in our case, the osculating planes of the characteristics always contact the associated integral surfaces. As a result, the characteristics are principal tangent curves.

Each line complex thus determines a three-fold infinitude of curves that will be enveloped by lines of the complex, and thus, possess the property that there are principal tangent curves each surface that includes a single infinitude of such curves, assuming that two consecutive curves among them always intersect. I will call these curves, which will play an important role in the theory of line complexes (cf., the third section of this treatise), *the principal tangent curves of the line complex.*

Klein has remarked to me that the lines of the complex that Plücker referred to as singular lines of the complex were principal tangent curves of it. Were the complex

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$$r + 2Ns + N^2t + M = 0,$$

where N and M depend upon x, y, z, p, q.

*) Darboux, to whom I communicated this theorem in 1870, knew this at the time. Cf., the beginning and conclusion of part two.

composed of the tangents of a surface or of lines that intersect a curve then all lines of the complex would be singular lines and thus the principal tangent curves of the complex, as well.

§ 4.

The system of equations $F_1(x, y, z, X, Y, Z) = 0$, $F_2(x, y, z, X, Y, Z) = 0$ determines a reciprocity between two spaces.

11. We now begin (cf., § 1) a study of the spatial reciprocity that is determined by the two equations:

$$F_1(x, y, z, X, Y, Z) = 0, \quad F_2(x, y, z, X, Y, Z) = 0, \tag{1}$$

when (x, y, z) and (X, Y, Z) are interpreted as the point coordinates of two spaces r and R *).

If one refers to two points (x, y, z) and (X, Y, Z) whose coordinate values satisfy the equations (1) as *conjugate* then one can say that the points (X, Y, Z) that are conjugate to a point (x, y, z) define a curve C; it will be represented by (1), assuming that one regards the x, y, z as parameters and the X, Y, Z, by comparison, as running coordinates. The points of the space r thus correspond to a three-fold infinitude of curves C, and likewise a curve complex appears in the space r whose curves c are in the same relation to the points of the space R. The points of a c thus have a common conjugate point in R, and thus their corresponding C is through this common point.

The two spaces r and R will be related to each other by equations (1) *in such a way that the points of one space correspond to the curves of a complex in the second space. Curves of the complex that go through a given point thus map to the points of the curve of the complex that corresponds to the given point.*

12. One may now show that equations (1) determine a general reciprocity between lines of the two complexes, and indeed between curves, in particular, that are enveloped by the curves of the complex c and C.

When two curves of the complex of the one space have a point in common – which obviously is not the case, in general – their image points lie on a curve of the complex of the second space; in particular, one must remark that two infinitely close intersecting curves of the complex correspond to points in the other space whose infinitesimal connecting line is a direction of the complex. One now considers a curve σ that is enveloped by a single infinitude of c and the C that correspond to the points of thus latter curve; it is obvious that any time two consecutive C intersect, their totality determines an enveloping curve Σ. If one performs the corresponding operation on Σ then one obtains a curve σ' that is enveloped by curves of the complex c, and indeed I assert that σ' is just the original curve σ.

*) Structures in the space r will be notated by small symbols; by comparison, I need large symbols for all of the ones that belong to the space R.

In order to prove this, one considers, on the one hand, a curvilinear polygon composed of curves of the complex c, and, on the other, the image points that are associated with the aforementioned curves, which obviously lie pair-wise on the curves of the complex C, which correspond to the vertices of the given polygon. This new polygon and the given one thus have a reciprocal relationship.

By passing to the limit, one obtains two curves that are enveloped by the curves of the complexes c and C that have a reciprocal relationship such that the points of the one correspond to those curves of the complex that envelop the second one.

Every curve that is enveloped by curves of the complex is thus mapped, in a two-fold way, to another curve that is likewise enveloped by curves of the complex that we refer to as *reciprocal* to the given one with respect to the system of equations (1), and obviously when the equations of the one curve are given, one can determine those of the reciprocal through simple operations – viz., differentiation and elimination. One must also remark that the *elementary directions of the complex* (dx, dy, dz) *and* (dX, dY, dZ) *are associated with each other pair-wise as reciprocals,* so two curves that are enveloped by curves of the complex, between which contact comes about, are mapped into the second space as just such curves.

13. Along with other spatial constructions, equations (1) also determine a correspondence, and indeed in a two-fold manner, which is not generally a *complete* reciprocity, through.

The elementary cone of the complex, whose vertex lies on a surface f, always intersects the corresponding tangent planes in n lines – n denotes the order of the cone of the complex – and thus associates each point of the surface with n elementary directions of the complex. The continuous sequence of these directions defines a family of curves that covers the surface n times and will be collectively enveloped by curves of the complex c. The corresponding curves Σ generate a surface F that we regard as the image surface of the given one. The fact that the reciprocity relationship is not complete resides in the fact that, in general, only one curve Σ goes through each point of the surface F. Moreover, a family of curves Σ' thus lives on F that is likewise enveloped by curves of the complex C, and indeed $(N - 1)$ such Σ' go through each point of our surface, assuming that N denotes the order of the elementary cone of the complex of the space R. The reciprocals that are associated with the curves Σ' define a surface φ that, to some extent, is associated with the given surface f.

As we said, n curves σ go through each point of the surface f, and thus this point is the image of a curve of the complex C that contacts n curves Σ; as a result, our C contacts the surface F in n points. On the other hand, only one Σ goes through each point of the surface F, compared to $(N - 1)$ such Σ', and thus our point is the image of a curve of the complex c that contacts f at one point, compared to $(N - 1)$ points. If we now introduce – by analogy with the terminology that is used for line systems – the terms "*curve congruence*" *and* "*focal surface* *) *of a curve congruence*" then we can summarize the aforementioned in the following way:

*) If one defines a curve congruence by a *linear* partial differential equation of first order then the focal surface is the one that one generally refers to as the *singular integral* of the differential equation in question.

The points of a surface f are mapped to R, as a two-fold infinitude of curves of the complex C, as a curve congruence whose focal surface F we regard as the image surface of the given one. Likewise, the points of the surface F are mapped as a two-fold infinitude of curves c, and indeed the associated focal surface includes the given surface f as a reducible subset.

14. The previous considerations are also valid when f and F are surface elements. *Our reciprocity determines a correspondence between surface elements.* A given element of the space r corresponds to n elements of the second space; on the other hand, each element of the space corresponds to N elements in r. Thus, should equations (1) be a complete reciprocity – that is, a one-to-one correspondence between surface elements – then the two numbers n and N must be equal to 1. As I have remarked in passing, the Ampère transformation (cf., § 7, 22.) belongs to them. By comparison, we have found that the correspondence between elementary directions of the complex is single-valued in general, and, in fact, *one obtains the clearest statement of our reciprocity when one regards each figure as composed of elementary directions of the complex. For example, we can say that the two-fold infinitude of elementary directions of the complex of a surface f is mapped to the elementary direction of the complex of the corresponding surface F.*

Now, let a curve k be given, which we shall regard as a *tubular surface of infinitesimal cross section*. All elementary directions of the complex that intersect k give two-fold infinitudes of directions of the complex (dX, dY, dZ) whose totality, in general, defines a surface F that is the image of k. This definition of the image surface is equivalent with the following two: F includes a single infinitude of curves C that get mapped to points of the curve k. All curves of the complex c that cut k are mapped to points of the surface F.

The equations $F_1(x, y, z, X, Y, Z) = 0$, $F_2(x, y, z, X, Y, Z) = 0$ thus convert arbitrary given structures into new ones, and they can thus serve to transform geometric theorems and problems. In the second section, we will give important applications of this transformation principle for a particular form of the mapping equations.

§ 5.

Determination of space transformations for which contact is an invariant relation.

15. As is well-known, transformations that can be expressed as follows:

$$X = f_1(x, y, z, p, q), \qquad Y = f_2(x, y, z, p, q), \qquad Z = f_3(x, y, z, p, q), \tag{1}$$

play an important role in the theory of partial differential equations. Here, p and q, as P and Q will later on, denote the partial derivatives $\partial z / \partial x$, $\partial z / \partial y$, $\partial Z / \partial X$, $\partial Z / \partial Y$. Worthy of particular notice is the case in which expressions for P and Q can be derived from equations (1) that likewise depend only upon (x, y, z, p, q). Then our transformation possesses the property that surfaces that contact each other will go to other such surfaces, and indeed in these paragraphs I will give an apparently new analytic-geometric classification of these transformations into three classes.

Equations (1) associate an arbitrary surface element (x, y, z, p, q) with a point (X, Y, Z), and thus the elements of a surface f all correspond to points of a surface F that certainly can degenerate into a curve, if not a point. If one lets a surface π vary in such a way that a surface element itself remains unchanged then the corresponding Π contact each other at one (or several) common elements E. When Π degenerates into a curve then, as a continuity argument shows, it includes two consecutive points of the element E. Finally, if Π were a point cone (infinitely small cone) then it would lie on E.

One now considers a three-fold infinitude of surfaces f that is chosen completely arbitrarily, along with the corresponding image F in the second space, which is either a surface, curve, or point cone. A surface φ will be enveloped by a two-fold infinitude of f, and, from the foregoing remarks, the corresponding two-fold infinitude of F will then contact the image surface Φ. In this, one finds a general definition of our transformations.

16. If we choose, in particular, the surfaces f to be all point cones of the space r then we find the analytical definition of the structure F when we eliminate p and q from the equations:

$$X = f_1(x, y, z, p, q), \qquad Y = f_2(x, y, z, p, q), \qquad Z = f_3(x, y, z, p, q).$$

Three cases are possible here: Either there exists only *one* relation between the (x, y, z, p, q), and then the transformation *) belongs to the general reciprocity that was presented by Plücker, or one finds two such relations that correspond to the reciprocity that I gave, or there exist three independent equations between the point coordinates of the two spaces, which is the case for all point transformations.

There are three distinct classes of transformations for which contact is an invariant: The first one corresponds to Plückerian reciprocity of spaces, and is defined by the aequatio directrix:

$$F(x, y, z, X, Y, Z) = 0.$$

The second one corresponds to the reciprocity presented by me, which is defined by two equations:

$$F_1(x, y, z, X, Y, Z) = 0, \qquad F_2(x, y, z, X, Y, Z) = 0.$$

Finally, the third one encompasses all point transformations that are determined by the three equations **)*:*

$$F_1(x, y, z, X, Y, Z) = 0, \qquad F_2(x, y, z, X, Y, Z) = 0, \qquad F_3(x, y, z, X, Y, Z) = 0.$$

*) Cf., Du Bois-Reymond, partielle Differential-Gleichungen. § 75-81.

**) For a space with n dimensions there are n distinct classes of transformations for which contact is an invariant relation. If m denotes an arbitrary whole, positive number that is not larger than n then we can say that each class will be defined by m equations between the coordinates of the new and transformed spaces.

The first two classes of transformations rest on the introduction of a new space element, and the last one, on the application of a new coordinate system.

§ 6.

Transformations of partial differential equations.

17. Legendre *) has given a general method of converting – in the language of modern geometry – a partial differential equation involving point coordinates x, y, z into one involving plane coordinates t, u, v. One can thus also regard t, u, v as point coordinates of a space that is reciprocally related to the given one. *When one introduces the curves or surfaces of a complex as space elements, it is likewise possible to transform a differential equation in the variables x, y, z into one in the coordinates X, Y, Z of the new elements.* Here, one must also remark that one can interpret X, Y, Z in the new equation as the point coordinates of the space R, and indeed this viewpoint will be predominant in our presentation.

The analytical proof for the truth of the foregoing assertion lies in the fact that the aforementioned exchange of space elements can be expressed by means of five relations between (x, y, z, p, q) and (X, Y, Z, P, Q). However, when one substitutes the values of x, y, z, p, q with X, Y, Z, P, Q in a partial differential equation:

$$\Pi(x, y, z, p, q) = 0$$

one obtains a new partial differential equation of first order. Geometrically, one can view this in the following way:

Let a partial differential equation of first order $\Pi(x, y, z, p, q) = 0$ be given, as well as all surfaces φ that define a so-called complete integral; here, one must recall that every other integral surface f will be enveloped by a single infinitude of φ. One further considers the associated image surfaces Φ and F. We will prove that every F is enveloped by a single infinitude of Φ. From the developments of the last paragraph, it follows that when two surfaces contact each other in a curve – i.e., they have a single infinitude of surface elements in common – the image surfaces have the same mutual relationship. Having assumed this, one considers an integral surface f_0, and then, all of the singly infinite number of φ_0 that contact it in a characteristic, and finally, the corresponding surfaces F_0 and Φ_0 . It is clear that each Φ_0 contacts the surface F_0 in a curve, so F_0 is the enveloping surface of the Φ_0 . We thus see that our transformation takes all integral surfaces of a partial differential equation $\Pi(x, y, z, p, q) = 0$ to the integral surfaces of a new partial differential equation $F(X, Y, Z, P, Q) = 0$, and that was our assertion precisely.

18. *We now consider a transformation that is defined by two curves of the complexes c and C that are related to each other. If we apply them to the partial differential*

*) The general picture of the so-called Legendre transformation that is given here is due to Plücker entirely. (Crelle IX. 1831)

equation of first order that is associated with the curve of the complex c (§ 3. 9) *then the corresponding differential equation in X, Y, Z decomposes into two equations, one of which corresponds to the curve of the complex C precisely.*

Namely, let a surface f be given that contacts the associated elementary cone of the complex at all of its points. At each surface element (§ 4, 13.), this cone determines n elementary directions of the complex, two of which always coincide. The curves on f that are enveloped by curves of the complex c thus separate into two families: The characteristics of the surface f and a family that covers it $(n - 2)$-fold. The points of our surface are thus mapped to a two-fold infinitude of curves in C whose focal surface divides into two surfaces F_1 and F_2, and indeed the curves in C contact the surface F_1 in two coincident points, while it contacts the surface F_2, by comparison, at $(n - 2)$ isolated points. Thus, from § 3.9, the surface F_1 satisfies the partial differential equation of first order that is associated with the curves of the complex C, while F_2 satisfies another partial differential equation. Our assertion is thus proved.

Our theorem is identical with the following one:

If a surface element of the space r contacts a elementary cone of the complex then it maps to R by $(n - 1)$ *elements, among which, one is counted twice, and thus contacts an elementary cone of the complex* *).

The curves of the complexes c and C determine two partial differential equations of first order whose characteristics are reciprocal curves with respect to the equations $F_1 = 0$, $F_2 = 0$.

19. The latter form of our theorem gives the following method for the transformation of partial differential equations of first order:

One determines the differential equation $f(x, y, z, dx, dy, dz) = 0$ of the characteristics and chooses an *arbitrary* relation:

$$\psi(x, y, z, dx, dy, dz, X) = 0,$$

in which X denotes a constant. Let:

$$F_1(x, y, z, X, Y, Z) = 0, \qquad F_2(x, y, z, X, Y, Z) = 0$$

be the integrals with two arbitrary constants Y and Z of the simultaneous systems:

$$f = 0, \quad \psi = 0.$$

The equations $F_1 = 0$ and $F_2 = 0$ give, by differentiation and elimination, the result:

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*) An element of the one space corresponds, in general, to n (N, resp.) elements of the second space. As a result, the elements of each space organize themselves into groups of n (N, resp.); we will say they are *associated* elements. If our transformation is defined by *two* equations $(F_1 = 0)$, $(F_2 = 0)$ then, from the development in the text, there is a four-fold infinitude of elements in any space that coincide with an associated element, and indeed it is noteworthy that such an element is mapped to a similar one in the second space. The theorem of the text is also true in this form for transformations that are defined by *one* equation $F(x, y, z, X, Y, Z) = 0$.

$$F_3(X, Y, Z, dX, dY, dZ) = 0,$$

which we regard as determining a partial differential equation:

$$F_4\left(X, Y, Z, \frac{\partial Z}{\partial X}, \frac{\partial Z}{\partial Y}\right) = 0$$

that can be found by conventional methods.

The given partial differential equation and the one just found ($F_4 = 0$) *are equivalent problems, in the sense that the associated characteristics are reciprocal curves relative to the system* $F_1 = 0$, $F_2 = 0$.

It is not difficult to recognize the truth of the following two assertions, which I present as examples:

If one transforms, according to the method given, the partial differential equation of first order that is associated with a line complex (§ 3, 10.) then the new equation $F_4 = 0$ is only of *second* degree. This originates in the fact that the lines of a line congruence contact the focal surface at *two* points [*]).

By comparison, if one transforms the partial differential equation (§ 3, 9) that is associated with a conic section complex then the new differential equation is generally of third degree. This rests upon the fact that when a two-fold infinitude of conic sections define a congruence each conic section contacts the focal surface in six points. As a result, the new elementary cone of the complex is of sixth order and thus of class thirty, etc.

20. The following remarks, whose proof I will not go into, might find a place here:

a. If the equation $f = 0$ in the last paragraph possesses the form:

$$X\,dx + Y\,dy + Z\,dz = 0 \quad {}^{**})$$

then the characteristics of the partial differential equation $F_1 = 0$ are exactly those curves that are represented by $F_1 = 0$, $F_2 = 0$, when one regards the x, y, z as parameters. We can conclude from this that the essential properties of the characteristics of a partial differential equation of first order are the following ones:

The curves of a given complex are characteristics of a partial differential equation of first order when each curve [***]) *in an arbitrary congruence that belongs to the complex contacts the focal surface at only one point. Thus, these focal surfaces will be omitted from all such congruences whenever possible.*

If the elementary cones of the complex in both spaces subdivide into pencils of lines then the three-fold infinitude of curves of the complex in each space lie on a *single* infinitude of surfaces; i.e., the equations:

[*]) The simple form that was suggested here that the partial differential equation that is associated with a line complex can assume is, from the explanation in § 2 (cf., also the third section of this treatise), based on the fact that one introduced the lines of the line complex in question as space elements.

[**]) X, Y, Z shall denote arbitrary functions of x, y, z here.

[***]) Any curve of such a congruence intersects only *one* infinitely neighboring curve.

$$f(x, y, z, dx, dy, dz) = 0, \qquad F_3(X, Y, Z, dX, dY, dZ) = 0,$$

which are linear with respect to the differentials, can be integrated. This theorem is, as I will show on another occasion, meaningful when one seeks all single-valued transformations especially.

b. All transformations for which contact is an invariant relation possess the characteristic property that the Monge-Ampère partial differential equation of second order:

$$A(rt - s^2) + Br + Cs + Dt + E = 0$$

goes over to a similar equation. If the given equation admits a general first integral then this is obviously also the case with the new one. (Cf., a treatise of Boole in Crelle-Borchardt's Journal, bd. 61.)

c. In general, under our transformations a given element corresponds to a finite number of elements of the second space. Meanwhile, there are excluded elements, which are mapped to an unbounded number of elements. Without going into a complete discussion of this important theory, I remark that when an element includes a singly infinite number of directions of the complex it must be an excluded element. This is immediately based upon the fact that in general an element is transformed into as many elements as the number of directions of the complex that it contains (§ 4, 13 and 14). If the elementary cone of the complex of the space r subdivides into a planar pencil then we obtain a *three-fold* infinitude of excluded elements in this space whose totality corresponds to a *four-fold* infinitude of elements that all envelop a cone of the complex of the space R.

Second section.

The Plücker line geometry *) may be transformed into a sphere geometry.

In this section, I will consider a special case of the previously-developed general theory, namely, the one in which the two curve complexes that are related to each other are line complexes. Thus, it is only a degenerate case that I will subject to a closer study. However, I believe that the examination of all possible special cases merits attention, as well as the general case.

§ 7.

The two curve complexes are line complexes.

21. If we assume that the two equations of reciprocity with respect to (x, y, z) and (X, Y, Z) are linear:

$$(1)\qquad \begin{cases} X(a_1x+b_1y+c_1z+d_1)+Y(a_2x+b_2y+c_2z+d_2)+ \\ \quad Z(a_3x+b_3y+c_3z+d_3)+(a_4x+b_4y+c_4z+d_4)=0, \\ X(\alpha_1x+\beta_1y+\gamma_1z+\delta_1)+Y(\alpha_2x+\beta_2y+\gamma_2z+\delta_2)+ \\ \quad Z(\alpha_3x+\beta_3y+\gamma_3z+\delta_3)+(\alpha_4x+\beta_4y+\gamma_4z+\delta_4)=0 \end{cases}$$

then obviously the points in the second space that are conjugate to a given point define a line. The two curve complexes are Plücker line complexes, and thus equations (1) determine a correspondence between r and R that possesses the following properties:

a) *In each space, one finds a line complex whose lines map to points in the second space.*

b) *When a point describes a line of the complex, it rotates the corresponding line around the image point of the line that it runs through.*

c) *Curves whose tangents belong to the two complexes and are pair-wise reciprocal to each other represent the idea that the tangents of one of them map to the points of the other.*

d) *A surface f will be associated with a surface F in a two-fold manner. On the one hand, F is the focal surface of the line congruence whose lines correspond to the points of f; on the other hand, the points of F are the image of all tangents to the surface f that belong to the line complex in question* **).

e) *On the aforementioned surfaces all curves are pair-wise conjugate.*

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*) With regard to the theory of complexes, I assume that one knows: Plücker, Neue Geometrie des Raumes…1868, 69; Klein, Zur Theorie der Complexe…Math. Ann. II, 2. I cannot cite Battaglini's works on line complexes, because they are inaccessible to me.

**) That the reciprocity is not complete comes from the fact that the surface *F* is the complete focal surface of the associated congruence; on the other hand, the lines of the second congruence, except for *f*, envelop yet a second surface.

f) *A curve whose tangents belong to the complex in question corresponds, as a conjugate, to a curve that is likewise enveloped by lines of the complex and indeed precisely that curve that we have referred to as reciprocal to the given one.*

The proof of the assertion made in e) is based in the fact that a line of the complex and a point that lies on it map to a point and a line of the complex that goes through it.

22. Each of equations (1) determines a homographic correspondence between the points and planes of the two spaces, and thus each of our complexes may be defined to be the totality of all lines of intersection of homographically corresponding planes or the connecting line of homographically associated points. The line complex of second degree that is determined by this is, from the investigations of Reye *), identical with the line systems that Binet first considered as the totality of all stationary rotational axes of a material body, and which several mathematicians, in particular Chasles and Reyes, have investigated later.

If the constants in equations (1) are specified then the complexes either occupy a special reciprocal position – for example, they can coincide, a case that Reye considered in his Geometrie der Lage, in which he maintained the same map of complexes that we examined here, along with the theorem that we presented in b) – or they will themselves be specified. Without going into a discussion of all the possible cases here, I propose the following two most important degenerate cases **):

The two complexes can be special linear complexes. This case leads to the well-known Ampère transformation, which then rests upon the fact that one introduces all lines that intersect a fixed line as the space element, instead of points or planes. The equations of the Ampère transformation, namely:

$$X = p, \qquad Y + y = 0, \qquad Z - z - px = 0$$

give the following two relations between (x, y, z, X, Y, Z):

$$Y + y = 0, \qquad Z + z + Xx = 0,$$

which obviously define a particular and symmetric form for equations (1).

The one complex can go over to the totality of all lines that intersect a fixed conic section; the second complex is then a linear one, in general ***). I will examine this degenerate case more closely in what follows under the assumption that the fundamental conic section is the infinitely distant circle.

*) Reye, Geometrie der Lage, second edition, 1868, pp. 116-172.

**) Lie, Repräsentation des Imaginären, Acad. zu Christiania. February and August 1869. The spatial relationship that is considered in this treatise § 17, § 25, § 27-29 is identical with the present one. In § 25, I discuss the first of the two degenerate cases.

***) Nöther has already occasionally given this map of the linear complex, which I meanwhile discovered independently. (Götting. Nachr. 1869: Zur Theorie der algebraische Functionen.) The viewpoint that is fundamental for us, that the two spaces contain a complex whose lines map to points of the second space is not mentioned in the note that we spoke of. I would like to further add that I have not found the idea of basing a correspondence between the surface elements of spaces on the map of a complex discussed anywhere.

23. We know that the two curve complexes are line complexes when the equations of reciprocity with respect to the two systems of variables (x, y, z) and (X, Y, Z) are linear, and we now pose the question of whether this sufficient condition is also necessary. As we shall now show, this is generally the case when one adds the condition that the correspondence shall be one-to-one. Meanwhile, I remark that this question, which is interesting in its own right, has no significance in the following theory.

If the one complex is a line complex then the elementary cone of the complex of the second complex, which is a curve complex in general, must decompose into a number of cones of second degree. In particular, were the one complex composed of tangents to a surface then the elementary cone of the second complex would decompose into a pencil of lines. If one now poses the requirement that the map should be one-to-one then only the following three cases are possible:

1) The two line complexes are general complexes of second degree.

2) One of the complexes is a special complex of second degree, and the other one is linear.

3) Both of the complexes are special linear.

We suggest, as one can show, that equations (1) *define the most general single-valued reciprocal map of two line complexes, so the case in which both line complexes are special linear ones is not taken into account completely* [*]).

If the two complexes are general complexes of second degree then, as one easily verifies, the associated singularity surfaces are not *curved* surfaces. Namely, two plane pencils of lines emanate from each point of the singularity surface whose lines map to the second space as the points of *one* line. All lines of a pencil thus transform to a single point. The totality of all lines that have no independent map cannot be a complex, but at best, a number of congruences. Since all plane pencils of lines whose vertices lie on a *curved* surface necessarily define a complex, the singularity surface consists of planes, and thus one may conclude that the two complexes are of second degree, of the type that Binet first considered.

If a special complex of second degree and a general linear one are mapped to each other then, *a priori*, two cases can be imagined. The lines of the complex of second degree can intersect a conic section or envelop a surface of second degree. By an argument that I will not go into here, I have proved that only the first case exists.

§ 8.

Reciprocity between a linear complex and the totality of all lines that intersect the infinitely distant imaginary circle.

24. We now turn to the equations:

[*]) One establishes a one-to-one correspondence between the planes of linear pencils. One then relates the lines of each plane dually with the points of the corresponding plane. In this way, one obtains a reciprocal map of two special linear complexes that is more general than the one that was determined by Ampère's equations.

(1) $$-Zz = x - (X + iY), \qquad (X - iY)\,z = y - Z,$$

which, as one sees, are linear with respect to (x, y, z) and (X, Y, Z), which thus determines a correspondence between two line complexes. We first seek the equations of this complex in terms of the Plücker line coordinates.

Plücker wrote the equations of the straight line in the form:

$$rz = x - \rho, \qquad sz = y - \sigma.$$

and considered the five quantities r, s, ρ, σ, $(r\sigma - s\rho)$ as line coordinates. Thus, when one regards the X, Y, Z in equations (1) as parameters, they represent a system of lines whose coordinates satisfy the following relations:

$$r = -Z, \qquad r = X + iY, \qquad s = X - iY, \qquad s = Z,$$

from which it emerges that:

(2) $$r + \sigma = 0.$$

The line complex in space r is therefore a linear one, and indeed, a general linear complex that includes the infinitely distant line *) of the xy-plane.

In order to determine the complex in the space R, one replaces the system (1) with the equivalent one:

$$\frac{1}{2}\left[z - \frac{1}{z}\right] Z = X - \frac{1}{2}\left[x + \frac{y}{z}\right],$$
$$\frac{1}{2i}\left[z + \frac{1}{z}\right] Z = Y - \frac{1}{2i}\left[x - \frac{y}{z}\right],$$

which, by comparing them with the general equations of a line:

(3) $$RZ = X - \mathrm{P}, \qquad SZ = Y - \Sigma,$$

gives the following relations:

$$R = \frac{1}{2}\left[z - \frac{1}{z}\right], \qquad \mathrm{P} = \frac{1}{2}\left[x + \frac{y}{z}\right],$$
$$S = \frac{1}{2i}\left[z + \frac{1}{z}\right], \qquad \Sigma = \frac{1}{2i}\left[x - \frac{y}{z}\right],$$

and we thus find for the equation of the complex in question:

(4) $$R^2 + S^2 + 1 = 0.$$

*) This line emerges as a fundamental construction of the map. I will sometimes refer to this line as the fundamental line of the space r.

Now, relations (3) give:

$$R = \frac{dX}{dZ}, \qquad S = \frac{dY}{dZ},$$

and thus (4) can also be written in the following form:

$$dX^2 + dY^2 + dZ^2 = 0.$$

The line complex of the space R will be defined by the imaginary lines whose length are equal to null [*]).

Equations (1) *relate the two spaces to each other in such a way that the points of the space r correspond to the imaginary lines whose lengths are null, while the points of the space R map to the lines of the linear complex* $r + \sigma = 0$.

One must remark that when a point runs through a line of the complex $r + s = 0$ the corresponding image line describes an infinitesimal sphere, viz., a point-sphere.

25. According to the general theory of reciprocal curves (§ 4, 12), when one knows a curve whose tangents belong to a complex, the image curve that is enveloped by lines of the second complex may be determined by simple operations – viz., differentiation and elimination. Now, Lagrange has found the general equation for all curves whose tangents intersect the imaginary circle, and thus *it is also possible to write down the general equation of those curves whose tangents belong to a linear complex.*

In order for us to not stray from our objective, we shall not go further into the simple geometric relations that exist between reciprocal curves of the two complexes [**]).

The correspondence that is determined by equation (1) between the surfaces of both spaces possesses some peculiarities, which I will briefly describe, while referring to the general developments of paragraph 4.

If the surface f has a general position in r then its tangents that belong to the linear complex envelop a second surface φ, besides. We will call the curves that lie on both of these surfaces, and whose tangents are lines of the complex, σ_f and σ_φ. The corresponding reciprocal curves Σ_f and Σ_φ generate *the same* surface F that is the image of the given surface f, as well as the image of the surface φ.

By comparison, if one takes an arbitrary surface Φ in the space R then the lines of the complex that contact Φ envelop no other surface. The curves Σ of our surface that are enveloped by lines of the complex define an *irreducible* family that covers Φ twice. The reciprocal curves σ generate the image surface, which includes no other curves, whose tangents belong to the linear complex.

Finally, I must point out that our map converts the following two problems into each other: *Determine the edge of regression of the developable surface on the focal surface of*

[*]) Following the more convenient French terminology, I refer to any line that intersects the infinitely distant imaginary circle as a *line of null length.*

[**]) I would like to remark on this that a vertex on the one curve corresponds to a stationary tangent in the second space. Generally, stationary tangents appear as *regular singularities* when one regards curves as line structures; that is, as enveloped by the lines of a *given* complex.

a congruence whose lines belong to a linear complex, and: *Find the geodetic curves on a given surface whose lengths are null.*

26. Later, I will sometimes lean on the following two theorems:

a. *A surface F of n^{th} order that contains the infinitely distant imaginary circle as a p-fold line is the image of a congruence whose order and class are equal to* $(n - p)$.

Namely, an imaginary line whose length is null cuts F in $(n - p)$ points that lie in finite space, and therefore just as many lines of the congruence in question go through each point of the space r. However, for a congruence that belongs to a complex it is known that that the class equals the order.

b. *A curve C of nth order that cuts the imaginary circle in p points maps to a line surface of* $(2n - p)^{\text{th}}$ *order.*

Namely, a line of the linear complex $r + \sigma = 0$ cuts the line surface in question in just as many points as the curve C and an infinitesimal cone have in common, while the infinitely distant points are not to be counted with them.

§ 9.

The Plücker line geometry can be transformed into a sphere geometry.

27. *In this paragraph, I will establish a fundamental connection between the Plücker line geometry and a geometry whose element is the sphere.*

Equations (1) of the last paragraph transform the lines of the space r into the spheres of the second space, and indeed in a two-fold manner. On the one hand, from a previous theorem (26. a), those lines of the linear complex $r + \sigma = 0$ that cut an arbitrary line l_1, and thus the associated reciprocal polar l_2 with respect to the complex, as well, go to the points of a sphere. On the other hand, the points of the lines l_1 and l_2 transform into the rectilinear generators of this sphere (26. b).

By the following analytical argument, one finds the relations that exist between the coordinates of the lines l_1 and l_2, and, on the other hand, the center coordinates X', Y', Z', and the radius H' of the corresponding sphere. If:

$$r z = x - \rho, \qquad s z = y - \sigma$$

are the equations of the lines l_1 or l_2, and one recalls that the lines of the complex $r + \sigma = 0$ can be represented as follows:

$$- Z z = x - (X + iY), \quad (X - iY) z = y - Y$$

then one sees that x, y, z must be eliminated from these four equations in order for the lines (1) to be subject to the condition that they cut l_1. In this way, one finds that the coordinates X, Y, Z of our lines of the complex, or, what comes down to the same thing, the coordinates X, Y, Z of the corresponding image point, satisfy the relation:

$$[X - (r + s)]^2 + [Y - (s - \rho)]^2 + [Z - (\sigma - r)]^2 = [\sigma + r]^2.$$

Our previous assertion is proved analytically by this. Likewise, we obtain the following formulas:

(2) $$X' = r + \sigma, \quad iY' = \rho - s, \quad Z' = \rho - r, \quad \pm H' = \sigma + r,$$

or the equivalent ones:

(3) $$\rho = \tfrac{1}{2}(X' + iY'),\ s = \tfrac{1}{2}(X' - iY'),\ \sigma = \tfrac{1}{2}(Z' \pm iH'),\ r = -\tfrac{1}{2}(Z' \mp iH'),$$

in which one can omit the primes with no further restrictions. *In our way of looking at things, a point of the space R is a sphere whose radius is infinitely small.*

Formulas *) (2) and (3) show that a given line in the space r goes to a completely determined sphere; *by contrast, a given sphere* $[X, Y, Z, H^2]$ *maps to two lines:*

$$(X, Y, Z, +H), \qquad (X, Y, Z, -H),$$

which are reciprocal polars with respect to the linear complex:

$$\pm H = r + \sigma = 0.$$

When one sets H equal to zero, equations like this obviously determine the single-valued relation between the point-spheres of the space R and lines of the complex $H = 0$.

A plane – that is, a sphere whose radius is infinitely large – transforms according to equations (2) into two lines l_1 and l_2, which cut the infinitely distant line of the *xy*-plane. Thus, from the foregoing, the points of the lines l_1 and l_2 are the image of all lines in our plane that go through the points of the associated imaginary circle. In particular, a plane that contacts the imaginary circle maps to a line of the complex that is parallel to the *xy*-plane.

28. *Two intersecting lines l and* λ *map to spheres, between which contact exists.*

The reciprocal polars of the given line with respect to the complex $H = 0$ then intersect, and thus contain the image spheres of two common lines that belong to different generators. However, if the intersection of two surfaces of second degree decompose into a conic section and a line pair then the surfaces contact each other in three points, namely, the double points of the intersection curve. The two spheres thus have three contact points, among which, two of them are imaginary and infinitely distant and do not come under consideration. Analytically, one proves our theorem in the following way: *The condition of intersection for the two lines:*

$$r_1 z = x - \rho_1, \quad r_2 z = x - \rho_2,$$
$$s_1 z = y - \sigma_1, \quad s_2 z = x - \sigma_2,$$

will be represented by:

*) As Klein has remarked to me, the relation between line and sphere geometry can be derived immediately from these formulas, when coupled with the analytically-based theorems of the next paragraph.

$$(r_1 - r_2)(\sigma_1 - \sigma_2) - (s_1 - s_2)(\rho_1 - \rho_2) = 0,$$

and, by an application of formulas (3), this equation goes to:

$$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 = (H_1 \pm H_2)^2,$$

that is, to the *condition that the two image spheres must contact each other.*

Two spheres that contact each other transform into two line pairs whose mutual position is such that each line of the one pair cuts a line of the second one.

§ 10.

Various maps.

29. One considers the lines of a planar pencil of lines in the space r, along with the associated reciprocal polars with respect to $H = 0$, which likewise defines a planar pencil of lines, and finally, the corresponding image spheres. It is easy to recognize that all of the these spheres contain two common lines, namely, the ones that correspond to the apexes of the two pencils of lines, and thus our spheres contact each other at the intersection point of these two lines. *The lines of a planar pencil of lines map to all spheres that contact each other in a common point.* From this, it follows that a surface f and all of its tangents at a given point map to a surface and all spheres that contact it at a point (§ 5, 15). This finds its simplest expression in the following theorem:

Under our map, all surface elements of the space r that contain two consecutive points of a line go to elements of the corresponding image sphere.

A lines that lies in the surface f that also contacts it at infinitely many points will be a sphere that contacts the image surface infinitely often – i.e., in a curve. From this, one may conclude that a line surface in a sphere envelope (a tubular surface). A developable surface is transformed into the enveloping surface of a singly infinite number of spheres that are subject to the condition that any two consecutive ones contact each other. One thus obtains the imaginary line surfaces that were considered by Monge, whose two families of lines of curvature coincide in the rectilinear generators. From the remark that a line surface goes to a sphere envelope, it follows that a surface of second degree, which, as is well-known, indeed contains two families of lines, transforms into a surface that can be regarded as a sphere envelope in two ways, and we indeed obtain the most general surface that possesses this property: the *Dupin cyclide*.

30. From a previous theorem, it follows immediately that all lines that cut a fixed line map to spheres that intersect a given sphere, and we thus know the map of the *special* linear complexes.

The equation of *general linear line complexes* is, from Plücker:

$$(1) \qquad (r\sigma - s\rho) + mr + n\sigma + p\rho + qs + t = 0,$$

and from this, the equation of the corresponding *linear sphere complexes* emerges:

$$(X^2 + Y^2 + Z^2 - H^2) + M X + N Y + P Z + Q H + T = 0 \text{ *)}.$$

Here, M, N, P, Q, T denote constants that depend upon m, n, p, q, t, *while* X, Y, Z, H *are regarded as inhomogeneous sphere coordinates.*

As one easily sees, the latter equation determines all spheres that intersect a fixed sphere:

$$(X^2 + Y^2 + Z^2) + M X + N Y + P Z + T = 0$$

at a constant angle, and indeed this sphere is the image of all lines that likewise belong to the given complexes (1) and $H = 0$. If the simultaneous invariant of these two complexes is equal to zero, when they are in involution, then, as one easily convinces oneself, the constant angle in question is equal to 90°.

The spheres that intersect a given sphere at a constant angle transform into the lines of two linear complexes that are conjugate to each other with respect to $H = 0$. *In particular, the spheres that are orthogonal to a given sphere are the images of the lines of a linear complex that is involution with respect to* $H = 0$.

An equation of the form:

$$(2) \qquad ar + b\sigma + c\rho + ds + e = 0$$

gives a linear equation between the sphere coordinates X, Y, Z, H, and therefore the sphere complex in question will be defined by all spheres that intersect a given plane at a constant angle. One can also conclude from this that the complex (2) contains the infinitely distant line of the *xy*-plane so the complexes (2) and $H = 0$ intersect each other along a linear congruence whose directrices are parallel with the *xy*-plane. Finally, when the complexes (2) and $H = 0$ are in involution, one obtains all spheres whose centers lie in a fixed plane.

As one easily verifies, the four complexes:

$$X = 0 = \rho + s, \qquad iY = 0 = \rho - s,$$
$$Z = 0 = \sigma - s, \qquad H = 0 = \sigma + r$$

lie pair-wise in involution, and thus contain a common line that is infinitely distant in the *xy*-plane. *Thus, the five complexes:*

$$X = 0, \quad Y = 0, \quad Z = 0, \quad H = 0, \quad \text{const.} = 0,$$

in which one counts the last one twice, because it is a special complex that is in involution with itself, define a system that can be regarded as a degenerate case of the six fundamental complexes of Klein. It is obvious that one can likewise regard the four sphere-coordinates X, Y, Z, H as inhomogeneous line coordinates.

Finally, it is also remarkable that each of the single infinitude of line complexes that can be represented by the equation:

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*) This equation may also be written as follows:

$$(X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2 + (iH - iH_0)^2 = \text{const.}$$

$$H = \text{const.}$$

map to all spheres of a given magnitude. These complexes contact each other along a common special congruence whose two directrices coincide in the infinitely distant lines of the *xy*-plane. The lines of this congruence map to all planes that contact the infinitely distant imaginary circle.

31. It is known to be an immediate consequence of the Plücker approach that when $l_1 = 0$ and $l_2 = 0$ are the equations of two linear complexes, the equation:

$$l_1 + u\, l_2 = 0,$$

in which u denotes a parameter, represents a family of linear complexes that contain a common linear congruence. Our map transforms this theorem into the following one:

The spheres K that intersect two fixed spheres S_1 and S_2 at given angles V_1 and V_2 have the same relationship to infinitely spheres S. There are two S that will contact all K, corresponding to the directrices of the aforementioned congruence.

The variable line complex $l_1 + u\, l_2 = 0$ intersects the complex $H = 0$ in a linear congruence whose two directrices describe a surface of second degree: the intersection surface of three complexes $l_1 = 0$, $l_2 = 0$, $H = 0$, and therefore the aforementioned sphere S envelops a Dupin cyclide, which certainly degenerates into a circle common to one of these spheres.

Here, I would like to point out that our map in interesting in that it takes discontinuous line groups into corresponding sphere groups. For example, from the well-known theory of the 27 lines on the surface of third degree, we conclude the existence of a group consisting of 27 spheres, such that any ten other spheres contact the group. In other words, sphere configurations go to singular line configurations.

§ 11.

Correspondence between problems that relate to spheres and ones that relate to lines.

32. In this paragraph, we address some well-known, and very simple, problems that relate to the spheres that are subject to certain conditions by considering the corresponding line problems.

a. *How many spheres contact four given spheres?*

The four given spheres transform into four line-pairs (l_1, λ_1), (l_2, λ_2), (l_3, λ_3), (l_4, λ_4). If we now take a line from each pair and assemble the four lines thus obtained into a group then one seeks the two transversals of this group. We find 16 different groups, which are meanwhile pair-wise conjugate with respect to $H = 0$, like, for instance:

$$l_1\ \lambda_2\ l_3\ l_4 \qquad\qquad \lambda_1\ l_1\ \lambda_3\ \lambda_4\,.$$

Obviously, the two transversal pairs:

$$t_1\, t_2 \qquad \tau_1\, \tau_2$$

of two such groups are self-conjugate, and thus map to just two spheres. *There are* 16 *spheres that contact four spheres, which organize into* 8 *pairs.*

b. *How many spheres intersect four given spheres in given angles?*

Spheres that intersect a given sphere at the same angle are images of the lines of two linear complexes that are conjugate to each other with respect to $H = 0$. We must therefore consider four pairs of linear complexes (l_1, λ_1), (l_2, λ_2), (l_3, λ_3), (l_4, λ_4), and first arrange into groups of four in all possible ways, in such a way that two complexes of a group never have the same index, then find all lines that belong to the four complexes of each group. Four linear complexes contain two common lines, and thus when one proceeds as in the first problem, one obtains 16 spheres grouped into 8 pairs as the solution to our problem.

Our problem simplifies when one or more of the given angles are equal to 90°, if the spheres orthogonal to one sphere map to the lines of *one* complex that is in involution with $H = 0$. Finally, if all four angles are equal to 90° then one asks about the common lines of four linear complexes that are in involution with $H = 0$. The two lines thus obtained are conjugate with respect to $H = 0$, and there is therefore *only one sphere that intersects four given spheres orthogonally*.

c. *Construct all spheres that intersect five given spheres at the same angle.*

Our map converts this problem into that of arranging five given pairs of lines (l_1, λ_1), (l_2, λ_2), …, (l_5, λ_5) in all possible ways into groups of 5, but with the restriction that no two lines of a group can have the same index, and then finding all linear complexes that always contains all the lines of one group. There are 32 different groups that are pair-wise conjugate with respect to $H = 0$. Corresponding to them, we obtain 32 pair-wise conjugate linear complexes that map to 16 linear sphere complexes. The 16 spheres, such that each of the spheres intersects such a complex at constant angles, are the solutions to our problem.

Two line groups such as:

$$l_1\, \lambda_2\, \lambda_3\, l_4\, l_5 \qquad l_1\, \lambda_2\, \lambda_3\, l_4\, \lambda_5$$

include four common lines (l_1 λ_2 λ_3 l_4), and thus the corresponding linear complexes intersect each other in a linear congruence whose directrices d_1, d_2 are the two transversals of the four lines in question. The complex $H = 0$ intersects this congruence in a surface of second degree that is the image of a circle, namely, the intersection circle of two of the spheres we seek, which are, however, also the two spheres that correspond to d_1, d_2 . These latter spheres may now also be defined by saying that they contact four of the five given ones, and one can therefore determine five circles on each of the sixteen spheres that intersect five given ones at the same angle, assuming that one can construct the spheres that contact four given ones.

§ 12.

Our map takes the principal tangent curves of a surface *f* to the lines of curvature of the image surface *F*.

33. The map that was discussed in the past paragraphs holds a great deal of interest due to the following remarkable theorem:

The lines of curvature of a surface F transform into line surfaces that contact the image surface f along principal tangent curves.

The tangents to the surface *f* are converted into spheres that contact *F* and thus the thought comes to mind that the principal tangents of the former surface map to the principal spheres of the latter one. This is, in fact, the case. Namely, the surface *f* will be intersected by a principal tangent at three coincident points, and three consecutive lines that lie on the image sphere contact the surface *F*. The intersection curve of these surfaces thus has a cusp at the point in question, and as a result, the sphere is a principal sphere. If one now further remarks that the direction of this cusp is the tangent to a line of curvature then one sees that two consecutive points of a principal tangent curve map to two imaginary lines that contact *F* in consecutive points of a line of curvature, so all points of the principal tangent curve transform into the generators of a line surface that contacts *F* along a line of curvature. However, our theorem follows from this (§ 7, 21c).

The following two examples confirm our theorem: A sphere transforms into a linear congruence whose focal surface can be regarded as the two directrices. Now, it is known that every curve that lies on a sphere is a line of curvature, and in fact, the directrices of a linear congruence are also principal tangent curves to each line surface that is associated with this congruence. On the other hand, we know that a hyperboloid *f* maps to a Dupin cyclide. Now the line surfaces of the complexes $H = 0$, which contact *f* along its principal tangent curves – viz., the rectilinear generators – are themselves of second degree, and we thus again find the well-known theorem that *the lines of curvature of the Dupin cyclide are two families of circles.*

As an interesting application of our theorem one can consider the following:

The Kummer surface of fourth order with sixteen nodes has algebraic principal tangent curves of sixteenth order that define the complete contact intersection of the surface with line surfaces of eighth order. [*])

The Kummer surface of fourth order with sixteen nodes is, as is well-known, the focal surface of the general congruence of second order and class. This line system (§ 8, 26a), when it belongs to $H = 0$, maps to a surface of fourth order F_4, and thus F_4 includes the infinitely distant imaginary circle as the double-conic section. Now, Darboux and Moutard have found that the lines of curvature of such a surface F_4 are curves of eighth order. They intersect the imaginary circle at eight points, and thus (§ 8, 26b) their image surfaces are line surfaces of eighth order whose generators are double tangents of the Kummer surface. Our theorem follows from this immediately.

[*]) Klein, to whom I communicated that the principal tangent curves are algebraic, found that they are identical with a curve system that he has already considered previously from a different standpoint (these Annals, II., pp. 219). Confer our joint paper in the Monats-Berichten der Berliner Akademie, December 1870.

It is obvious that also the degenerate cases of the Kummer surface – e.g., the *wave surface*, the Plücker complex surfaces, the Steiner [*]) surface of fourth order and third class, some line surfaces of fourth order with two double lines that can coincide, the line surfaces of third order, etc. – have algebraic principal tangent curves.

34. Darboux has shown that on any surface, in general, one can give a line of curvature that lies in finite space, namely, the contact curve with the imaginary developable that is circumscribed by the given surface and the imaginary circle, as well.

Correspondingly, in general, a principal tangent curve can be given on the focal surface of any congruence that belongs to a linear complex that is the geometric locus of all points for which the tangent plane is likewise the plane associated with the linear complex at the point in question.

The infinitely distant spheres that contact a surface F divide into two systems: first, the points of the surface, and second, the points of the aforementioned imaginary developables.

Thus, the lines of the linear complex $H = 0$, which contacts the image surface f, also divide into a system of double tangents and the totality of all lines that contact f at the points of a curve c. This curve is, however, as the image of an imaginary line surface that contacts F along a line of curvature, a principal tangent curve on f. This determination will thus be illusory if not the congruence, but its focal surface – or really, a reducible subset of it – is given, in general. Namely, on a surface one generally finds only a finite number of points whose tangent planes are associated with the point in question through a linear complex. The foregoing finds its simplest expression in the following theorem:

If a surface is its own reciprocal polar with respect to a linear complex then it includes a principal tangent curve whose tangents belong to the complex. This distinguished curve can be determined by differentiation and elimination. [**])

One remarks that each line surface whose generators belong to a linear complex is its own reciprocal polar with respect to the complex, so one may state the following theorem:

On any line surface that is included in a linear complex there generally lies a principal tangent curve [***]) *whose tangents belong to the complex. It can always be found without integration.*

Clebsch has now shown (Borchardt's Journal, Bd. 68) that when a principal tangent curve is known on a line surface, besides the generators, the determination of the remaining ones comes down to a quadrature. *Thus, finding the principal tangent curves on a ruled surface that belongs to a linear complex depends only upon a quadrature.*

If we apply our transformation method to this theorem of Clebsch, as well as to the consequence it leads to, then we obtain the following two theorems:

[*]) Clebsch has found the principal tangent curves of the Steiner surface; they are curves of fourth order (Borchardt's Journal, Bd. 67).

[**]) This curve is, from a remark of Klein, likewise a curve of four-point of the surface.

[***]) An interesting application of this theorem is the following one: According to Plücker, the lines of a linear congruence belong a singly infinite number of linear complexes. *As a result, every ruled surface whose lines intersect two fixed lines contains a single infinitude of algebraic principal tangent curves, among which, each will be enveloped by the lines of a linear complex* (cf., also Cremona, Annali di mat. Ser. II, t. 1).

If a non-circular line of curvature is known on a tubular surface (sphere envelope) then the remaining ones can be found by quadrature.

If a single infinitude of spheres intersect a sphere S at a constant angle then one can first find a line of curvature by differentiation and elimination, and then determine the other ones by quadrature.

The fact that one can find a line of curvature on the tubular surface in question can be concluded immediately from the fact that this surface and the sphere S intersect each other at a constant angle. The intersection curve is now a line of curvature on the sphere S and thus, from a well-known theorem, has the same relationship with the tubular surface.

§ 13.

Correlations between transformations of the two spaces.

35. As we know (§ 5), our map can be expressed by equations that determine each quantity in one of the groups:

$$(x, y, z, p, q), \qquad (X, Y, Z, P, Q)$$

as functions of the quantities in the second group. Now, if a space – e.g., r – were subjected to a transformation under which surfaces that contact each other go to other such surfaces then the corresponding conversion of the second space possesses the same property. The aforementioned transformation is expressed by five equations between (x_1, y_1, z_1, p_1, q_1) and (x_2, y_2, z_1, p_2, q_2), where the two indices refer to the two states of the space r, and by means of the mapping equations these relations go to five relations between (X_1, Y_1, Z_1, P_1, Q_1) and (X_2, Y, Z_1, P_2, Q_2), from which, our assertion is proved.

If we restrict ourselves to linear transformations of the space r then we find amongst the corresponding conversions of the second space: all motions (translations, rotations, and screw motions), parallel transformations [*]) – which we understand to mean the transition from a surface to a parallel surface – the reciprocal transformations that were given by Bonnet, reciprocal transformation with respect to a cyclide, etc., which all, because they correspond to linear transformations of the space r, possess the property of taking lines of curvature to lines of curvature in the transformed surface. Finally, I prove that these transformations of the space R are the only ones that take surfaces that contact each other into other such surfaces.

36. If we next consider those linear point transformations of the space r that correspond to *linear* conversions of the second spaces then it is clear that we can look at linear transformations of the space R for which the infinitely distant imaginary circle preserves its position, and conversely we obtain all of them. Namely, such a transformation takes, on the one hand, a line that intersects the infinitely distant circle to just such lines, and, on the other hand, spheres to spheres, and thus the corresponding

[*]) Cf., Bonnet's dilatation.

conversion of space r is likewise a point and line transformation; i.e., a linear point transformation, which was to be proved.

The general linear transformation for which the imaginary circle retains its position contains seven essential constants, and, as is well-known, can be composed of translations, rotations, and similarity transformations. The corresponding conversion of the space r, which likewise depends upon seven constants, can then be characterized by saying that it takes the linear complex $H = 0$ and a line in it (const. = 0) to themselves. This transformation is likewise *the most general one that transforms a special linear congruence to itself*.

By an analytical argument, one finds, in the following way, that a *translation* of the space R corresponds to a linear point-transformation of the space r. A translation, when regarded as a sphere transformation, is expressed by the equations:

$$X_1 = X_2 + A, \quad Y_1 = Y_2 + B, \quad Z_1 = Z_2 + C, \quad H_1 = H_2 ,$$

and they give, by using formulas (2) in § 9:

$$r_1 = r_2 + a, \qquad s_1 = s_2 + b, \qquad \rho_1 = \rho_2 + c, \qquad \sigma_1 = \sigma_2 + d.$$

By substituting this expression in the equations for a straight line:

$$r_1 z = x - \rho_1, \quad s_1 z = y - \sigma_1,$$

one finds the definition of the transformation in question:

$$z_1 = z_2 , \qquad x_1 = x_2 + a\, z_2 + c, \qquad y_1 = y_2 + b\, z_2 + d.$$

Likewise, it is easy to determine a conversion of the space r that corresponds to similarity transformation. The equations:

$$X_1 = m\, X_2 , \qquad Y_1 = m\, Y_2 , \qquad Z_1 = m\, Z_2 , \qquad H_1 = m\, H_2$$

give, by an application of the formulas (2) of § 9:

$$r_1 = m\, r_2 , \qquad s_1 = m\, s_2 , \qquad \rho_1 = m\, \rho_2 , \qquad \sigma_1 = m\, \sigma_2 ,$$

equations that can define a transformation that can also be determined by:

$$z_1 = z_2 , \qquad x_1 = m\, x_2 , \qquad y_1 = m\, y_2 .$$

The latter equations define a linear point transformation under which the points of two lines preserve their position.

By a geometric argument, I will prove that rotational motions of the space R also go to transformations of just that sort. Let A be the rotational axis, and let M, N be the two points of the imaginary circle that are not displaced by the rotation. It is then obvious that all imaginary lines that cut A, and likewise go through M or N, preserve their position under rotation, and consequently the image points that are associated with these lines, which lie on two lines, also remain unchanged under the corresponding conversion of the space r.

37. *A transformation by reciprocal radii* of the space R transforms points into points, spheres into spheres, and finally, lines that intersect the imaginary circle into similar lines, and thus the corresponding conversion of the space r is a linear point transformation that takes the linear complex $H = 0$ to itself. If one further remarks that each transformation by reciprocal radii leaves the points and rectilinear generators of a sphere unchanged then one sees that under the corresponding reciprocal point transformation of the space r the points of two lines keep their position. Klein has remarked that this transformation can be composed of two reciprocal transformations with respect to two linear complexes that are in involution, and indeed, in our case $H = 0$ is the one complex, while the other one is mapped to the system of spheres that intersect perpendicularly under the transformation that is based on reciprocal radii.

A surface F that goes to itself under a transformation by reciprocal radii is thus mapped to a congruence in the space r that is its own polar with respect to a linear complex that is involution with respect to $H = 0$. The associated focal surface is its own reciprocal polar with respect to each of the two complexes that lie in involution, and consequently the system of its double tangents decomposes into three congruences, one of which the complex $H = 0$ belongs to, while the latter is included in the second complex.

38. One considers the most general line transformations of the space r under which intersection of two lines is an invariant relation, and on the other hand, in the space R, the corresponding conversion that obviously takes spheres into spheres, and spheres that contact each other into other ones of that type. Under the aforementioned line transformations, all tangents of a surface transform into those of a second one, and thus, in particular, the principal tangents of the two surfaces correspond to each other, whether the transformation in question is a linear point transformation or a linear dualistic transformation. Under the corresponding transformation of the space R all three-fold infinitude of spheres that contact a surface F_1 go to all spheres that have the same relationship to another surface F_2, and, in particular, the principal spheres of the two surfaces correspond to each other. From this, it follows that the lines of curvature of the surface correspond to each other, in the sense that if an equation:

$$F(X_1, Y_1, Z_1, P_1, Q_1) = 0 \tag{1}$$

is true for all points of a line of curvature on F_1 then the equation that one obtains when one replaces the values of the quantities $(X_1, Y_1, Z_1, P_1, Q_1)$ in (1) with $(X_2, Y_2, Z_2, P_2, Q_2)$ is also true for all points of line of curvature on F_2.

I will now prove that we obtain all transformations for which, on the one hand, contact is an invariant relation, and, on the other, lines of curvature are covariant curves when we apply our map to all linear point transformations (or linear dualistic conversions) of the space R.

To prove this, I remark that there are two types of surfaces whose curves are all lines of curvature: spheres and imaginary developable surfaces that contain the infinitely distant imaginary circle. It is clear that the desired transformation must take each such surface into one that likewise belongs to one of these categories, and indeed, one then conjectures that, in particular, spheres must go to spheres. This is also the case. Namely, the aforementioned imaginary developables satisfy the partial differential equation:

$$1 + P^2 + Q^2 = 0,$$

and thus the corresponding surfaces in r also satisfy a partial differential equation of first order:

$$F(x, y, z, p, q) = 0.$$

Among the integral surfaces themselves, one can find at most a three-fold infinitude of spheres, and thus spheres cannot generally go to imaginary developables.

Our transformation is thus a sphere transformation, and indeed by our assumptions, one for which spheres that contact each other go to other such spheres. The corresponding conversion of space r is thus a *line* transformation for which the intersection of lines is an invariant relation, and this is well-known to be the case for the linear point transformations and the linear dualistic transformations.

If one remarks that all point transformations for which lines of curvature are covariant curves take *infinitesimal* spheres to *infinitesimal* curves, and that consequently these conversions are the most general point transformations for which similarity at the smallest scale is preserved then one can state the following theorem:

Under our map, all linear transformations of the space r that take the complex H = 0 *to itself go to all point transformations for which similarity at the smallest scale is preserved.*

Likewise, upon considering our previous theorem, we again find without difficulty the following theorem that was first proved by Liouville:

Each point transformation for which similarity at the smallest scale is preserved may be the composition of a transformation through reciprocal radii and a motion.

39. As is well-known, *parallel transformations* – by which, we mean transitions from one surface to a parallel surface – take lines of curvature to lines of curvature, and in fact it is easy to recognize that they are the images of linear transformations of the space r. From (36), the equations:

$$X_1 = X_2, \qquad Y_1 = Y_2, \qquad Z_1 = Z_2, \qquad H_1 = H_2 + A$$

correspond to relations of the following form:

$$z_1 = z_2, \qquad x_1 = x_2 + a\, z_2 + b, \qquad y_1 = y_2 + b\, z_2 + d,$$

from which my assertion is proved.

Bonnet has often considered a transformation that he defined by the equations:

$$Z_1 = iZ_2\sqrt{1+P_2^2+Q_2^2}\,, \qquad X_1 = X_1 + P_2\,X_2, \qquad Y_1 = Y_1 + Q_2\,Z_2\,.$$

Bonnet showed that this transformation is a reciprocal one, that the lines of curvature go to lines of curvature, and finally, that the relations:

$$(1) \qquad \zeta_1 = i\,H_2, \qquad H_1 = -\,i\,\zeta_2$$

come about, assuming that H_1 and H_2 denote the curvature radii of corresponding points, and that ζ_1 and ζ_2 are the z-ordinates of the associated curvature centers.

The Bonnet transformation is, as we will likewise prove, the image of a reciprocal conversion of the space r with respect to the linear complex:

$$Z + i\,H = 0.$$

From Klein, the coordinates of two lines (X_1, Y_1, Z_1, H_1), (X_2, Y_2, Z_2, H_2) that are conjugate to each other with respect to this complex satisfy the relations:

$$X_1 = X_2\,, \qquad Y_1 = Y_2\,, \qquad Z_1 = i\,H_2, \qquad H_1 = -\,i\,Z_2\,.$$

However, when X, Y, Z, H are regarded as sphere coordinates, these equations determine a correspondence between all spheres of the space, and, in fact, the same one as the Bonnet transformation. *)

As is well-known, among the linear transformations of space, the reciprocal conversions with respect to surfaces of second degree play a fundamental role, and this is closely related to the consideration of the corresponding linear sphere transformations. The same can be said of the two sphere groups of a Dupin cyclide, and indeed in the following: A given sphere Q_1 contacts two spheres from each group S_1, S_2 and Σ_1, Σ_2 ; Now, as is known, there are fifteen spheres besides Q_1 that contact these S and Σ, and among them, one chooses the one Q_2 that is associated with Q_1 in the well-known sense. Q_1 and Q_2 are associated with each other by the sphere transformation in question. What is especially noteworthy is the case in which the generators of the one system on the original surface of second degree that we assumed belong to the linear complex $H = 0$. In that case, the cyclide reduces to a circle, and furthermore, the sphere transformation is a point transformation. Here, we thus find a distinguished conformal point transformation under which a circle that lies in finite space enters in as a fundamental structure.

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*) The Bonnet transformation associated the points of space with spheres whose centers lie in a plane. Here, if one replaces the sphere each time with the intersection circle of it with each plane then one finds an interesting connection between the Bonnet transformation and an idea that originated with Möbius (Abhandlungen der Sächs. Akad. 1854). Here, this suggests the idea of an element of a three-dimensional geometry of circles in the plane, where one uses the center coordinates and radii for those coordinates.

I summarize the most important result of these paragraphs in the following manner:

Under my map, one finds the correspondences:

a. All linear point transformations and linear dualistic transformations of space.	*a. All transformations for which contact along lines of curvature is an invariant relation.*
b. All ∞^{10} *linear transformations for which a linear line complex goes to itself.*	*b. All conformal point transformations of space.*
c. All linear ∞^{10} *transformations that take a special linear congruence into itself.*	*c. All conformal point transformations that are homographic transformations for which the infinitely distant imaginary circle preserves its position.*

These developments give rise to some important theories. For example, let us cite:

All transformations for which lines that intersect go to other such lines can, from a remark of Klein, be composed of reciprocal transformations with respect to a linear complex. Correspondingly, one finds that all of our transformations for which lines of curvature are covariant structures can be composed of transformations through reciprocal radii and parallel transformations (dilatations).

If one, as Klein has proposed, considers the line or sphere complexes upon basing the coordinates X, Y, Z, H on a metric geometry in four variables then one easily finds that my linear sphere transformation is identical with the totality of all conformal point transformations of this sphere space. [*])

Meanwhile, I hope to be able to give an exhaustive representation of the latter theory in another treatise whose main objective will be the geometry of a space of *n* dimensions, and indeed to give it for such a space. [**])

Christiania, 10 October 1871.

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[*]) I have presented the corresponding theorem in the Göttinger Nachrichten (1871, no. 7) for a space of *n* dimensions. Moreover, let it be remarked that all conformal point transformations of a space R_n can be composed of motions, similarity transformations, and transformations through reciprocal radii. Thus, in connection with this, one has the theorem that when $x_1, x_2, \ldots, x_n$ are given functions of $y_1, y_2, \ldots, y_n$ such that $\sum dx^2 = \Phi(y_1, y_2, \ldots, y_n) \sum dy^2$, one also has an equation of the form:

$$\sum_{i=1}^{i=n} (x_i - x_i')^2 = \Pi(y_1, y_2, \ldots, y_n) \cdot (y_1', \cdots, y_n') \sum_{i=1}^{i=n} (y_i - y_i')^2 .$$

[**]) I further suggest that every theorem in line or sphere geometry can be transformed in an interesting way into a theorem about surfaces that emerge from an arbitrarily chosen one by the application of all translations and parallel transformations. The following two remarks are also important, which were presented to me too late to find a place in the text: 1) In the line space r, there are, as is known, two types of transformation for which lines that intersect go to other such lines. The corresponding transformations of the space R *do not* divide into two classes when point coordinates are the basic ones. 2) Line transformations for which (const. = 0) keeps its position all give transformations of R for which surfaces with a common spherical image go to other such surfaces. The new spherical image arises from the old one by a conformal point transformation of the image sphere. The Bonnet transformation belongs to them.

Part II.

In the first part of this treatise, I believe that I gave the first complete analytic-geometric interpretation of all space transformations for which contact is an invariant relation. In particular, I considered a singular relationship – I call it a *sphere map*, for the sake of brevity – that takes the lines in a space r to spheres in a space R, which was understood to mean that all surface elements that contained two consecutive points of line went to elements of a sphere. On this, I based a precise and – in my opinion – fundamental connection between line geometry and sphere geometry, and as a consequence, between several projective and metric theories. In particular, I showed that the principal tangent curves of surface f transform into the lines of curvature of the image surface F.

Since I have just used the words "sphere geometry," I must further remark that my understanding of such a geometry up to now would not exist unless I had already addressed many particular problems and theories that relate to spheres. However, after I sent the following treatise to Darboux [*]), in a somewhat different form that appeared for the first time in the Berichten der Akademie zu Christiania in the summer of 1871, I learned that in 1868 he submitted an as-yet-unpublished treatise to the Paris Academy, in which he addressed the same sphere systems that I have called sphere complexes. On the same occasion, he communicated to me that he had just prepared a note in which he treated several problems that I had considered in paragraphs 15 and 24 of my recent treatise. In the following, when it is at all possible, I will make citations in which I will refer to Darboux's relevant papers, which will hopefully appear soon.

However, if sphere geometry exists on a par with line geometry then the peculiar connection between these two disciplines seems to have been first pointed out by myself. Plücker, whom one must thank exclusively for the idea of giving a tangible representation of an algebra with four or more variables, chose the line to be the element of the space R_4. The validity of his choice is not in doubt, but it would, in my opinion, be likewise appropriate to use the sphere. Certainly, line geometry possesses advantages that sphere geometry lacks; the converse is, however, also true. This comes from the fact that the line, as well as the sphere, appeal to the imagination in a special way, and, on the other hand, the fact that there is a simple cycle of sphere transformations that correspond to those line transformations for which intersection is an invariant relation. *It would thus be profitable to develop line geometry and sphere geometry side by side, as I have begun to do, when one always evaluates the results in one geometry in terms of the other one using my map.* If I have sometimes arrived at difficult problems then this is essentially due to the fact that I *alternately* established the presentation in terms of lines and spheres. I have retained this method, which was, for me, the path of discovery, in my present presentation, although I fear that the frequent switching of geometric pictures will create difficulties for the reader.

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[*]) I learned of the recent relations between Darboux's work and my own from the editor. Cf., the conclusion.

Third Section

On the theory of partial differential equations in three variables.

In this section, I will seek to apply, on the one hand, the geometric concepts that Plücker introduced in his last works, and, on the other, the aforementioned developments in the theory of partial differential equations. One easily recognizes that under the aforementioned transformation a partial differential equation of arbitrary order whose characteristics are *principal tangent curves* on the integral surfaces will go to a differential equation of the same order whose characteristics are *lines of curvature*. One can base an interesting parallelism between several important classes of partial differential equations. Along with them, I include, as we will see later, certain differential equations whose characteristics are *geodetic curves.*

The following developments will have, to some extent, a singular character, insofar as I will only be concerned with *special* classes of differential equations. Thus, I must suggest that the path that we follow here, namely, the treatment of partial differential equations in connection with an extended geometric notion, seems to be a method by which one can expect progress in the direction that Monge set out in.

On some partial differential equations of first order.

First, I consider three classes of partial differential equations of first order that can be transformed into each other, and which I will denote by the symbols D_{11}, D_{12}, D_{13}, for brevity.

1) D_{11}. The characteristics are principal tangent curves on the integral surfaces. To a certain extent, as I will show later, the equations D_{11} correspond to line complexes and line congruences.

2) D_{12}. The characteristics are lines of curvature. Each such D_{12} corresponds to either a sphere complex or a sphere congruence.

3) D_{13}. The characteristics are geodetic curves. If H denotes an arbitrary, known function of x, y, z, and, as usual, p, q denote the partial derivatives of z with respect to x and y then each D_{13} can be written as follows:

$$\frac{\partial H}{\partial x}p+\frac{\partial H}{\partial y}q-\frac{\partial H}{\partial z}=\sqrt{1+p^2+q^2}\cdot\sqrt{\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2+\left(\frac{\partial H}{\partial z}\right)^2-1}\,.$$

These equations are, we remark, only of *second* degree with respect to p and q.

From the contents of the following chapter, I further suggest that the determination of geodetic curves on a surface, to some extent, comes from the integration of a particular D_{12} or D_{11} . As a result, the theory of geodetic curves up to now can be realized in the new theory of Plücker complexes.

§ 14.

Partial differential equations of first order whose characteristics are principal tangent curves on the integral surfaces.

40. We have found (§3.10) that when the characteristic curves of a partial differential equation of first order are enveloped by the lines of a line complex, the characteristics are principal tangent curves on the integral surfaces. *) On the other hand, it is easy to recognize that each line congruence corresponds to a *linear* partial differential equation of first order whose double infinitude of characteristics – namely, the lines of the congruence – appear as principal tangent curves in the integral surfaces. Conversely, we will prove that there are no other partial differential equations of first order that possess this property than those of the two types.

If we write a general partial differential equation of first order in the form:

$$F(x, y, z, p, q) = 0$$

then we must regard F as a function of x, y, z, p, q that is determined in a general manner such that at any arbitrary point of an integral surface the direction of the characteristic coincides with that of the trajectory. Namely, the two aforementioned directions lie harmonically with respect to the two corresponding principal tangents, and if they thus coincide then they will likewise be identical with a principal tangent. Following Monge, however, the equations:

$$\frac{\partial F}{\partial p}dy - \frac{\partial F}{\partial q}dx = 0, \qquad \left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right)dx + \left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right)dy = 0$$

determine the directions of the characteristics and the trajectory, respectively, and thus our problem comes down to that of determining the general integral of the partial differential equation:

$$\text{(1)} \qquad \frac{\partial F}{\partial p}\left(\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}\right) + \frac{\partial F}{\partial q}\left(\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}\right) = 0.$$

This equation may be integrated by the usual methods; since we would then obtain the solution in a form that is not easy to interpret, we will find it preferable to approach it in an indirect way.

41. If we first assume that the desired partial differential equation $F = 0$ is nonlinear then, as is well-known, it corresponds to a three-fold infinitude of characteristics, and thus these curves satisfy only *one* equation of the form:

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*) In the course of a conversation in the summer of 1870, Darboux communicated to me that he was aware at that time of the theorem that every line complex determines a partial differential equation of first order whose characteristics are principal tangent curves.

$$f(x, y, z, dx, dy, dz) = 0.$$

Here, if we re place y and z with the equivalent expressions:

$$y = \frac{(ydx - xdy) + xdy}{dx}, \qquad z = \frac{xdz - (xdz - zdx)}{dx}$$

then the equation of the characteristics ($f = 0$) takes the form:

$$\chi[dx, dy, dz, (x\,dz - z\,dx), (y\,dx - x\,dy), \varphi(x)] = 0,$$

and indeed we know that when $\varphi(x)$ is a constant, and only then, the characteristics will be enveloped by the lines of a line complex. When one now eliminates the quantities dx, dy, dz from the equations:

$$\chi = 0, \quad \chi'_{dx} = \rho\, p, \qquad \chi'_{dy} = \rho\, q, \qquad \chi'_{dz} = -\rho,$$

according to the ordinary rules, in which ρ falls away, one obtains the original partial differential equation ($F = 0$) in the form:

$$\pi[x, y, z, p, q, \varphi(x)] = 0,$$

and indeed one wonders here whether the expression π can satisfy equation (1) in other cases than when $\varphi(x)$ is a constant.

If one carries out the operations on π that are given by (1) then what remains is a reduction that is based on the fact that in the given case π satisfies equation (1), namely:

$$\frac{d\pi}{dp} \cdot \frac{d\pi}{d\varphi} \cdot \frac{d\varphi}{dx} = 0,$$

an equation that splits into three equations:

$$\frac{\partial \pi}{\partial p} = 0, \qquad \frac{\partial \pi}{\partial \varphi} = 0, \qquad \frac{\partial \varphi}{\partial x} = 0.$$

If $\partial\pi / \partial p$ is equal to zero then p does not enter into the equation $\pi = 0$ at all, and it can then be put into the *linear* form:

$$q = \Phi(x, y, z);$$

however, we have casually excluded this case. The equations $\partial\pi / \partial\varphi = 0$ and $\partial\pi / \partial x = 0$ say, respectively, that π does not contain the quantity φ and that φ is a constant, and thus my initial assertion is proved, insofar as it relates to nonlinear equations.

We now go on to the case in which $F = 0$ is a linear partial differential equation. There is then a two-fold infinitude of characteristics, and it is easy to recognize that they must be straight lines if the property in question is to enter the picture. Namely, if we consider a point p that goes through a characteristic c, and finally, a variable infinitely close characteristic c'. It is clear that the tangent plane of the corresponding integral surface at the point p varies with c'; however, this plane shall osculate the curve c at this point, and thus c must possess the property that its points correspond to an indeterminate osculating plane. This is, however, the case only for straight lines.

The results derived thereby may be summarized in the following manner:

There are two distinct classes of partial differential equations of first order whose characteristics are principal tangent curves on the integral surfaces: One of them consists of linear differential equations whose rectilinear characteristics define a congruence. The other class corresponds to the Plücker line complex, in the sense, that the characteristics of such a differential equation will be enveloped by the lines of the complex. Thus, the problem of integration comes down to this: Find the most general surface whose two-fold infinitude of principal tangents belong to a system in a given line complex. We denote the totality of these two classes by the symbol D_{11}. [*])

If D_{11} corresponds to a line congruence then the associated focal surface is its singular integral. In the second case, as Klein has remarked to me, one finds the developable surfaces of singular lines amongst the integral surfaces, along with the singularity surfaces of the line complex in question.

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42. In connection with the contents of this paragraph, one has the following theorem:

On a surface, there are a single infinitude of curves whose tangents belong to a given line complex. If these curves have an enveloping curve – which is not true, in general – then two cases are possible: Either the surface contacts the cone of the complex whose vertex lie on the enveloping curve, and then it is principal tangent curve, or the tangents to the enveloping curve are double edges of the cone of the complex in question; in the latter case, one can conclude nothing.

It was by an application of this theorem that I determined the principal tangent curves of the tetrahedrally-symmetric surfaces (Götting. Nachr., Jan., 1870). [Clebsch, by another method that I will discuss later, was already led to this determination without having published anything about it. Later (Bulletin, Nov., 1870, pp. 8), Darboux obtained the same result as a corollary to a general determination.] Each tetrahedrally-symmetric surface has the aformentioned relationship with a line complex whose lines intersect the fundamental tetrahedron associated with the surface with constant double ratios.

[*]) Klein remarked to me that the results of this paragraph can be proved quite easily and lucidly by a geometric argument.

§ 15.

Partial differential equations of first order whose characteristics are lines of curvature on the integral surfaces.

43. We know that our sphere map takes the elements of one surface f to the elements of a surface F. Obviously, a pair-wise union between all curves of the two surfaces is thus established, and indeed, the principal tangent curves on f, in particular, correspond to the lines of curvature on F. From this, it follows that two surfaces f_1 and f_2 that contact each other along a principal tangent curve generally go to surfaces F_1 and F_2 that have the same relationship along a line of curvature. This theorem thus suffers an exception (which admittedly will not come under consideration in the sequel), and in order to make everything as clear as possible, I will not go into it further here. *)

From § 6.18, 20, we known that when two spaces r and R are reciprocally related to each other by a system of equations:

$$F_1(x, y, z, X, Y, Z) = 0, \qquad F_2(x, y, z, X, Y, Z) = 0,$$

a surface element of one space that contacts an elementary cone of the complex is mapped to a similar element in the other space. Now, in general, in any space there is obviously a *four-fold* infinitude of elements in this distinguished position. However, if the elementary cone of the complex in the space r is a *plane* pencil of lines then there is only a *three-fold* infinitude of such elements in r – I denote it with the notation e – that corresponds to the *four-fold* infinitude of distinguished elements E in the space R. *An element e then corresponds to a single infinitude of elements E.*

This is true especially for our sphere map. Only *one* distinguished element goes through each point of r, namely, the one that is associated with the point in question by the linear complex $H = 0$. On the other hand, the distinguished elements E are the ones that satisfy the equation:

$$1 + P^2 + Q^2 = 0,$$

where P and Q shall denote the partial derivatives of Z with respect to X and Y, as before. As a geometric argument shows, *it is always the single infinitude of elements E that are connected with a line of null length that correspond to these elements e in the space r.*

44. From the developments above, we can state the following theorem:

If two surfaces f_1 and f_2 contact each other along a principal tangent curve, and the tangents to this curve do not belong to the linear complex then the image surfaces F_1 and F_2 contact each other along a line of curvature. In the excluded case, all tangents to the surfaces f_1 and f_2 that go through a point of the common principal tangent curve belong to the complex $H = 0$, and then one can only conclude that the image surfaces F_1 F_2 are

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*) We must defer to another work a thorough discussion of all of the peculiarities that come about under the Nöther map of linear complexes and my own sphere map that was based upon it with respect to the fundamental structures of the two spaces.

inscribed in a common developable surface that includes the infinitely distant imaginary circle (§ 26, 84b).

If one considers the fact that there is no partial differential equation whose curvilinear characteristics are all enveloped by lines of the *linear* complex $H = 0$ then one may conclude (§ 6.17) that our sphere map takes each D_{11} into a partial differential equation whose characteristics are lines of curvature. On the other hand, we obtain all differential equations with this property in this way, because the following theorem is true without exception:

Two surfaces F_1 and F_2 that contact each other along a line of curvature give image surfaces in r that contact each other along a common principal tangent curve.

The result obtained in the foregoing paragraphs goes over to the following one:

There are two distinct classes of partial differential equations of first order whose characteristics are lines of curvature on the integral surfaces. Equations of the first class can thus be characterized by saying that they admit the totality of the two-fold infinitude of spheres – a sphere congruence – as a complete integral. The general integral will thus consist of a tubular surface, and its circular lines of curvature are the characteristics. The second corresponds to the sphere complexes. Geometrically, the problem of integrating such a differential equation comes down to this: Find the most general surface whose two-fold infinitude of principal spheres belong to a system in a given complex. I will denote the totality of these two classes by the symbol D_{12}. [*])

45. In his work: *Partielle Differential-Gleichungen*, pages 127-129, Du Bois-Reymond has posed the problem that we just raised. About it, he remarked that if the characteristics are lines of curvature then this also the case with the trajectories. However, characteristics and trajectories then intersect each other orthogonally, and therefore the stated problem (§ 14.40) will come down to the integration of the partial differential equation:

$$\frac{\partial F}{\partial x}\left[\frac{\partial F}{\partial p}+q\left(p\cdot\frac{\partial F}{\partial p}+q\frac{\partial F}{\partial q}\right)\right]-\frac{\partial F}{\partial y}\left[\frac{\partial F}{\partial p}+p\left(p\cdot\frac{\partial F}{\partial p}+q\frac{\partial F}{\partial q}\right)\right]+\frac{\partial F}{\partial z}\left[p\frac{\partial F}{\partial p}-q\frac{\partial F}{\partial p}\right]=0.$$

Du Bois-Reymond carried out this integration in some very simple cases [**]), and thus raised the conjecture that no substantial analytical difficulties would arise in the general case, as well. In any case, the present solution has a certain special interest.

Here, it might also be appropriate to remark that if two surfaces f_1 and f_2 have contact of n^{th} order with each other along a principal tangent curve then the image surfaces in general have the same relationship along a line of curvature. As a consequence, under our sphere map, a partial differential equation of n^{th} order whose characteristics are a system of principal tangent curves on the integral surfaces will go to an equation of the same order whose characteristics are a system of lines of curvature.

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[*]) Darboux has communicated to me that he has also remarked that the problem of determining a surface by a property of the principal spheres, in a certain sense, leads only to a partial differential equation of *first* order. (Cf., the citation to Darboux in § 24.)

[**]) The first two types of D_{12} for Du Bois-Reymond were: The developable surfaces that circumscribe a surface and the surfaces of rotation that are associated with a given axis, which correspond to *congruences* in a *linear* sphere complex. The geometric meaning of his third type is not clear to me.

46. I will mention here how some metric theories can be generalized.

As is known, the concept of the lines of curvature of a surface can be extended in the following way: Let there be given a four-fold infinitude of surfaces U, an arbitrary surface F, and a point p on it. There is always some U that has a stationary contact with F at p, and thus F intersects in a curve with a cusp. If one considers the tangent to this cusp then the continuous sequence of mutually intersecting associated directions define curves that I will refer to as the *U-lines of curvature* of the surface F.

The case in which all U can be derived from a certain surface U_0 by the application of all translations and parallel transformations is worthy of note. All theories about lines of curvature, and especially the ones that are given in this treatise, can be extended, in a certain sense, to this case (§ 13, conclusion). It is then, by way of example, always possible to find all surfaces whose U-lines of curvature possess the property that the surface normals that are erected at its points are parallel with a plane. Namely, by a certain transformation, this problem corresponds to one that Bonnet solved: Find all surfaces with planar lines of curvature. One further has the validity of the theorems:

a) If a surface is mapped to a sphere by the ordinary methods then the U-lines of curvature go to a family of *orthogonal* curves.

b) The determination of the U-lines of curvature of a surface F can come down to finding the lines of curvature of a certain other surface Φ.

In particular, if U_0 and F are minimal surfaces or surfaces that are parallel to them then Φ is also a surface of that type, and thus the U-lines of curvature of F can be determined.

The case in which all surfaces are similar to U and similarly located also deserves a special examination. The directions of the U-lines of curvature that are associated with a point then have a harmonic position pair-wise with respect to the principal tangents of the surface, as they do with respect to the principal tangents of the stationary contacting U.

Finally, if one assumes that all U are an infinitely cylinders – i.e., a straight lines – then the U-lines of curvature are identical with the principal tangent curves to the surface in question.

§ 16.

Partial differential equations of first order whose characteristics are geodetic curves on the integral surfaces.

47. The developments of this paragraph may be based upon the well-known theorem: If two surfaces I and U have a contact of n^{th} order with each other along a line of curvature then their center surfaces C_i and C_u have the same relationship with respect to a common geodetic curve.

If one considers, on the one hand, the integral surfaces I of a D_{12} and among them, the surfaces U of a complete integral, and on the other hand, the associated center surfaces C_i and C_u then one easily sees (§ 6, 17) that the surfaces C_i satisfy a partial differential equation of first order D_{13} whose characteristics are geodetic curves and thus the surfaces C_n define a complete integral.

In general, we can say that the integral surfaces of a partial differential equation of n^{th} order D_{n2} whose characteristics are a system of lines of curvature correspond to center surfaces that satisfy a differential equation of the same order D_{n3}, and indeed the characteristics of a system are geodetic curves. *)

From the above, any D_{n2} corresponds to a differential equation whose characteristics are geodetic curves; the converse, by comparison, is not true, and as a consequence the D_{n3} are not the only partial differential equations that possess the property that their characteristics are geodetic curves.

48. For the determination of the general form of the equations D_{13}, we choose another path, in which we lean on the theory of reciprocal curve complexes that was developed in section one.

Namely, let an arbitrary line complex be given in the space r and let a corresponding sphere complex be given in R that may be represented by an equation:

$$F(X, Y, Z, H) = 0.$$

Thus, on the one hand one must regard X, Y, Z, H as line coordinates (§ 10.30) with respect to four linear complexes that lie pair-wise in involution, and, on the other hand, as sphere coordinates. *If we now regard the center* (X, Y, Z) *of an arbitrary sphere* (X, Y, Z, H) *of the aforementioned sphere complex as the image of the lines* (X, Y, Z, H) *then we obtain a map of the line complex* $F(X, Y, Z, H) = 0$ *to the point space* R under which each line of the complex corresponds to a definite point, while there is a number of lines in the complex that map to the same point, namely, there are as many lines as the degree of the equation $F(X, Y, Z, H) = 0$ with respect to H. The lines of the complex that go through a point correspond to the points of curve C, and it is clear that *all C define a curve complex that has a reciprocal relationship to our line complex,* as we considered in the first section. Here, we must recall that the partial differential equations of first order that are associated with two reciprocal curve complexes are equivalent to each other in the sense that the solution of the one gives that of the other.

In the equations of a line:

$$r\,z = x - \rho, \qquad s\,z = y - \sigma,$$

if one substitutes (§ 9.27) the values:

$$\rho = \tfrac{1}{2}(X + iY), \qquad s = \tfrac{1}{2}(X - iY),$$
$$\sigma = \tfrac{1}{2}(Z \pm iH), \qquad r = -\tfrac{1}{2}(Z \mp iH)$$

then the relations:

$$-(Z \mp iH)\,z = 2x - (X + iY),$$
$$(X - iY)\,z = 2y - (Z \pm iH),$$

*) Integral surfaces that correspond to a differential equation of n^{th} order whose center surfaces satisfy a certain equation of $(n + 1)^{\text{th}}$ order. Besides the center surfaces, the latter equation possesses still more integral surfaces in the general case.

in which one regards H as the function of X, Y, Z that is determined by $F(X, Y, Z, H) = 0$, determine the aforementioned map of the two spaces. If one now (§ 3.6) differentiates with respect to X, Y, Z:

$$-(dZ \mp dH)\, z = -(dX + i\, dY),$$
$$(dX - i\, dY)\, z = -(dZ \pm dH),$$

and eliminates x, y, z between these two (and the original) equations then one obtains the *differential equation of the curve complex in R:*

$$dX^2 + dY^2 + dZ^2 + (i\, dH)^2 = 0,$$

which one can also write:

$$dX^2 + dY^2 + dZ^2 = dH^2,$$

an equation whose geometric meaning is that the two spheres (X, Y, Z, H) *and* $(X + \Delta X, Y + \Delta Y, Z + \Delta Z, H + \Delta H)$ *contact each other, so the corresponding lines intersect each other.*

The elementary cones of the complex:

$$dX^2 + dY^2 + dZ^2 = \left(\frac{\partial H}{\partial X}dX + \frac{\partial H}{\partial Y}dY + \frac{\partial H}{\partial Z}dZ\right)^2,$$

as their equation shows, contact the infinitely distant imaginary circle in its two intersection points with the plane:

$$\frac{\partial H}{\partial X}dX + \frac{\partial H}{\partial Y}dY + \frac{\partial H}{\partial Z}dZ = 0,$$

and thus they are the cones of revolution whose axis possesses the direction cosines: $\partial H / \partial X$, $\partial H / \partial Y$, $\partial H / \partial Z$. We thus obtain the following obvious presentation of this curve complex:

The elementary cones of the complexes, whose apexes lies on an arbitrary surface in the family $H = \text{const}$, *are cones of revolution whose axes are the corresponding normals to the surface in question. The angles of these cones vary, as the equation:*

$$dX^2 + dY^2 + dZ^2 = dH^2$$

shows, in such a way that the infinitely close surfaces $H = C$ *and* $H = C + \Delta C$ *cut out segments of the same magnitude from the generators of this cone. From this, it follows, as we will likewise show, that the surfaces* $H = C$ *intersect the integral surfaces of our* D_{13} *in equidistant curves; thus, the associated orthogonal curves are, as is known, geodetic lines and likewise characteristics with respect to the* D_{13}. *This geometric interpretation of a* D_{13} *easily gives the general formula for them:*

$$\frac{\partial H}{\partial X}P+\frac{\partial H}{\partial Y}Q-\frac{\partial H}{\partial Z}=\sqrt{1+P^2+Q^2}\cdot\sqrt{\left(\frac{\partial H}{\partial X}\right)^2+\left(\frac{\partial H}{\partial Y}\right)^2+\left(\frac{\partial H}{\partial Z}\right)^2-1},$$

an equation that we find later by an analytical method.

One considers an arbitrary curve k that lies on $H = C$ and the points of the elementary cones of the complex that are associated with it, whose infinitesimal intersection curve with the surface $H = C + \Delta C$ determines two enveloping curves k', among which we choose one of them; it is known that the surface strip that lies between k and k' is an integral surface. By repeating this operation, one finds a family of curves k in the successive surfaces $H = C$ whose totality defines an integral surface, and thus it follows from the foregoing that all of the k are equidistant curves. Now, the tangents to a k and the axis of the associated cone of the complex are always perpendicular to each other, and this cone contacts the integral surface in question along a direction that likewise intersects the tangent to k orthogonally. *The characteristics and the curves k define an orthogonal system, as we asserted earlier.* The curves k are, however, equidistant, and thus we again find the theorem that the characteristics of a D_{13} are geodetic curves on the integral surfaces.

In order to determine the partial differential equation that is associated with the differential equation:

$$W = dX^2 + dY^2 + dZ^2 - \left(\frac{\partial H}{\partial X}dX+\frac{\partial H}{\partial Y}dY+\frac{\partial H}{\partial Z}dZ\right)^2 = 0,$$

one must eliminate the quantities dX, dY, dZ from the equations:

$$\frac{\partial W}{\partial X}=\rho P, \qquad \frac{\partial W}{\partial Y}=\rho Q, \qquad \frac{\partial W}{\partial Z}=-\rho,$$

and one thus finds for *the general form of the partial differential equations* D_{13} :

$$\frac{\partial H}{\partial X}P+\frac{\partial H}{\partial Y}Q-\frac{\partial H}{\partial Z}=\sqrt{1+P^2+Q^2}\cdot\sqrt{\left(\frac{\partial H}{\partial X}\right)^2+\left(\frac{\partial H}{\partial Y}\right)^2+\left(\frac{\partial H}{\partial Z}\right)^2-1},$$

assuming that H denotes an arbitrary known function of X, Y, Z.

From our previous developments (§ 6.18), it follows that the integration of a D_{13} can come down to the determination of the principal tangent curves of the corresponding line complexes. The characteristics in question are then indeed reciprocal curves with respect to the mapping equations:

$$-(Z \mp H)\,z = 2x - (X + i\,Y),$$
$$(X - i\,Y)\,z = 2y - (Z \pm H),$$

and if one thus knows the general equation of a system of curves then one finds the general equation of the other one by differentiation and elimination. On the other hand, if we say that the integration of a D_{13} is equivalent to that of a D_{12} then what emerges from this, geometrically speaking, is that instead of determining a surface by a property of the principal cone, we look for the center surface. Bonnet employed such a transformation for his determination of all surfaces with planar or spherical lines of curvature.

I communicated the aforementioned form for a D_{13} to the Akademie zu Christiania (October, 1870) in a note in which I presented the three classes D_{11}, D_{12}, D_{13}, among other things. One obtains a symmetric form when one poses the problem in the following manner: Find an equation $\Phi(X, Y, Z, H) = 0$ such that when it is combined with that of the sphere complex in question $\Pi(X, Y, Z, H) = 0$ it gives Z as the required function of X and Y. I thank Klein for this remark (or really something that is equivalent to it), who was led to it during his own line-geometric investigations (cf., the second paper that follows this one). On the other hand, Darboux just communicated to me (October, 1871) that he had found a corresponding form by investigating sphere complexes.

I shall summarize the most important result of the three foregoing paragraphs as follows:

Partial differential equations of n^{th} order whose characteristics are principal tangent curves or lines of curvature and a class whose characteristics are geodetic lines define equivalent problems in the sense that they can be transformed into each other inversely. In particular, if n equals 1 *then these problems correspond to examinations of congruences and complexes whose elements are straight lines or spheres.*

Here, it should also be remarked that just as there is a cycle of conversions that take the equations D_{n1} into equations of the same type – namely, all linear point transformations, together with all dualistic deformations of space – there is also a cycle of conversions that preserves the character of the equations D_{n2} and D_{n3} .

§ 17.

On line complexes that possess infinitesimal linear transformations into themselves. *)

49. Line complexes that can be represented by an equation of the form:

$$F(X, Y, Z) = 0$$

map to the spheres whose centers lie on the surface $F(X, Y, Z) = 0$. This sphere complex will now obviously be taken to itself by an *arbitrary* parallel transformation, or, what amounts to the same thing, an *infinitesimal* one, and thus we can, from § 13.38, characterize the line complex $F(X, Y, Z) = 0$ by saying that it admits an infinitesimal transformation of the form:

$$z_1 = z_2, \qquad x_1 = x_2 + a\, z_2 + b, \qquad y_1 = y_2 + c\, z_2 + d.$$

*) Cf., *Sur une certaine famille de courbes et de surfaces,* by Klein and Lie, Comptes rendus, 1870. *Ueber vertauschbare lineare Transformationen* by Klein and Lie, Math. Ann. Bd. 4.

One now remarks that the problem of finding the general surface whose center of curvature lies on a given surface comes down to the problem of finding the geodetic curves of that surface. Our previous theories thus give the following interesting theorem: *The determination of the principal tangent curves of the line complex* $F(X, Y, Z) = 0$ *and the search for the geodetic curves on the surface* $F(X, Y, Z) = 0$ *are equivalent problems.* One must remark that the degree of the line complex is equal to the order of the surface; however, while the surface is arbitrary, the complex must possess the aforementioned infinitesimal transformation into itself.

Among the linear tangent complexes of the sphere complex $F(X, Y, Z) = 0$, I consider the following one:

$$\frac{\partial F_0}{\partial X_0}(X - X_0) + \frac{\partial F_0}{\partial Y_0}(Y - Y_0) + \frac{\partial F_0}{\partial Z_0}(Z - Z_0) = 0,$$

whose spheres intersect a tangent plane of the surface $F(X, Y, Z) = 0$ orthogonally (§ 10.30). An arbitrary parallel transformation takes the given complex, as well as the tangent complex to itself, and thus we see that these complexes contact each other in a single infinitude of common spheres. As a result, the complex $F(X, Y, Z) = 0$ may be regarded as the envelope of a two-fold infinitude of linear complexes. If we turn to the representation of lines then we can define corresponding line complexes to be envelopes of a two-fold infinitude of linear complexes that are in involution with each other with respect to a given linear complex $H = 0$, and, in addition, contain a common line in it (viz., the fundamental line of the space r) (cf., § 10.30).

A two-fold infinitude of linear complexes that lie in involution and contain a line of the latter complex, moreover, envelopes a line complex whose principal tangent curves may be determined by seeking the geodetic curves of a certain surface.

In the next section, I will come back to the contents of this paragraph.

50. By the developments of the foregoing paragraph, one will be led to pose the problem of whether the determination of the principal tangent curves can always be simplified when the line complex in question admits an infinitesimal linear transformation. The answer lies immediately in the aforementioned papers of Klein and myself (cf., especially these Annalen, Bd. 4, pp. 80). Namely, we have directed our attention to the fact that when an infinitesimal transformation is known for a structure, the determination of other structures that have an unperturbed relationship with the given one under the transformation in question can be simplified by a suitable choice of coordinates, in general.

Thus, one must apply those curves that define the geometric locus for the infinitesimal path that all points of space describe under the transformation in question. In particular, if we assume that the known transformation is a linear one then these curves will be just the ones that Klein and myself examined in the form of space curves W. One arranges the two-fold infinitude of curves W in question in two ways into families of surfaces:

$$U_1 = A, \qquad U_2 = B.$$

Thus, each surface U_1 or U_2 goes to itself under the associated transformation. One further chooses a third family: namely, those surfaces:

$$V = C$$

that emerge from an arbitrary one by the continuous application of the transformation in question, and therefore C shall be the parameter of the transformation.

If one now introduces U_1, U_2, and V as the point coordinates then, by way of example, the equation of each surface that admits a transformation assumes the form:

$$F(U_1, U_2) = 0.$$

Likewise, one can write a partial differential equation of first order whose totality of elementary cones of the complex remains unchanged under the transformation, in the following way:

$$F\left(U_1, U_2, \frac{\partial V}{\partial U_1}, \frac{\partial V}{\partial U_2}\right) = 0,$$

which is, as is well-known, a step forward. In particular, this is the case with the D_{11} of a line complex that remains unchanged in its own right.

If we consider, for example, the four linear complexes $X = 0$, $Y = 0$, $Z = 0$, $H = 0$ that are pair-wise in involution, and a line complex whose equation is the following one:

$$F\left(\frac{X}{H}, \frac{Y}{H}, \frac{Z}{H}\right) = 0$$

then it is clear that each transformation amongst the infinitude:

$$X_1 = m\,X_2, \qquad Y_1 = m\,Y_2, \qquad Z_1 = m\,Z_2, \qquad H_1 = m\,H_2$$

takes our complex into itself, and thus the associated D_{11} assumes the form above. As will be shown in the next section, a complex of second degree with 17 constants belongs to them, and indeed the same is true for the general complex of second degree whose singularity surface is a ruled surface. The complexes of second degree with 18 and 19 constants admit no infinitesimal linear transformation. *)

51. Likewise, for the investigation of spatial structures that admit two infinitesimal and permutable linear transformations it is preferable to make a special choice of coordinates. First, one takes the single infinitude of surfaces:

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*) The line complex $F\left(\frac{X}{H}, \frac{Y}{H}, \frac{Z}{H}\right) = 0$ may also be defined as the envelope of a two-fold infinitude of linear complexes that are in involution with two given complexes. Among the principal tangent curves of such a complex, there is a single infinitude of distinguished families. Each of them consists of a single infinitude of curves of a linear complex $X^2 + Y^2 + Z^2 - H^2 = 0$. The aforementioned theorem is, at the same time, a transformation and a generalization of the well-known theorem: Among the geodetic curves of a surface there is a single infinitude of them whose tangents intersect the imaginary circle. The line complex that is considered here possesses the characteristic property that its singularity surfaces are ruled surfaces with two straight guide lines. Each curved principal tangent curve of such a ruled surface will also be enveloped by the lines of a linear complex $X^2 + Y^2 + Z^2 - H^2 = \text{const.}$

$$V = A$$

that remain unchanged under our transformations. One further chooses two distinct *infinitesimal* transformations β, γ from our closed system, and finally, two surfaces B_0 and C_0. By a continuous application of the transformations β and γ to these surfaces, one obtains two families of surfaces:

$$U_1 = B, \qquad U_2 = C,$$

where B and C denote transformation constants. If one now chooses V, U_1, U_2 for point coordinates then the D_{11} of a line complex that remains unchanged under our transformations takes on the form:

$$F\left(V, \frac{\partial V}{\partial U_1}, \frac{\partial V}{\partial U_2}\right) = 0.$$

As is known, the integration of this equation comes down to a quadrature.

We thus come to a class of complexes whose principal tangent curves can be determined. Among them, one finds, e.g., the complex of second degree whose singularity surface decomposes into two surfaces of second degree, which then necessarily have four generators in question.

By way of example, one obtains a general determination of the geodetic curves on any screw surface here. The totality of all spheres whose centers lie on such a surface admits two permutable, infinitesimal transformations into itself, namely, a screw motion and a parallel transformation, and, from § 13, such deformations of the sphere space R correspond to linear point transformations of the line space r. Thus, the D_{11} whose integration is equivalent to the determination of each geodetic curve belong to the category that was mentioned in no. 51, from which, my assertion is proved.

If one seeks the geodetic curves on a surface that admit an infinitesimal linear transformation for which the imaginary sphere circle keeps its position then one can, by the method that was set down in no. 50, convert this problem into the integration of an ordinary differential equation of first order in two variables. The lines of curvature and principal tangent curves of this surface can be determined (Cf., the paper cited, Math. Ann., Bd. 4, pp. 84.)

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52. As a last example, I consider the well-known problem: Find all surfaces whose normals belong to a given line complex. [*]) Abel Transon has shown that this problem, which leads immediately to an equation of the form:

$$F(x, y, z, p, q) = 0,$$

[*]) It is, perhaps, nowhere stated explicitly that this problem is, in a certain sense, equivalent to the following one: Find all surfaces that contain a family of geodetic curves whose tangents belong to a given line complex. If one omits the developable surfaces from the line complex in question then one expresses the aforementioned problem *immediately* by a partial differential equation of first order.

always admits a simplification. *) The same thing can be supported by the following simple remark of Darboux (Bulletin, Nov., 1870, pp. 3): The parallel surfaces to an integral surface are themselves integral surfaces. The totality of all integral surfaces thus admits an infinitesimal parallel transformation, and thus the equation $F = 0$ belongs to the category in number 50.

If we now assume that the line complex admits an infinitesimal motion then $F = 0$ will be integrable. This is especially the case when the complex can be described by the rotation of a line congruence around a fixed axis. To them belong the linear complexes, and in fact, it is also known – although it is, perhaps, nowhere stated explicitly – that all screw surfaces that correspond to a certain screw motion satisfy the specified equation $F = 0$. **) To them, belongs a complex of second degree whose singularity surface consists of a sphere and two parallel tangent planes to it, and finally, the well-known complex whose lines intersect a tetrahedron with constant double ratios, under the assumption that two tetrahedral vertices lie on the imaginary sphere circle.

Darboux found, by means of his aforementioned remark, that there is another homographic particularization of this complex that Binet and Chasles considered, namely, the ones whose associated equation $F = 0$ can be integrated. This also allows one to conclude that in this case, as Reye has remarked, one can give a two-fold infinitude of surfaces of second degree whose normals are lines of the complex.

§ 18.

Trajectory circle. Trajectory curve.

53. On any sphere of a sphere complex there lies a distinguished circle that can be regarded, to some extent, as the neighboring sphere. In order to clarify the meaning of this statement, and to find the equation of this circle, moreover, it will be preferable to resort to line geometry.

Following Plücker, there is a single infinitude of linear complexes that contact a given complex $F(X, Y, Z, H) = 0$ in a line (X_0, Y_0, Z_0, H_0) that belongs to it; we understand this to mean that the infinitely close lines are all common to these complexes. A tangential complex is distinguished in our coordinate system, namely, the following one:

$$H - H_0 = \frac{\partial H_0}{\partial X_0}(X - X_0) + \frac{\partial H_0}{\partial Y_0}(Y - Y_0) + \frac{\partial H_0}{\partial Z_0}(Z - Z_0) .$$

When we revert to the sphere picture, this consists of all spheres that intersect the plane:

*) Journal de l'École Polytéchnique, 1861.

**) A linear complex admits two infinitesimal and permutable motions. As a consequence, the associated $F = 0$ possesses *three* infinitesimal and permutable transformations into itself. The corresponding partial differential equation of the line space r admits three such transformations, which are *linear* point transformations. From this, one may conclude, e.g., that each screw motion gives a two-fold infinitude of screw surfaces on which the screw lines in question are lines of curvature.

$$-H_0 = \frac{\partial H_0}{\partial X_0}(X - X_0) + \frac{\partial H_0}{\partial Y_0}(Y - Y_0) + \frac{\partial H_0}{\partial Z_0}(Z - Z_0)$$

like the sphere:

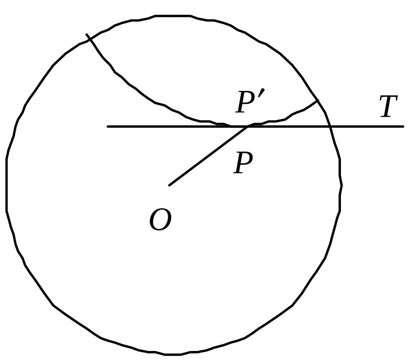
$$H_0^2 = (X - X_0)^2 + (Y - Y)^2 + (Z - Z_0)^2.$$

One sees that the last two equations, or the circle that they represent, defines the neighboring spheres. In particular, the sphere (X_0, Y_0, Z_0, H_0) will contact a single infinitude of neighboring spheres at the points of this circle. One can remark that our circle likewise lies on the elementary cone of the complex:

$$(X - X_0)^2 + (Y - Y)^2 + (Z - Z_0)^2 = \left[\frac{\partial H_0}{\partial X_0}(X - X_0) + \frac{\partial H_0}{\partial Y_0}(Y - Y_0) + \frac{\partial H_0}{\partial Z_0}(Z - Z_0)\right]^2$$

that is associated with the sphere H_0, and this is geometrically evident since this cone defines all directions from which one must go when starting at the point (X_0, Y_0, Z_0) if the associated sphere should contact the original one (X_0, Y_0, Z_0, H_0).

One now recalls the geometric meaning of the problem: Integrate the D_{12} that is associated with a sphere complex, so one sees that each integral surface that has a stationary contact with the cone H_0 contacts it at a point of a circle; Thus, as I will likewise prove, the associated tangent *PT* (see the figure) to the circle is the always the corresponding trajectory direction of the integral surface.



The line *PO*, where *O* is the center of our sphere, contacts – namely, at *O* – a geodetic curve that lies on the center surface of our integral curve whose tangents meet the integral surface at the points of characteristic (viz., a line of curvature). If we now let P' denote one of these points that lies infinitely close to *P* then the plane OPP' osculates the aforementioned geodetic curve at *O*, and consequently is perpendicular to the plane *OPT*, which likewise contacts the elementary cone of the complex:

$$(X - X_0)^2 + (Y - Y)^2 + (Z - Z_0)^2 = \left[\frac{\partial H_0}{\partial X_0}(X - X_0) + \frac{\partial H_0}{\partial Y_0}(Y - Y_0) + \frac{\partial H_0}{\partial Z_0}(Z - Z_0)\right]^2$$

along the line *OP* and the center surface at the point *O*. The elementary line *PT* thus intersects the direction of curvature PP' orthogonally; *PT* is the direction of the trajectory. We shall then call our circle the trajectory circle of the sphere H_0 .

Any sphere of a sphere complex will contact neighboring spheres of the complex at points of a certain circle. All integral surfaces of the associated D_{12}*, for which the sphere is a principal sphere, contact it at a point of a circle, and thus the corresponding tangent to the circle is always a trajectory direction. This circle, which I refer to as the trajectory circle, plays a meaningful role in the investigation of sphere complexes.*

All spheres of a space that contact a given sphere of a complex at the points of the trajectory circle define a sphere congruence that is the image of that special linear line congruence that is common to all of the linear tangential complexes that are associated with a line of the complex.

54. One obtains a tangible statement of the problem of integrating a given D_{12} in the following way: Every partial differential equation of first degree:

$$F(x, y, z, p, q) = 0$$

selects a four-fold infinitude of surface elements of space from the five-fold infinitude of all of them. In particular, the surface elements that correspond to a D_{12} distribute themselves into a three-fold infinitude of families, each of which is defined by a single infinitude of elements that lie on a sphere of the given complex and connect with the trajectory circle on it.

Here, the following remark might find a place, that one can find the differential equation between X, Y, Z, dX, dY, dZ that the trajectories of the associated D_{12} satisfy from the equation of a sphere complex $H = F(X, Y, Z)$ in the following way: One eliminates X_0, Y_0, Z_0 from the two equations of the trajectory circles:

$$U = (X - X_0)^2 + (Y - Y_0)^2 + (Z - Z_0)^2 + H_0^2 = 0,$$
$$V = H_0 + \frac{\partial H_0}{\partial X_0}(X - X_0) + \frac{\partial H_0}{\partial Y_0}(Y - Y_0) + \frac{\partial H_0}{\partial Z_0}(Z - Z_0) = 0,$$

and the corresponding differential equations:

$$\frac{\partial U}{\partial X} dX + \frac{\partial U}{\partial Y} dY + \frac{\partial U}{\partial Z} dZ = 0,$$
$$\frac{\partial V}{\partial X} dX + \frac{\partial V}{\partial Y} dY + \frac{\partial V}{\partial Z} dZ = 0,$$

and the desired equation emerges.

Finally, in order to find the partial differential equation D_{12} itself from the equation of the sphere complex one can proceed in the following way: The trajectory circle satisfies the equation:

$$(1) \qquad H_0 + \frac{\partial H_0}{\partial X_0}(X - X_0) + \frac{\partial H_0}{\partial Y_0}(Y - Y_0) + \frac{\partial H_0}{\partial Z_0}(Z - Z_0) = 0,$$

so the surface elements of our sphere that connect to the circle, which therefore satisfy the equation D_{12}, further satisfy the following relations:

$$(2) \qquad X - X_0 = \frac{H_0 P}{\sqrt{1 + P^2 + Q^2}}, \quad Y - Y_0 = \frac{H_0 Q}{\sqrt{1 + P^2 + Q^2}}, \quad Z - Z_0 = \frac{-H_0}{\sqrt{1 + P^2 + Q^2}}.$$

Upon substituting these values in (1), one finds that:

$$\sqrt{1+P^2+Q^2}+P\frac{\partial H_0}{\partial X_0}+Q\frac{\partial H_0}{\partial Y_0}-\frac{\partial H_0}{\partial Z_0} = 0,$$

and in this equation one must replace the X_0, Y_0, Z_0 with the values of these quantities that one takes from (2), as expressed in terms of X, Y, Z, P, and Q.

In the latter analytical developments, we always thought of H_0 as being a given function of X_0, Y_0, Z_0.

55. Among the elementary cones of the complex of a D_{12} whose apexes lie in a plane, there is a single infinitude of them that contact this plane. The locus of the apexes in question is a curve c whose tangent (as a trajectory direction) is always perpendicular to the contact direction of the corresponding cones of the complex (viz., the direction of the characteristic). The curve c may also be defined as the *geometric locus of all surface elements of our plane that satisfy the given* D_{12}. One can likewise determine all elementary cones of the complex whose apexes lie on an arbitrary sphere and look for the locus of points whose associated cone contacts the sphere. I also assert that *the tangents of this curve and the corresponding contact direction to the cone are orthogonal.* In order to prove this, all that is required is to perform a transformation through reciprocal radii in such a way that the sphere goes to a plane and the sphere complex goes to another sphere complex. We call the aforementioned curve the *trajectory curve* of our sphere, and it is clear that when the sphere belongs to the complex the trajectory curve splits into the trajectory circle and a second curve. We can also say that *the trajectory curve of a sphere is the geometric locus of all surface elements on it that satisfy the given* D_{12}.

If the sphere is infinitesimal then the surface elements of the sphere that connect with the trajectory curve envelop the elementary cone of the complex in question.

The cone whose apex lies at the center of an arbitrary sphere, and which contains the trajectory curves of it, goes to the corresponding *normal cone* when the sphere is infinitesimal; that is, to a cone whose generators are normals to all integral surfaces that go through the point in question.