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Perturbation theory and contact transformations

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In perturbation theory, one addresses the solution of the following problem:

Problem I: *Determine the most general transformation:*

$$x'_k = X_k(x_1, \ldots, x_n, p_1, \ldots, p_n),$$
$$p'_k = P_k(x_1, \ldots, x_n, p_1, \ldots, p_n) \qquad (k = 1, \ldots, n)$$

that simultaneously takes ***all*** *simultaneous systems of the form:*

$$dx_k = \frac{dF}{dp_k}dt, \qquad dp_k = -\frac{dF}{dx_k}dt \qquad (k = 1, \ldots, n)$$

into system of the same form in the new variables.

As is known, **Jacobi** and **Bour** have found that the most general transformation of the desired type is defined by the equations:

$$(X_k\,X_i) = (X_k\,P_i) = (P_k\,P_i) = 0, \qquad (P_k\,X_i) = 1. \tag{0}$$

On the other hand, in my opinion, the following problem is at the basis for the theory of contact transformations:

Problem II: *Determine* $2n$ *quantities* $X_1, \ldots, X_n, P_1, \ldots, P_n$ *as functions* $x_1, \ldots, x_n, p_1, \ldots, p_n$ *in the most general way such that a relation of the form:*

$$P_1\,dX_1 + \ldots + P_n\,dX_n = p_1\,dx_1 + \ldots + p_n\,dx_n + dV$$

exists, in which one assumes that V *is regarded as an undetermined function of* $x_1, \ldots, p_n$.

For me, one will obtain the most general solution to that problem when one takes an *arbitrary* system of quantities X_k, P_k that fulfills the relations (0) ([1]).

With that, one discovers a more precise connection between two apparently-different problems. That connection was so clear *a priori* in my *synthetic* way of looking at things that I have referred to Problem II as only a different form of Problem I on a different occasion ([2]). However, it has been my experience that even outstanding mathematicians have yet to clearly see the intrinsic basis for that connection. Thus, I regard it as useful to thoroughly prove, by *analytical* considerations, that the problems in question can actually be converted into each other in a reciprocal way. At the same time, I will show that my prior investigations into contact transformations will solve two general problems that can be regarded as generalizations of Problem I.

In connection with the foregoing, I will then prove, by some new considerations, that the differential equations of mechanics, as well as those of the calculus of variations, can be brought into the canonical form. Perhaps the celebrated **Hamilton-Jacobi** theory will take on a greater simplicity than before in that way.

In the last section, I will solve the following problem:

Problem III: *Determine the most general transformation that takes* ***a given*** *system of the form:*

$$dx_k = \frac{dF}{dp_k}\,dt\,, \qquad dp_k = -\frac{dF}{dx_k}\,dt \qquad (k = 1, \ldots, n)$$

into a similar system.

The transformations in question, which are no longer independent of the form of F, *are not contact transformations, in general*.

Finally, I shall give (without proof) a general case in which the integral of a given simultaneous system will admit some simplifications that correspond to those of a canonical system.

§ 1. – General canonical system.

1. – $2n$ equations of the form:

$$\text{(1)} \qquad \begin{cases} x_i' - x_i = \delta x_i = Y_i\,(x_1, \ldots, x_n, p_1, \ldots, p_n)\,\delta t\,, \\ p_i' - p_i = \delta p_i = Q_i\,(x_1, \ldots, x_n, p_1, \ldots, p_n)\,\delta t \end{cases} \qquad (i = 1, \ldots, n),$$

in which δt denotes an arbitrary infinitesimal quantity, determine an *infinitesimal* transformation between the variables $x_1, \ldots, x_n, p_1, \ldots, p_n$.

([1]) **Jacobi** considered Problem II, and added the further demand that the equations $X_1 = a_1, \ldots, X_n = a_n$ can be solved for $p_1, \ldots, p_n$. He recognized the necessity of the relations (0) in his statement of the problem, but their existence is not sufficient.

([2]) [Art. VII, pp. 49, nos. 1-7.]

I now demand that, in particular, this transformation should be an infinitesimal contact transformation, so analytically speaking, that the difference:

$$p_1' \, dx_1' + \cdots + p_n' \, dx_n' - (p_1 \, dx_1 + \cdots + p_n \, dx_n)$$

should be a complete differential $d\,\Omega$. That gives the condition equation:

$$\frac{\delta}{\delta t} \sum p_i \, dx_i = d\,\Omega ,$$

or when written out:

$$\sum_i \left(\frac{\delta p_i}{\delta t} dx_i + p_i \frac{\delta}{\delta t} (dx_i) \right) = d\,\Omega ,$$

from which, when one switches the symbols δ and d, one will get:

$$\sum_i \left(\frac{\delta p_i}{\delta t} dx_i + p_i \, d \frac{\delta x_i}{\delta t} \right) = d\,\Omega .$$

When we replace the values of δx_i and δp_i in (1) here, we will find the equation:

$$\sum_i (Q_i \, dx_i + p_i \, dY_i) = d\,\Omega ,$$

which is equivalent to the $2n$ following ones:

$$\frac{d\Omega}{dx_r} = Q_r + \sum_i p_i \frac{dY_i}{dx_r} , \qquad \frac{d\Omega}{dp_\rho} = \sum_i p_i \frac{dY_i}{dp_\rho} .$$

That will give:

$$\frac{d}{dx_\rho} \left(Q_r + \sum_i p_i \frac{dY_i}{dx_r} \right) = \frac{d}{dx_r} \left(Q_\rho + \sum_i p_i \frac{dY_i}{dx_\rho} \right) ,$$

$$\frac{d}{dp_\rho} \left(Q_r + \sum_i p_i \frac{dY_i}{dx_r} \right) = \frac{d}{dx_r} \sum_i p_i \frac{dY_i}{dp_\rho} ,$$

$$\frac{d}{dp_\rho} \sum_i p_i \frac{dY_i}{dx_r} = \frac{d}{dp_r} \sum_i p_i \frac{dY_i}{dp_\rho} ,$$

and after dropping the terms that cancel:

$$\frac{dQ_r}{dx_\rho} = \frac{dQ_\rho}{dx_r} , \qquad \frac{dQ_r}{dp_\rho} = -\frac{dY_\rho}{dx_r} , \qquad \frac{dY_\rho}{dp_r} = \frac{dY_r}{dp_\rho} ,$$

from which it will follow that Y_r and Q_ρ are the partial derivatives with respect to p_r and $-x_\rho$ of a function of $x_1, \ldots, x_n, p_1, \ldots, p_n$:

$$Y_r = \frac{dF}{dp_r}, \qquad Q_\rho = -\frac{dF}{dx_\rho}.$$

That gives:

Theorem 1:

Any infinitesimal contact transformation between x, p will possess the form:

$$\delta x_i = \frac{dF}{dp_i}\delta t, \qquad \delta p_i = -\frac{dF}{dx_i}\delta t \qquad (i = 1, \ldots, n),$$

in which F denotes an arbitrary function of $x_1, \ldots, x_n, p_1, \ldots, p_n$ [1].

2. – Conversely, I shall now seek the most general expression:

$$W = \sum_{k=1}^{n} X_k(x_1, \ldots, p_n)\, dx_k + \sum_{k=1}^{n} P_k(x_1, \ldots, p_n)\, dp_k$$

that possesses the property that the expression:

$$\frac{\delta W}{\delta t} = \sum_k \left(\frac{dW}{dx_k}\frac{dF}{dp_k} - \frac{dW}{dp_k}\frac{dF}{dx_k}\right)$$

is always a complete differential, no matter what the function F might be.

When one expands the condition equation:

$$\frac{\delta W}{\delta t} = d\Omega\,(x_1, \ldots, x_n, p_1, \ldots, p_n),$$

it will take the form:

$$d\,\Omega = \sum_k X_k\, d\frac{dF}{dp_k} + \sum_k (F\, X_k)\, dx_k - \sum_k P_k\, d\frac{dF}{dx_k} + \sum_k (F\, P_k)\, dp_k,$$

which will give:

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[1] By means of that theorem, Problem I will take the form: *Determine the most general analytical conversion under which all infinitesimal contact transformations will remain the same.* Obviously, any contact transformation will be such a conversion.

$$\frac{d\Omega}{dx_u} = \sum_k X_k \frac{d^2F}{dp_k\,dx_u} - \sum_k P_k \frac{d^2F}{dx_k\,dx_u} + (F\;X_u)\,,$$

$$\frac{d\Omega}{dp_u} = \sum_k X_k \frac{d^2F}{dp_k\,dp_v} - \sum_k P_k \frac{d^2F}{dx_k\,dp_v} + (F\;P_v)\,.$$

We now define the identity:

$$\frac{d}{dp_v}\frac{d\Omega}{dx_u} = \frac{d}{dx_v}\frac{d\Omega}{dp_u}\,,$$

and when we drop the terms that cancel, we will then find that:

$$\sum_k \frac{dX_k}{dp_v}\frac{d^2F}{dp_k\,dx_u} - \sum_k \frac{dP_k}{dp_v}\frac{d^2F}{dx_k\,dx_u} + \left(\frac{dF}{dp_v}, X_u\right) + \left(F, \frac{dX_u}{dp_v}\right)$$
$$- \sum_k \frac{dX_k}{dx_u}\frac{d^2F}{dp_k\,dp_v} + \sum_k \frac{dP_k}{dx_u}\frac{d^2F}{dx_k\,dp_v} - \left(\frac{dF}{dx_u}, P_v\right) - \left(F, \frac{dP_v}{dx_u}\right) = 0\,.$$

That relation must be true for any F. If we then combine those terms that include the same differential quotients of F then the coefficients that emerge for each such differential quotient must be zero. That will give the following equations:

(2) $$\frac{dX_k}{dp_v} - \frac{dP_v}{dx_k} = 0 \qquad \text{for} \qquad k \neq v\,,$$

(3) $$\frac{dX_v}{dp_v} - \frac{dP_v}{dx_v} = \frac{dX_u}{dp_u} - \frac{dP_u}{dx_u}\,,$$

(4) $$\frac{dX_u}{dx_k} - \frac{dX_k}{dx_u} = 0\,,$$

(5) $$\frac{dP_u}{dp_k} - \frac{dP_k}{dp_u} = 0\,,$$

(6) $$\frac{d}{dx_k}\left(\frac{dX_u}{dp_v} - \frac{dP_v}{dx_u}\right) = 0\,, \qquad \frac{d}{dx_k}\left(\frac{dX_u}{dp_v} - \frac{dP_v}{dx_u}\right) = 0\,.$$

The last two equations show that the quantity:

$$\frac{dX_v}{dp_v} - \frac{dP_v}{dx_v}$$

is constant, and at the same time, independent of the number v, due to (3). If we then let A denote an absolute constant then we can set:

$$\frac{dX_v}{dp_v} - \frac{dP_v}{dx_v} = A ,$$

from which, it will follow that:

$$\text{(7)} \qquad \frac{d\,(X_v - A\,p_v)}{dp_v} - \frac{dP_v}{dx_v} = 0 \qquad (v = 1, \ldots, n).$$

On the other hand, it is clear that equations (2) and (4) can be written as follows:

$$\frac{d\,(X_k - A\,p_k)}{dp_v} = \frac{dP_v}{dx_k} \qquad k \neq v ,$$

$$\frac{d\,(X_k - A\,p_k)}{dx_v} = \frac{d\,(X_v - A\,p_v)}{dx_k} .$$

Those equations, together with (5) and (7), show that the quantities $X_k - A\,p_k$ and P_i are the partial derivatives with respect to x_k and p_i, resp., of a function of $x_1, \ldots, x_n, p_1, \ldots, p_n$:

$$X_k - A\,p_k = \frac{dU}{dx_k}, \quad P_k = \frac{dU}{dp_k} \qquad (k = 1, \ldots, n).$$

That will imply that the desired expression W possesses the form:

$$\sum_k \left(A\,p_k + \frac{dU}{dx_k} \right) dx_k + \sum_k \frac{dU}{dp_k}\,dp_k ,$$

or what amounts to the same thing, the form:

$$A \sum_k p_k\,dx_k + dU .$$

Conversely, one easily proves that this expression will always possess the desired property, no matter what the constant A and the function U might be. That is because:

$$\begin{aligned} \frac{\delta}{\delta t} \sum_k p_k\,dx_k &= \sum_k \frac{\delta p_k}{\delta t}\,dx_k + \sum_k p_k\,d\,\frac{\delta x_k}{\delta t} \\ &= -\sum_k \frac{dF}{dx_k}\,dx_k + \sum_k p_k\,d\,\frac{dF}{dp_k} , \end{aligned}$$

so:

$$\frac{\delta}{\delta t}\sum_k p_k\,dx_k = d\left(-F+\sum_k p_k\frac{dF}{dp_k}\right).$$

On the other hand:

$$\frac{\delta}{\delta t}\,dU = d\,\frac{\delta U}{\delta t}\,.$$

We can then express the following theorem:

Theorem 2:

If a given expression:

$$W = \sum_k X_k\,dx_k + \sum_k P_k\,dp_k$$

possesses the property that (F W) *is always a complete differential in* $x_1, \dots, x_n, p_1, \dots, p_n$*, which might also be the function F, then W will possess the form* $A\sum_k p_k\,dx_k + dU$.

3. – Assuming that, I would like to think that one has introduced new variables into the simultaneous system:

$$\delta x_k = \frac{dF}{dp_k}\delta t\,, \qquad \delta p_k = -\frac{dF}{dx_k}\delta t \qquad (k = 1, \dots, n) \tag{8}$$

and the expression:

$$p_1\,dx_1 + \dots + p_n\,dx_n$$

in place of $x_1, \dots, x_n, p_1, \dots, p_n$, say, $y_1, \dots, y_n, q_1, \dots, q_n$. In so doing, y_k and q_k shall initially be subject to no other restriction than the obvious one that they should be independent functions of $x_1, \dots, x_n, p_1, \dots, p_n$. Let:

$$\delta y_k = \eta_k\,\delta t\,, \qquad \delta q_k = \kappa_k\,\delta t \qquad (k = 1, \dots, n)$$

be the new form of the simultaneous system (8), and let:

$$\sum_k p_k\,dx_k = \sum_k Y_k\,dy_k + \sum_k Q_k\,dq_k = W\,,$$

where Y_k and Q_k are certain functions of $y_1, \dots, y_n, q_1, \dots, q_n$. Now, from the foregoing:

$$\frac{\delta W}{\delta t} = \sum_i\left(\frac{dW}{dx_i}\frac{dF}{dp_i} - \frac{dW}{dp_i}\frac{dF}{dx_i}\right) = d\,\Omega\,.$$

When we also introduce the new variables here, that will give:

$$\sum_i \left(\frac{dW}{dy_i}\eta_i + \frac{dW}{dq_i}\kappa_i \right) = d\,\Omega\ .$$

If we then demand, in particular, that η_k and κ_k should possess the form:

$$\eta_k = \frac{d\Phi}{dx_k}\,, \qquad \kappa_k = -\,\frac{d\Phi}{dy_k} \qquad\qquad (k = 1, \ldots, n),$$

no matter what the form of the function F might be, then from Theorem 2, W must be regarded as a function of $y_1, \ldots, y_n, q_1, \ldots, q_n$ that possesses the form:

$$W = A \sum_i q_i\, dy_i + dV\ ,$$

and one then has:

$$\sum_k p_k\, dx_k = A \sum_i q_i\, dy_i + dV\ ,$$

which comes from the fact that our transformation must be a contact transformation between $x_1, \ldots, x_n, p_1, \ldots, p_n$ and $y_1, \ldots, y_n, q_1, \ldots, q_n$. Thus:

Theorem I:

If a given transformation between $x_1, \ldots, x_n, p_1, \ldots, p_n$ *and* $y_1, \ldots, y_n, q_1, \ldots, q_n$ *possesses the property that it takes every simultaneous system of the form:*

$$\delta x_k = \frac{dF}{dp_k}\,\delta t\,, \qquad \delta p_k = -\frac{dF}{dx_k}\,\delta t \qquad\qquad (k = 1, \ldots, n)$$

into a similar system in $y_1, \ldots, y_n, q_1, \ldots, q_n$ *then it will be a contact transformation, and there will then exist a relation of the form:*

$$\sum_k p_k\, dx_k = A \sum_i q_i\, dy_i + dV\ .$$

4. – I shall now postulate, in particular, the most general contact transformation between $x_1, \ldots, x_n, p_1, \ldots, p_n$ and $y_1, \ldots, y_n, q_1, \ldots, q_n$ that takes a given canonical system:

$$\delta x_k = \frac{dX_1}{dp_k}\,\delta t\,, \qquad \delta p_k = -\frac{dX_1}{dx_k}\,\delta t \qquad\qquad (k = 1, \ldots, n)$$

to another well-defined system:

$$\delta y_k = \frac{dY_1}{dq_k}\delta t\,, \qquad \delta q_k = -\frac{dY_1}{dy_k}\delta t \qquad\qquad (k = 1, \ldots, n).$$

Otherwise speaking, I will look for the most general constant transformation that takes the expression:

$$(X_1\, f)$$

to

$$(Y_1\, f)\,.$$

From my theory of contact transformations, that comes down to the search for the most general contact transformation that takes X_1 to Y_1 . One will find the same thing when one looks for two maximally-general canonical groups:

$$X_1, \ldots, X_n, P_1, \ldots, P_n,$$
$$Y_1, \ldots, Y_n, Q_1, \ldots, Q_n$$

in the variables x, p and y, q, respectively, and into which X_1 and Y_1 enter. If one sets:

$$X_k = Y_k\,, \qquad P_k = Q_k \qquad\qquad (k = 1, \ldots, n)$$

then those equations will define the most general transformation of the desired type.

In particular, one can demand that Y_1 should be the same function of the y_k, q_k that X_1 is of the x_k, p_k . The solution to that special problem will follow from what was just said with no further discussion.

5. – When several equations of the form:

$$(9) \qquad (F_1\, F) = 0\,, \quad \ldots, \quad (F_r\, F) = 0 \qquad\qquad (x_1, \ldots, p_n)$$

are given at the same time, one can look for the most general contact transformation that takes them to:

$$(\Phi_1\, F) = 0\,, \quad \ldots, \quad (\Phi_r\, F) = 0 \qquad\qquad (y_1, \ldots, q_n),$$

respectively. That comes down to the search for the most general contact transformation that takes $F_1, \ldots, F_r$ to $\Phi_1, \ldots, \Phi_r$, respectively. In my invariant theory of contact transformations (Math. Ann., Bd. VIII, pp. 272) ([1]), I showed that one can decide whether it is possible to solve a given problem of this kind by operations that can be performed. If that is the case then one will find the desired transformation by integrating ordinary differential equations.

In particular, if equations (9) define a complete system then one can look for the most general contact transformation that takes it to another complete system:

([1]) [*Ges. Abh.*, vol. IV, art. I, **§ 16**, no. 36.]

$$(\Phi_1 F) = 0\,, \qquad (\Phi_2 F) = 0\,, \ldots$$

in the $y_1, \ldots, y_n, q_1, \ldots, q_n$. As I said (Math. Bd. VIII, pp. 251 *et seq.*)([1]), $F_1, \ldots, F_r$ and $\Phi_1, \Phi_2, \ldots$ must define groups with just as many terms and just as many distinguished functions. If that requirement is fulfilled then one can put those two groups into their canonical forms:

$$X_1, \ldots, X_\rho, P_1, \ldots, P_{r-\rho}\,,$$
$$Y_1, \ldots, Y_\rho, Q_1, \ldots, Q_{r-\rho}\,,$$

and then look for two canonical systems of quantities:

$$X_1, \ldots, X_n, P_1, \ldots, P_n\,,$$
$$Y_1, \ldots, Y_n, Q_1, \ldots, Q_n$$

in the most general way. The equations:

$$X_k = Y_k\,, \qquad P_k = Q_k \qquad\qquad (k = 1, \ldots, n)$$

will then define the most general transformation of the required kind.

§ 2. – Canonical systems whose characteristic functions possess the form $p + f(x, x_1, \ldots, x_n, p_1, \ldots, p_n)$.

I will now turn to the case that is important in the applications to mechanics and the calculus of variations in which the characteristic function possesses the form:

$$p + f(x, x_1, \ldots, x_n, p_1, \ldots, p_n)\,.$$

In the corresponding simultaneous system:

$$\frac{\delta x}{1} = \frac{\delta x_k}{\dfrac{df}{dp_k}} = \frac{\delta p}{-\dfrac{df}{dx}} = \frac{\delta p_k}{-\dfrac{df}{dx_k}} = \delta t\,,$$

we do not need to include the term:

$$\frac{\delta p}{-\dfrac{df}{dx}} = \frac{\delta p_k}{-\dfrac{df}{dx_k}}\,,$$

([1]) [*Ges. Abh.*, vol. IV, art. I, **§§ 10-13**, nos. 23-29.]

since the remaining terms do not include p at all. Moreover, it should be noted that the auxiliary variable t is equal to x now.

6. – We seek the most general system of equations:

$$\begin{aligned} \delta x_k &= \xi_k (x, x_1, \ldots, x_n, p_1, \ldots, p_n)\, \delta x\,, \\ \delta p_k &= \eta_k (x, x_1, \ldots, x_n, p_1, \ldots, p_n)\, \delta x \qquad (k = 1, \ldots, n) \end{aligned}$$

by means of which the expression:

$$\frac{\delta}{\delta x}(p_1\, dx_1 + \cdots + p_n\, dx_n)$$

will assume the form:

$$d\,\Phi + \omega\,(x, x_1, \ldots, x_n, p_1, \ldots, p_n)\, dx\,.$$

That demand will be expressed by the equation:

$$\sum_{k=1}^{n} p_k\, d\xi_k + \sum_{k=1}^{n} \eta_k\, dx_k = d\,\Phi + \omega\, dx\,,$$

from which:

$$\begin{aligned} \sum_k p_k \frac{d\xi_k}{dx_r} + \eta_r &= \frac{d\Phi}{dx_r} && (r = 1, \ldots, n), \\ \sum_k p_k \frac{d\xi_k}{dp_\rho} &= \frac{d\Phi}{dp_\rho} && (\rho = 1, \ldots, n), \\ \sum_k p_k \frac{d\xi_k}{dx} &= \frac{d\Phi}{dx} + \omega\,. \end{aligned}$$

When we proceed as before, that will yield:

$$\frac{d\eta_k}{dx_i} = \frac{d\eta_i}{dx_k}, \qquad \frac{d\eta_k}{dp_i} = -\frac{d\xi_i}{dx_k}, \qquad \frac{d\xi_k}{dp_i} = \frac{d\xi_i}{dp_k}, \qquad \frac{d\eta_k}{dx} = -\frac{d\omega}{dx_k}, \qquad \frac{d\xi_k}{dx} = \frac{d\omega}{dp_k}\,.$$

There is then a function U of $x, x_1, \ldots, x_n, p_1, \ldots, p_n$ such that:

$$\xi_k = \frac{dU}{dp_k}\,, \qquad \eta_k = -\frac{dU}{dx_k} \qquad (k = 1, \ldots, n), \qquad \omega = \frac{dU}{dx}\,.$$

Thus:

Theorem 3:

If the expression $\frac{\delta}{\delta x}(p_1\,dx_1 + \cdots + p_n\,dx_n)$, *which is defined by means of the equations:*

$$\delta x_k = \xi_k\,\delta x\,, \qquad \delta p_k = \eta_k\,\delta x \qquad (k = 1, \ldots, n),$$

possesses the form $dW + \omega\,dx$ *then there will be a function U of* $x, x_1, \ldots, x_n, p_1, \ldots, p_n$ *such that:*

$$\xi_k = \frac{dU}{dp_k}\,, \qquad \eta_k = -\frac{dU}{dx_k} \qquad (k = 1, \ldots, n), \qquad \omega = \frac{dU}{dx}\,.$$

7. – We then look for the most general expression:

$$\sum_{k=1}^{n} X_k\,dx_k + \sum_{k=1}^{n} P_k\,dp_k + X\,dx = W$$

whose differential quotient with respect to x (viz., $\delta W / \delta x$), which is defined by the equations:

$$\delta x_k = \frac{dK}{dp_k}\,\delta x\,, \qquad \delta p_k = -\frac{dK}{dx_k}\,\delta x \qquad (k = 1, \ldots, n),$$

possesses the form $d\,\Omega + \omega\,dx$. It is not assumed that K is a well-defined quantity in that, but rather, it is regarded as an undetermined function of $x, x_1, \ldots, x_n, p_1, \ldots, p_n$.

When one expands the condition equation:

$$\frac{\delta W}{\delta x} = d\,\Omega + \omega\,dx\,,$$

it will assume the form:

$$d\,\Omega + \omega\,dx = \sum_k X_k\,d\,\frac{dK}{dp_k} - \sum_k P_k\,d\,\frac{dK}{dx_k} + \sum_k (p+K, X_k)\,dx_k + \sum_k (p+K, P_k)\,dp_k + (p+K, X)\,dx\,,$$

from which, one will get:

$$\frac{d\Omega}{dx_u} = \sum_k X_k\,\frac{d^2K}{dp_k\,dx_u} - \sum_k P_k\,\frac{d^2K}{dx_k\,dx_u} + (p+K, X_u)\,,$$
$$\frac{d\Omega}{dp_v} = \sum_k X_k\,\frac{d^2K}{dp_k\,dp_v} - \sum_k P_k\,\frac{d^2K}{dx_k\,dp_v} + (p+K, P_v)\,,$$

$$\frac{d\Omega}{dx} = \sum_k X_k \frac{d^2K}{dp_k\,dx} - \sum_k P_k \frac{d^2K}{dx_k\,dx} + (p+K, X) - \omega .$$

We now establish the identity:

$$\frac{d}{dp_v}\frac{d\Omega}{dx_u} = \frac{d}{dx_u}\frac{d\Omega}{dp_v}$$

and then find, when we drop the terms that cancel, that:

$$\sum_k \frac{dX_k}{dp_v}\frac{d^2K}{dp_k\,dx_u} - \sum_k \frac{dP_k}{dp_v}\frac{d^2K}{dx_k\,dx_u} + \left(p+K, \frac{dX_u}{dp_v}\right) + \left(\frac{dK}{dp_v}, X_u\right)$$
$$- \sum_k \frac{dX_k}{dx_u}\frac{d^2K}{dp_k\,dp_v} + \sum_k \frac{dP_k}{dx_u}\frac{d^2K}{dx_k\,dp_v} - \left(p+K, \frac{dP_v}{dx_u}\right) - \left(\frac{dK}{dx_u}, P_v\right) = 0 .$$

That relation should exist no matter what the function K might be. If we then combine the terms that contain the same differential quotients of K then the coefficient of each such differential quotient that arises in that way must be equal to zero. That will give the relations:

$$\frac{dX_k}{dp_v} - \frac{dP_v}{dx_k} = 0 \qquad \text{when} \quad k \neq v ,$$

$$(\alpha) \qquad \frac{dX_v}{dp_v} - \frac{dP_v}{dx_v} = \frac{dX_u}{dp_u} - \frac{dP_u}{dx_u} ,$$

$$\frac{dX_u}{dx_k} - \frac{dX_k}{dx_u} = \frac{dP_u}{dp_k} - \frac{dP_k}{dp_u} = 0 ,$$

$$\frac{d}{dx_k}\left(\frac{dX_u}{dp_v} - \frac{dP_v}{dx_u}\right) = 0 , \qquad \frac{d}{dp_k}\left(\frac{dX_u}{dp_v} - \frac{dP_v}{dx_u}\right) = 0 , \qquad \frac{d}{dx}\left(\frac{dX_u}{dp_v} - \frac{dP_v}{dx_u}\right) = 0 .$$

The last three equations show that the quantity:

$$\frac{dX_v}{dp_v} - \frac{dP_v}{dx_v}$$

is constant, and indeed, due to (α), that constant is independent of the number v. One will then have:

$$\frac{dX_v}{dp_v} - \frac{dP_v}{dx_v} = A = \text{const.}$$

or

$$\frac{d\,(X_v - A\,p_v)}{dp_v} = \frac{dP_v}{dx_v} \qquad (v = 1, \ldots, n).$$

When we then proceed as in number 2, we will see that the quantities $X_k - A\,p_k$ and P_i are the partial derivatives with respect to x_k and p_i, resp., of a function of x, x_1, …, x_n, p_1, …, p_n :

$$X_k - A\,p_k = \frac{dU}{dx_k}, \qquad P_i = \frac{dU}{dp_i},$$

which makes:

$$X_k = A\,p_k + \frac{dU}{dx_k}, \qquad P_i = \frac{dU}{dp_i},$$

and therefore W will possess the form:

$$A\sum_k p_k\,dx_k + dU + \varphi\,dx\,.$$

Conversely, it is easy to see that this expression always possesses the desired property (that is, no matter what the constant A and the functions U and φ might be). That is because:

$$\begin{aligned}
\frac{\delta}{\delta x}\sum_k p_k\,dx_k &= d\left(-K + \sum_k p_k \frac{dK}{dp_k}\right) + \frac{dK}{dx}\,dx\,, \\
\frac{\delta}{\delta x}dU &= d\,\frac{\delta U}{\delta x}\,, \\
\frac{\delta}{\delta x}(\varphi\,dx) &= \frac{\delta\varphi}{\delta x}dx\,.
\end{aligned}$$

We can then express the following theorem:

Theorem 4:

If the expression:

$$(p + K, \sum_k X_k\,dx_k + \sum_k P_k\,dp_k + X\,dx)\,,$$

in which X_k, P_k, *and* X *denote given functions of* $x, x_1, \dots, x_n, p_1, \dots, p_n$, *while* K *is an undetermined function of the same quantities, always possesses the form* $d\Omega + \omega\, dx$, *no matter what the function* K *is, then* $\sum_k X_k\, dx_k + \sum_k P_k\, dp_k + X\, dx$ *can take the form:*

$$A \sum_k p_k\, dx_k + dU + \varphi\, dx .$$

In that, A *is an arbitrary constant, while* U *and* φ *are arbitrary functions of* $x, x_1, \dots, x_n, p_1, \dots, p_n$.

8. – I will now imagine that one has introduced new variables, say, $x, y_1, \dots, y_n, q_1, \dots, q_n$, in place of $x, x_1, \dots, x_n, p_1, \dots, p_n$ in the simultaneous system:

$$\delta x_k = \frac{dK}{dp_k}\,\delta x, \qquad \delta p_k = -\frac{dK}{dx_k}\,\delta x \qquad (k = 1, \dots, n)$$

and in the expression $W = p_1\, dx_1 + \dots + p_n\, dx_n$. In so doing, the quantities y_k and q_k shall initially be subject to only the restriction that they are independent with respect to $x, x_1, \dots, x_n, p_1, \dots, p_n$. Let:

$$(10) \qquad \delta y_k = \eta_k\, \delta x, \qquad \delta q_k = \zeta_k\, \delta x \qquad (k = 1, \dots, n)$$

be the new form of our simultaneous system, and let:

$$\sum_k p_k\, dx_k = \sum_k Y_k\, dy_k + \sum_k Q_k\, dq_k + Y\, dx = W,$$

in which Y_k, Q_k, and Y are certain functions of the new variables.

Due to the form of W in the old variables, there exists an equation of the form:

$$\frac{\delta W}{\delta x} = d\,\Omega + \omega\, dx .$$

If we introduce the new variables here then that will give:

$$\sum_i \left(\frac{dW}{dy_i}\eta_i + \frac{dW}{dq_i}\zeta_i \right) = d\,\Omega + \omega\, dx ,$$

in which the expression on the left is understood to mean what it usually does.

Now, if the transformed system (10) always possesses the canonical form:

$$\delta y_k = \frac{d\Psi}{dq_k}\,\delta x\,, \qquad \delta q_k = -\frac{d\Psi}{dy_k}\,\delta x \qquad (k = 1, \ldots, n)$$

in particular, no matter what the function *K* might be, then, from the foregoing theorem, *W* must possess the form:

$$A\sum_k q_k\,dy_k + dV + \varphi\,dx$$

in the new variables. One then has:

$$\sum_k p_k\,dx_k = A\sum_k q_k\,dy_k + dV + \varphi\,dx\,.$$

If we add the quantity $p\,dx$ to the right-hand side and the left-hand side and then denote the sum $\varphi + p$ by $A\,q$ then that will give:

$$p\,dx + p_1\,dx_1 + \ldots + p_n\,dx_n = A\,(q\,dx + q_1\,dy_1 + \ldots + q_n\,dy_n) + dV\,.$$

With that, we have proved that our transformation can be regarded as a contact transformation.

Theorem II:

If a given transformation between $x, x_1, \ldots, x_n, p_1, \ldots, p_n$ *and* $x, y_1, \ldots, y_n, q_1, \ldots, q_n$ *possesses the property that any system of the form:*

$$\delta x_k = \frac{dK}{dp_k}\,\delta x\,, \qquad \delta p_k = -\frac{dK}{dx_k}\,\delta x \qquad (k = 1, \ldots, n)$$

will go to a similar system in the new variables, in which one assumes that K denotes an arbitrary function of $x, x_1, \ldots, x_n, p_1, \ldots, p_n$, *then our transformation will be a contact transformation, that is, it will consist of a relation of the form:*

$$p\,dx + p_1\,dx_1 + \ldots + p_n\,dx_n = A\,(q\,dx + q_1\,dy_1 + \ldots + q_n\,dy_n) + dV\,.$$

9. – Now let a *well-defined* system be given:

$$\delta x_k = \frac{dX}{dp_k}\,\delta x\,, \qquad \delta p_k = -\frac{dX}{dx_k}\,\delta x \qquad (x, x_1, \ldots, x_n, p_1, \ldots, p_n)$$

that one wishes to transform into another well-defined system:

$$\delta y_k = \frac{dY}{dq_k}\,\delta x\,, \qquad \delta q_k = -\frac{dY}{dy_k}\,\delta x \qquad (x, y_1, \ldots, y_n, q_1, \ldots, q_n)$$

by means of a contact transformation. Under the desired transformation, the equation:

$$\frac{df}{dx} - \sum_k \left(\frac{dX}{dx_k}\frac{df}{dp_k} - \frac{dX}{dp_k}\frac{df}{dx_k} \right) = 0 = (p + X, f)$$

will go to:

$$\frac{df}{dx} - \sum_k \left(\frac{dY}{dy_k}\frac{df}{dq_k} - \frac{dY}{dq_k}\frac{df}{dy_k} \right) = 0 = (q + Y, f),$$

in which f denotes an unknown function of $x, x_1, \ldots, x_n, p_1, \ldots, p_n$ or also $x, y_1, \ldots, y_n, q_1, \ldots, q_n$.

Here, one can conclude with no further analysis that $p + X$ goes to $q + Y$ under the transformation. Therefore, let:

$$q + U\,(x, y_1, \ldots, y_n, q_1, \ldots, q_n)$$

be the function into which $p + X$ converts. From the theorem of contact transformations, $(p + X, f)$ will then go to $(q + U, f)$. Thus:

$$(q + Y, f) = (q + U, f)\,,$$

from which, it will follow that:

$$(Y - U, f) = 0\,.$$

This equation must be true when f is set equal to an arbitrary function of $x, x_1, \ldots, x_n, p_1, \ldots, p_n$ that is in involution with $p + X$. We can then conclude that $Y - U$ is a constant:

$$U = Y + A\,.$$

The desired transformation then takes $p + X$ to $q + Y$.

In order to determine it in the most-general way, one defines two canonical groups is the most-general way:

(11) $$x, X_1, \ldots, X_n, p + X, \quad P_1, \ldots, P_n\,,$$

(12) $$x, Y_1, \ldots, Y_n, q + Y + A, Q_1, \ldots, Q_n\,,$$

in which the X_k, P_k are functions of $x, x_1, \ldots, x_n, p_1, \ldots, p_n$, while the Y_k, Q_k are functions of $x, y_1, \ldots, y_n, q_1, \ldots, q_n$. The equations:

$$x = x\,, \quad p + X = q + Y + A\,, \quad P_k = Q_k\,, \quad X_k = Y_k \qquad (k = 1, \ldots, n)$$

will then define the desired transformation.

Moreover, it should be remarked that the quantities (12) will always define a canonical group when A is set equal to zero. One will then find the desired most-general transformation between x, $x_1, \dots, x_n, p_1, \dots, p_n$ and $x, y_1, \dots, y_n, q_1, \dots, q_n$ when one takes $p + X$ to $q + Y$ in the most-general way by means of a contact transformation.

§ 3. – Applications to mechanics and the calculus of variations.

It is known that **Jacobi** was the first to show that the integration of the so-called simultaneous canonical system:

$$\delta x_k = \frac{dF}{dp_k}\,\delta t\,, \qquad \delta p_k = -\frac{dF}{dx_k}\,\delta t \qquad (k = 1, \dots, n) \tag{13}$$

admits some specialized simplifications. After that, **Weiler**, **Mayer**, and myself developed even simpler methods for integrating such systems.

10. – Therefore, if any simultaneous system is given then it would be natural to ask the question of whether one can put it into canonical form. It is known that **Hamilton** had put the differential equations of mechanics into that form in a far-reaching class of cases. **Jacobi** pointed out the importance of that reduction and, at the same time, showed that there exists an even-more-general category of mechanical problems that can take the form in question.

I will now derive that **Hamilton-Jacobi** theory in a new way that is based upon the foregoing developments. In that way, I will first consider the simple case of a number of *free* points that move as a result of their mutual attraction or also as a result of their attraction to a fixed point.

Let $x, x_1, \dots, x_n$ be the coordinates of our point. Let U be the force function, which might also include time. As is known, the motion will then be determined by the equations:

$$\frac{\delta}{\delta t}\frac{\delta x_k}{\delta t} = \frac{dU}{dx_k} \qquad (k = 1, \dots, n).$$

If we set:

$$\frac{\delta x_k}{\delta t} = y_k \qquad (k = 1, \dots, n) \tag{14}$$

then that will give:

$$\frac{\delta y_k}{\delta t} = \frac{dU}{dx_k} \qquad (k = 1, \dots, n). \tag{15}$$

In order to put equations (14) and (15) into canonical form, as one can see in this simplest of cases with no further analysis, it will only be necessary to set:

$$\tfrac{1}{2}(y_1^2 + \cdots + y_n^2) - U = T\,.$$

Our equations will then, in fact, assume the form:

$$\frac{\delta x_k}{\delta t} = \frac{dT}{dy_k}, \qquad \frac{\delta y_k}{\delta t} = -\frac{dT}{dx_k} \qquad (k = 1, \ldots, n).$$

That is just how **Jacobi** arrived at the first result.

Now, in order to generalize that theory, it is useful to look for the intrinsic basis for what was found already.

Since the introduction of the quantities $x_1, \ldots, x_n, p_1, \ldots, p_n$ as independent variables of the given simultaneous system will put it into canonical form, from the foregoing section, the expression:

$$\frac{\delta}{\delta t}(y_1\,dx_1 + \cdots + y_n\,dx_n)$$

must possess the form $d\,\Omega + \rho\,dt$. One verifies that as follows: One has:

$$\frac{\delta}{\delta t}\sum_k y_k\,dx_k = \sum_k \frac{\delta y_k}{\delta t} dx_k + \sum_k y_k\,d\,\frac{\delta x_k}{\delta t},$$

so, from (14), (15):

$$\begin{aligned}\frac{\delta}{\delta t}\sum_k y_k\,dx_k &= \sum_k \frac{dU}{dx_k} dx_k + \sum_k y_k\,dy_k \\ &= d\{U + \tfrac{1}{2}\sum y_k^2\} - \frac{dU}{dt}dt,\end{aligned}$$

which will lead to the proof.

We shall now turn to the general case in which the coordinates $x_1, \ldots, x_n$ are constrained by several relations *that might also include time* t :

(16) $$f_1(x_1, \ldots, x_n, t) = 0, \qquad \ldots, \qquad f_q(x_1, \ldots, x_n, t) = 0.$$

We always assume the existence of a force function U in that. According to **Lagrange**, the motion will be determined by the equations:

(17) $$\frac{\delta}{\delta t}\frac{\delta x_k}{\delta t} = \frac{dU}{dx_k} + \sum_i \lambda_i \frac{df}{dx_k} \qquad (k = 1, \ldots, n),$$

together with (16).

It is now only natural to examine whether:

$$\frac{\delta}{\delta t}\sum_k y_k\,dx_k$$

can also take the form $d\,\Omega + \rho\,dt$. One finds that:

$$\frac{\delta}{\delta t}\sum_k y_k\,dx_k = \sum_k \frac{\delta y_k}{\delta t}dx_k + \sum_k y_k\,d\,\frac{\delta x_k}{\delta t},$$

so from (17):

$$\frac{\delta}{\delta t}\sum_k y_k\,dx_k = \sum_k \left(\frac{dU}{dx_k} + \sum_i \lambda_i \frac{df_i}{dx_k}\right)dx_k + \sum_k y_k\,d\,\frac{\delta x_k}{\delta t},$$

$$= d\,(U + \tfrac{1}{2}\sum y_k^2) + \sum_k \lambda_i\,df_i - \left(\frac{dU}{dt} + \sum_i \lambda_i \frac{df_i}{dt}\right)dt\,.$$

However, all df_i vanish, such that one will now find that:

$$\frac{\delta}{\delta t}\sum_k y_k\,dx_k = d\,(U + \tfrac{1}{2}\sum y_k^2) - \left(\frac{dU}{dt} + \sum_i \lambda_i \frac{df_i}{dt}\right)dt,$$

which justifies our conjecture.

Now, in the expression $\sum y_k\,dx_k$, x_k and $y_k = \delta x_k / \delta t$ are coupled by the equations (16). We will, in fact, dispose of the dependent quantities y_k and dx_k by introducing the quantities t and dt. It is convenient to think of the $f_i = 0$ as being solved for q of the quantities x, say, $x_{n-q+1}, \ldots, x_n$:

$$x_k = \varphi_k\,(x_1, \ldots, x_{n-q}, t) \qquad (k = n - q + 1, \ldots, n).$$

That will give:

(18) $$dx_k = \sum_{r=1}^{n-q} \frac{d\varphi_k}{dx_r}\,dx_r + \frac{d\varphi_k}{dt}\,dt \qquad (k = n - q + 1, \ldots, n)$$

and

(19) $$y_k = \sum_{\rho=1}^{n-q} \frac{d\varphi_k}{dx_\rho}\,y_\rho + \frac{d\varphi_k}{dt} \qquad (k = n - q + 1, \ldots, n).$$

If one substitutes those values in $\sum y_k\,dx_k$ then that will make:

$$\sum_{k=1}^{n} y_k\,dx_k = \sum_{r=1}^{n-q} y_r\,dx_r + \sum_{k=n-q+1}^{n}\left(\sum_{\rho=1}^{n-q}\frac{d\varphi_k}{dx_\rho}\,y_\rho + \frac{d\varphi_k}{dt}\right)\left(\sum_{r=1}^{n-q}\frac{d\varphi_k}{dx_r}\,dx_r + \frac{d\varphi_k}{dt}\,dt\right)$$

or

$$\sum_{k=1}^{n} y_k\,dx_k = dt\sum_{k=n-q+1}^{n}\frac{d\varphi_k}{dt}\left(\sum_{\rho=1}^{n-q}\frac{d\varphi_k}{dx_\rho}\,y_\rho + \frac{d\varphi_k}{dt}\right) + \sum_{r=1}^{n-q} dx_r\left\{y_r + \sum_{k=n-q+1}^{n}\frac{d\varphi_k}{dx_r}\left(\sum_{\rho=1}^{n-q}\frac{d\varphi_k}{dx_\rho}\,dx_\rho + \frac{d\varphi_k}{dt}\,dt\right)\right\}.$$

With that, one will then find an equation of the form:

$$\sum_{k=1}^{n} y_k \, dx_k = Y_1 \, dx_1 + \ldots + Y_{n-q} \, dx_{n-q} + Y \, dt \, .$$

If we then determine the quantities λ_i by means of equations (16), (17), (19) as functions of the x_k, y_k, and t and then introduce the quantities:

$$x_1, \ldots, x_{n-q}, Y_1, \ldots, Y_{n-q}, t$$

into our simultaneous system as variables then, from Theorem 3, it will assume the canonical form:

$$\delta x_k = \frac{dW}{dY_k} \delta t \, , \qquad \delta Y_k = -\frac{dW}{dx_k} \delta t \qquad (k = 1, \ldots, n - q) \, .$$

The function W can obviously be determined in each individual case.

The new variables Y_i are the partial derivatives of a certain quantity. In fact, if one sets:

$$\tfrac{1}{2}(y_1^2 + \cdots + y_n^2) = \Omega$$

then that will give:

$$\Omega = \tfrac{1}{2} \sum_{k=1}^{q} y_k^2 + \tfrac{1}{2} \sum_{k=n-q+1}^{n} \left\{ \sum_{\rho=1}^{n-q} \frac{d\varphi_k}{dx_\rho} y_\rho + \frac{d\varphi_k}{dt} \right\}^2 ,$$

from which, one will have:

$$\frac{d\Omega}{dy_r} = y_r + \sum_{k=n-q+1}^{n} \frac{d\varphi_k}{dx_r} \left\{ \sum_{\rho=1}^{n-q} \frac{d\varphi_k}{dx_\rho} y_\rho + \frac{d\varphi_k}{dt} \right\}$$

for $r = 1, \ldots, n - q$, such that:

$$Y_1 = \frac{d\Omega}{dy_1}, \ldots, Y_{n-q} = \frac{d\Omega}{dx_{n-q}} .$$

11. – If one seeks to determine the quantities $x_1, \ldots, x_n$ as functions of t in such a way that the integral:

$$\int \varphi(t, x_1, \ldots, x_n', \ldots, x_n') \, \delta t \, ,$$

in which $x_k' = dx_k / dt$, will become a minimum then, as is known, it will be necessary that the equations:

(20) $$\frac{d\varphi}{dx_k} \delta t - \delta \frac{d\varphi}{dx_k'} = 0 \qquad (k = 1, \ldots, n)$$

should exist. Those equations, together with:

$$\frac{\delta x_k}{\delta t} = x_k' \qquad (k = 1, \ldots, n),$$

define a simultaneous system with $2n$ terms, and according to **Jacobi**, it will assume the canonical form when one introduces the quantities:

(21) $$x_k,\ y_k = \frac{d\varphi}{dx_k'} \ (k = 1, \ldots, n),\ t$$

as variables.

In order to verify that fundamental theorem in a simple way, I shall form the differential quotients of $\sum y_k \, dx_k$ with respect to t :

$$\frac{\delta}{\delta t}\sum y_k \, dx_k = \sum \frac{\delta y_k}{\delta t} dx_k + \sum y_k \, d\frac{\delta x_k}{\delta t},$$

from which, due to (21) and (20):

$$\frac{\delta}{\delta t}\sum y_k \, dx_k = \sum \frac{d\varphi}{dx_k} dx_k + \sum \frac{d\varphi}{dx_k'} dx_k',$$

or

$$\frac{\delta}{\delta t}\sum y_k \, dx_k = d\varphi - \frac{d\varphi}{dt} dt,$$

which will lead to the verification.

§ 4. – Solution to Problem III.

12. – Allow me to now assume that a *well-defined* canonical system:

(22) $$\delta x_k = \frac{dF_1}{dp_k}\delta t, \qquad \delta p_k = -\frac{dF_1}{dx_k}\delta t \qquad (k = 1, \ldots, n)$$

can, upon introducing the variables $y_1, \ldots, y_n, q_1, \ldots, q_n$, where:

$$y_k = y_k(x_1, \ldots, x_n, p_1, \ldots, p_n),$$
$$q_k = q_k(x_1, \ldots, x_n, p_1, \ldots, p_n) \qquad (k = 1, \ldots, n)$$

assume the form:

$$\delta y_k = \frac{d\Phi_1}{dq_k}\delta t, \qquad \delta q_k = -\frac{d\Phi_1}{dy_k}\delta t \qquad (k = 1, \ldots, n).$$

If that transformation is not a contact transformation then let:

$$\sum q_k\, dy_k = \sum X_k\, dx_k + \sum P_k\, dp_k = W .$$

There exists (Theorem 1) a relation of the form:

$$\frac{\delta}{\delta t} \sum q_k\, dy_k = (\Phi_1, \sum q_k\, dy_k) = d\,\Omega ,$$

so one will have:

$$\frac{\delta}{\delta t} \left\{ \sum_k X_k\, dx_k + \sum_k P_k\, dp_k \right\} = d\,\Omega .$$

On the other hand, let an arbitrary expression be given:

$$\sum X_k'\, dx_k + \sum P_k'\, dp_k \tag{23}$$

that has a normal form with n terms, and whose differential quotient with respect to t is a complete differential:

$$\frac{\delta}{\delta t} \left\{ \sum X_k'\, dx_k + \sum P_k'\, dp_k \right\} = (F_1, \sum X_k'\, dx_k + \sum P_k'\, dp_k) = d\,\Omega . \tag{24}$$

If one then puts $\sum X_k'\, dx_k + \sum P_k'\, dp_k$ into its normal form:

$$\sum X_k'\, dx_k + \sum P_k'\, dp_k = q_1'\, dy_1' + \cdots + q_n'\, dy_n' + d\Theta$$

then when one introduces the y_k', q_k' (which are assumed to be independent) as variables, the system (22) will obviously assume the canonical form:

$$\delta y_k' = \frac{d\Psi}{dq_k'}\, \delta t , \qquad \delta q_k' = -\frac{d\Psi}{dy_k'}\, \delta t \qquad (k = 1, \ldots, n),$$

in turn.

If one would then like to find the most general transformation that lets the system (22) take on its canonical form then one must look for the most general expression (23) that fulfills a relation of the form (24), and then put that expression into its normal form in the most general way. After that, the transformation in question can be exhibited with no further discussion.

I shall next look for a $2n$-parameter canonical group that includes F_1:

$$F_1, \ldots, F_n, G_1, \ldots, G_n$$

and then introduce those quantities as variables. One then deals with the search for the most general expression:

(25) $$\sum L_k\, dF_k + \sum M_k\, dG_k$$

that fulfills a relation of the form:

$$(F_1, \sum L_k\, dF_k + \sum M_k\, dG_k) = d\,\Omega\ .$$

However, that equation must assume the form:

$$\sum \frac{dL_k}{dG_1} dF_k + \sum \frac{dM_k}{dG_1} dG_k = d\,\Omega\ ,$$

from which:

$$\frac{dL_k}{dG_1} = \frac{d\Omega}{dF_k}\ , \qquad \frac{dM_k}{dG_1} = \frac{d\Omega}{dG_k} \qquad (k = 1, \ldots, n),$$

and upon integrating over G_1 :

(26) $$L_k = \int \frac{d\Omega}{dF_k} dG_1\ ,\ M_k = \int \frac{d\Omega}{dG_k} dG_1 \qquad (k = 1, \ldots, n).$$

In those expressions for the quantities L_k and M_k, the integration constants are arbitrary functions of $F_1, \ldots, F_n, G_2, \ldots, G_n$, while Ω denotes an arbitrary function of all F_k and G_k. If one then expresses F_k and G_k as functions of the x_k and p_k in (25) then one will get the most general expression:

$$\sum X_k\, dx_k + \sum P_k\, dp_k$$

that fulfills a relation of the form:

$$(F_1, \sum X_k\, dx_k + \sum P_k\, dp_k) = d\,\Omega\ .$$

After that, one will find the desired transformation from the rules that were set down before.

13. – In order to explicitly verify that the transformations that are found in that way are not contact transformations, in general, I shall make the following argument:

When I denote arbitrary functions of $G_2, \ldots, G_n, F_2, \ldots, F_n$, by λ_k and μ_k, formulas (26) can be written as follows:

(27) $$L_k = \frac{d}{dF_k}\left(\int \Omega\, dG_1\right) + \lambda_k\ , \qquad M_k = \frac{d}{dG_k}\left(\int \Omega\, dG_1\right) + \mu_k \qquad (k = 1, \ldots, n).$$

Now should the transformation in question be a contact transformation then the relation:

$$\sum L_k\, dF_k + \sum M_k\, dG_k = \sum p_k\, dx_k + d\Psi = \sum G_k\, dF_k + d\,\Pi$$

must exist, so:

(28) $$L_k = G_k + \frac{d\Pi}{dF_k}, \qquad M_k = \frac{d\Pi}{dG_k} \qquad (k = 1, \ldots, n).$$

When we set:

$$\Pi - \int \Omega \, dG_1 = S,$$

those formulas, together with (27), will give the equations:

$$\lambda_k = G_k + \frac{dS}{dF_k}, \qquad \mu_k = \frac{dS}{dG_k} \qquad (k = 1, \ldots, n).$$

However, since λ_k and μ_k are generally arbitrary functions of $G_2, \ldots, G_n$, $F_2, \ldots, F_n$, we have actually proved that our transformations are contact transformations only in exceptional cases. That gives:

Theorem III:

In order to convert any canonical system:

$$\delta x_k = \frac{dF_1}{dp_k}\,\delta t, \qquad \delta p_k = -\frac{dF_1}{dx_k}\,\delta t \qquad (k = 1, \ldots, n)$$

into a similar system in the most general way, one proceeds as follows: One satisfies the equation:

$$\sum p_k \, dx_k = \sum G_k \, dF_k + dV$$

in the most general way and then sets:

$$L_k = \lambda_k + \frac{dU}{dF_k}, \qquad M_k = \mu_k + \frac{dU}{dG_k} \qquad (k = 1, \ldots, n),$$

in which U is an arbitrary function of the F_k and G_k, while the λ_k and μ_k denote arbitrary functions of $G_2, \ldots, G_n$, $F_1, \ldots, F_n$. One will then put:

$$\sum L_k \, dF_k + \sum M_k \, dG_k$$

into the form:

$$Q_1 \, dY_1 + \ldots + Q_n \, dY_n + dY$$

in the most general way. The equations:

$$q_k = Q_k\,, \qquad y_k = Y_k \qquad (k = 1, \ldots, n)$$

will determine the most general transformation of the required kind.

Note.

14. – If any **Pfaff** expression:

$$X_1\, dx_1 + \ldots + X_m\, dx_m = \sum X\, dx$$

is given then one can pose the problem of finding the most general infinitesimal transformation:

$$A\,f = \xi_1 \frac{\delta f}{\delta x_1} + \cdots + \xi_m \frac{\delta f}{\delta x_m}$$

that fulfills a relation of the form:

$$A\left(\sum X\, dx\right) = d\,\Omega\,,$$

or also gives:

$$A\left(\sum X\, dx\right) = 0\,.$$

Those problems can always be solved. If $m = 2n$, in particular, and the normal form of $\sum X\, dx$ has n terms in $2n$ *independent* functions, as a result, then the first problem will require only *performable* operations.

Conversely, let a complete system be given:

$$A_1\, f = 0\,, \qquad \ldots, \qquad A_q\, f = 0\,.$$

I shall assume that I know an expression:

$$X_1\, dx_1 + \ldots + X_m\, dx_m$$

that fulfills q relations of the form:

$$A_i\left(\sum X\, dx\right) = d\,\Omega_i \quad (\text{or} = 0).$$

I now pose the problem of exploiting that situation as much as possible. In particular, if $q = 1$, $m = 2n$ and the normal form of $\sum X\, dx$ then includes $2n$ independent functions then the integration of $A_i\, f = 0$ will require only $2n - 2$, $2n - 4$, …, 6, 4, 2 operations.

On another occasion, I will extend all of my investigations into first-order partial differential equation to the **Pfaff** problem.

Christiania, January 1877.

Voluntary disclosures about this article.

1. – Repertorium, Bd. II, pp. 408. Leipzig 1879.

The most general transformation:

$$(1)\qquad x'_k = X_k(x_1, \ldots, x_n, p_1, \ldots, p_n), \qquad p'_k = P_k(x_1, \ldots, x_n, p_1, \ldots, p_n)$$

that simultaneously converts all simultaneous systems of the form:

$$(2)\qquad dx_k = \frac{dF}{dp_k}\, dt, \qquad dp_k = -\frac{dF}{dx_k}\, dt$$

into systems of the same form, was determined by **Jacobi** and **Bour** from the equations:

$$(3)\qquad (X_i\, X_k) = (X_i\, P_k) = (P_i\, P_k) = 0, \qquad (P_k\, X_k) = 1 .$$

From the author's investigations of contact transformations, the relations that were just written down likewise determine the most general system of quantities X_i, P_i that fulfill a condition equation of the form:

$$P_1\, dX_1 + \ldots + P_n\, dX_n = p_1\, dx_1 + \ldots + p_n\, dx_n + d\,\Omega .$$

The treatise seeks the intrinsic basis for that connection between perturbation theory and the theory of contact transformations.

If one requires the most general transformation that converts only one system (2) into a similar system then the relations (3) will no longer be necessary. All transformations that fulfill such a requirement will then be determined.

This voluntary disclosure agrees almost exactly with one that was written in French in volume XIV of the Bulletin des Sciences mathématiques et astronomiques [Ser. (2), t. III], Sec. 2, pp. 185-186, Paris, November 1879, except that the sentence "The treatise seeks, etc." reads as:

"The present treatise explains the deep reason for that dependency between the theory of perturbations and that of contact transformations."

2. – F. d. M., Bd. IX, Jahrg. 1877, pp. 259-261. Berlin 1880.

In the theory of perturbations, one solves the following problem:

Problem I: *Determine the most general transformation:*

$$x'_k = X_k(x_1, \ldots, x_n, p_1, \ldots, p_n), \qquad p'_k = P_k(x_1, \ldots, x_n, p_1, \ldots, p_n)$$

that simultaneously takes all simultaneous systems of the form:

$$dx_k = \frac{dF}{dp_k}\,dt\,, \qquad dp_k = -\frac{dF}{dx_k}\,dt$$

to systems of the same form.

Jacobi and **Bour** have shown that the most general transformation of the desired kind is defined by:

$$(1) \qquad (X_i\,X_k) = (X_i\,P_k) = (P_i\,P_k) = 0\,, \qquad (P_k\,X_k) = 1\,.$$

On the other hand, from the author's previous work, the following problem is at the basis for the theory of contact transformations:

Problem II: *Determine* $2n$ *quantities:*

$$X_1\,, \ldots, X_n, P_1\,, \ldots, P_n$$

as functions of:

$$x_1\,, \ldots, x_n, p_1\,, \ldots, p_n$$

in the most general way such that a relation of the form:

$$P_1\,dX_1 + \ldots, P_n\,dX_n = p_1\,dx_1 + \ldots, p_n\,dx_n + dV$$

exists, in which one assumes that V *is regarded as an undetermined function of* $x_1\,, \ldots, p_n$.

As is known, one gets the most general solution of that problem when one takes an arbitrary system of quantities X_k , P_k that fulfill the relations (1).

With that, one verifies that there is a more precise connection between two apparently-different problems. In the present article, the intrinsic basis for that identity is present by way of analytical considerations. At the same time, several analogous problems are presented and resolved. In particular, the following new problem was solved:

Problem III: *Determine the most general transformation that takes a given system of the form:*

$$dx_k = \frac{dF}{dp_k}\,dt\,, \qquad dp_k = -\frac{dF}{dx_k}\,dt$$

into a similar system.

It was shown that the transformations in question, which were all determined, are not contact transformations, in general.