"Champ électrique et magnétique produit par une charge électrique concentrée en un point et animée d'un mouvement quelconque," L'Éclairage Électrique, Revue Hebdomadaire d'Électricité **16** (1898), 5-14, 53-59, 106-112.

Electric and magnetic field produced by an electric charge that is concentrated at a point and animated with an arbitrary motion

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Translated by D. H. Delphenich

Assume that an electric mass in motion has a density ρ and a velocity u and produces the same field as a conduction current of intensity $u\,\rho$ at each point. Upon preserving the notations of a preceding article ([1]), we will get the following equations for determining the field:

$$\frac{1}{4\pi}\left(\frac{d\gamma}{dy}-\frac{d\beta}{d\gamma}\right)=\rho\,u_x+\frac{df}{dt}, \tag{1}$$

$$V^2\left(\frac{dh}{dy}-\frac{dg}{d\gamma}\right)=-\frac{1}{4\pi}\frac{d\alpha}{dt}, \tag{2}$$

with the analogues that are deduced by cyclic permutation, and in addition, the following ones:

$$\rho=\frac{df}{dx}+\frac{dg}{dy}+\frac{dh}{dz}, \tag{3}$$

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0. \tag{4}$$

One easily deduces the following relations from that system of equations:

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)f=V^2\frac{d\rho}{dx}+\frac{d}{dt}(\rho u_x), \tag{5}$$

$$\left(V^2\Delta-\frac{d^2}{dt^2}\right)\alpha=4\pi V^2\left[\frac{d}{dz}(\rho u_y)-\frac{d}{dy}(\rho u_z)\right], \tag{6}$$

Now let four functions ψ, F, G, H be defined by the conditions:

([1]) "La théorie de Lorentz," L'Éclairage Électrique **14**, pp. 417. α, β, γ are the components of the magnetic force, and f, g, h are those of the displacement in the ether.

$$\left(V^2\Delta - \frac{d^2}{dt^2}\right)\psi = -\,4\pi V^2\rho\,, \tag{7}$$

$$\left.\begin{aligned}
\left(V^2\Delta - \frac{d^2}{dt^2}\right)F &= -4\pi V^2\rho u_x\,,\\
\left(V^2\Delta - \frac{d^2}{dt^2}\right)G &= -4\pi V^2\rho u_y\,,\\
\left(V^2\Delta - \frac{d^2}{dt^2}\right)H &= -4\pi V^2\rho u_z\,,
\end{aligned}\right\} \tag{8}$$

One satisfies the conditions (5) and (6) by taking:

$$4\pi f = -\,\frac{d\psi}{dx} - \frac{1}{V^2}\frac{dF}{dt}\,, \tag{9}$$

$$\alpha = \frac{dH}{dy} - \frac{dG}{dz}\,. \tag{10}$$

As for equations (1) to (4), in order for them to be satisfied, it is necessary that, along with (7) and (8), one must have the condition:

$$\frac{d\psi}{dt} + \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz} = 0\,. \tag{11}$$

We first address equation (7). We know that the most general solution is the following one:

$$\psi = \int \frac{\rho\left[x', y', z', t - r/V\right]}{r}\,d\omega'\,, \tag{12}$$

in which r is the distance from a point $M'(x', y', z')$ of the volume element $d\omega'$ to the point M (x, y, z) where we would like to find the value of ψ at the instant t, the density ρ is not taken at the instant t, but at the time $t - r / V$, and the integral is extended over all space ([1]).

Suppose that we have only one electrified mass that occupies the volume Ω. Outside of it, ρ is zero, and the corresponding integral elements are zero. It then seems that it will suffice to reduce the field of integration to the volume Ω of the electric mass. However, one must note that the value of r at the different points in space must be taken at different epochs.

Imagine a sphere S of arbitrary radius r and center M (Fig. 1), and let Ω be the corresponding position of the electrified mass, i.e., its position at the instant $t - r / V$. If S and Ω do not meet then the integral elements will be zero for all the points of S, but if a region AB of S is found inside of

([1]) LORENTZ, Archives néerlandaises, 1892.

Ω then the integral elements will be non-zero for all points of *AB*. The domain of integration will then be the volume generated by *AB*, when one varies the radius of the sphere *S*, and at the same time, the position of Ω.

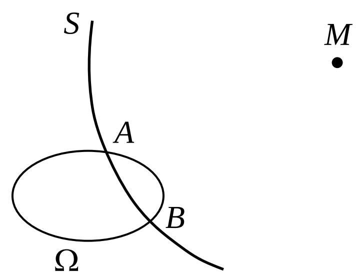

Figure 1.

Let *e* be the total charge of Ω, which we assume has very small dimensions, in such a way that *u* has essentially the same value at all points, and that *r* has essentially the same value and direction everywhere in the domain of integration. For an increase *dr* in *r*, the corresponding displacement in Ω in the normal direction to the sphere and towards the interior will be equal to $-(dr/V)\,u\cos(u,r)\,u$, if one takes the positive direction for *r* to be the direction from Ω to *M*.

As a result, whereas the elementary volume that is actually swept out by *AB* will be equal to the area *AB* × *dr*, the volume that is swept out by Ω will be equal to only:

$$\text{area } AB \times dr \left[1 - \frac{u}{V}\cos(u,r)\right] .$$

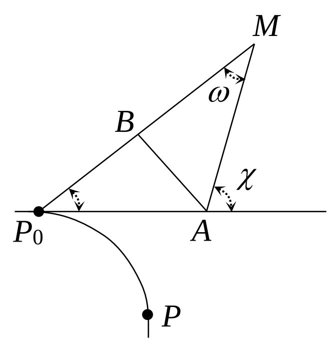

Figure 2.

As a result, the domain of integration will not be equal to Ω, but to $\Omega \Big/ \left[1 - \frac{u}{V}\cos(u,r)\right]$, when one takes *u* and *r* to have a mean value, and one will get the following value for ρ:

$$\psi = \int \frac{\rho}{r}\,d\omega' = \frac{1}{r}\int \rho\,d\omega' = \frac{e}{r\left[1 - \frac{u}{V}\cos(u,r)\right]} .$$

We can now suppose that all of the charge is concentrated at a point. I shall represent its position at the instant t by P (Fig. 2), and its position at the instant $t - \theta$ by P_0 (x_0, y_0, z_0), in which θ is such that $P_0M = r = V\,\theta$. x_0, y_0, z_0 will functions of $(t - \theta)$, and one will have:

$$u_x = \frac{\partial x_0}{\partial t}, \qquad u_y = \frac{\partial y_0}{\partial t}, \qquad u_z = \frac{\partial z_0}{\partial t}.$$

$u \cos(u, r)$ will then be equal to:

$$\frac{1}{r}\,[u_x(x-x_0)+u_y(y-y_0)+u_z(z-z_0)]\,,$$

and one will finally have:

$$\psi = \frac{e}{\left[r - \frac{1}{V}\sum u_x(x-x_0)\right]}. \tag{12}$$

Trace out a length P_0A that is equal to $u\,\theta$ along the tangent to the trajectory of the point P at P_0 that points in the direction of motion. In order to represent the position of the point P_0 at time t, upon starting at time $t - \theta$, one must maintain a uniform rectilinear motion. Draw AM and let B be the projection of A onto P_0M. One will immediately have:

$$P_0B = \sum (u_x\,\theta)\frac{x-x_0}{r} = \frac{\theta}{r}\sum u_x\,(x-x_0) = \frac{1}{V}\sum u_x\,(x-x_0)\,,$$

and as a result, (12) can be written:

$$\psi = \frac{e}{(BM)}. \tag{12′}$$

Equations (8) differ from (7) by only the change from ρ to $\rho\,u_x$, $\rho\,u_y$, $\rho\,u_z$. Moreover, in order to solve (7), we will not have to make any assumption on the constancy or variation of e in time, and from the algebraic viewpoint, the value (12) of ρ will be variable, even if e is variable, on the condition that we must take its value at the instant $t - \theta$. We will then obtain the solutions to (8) by simply changing e into $e\,u_x$, $e\,u_y$, $e\,u_z$, which will give:

$$F = \frac{e\,u_x}{\left[r - \frac{1}{V}\sum u_x(x-x_0)\right]}, \tag{13}$$

and two analogous expressions for G and H.

We must now calculate f, g, h, α, β, γ using equations (9) and (10), but first, we must verify that the condition (11) is satisfied, and in order to do that, we shall calculate the derivatives $\frac{d\psi}{dt}$, $\frac{dF}{dx}$, $\frac{dG}{dy}$, $\frac{dH}{dz}$.

ψ, F, G, H depend upon x directly and by the intermediary of θ. Indeed, θ is determined by the condition that $P_0 M$ is equal to $V\,\theta$, so one has the relation:

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = V^2\,\theta^2 . \tag{14}$$

Differentiate this while supposing that y, z, and t are constant, but x, and as a result, θ, are variable. Upon suppressing a factor of 2, one will get:

$$(x-x_0)\,dx + d\theta \sum (x-x_0)\left(-\frac{\partial x_0}{\partial \theta}\right) = V^2\,\theta\,d\theta ,$$

or since:

$$-\frac{\partial x_0}{\partial \theta} = \frac{\partial x_0}{\partial t} = u_x ,$$

$$(x-x_0)\,dx = [V^2\,\theta - \sum u_x\,(x-x_0)]\,d\theta = V\left[V\theta - \frac{1}{V}\sum u_x (x-x_0)\right] d\theta = (B\,M)\,V\,d\theta ,$$

or rather:

$$\frac{d\theta}{dx} = \frac{x-x_0}{V(B\,M)} . \tag{15}$$

One will get $d\theta / dy$ and $d\theta / dz$ similarly. In order to get $d\theta / dt$, it will suffice to differentiate (14) while keeping x, y, z constant, but recalling that x_0, y_0, z_0 are functions of $t - \theta$:

$$-(dt - d\theta) \sum u_x\,(x-x_0) = V^2\,\theta\,d\theta ,$$

so:

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \frac{-\frac{1}{V}\sum u_x (x-x_0)}{V\,\theta - \frac{1}{V}\sum u_x (x-x_0)} = -\frac{(P_0\,B)}{(B\,M)}, \\ \text{and} \quad \frac{d(t-\theta)}{dt} &= \frac{V\,\theta}{V\,\theta - \frac{1}{V}\sum u_x (x-x_0)} = \frac{(P_0\,M)}{(B\,M)}, \end{aligned}\right\} \tag{16}$$

Let W be the acceleration of P at the instant $(t - \theta)$. One can write the following equalities:

$$-\frac{\partial u_x}{\partial \theta} = \frac{\partial u_x}{\partial t} = w_x, \qquad \frac{dr}{d\theta} = \frac{d(V\theta)}{d\theta} = V,$$

$$\frac{\partial (P_0 B)}{\partial \theta} = \frac{1}{V}\frac{\partial}{\partial \theta}\sum u_x (x - x_0) = \frac{-1}{V}\sum w_x (x - x_0) + \frac{1}{V} + \sum u_x^2 = \frac{u^2}{V} - \frac{1}{V}\sum w_x (x - x_0),$$

$$\frac{\partial (BM)}{\partial \theta} = \frac{\partial}{\partial \theta}(r - P_0 B) = \frac{1}{V}[V^2 - u^2 + \sum w_x (x - x_0)],$$

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -V\frac{(P_0 B)}{(BM)},$$

$$\frac{d(P_0 B)}{dt} = \frac{d(P_0 B)}{d(t-\theta)}\frac{d(t-\theta)}{dt} = -\frac{\partial (P_0 B)}{\partial \theta}\frac{d(t-\theta)}{dt} = -\frac{1}{V}[u^2 - \sum w_x (x - x_0)]\frac{(P_0 M)}{(BM)}.$$

As a result:

$$\frac{4\pi}{e}\frac{d\psi}{dt} = \frac{d}{dt}\frac{1}{BM} = -\frac{1}{(BM)^2}\frac{d(BM)}{dt} = -\frac{1}{(BM)^2}\frac{d}{dt}[r - P_0 B]$$

$$= \frac{1}{V(BM)^2}\{V^2 (P_0 M) - [u^2 - \sum w_x (x - x_0)](P_0 B)\},$$

$$\frac{4\pi}{e}\frac{dF}{dt} = -\frac{d}{dx}\frac{u_x}{BM} = \frac{d\theta}{dx}\left[\frac{\partial u_x / \partial \theta}{BM} - \frac{u_x}{(BM)^2}\frac{\partial BM}{\partial \theta}\right] - \frac{u_x}{(BM)^2}\frac{\partial}{\partial x}[r - \frac{1}{V}\sum u_x (x - x_0)],$$

$$= \frac{-(x - x_0)}{V^2 (BM)^3}\left\{V(BM) w_x + u_x [V^2 - u^2 + \sum w_x (x - x_0)]\right\} + \frac{u_x^2}{V(BM)^2},$$

so

$$\frac{4\pi}{e}\sum\frac{dF}{dt} = -\frac{\sum w_x (x - x_0)}{V(BM)^2} - [V^2 - u^2 + \sum w_x (x - x_0)],$$

$$\frac{\sum u_x (x - x_0)}{V^2 (BM^3)} + \frac{u^2}{V(BM)^2} = -\frac{V(P_0 B)}{(BM)^3} + \frac{(P_0 B)[u^2 - \sum w_x (x - x_0)]}{V(BM)^3},$$

i.e., $-\dfrac{4\pi}{e}\dfrac{dF}{dt}$, precisely. The relation (11) is then established.

All that remains is to calculate the values of f and α that are given by (9) and (10):

$$\frac{4\pi f}{e} = -\frac{d}{dx}\frac{1}{BM} - \frac{1}{V^2}\frac{d}{dt}\frac{u_x}{BM},$$

$$\left[\frac{\partial (BM)}{\partial \theta}\frac{d\theta}{dx} + \frac{\partial (BM)}{\partial x}\right] - \frac{1}{V^2 (BM)}\frac{du_x}{dt} + \frac{u_x}{V^2 (BM)^2}\frac{d\,BM}{dt}$$

$$= \frac{[V^2 - u^2 + \sum w_x (x - x_0)](x - x_0)}{V^2 (BM)^3} - \frac{u_x}{V (BM)^2} - \frac{w_x}{V^2 (BM)}\frac{d\,(t-\theta)}{dt}$$

$$+ \frac{u_x}{V^2 (BM)^2}\left\{-V\frac{(P_0 B)}{(BM)} + \frac{u^2 + \sum w_x (x - x_0)}{V (BM)} P_0 M\right\} \tag{17}$$

$$= \frac{[V^2 - u^2 + \sum w_x (x - x_0)]\left[x - x_0 - u_x \frac{P_0 M}{V}\right] - w_x (P_0 M)(BM)}{V^2 (BM)^3} \tag{17}$$

$$\frac{\alpha}{e} = \frac{[V^2 - u^2 + \sum w_x (x - x_0)][u_y (z - z_0) - u_z (y - y_0)] - V(BM)[w_y (z - z_0) - w_z (y - y_0)]}{V^2 (BM)^3}\ . \tag{18}$$

We remark that $u_x (P_0 M) / V$ is equal to $\theta\, u_x$ and represents the projection of $P_0 A$ onto the x-axis. As a result, $x - x_0 - u_x (P_0 M) / V$ represents the projection of AM onto O_x . Geometrically, upon letting $\bar{D}$ represent the displacement and letting $\bar{H}$ represent the magnetic force, one can write:

$$\bar{D} = \frac{[V^2 - u^2 + \bar{w}\,\overline{P_0 M}\,]\,\overline{AM} - (P_0 M)(BM)\bar{w}}{4\pi V^2 (BM)^3} e, \tag{17′}$$

$$\bar{H} = \frac{[V^2 - u^2 + \bar{w}\,\overline{P_0 M}\,]\,\bar{u}\cdot\overline{P_0 M} + V(BM)\,\overline{\overline{\bar{w}\,\overline{P_0 M}}}}{V^2 (BM)^3} e\ . \tag{18′}$$

More generally, we shall let $\bar{X}\,\bar{Y}$ represent the scalar geometric product of the two vectors $\bar{X}$, $\bar{Y}$, i.e., the quantity $X\,Y \cos (X, Y)$, and let $\overline{\overline{\bar{X}\,\bar{Y}}}$ represent the vectorial product, i.e., a vector that is equal to $X\,Y \sin (X, Y)$ and is perpendicular to the two vectors $\bar{X}$ and $\bar{Y}$, in a sense that makes the projection of $\bar{Y}$ onto a plane normal to $\bar{X}$ coincide with the direction of the vectorial product under a rotation of 90° in the direct sense [here, that would be right to left, from the way that the axes were supposed to be oriented in formulas (1) and (2)].

Formula (18′) shows, first of all, that the magnetic force $\bar{H}$ is normal to $\overline{P_0 M}$. If we consider the sphere with its center at P_0 that passes through M then the point P_0 will play the same role as

M for all points on that sphere. Therefore, at the instant t, the magnetic force flux across that sphere will be zero.

We shall likewise look for the displacement flux across that sphere. In order to get the component of $\bar{D}$ that is normal to the sphere (i.e., along $P_0 M$), it suffices to replaces $\overline{AM}$ and $\bar{W}$ with their projections (BM) and W cos (w, r), resp., onto the radius in the expression for $\bar{D}$. As a result:

$$D_x = \frac{[V^2 - u^2 + \bar{w}\cdot\overline{P_0M}](BM) - (P_0M)(BM)w\cos(w,r)}{4\pi V^2 (BM)^2} e = \frac{V^2 - u^2}{4\pi V^2 (BM)^2} e$$

$$= \frac{V^2 - u^2}{4\pi r^2 [V - u\cos\varphi]^2} e \, ,$$

when we let φ denote the angle $AP_0 M$. We shall take the surface element dS of the sphere to be the zone that is bounded by the two cones with half-openings of φ and $\varphi + d\varphi$. We will have:

$$\int D_n \, dS = \int_0^\pi \frac{(V^2 - u^2)e}{4\pi r^2 [V - u\cos\varphi]^2} 2\pi r^2 \sin\varphi \, d\varphi = \frac{V^2 - u^2}{2u} e \left[\frac{-1}{V - u\cos\varphi} \right]_0^\pi = e \, .$$

Moreover, that result will an obligatory consequence of (3).

The two vectors D and H are at right angles, and the ratio of H to the projection D_t of D onto the plane normal to P_0M is constant and equal to $4\pi V$.

In order to get D_t, it suffices to replace the vectors $\overline{AM}$ and $\bar{W}$ with the components normal to P_0M in (18′):

$$\bar{D}_t = \frac{[\cdots]\overline{AB} - (P_0M)(BM)\,\bar{w}_t}{4\pi V^2 (BM)^2} \, .$$

Moreover:

$$\overline{\bar{u}\cdot\overline{P_0M}} = -\overline{\overline{P_0M}\cdot\bar{u}} = -\frac{1}{\theta}\overline{\overline{P_0M}\cdot\overline{P_0A}} = \frac{1}{\theta}\overline{\overline{P_0M}\cdot\overline{AB}} \, ,$$

because one can replace the second factor $-\overline{P_0A}$ with its projection $\overline{AB}$ onto a plane that is normal to the first one. The vectorial product considered is then numerically equal to ($P_0 M$) / θ (AB) = V(AB), and the direction of AB turns through a right angle around $P_0 M$ in the direct sense.

Similarly:

$$V\,\overline{\bar{w}\cdot\overline{P_0M}} = -V\,\overline{\overline{P_0M}\cdot\bar{w}} = - - V\,\overline{\overline{P_0M}\cdot\bar{w}_t}$$

is the vectorial product whose numerical value is – V ($P_0 M$) w_t , and the direction is that of $\bar{w}_t$ turning through a right angle around $P_0 M$ in the direct sense.

Having established those relations, it will result immediately from a comparison of the values of H and D_t that H is equal to $4\pi\, V\, D_t$, and the direction of $\bar{D}_t$ turns through a right angle around $P_0\, M$ in the direct sense, i.e., from right to left for an observer that has his feet at P_0 and his head at M.

Since H is perpendicular to the two components D_t and D_n of D, it will be perpendicular to D itself *a fortiori*.

In the case where the motion of P is uniform and rectilinear, formulas (17) and (18) will simplify. First, the terms in W will disappear. In addition, the points P and A will coincide, and it will be possible to express the values $\bar{D}$ and $\bar{H}$ in terms of the current position P or A without introducing P_0 . Indeed, one first has:

$$\overline{\bar{u}\cdot\overline{P_0M}} = \overline{\bar{u}\cdot\overline{P_0A}} + \overline{\bar{u}\cdot\overline{AM}} = \overline{\bar{u}\cdot\overline{AM}}\,,$$

because the vectorial product $\overline{P_0A}$ is zero, since its two factors have the same direction. In addition, if one lets ω and χ denote the angle P_0MA and the angle between AM and the prolongation of P_0A, resp., then one will have the equalities:

$$\frac{\sin\omega}{\sin\chi} = \frac{P_0A}{P_0M} = \frac{n}{V}\,,$$

$$BM = AM\cos\omega = AM\sqrt{1-\sin^2\omega} = AM\sqrt{1-\frac{u^2\sin^2\chi}{V^2}}\,,$$

and (17′) and (18′) will become:

$$D = \frac{V(V^2-u^2)}{4\pi(AM)^2[V^2-u^2\sin^2\chi]^{3/2}}\,e\,, \tag{17″}$$

$$H = \frac{V(V^2-u^2)\,u\sin\chi}{(AM)^2[V^2-u^2\sin^2\chi]^{3/2}}\,e\,, \tag{18″}$$

where the first vector points along AM and the second one points along the common perpendicular to AM and u.

Equations (17′) and (18′) will again reduce to (17″) and (18″) when one supposes that the points P_0 and M are infinitely close and that one confines oneself to principal values.

On the contrary, at a great distance from the point P_0, those are the most important terms in w, and D will essentially reduce to its component that is normal to $P_0\, M$.

APPLICATIONS OF THE PRECEDING FORMULAS

I. – As a first application, we shall calculate the electric and magnetic field that are produced by a circuit that carries a constant linear charge density and slides along itself with a constant velocity u.

Let S be a point of the circuit C that is defined by the length S of an arc of the circuit when measured by starting from a fixed origin O (Fig. 3) in the sense of the velocity u (1). The action of the charge $\rho\, ds$ of an element of length ds that neighbors on the point P will depend upon its position at P_0 (s_0) at time that is θ earlier.

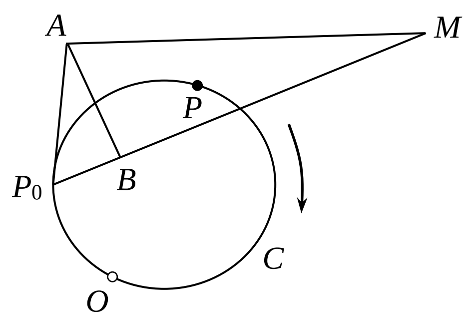

Figure 3.

Upon always letting x, y, z denote the coordinates of M, while x_0, y_0, z_0 denote those of P_0, which are supposed to be given as functions of s_0, θ will be equal to $(s - s_0) / u$ and will be given by the condition that $V\theta = P_0 M$, or:

$$\frac{V^2}{u^2}(s - s_0)^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 . \tag{19}$$

The components of the velocity u at P_0 will be $u\frac{dx_0}{ds_0}$, $u\frac{dy_0}{ds_0}$, $u\frac{dz_0}{ds_0}$, and those of the acceleration will be $u\frac{d^2x_0}{ds_0^2}\frac{\partial s_0}{\partial t} = u^2\frac{d^2x_0}{ds_0^2}$, and similarly $u^2\frac{d^2y_0}{ds_0^2}$, $u^2\frac{d^2z_0}{ds_0^2}$, because u is constant.

If one takes that into account then one will have:

$$d\alpha = \frac{u\left\{V^2 - u^2\left[1 - \sum \frac{d^2x_0}{ds_0^2}(x - x_0)\right]\right\}\left[\frac{dy_0}{ds_0}(z - z_0) - \frac{dz_0}{ds_0}(y - y_0)\right] + V\,u^2 (BM)\left[\frac{d^2z_0}{ds_0^2}(z - z_0) - \cdots\right]}{V^2 (BM)^3}\,\rho\, ds .$$

However:

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(1) In the figure, the line $P_0 A$ must be tangent to the arc OP_0, and the line AB is perpendicular to P_0M.

$$\frac{d\,(BM)}{ds_0} = \frac{d\left[r - \frac{u}{V}\sum \frac{dx_0}{ds_0}(x-x_0)\right]}{ds_0} = -\frac{1}{r}\sum \frac{dx_0}{ds_0}(x-x_0) + \frac{u}{V}\left[1 - \sum \frac{d^2x_0}{ds_0^2}(x-x_0)\right],$$

because:

$$\sum \frac{d^2 x_0}{ds_0^2} = \frac{dx_0^2 + dy_0^2 + dz_0^2}{ds_0^2} = 1 .$$

If one infers $1 - \sum \frac{d^2x_0}{ds_0^2}(x - x_0)$ from that equation and substitutes it in the expression for $d\alpha$ then it will become:

$$d\alpha = \frac{u\left\{V^2 - \frac{V\,u}{r}\sum \frac{dx_0}{ds_0}(x-x_0) - V\,u\,\frac{d\,BM}{ds_0}\right\}\left[\frac{dy_0}{ds_0}(z-z_0) - \cdots\right] + V\,u^2(BM)\left[\frac{d^2z_0}{ds_0^2}(z-z_0) - \cdots\right]}{V^2(BM)^3}\,\rho\,ds$$

$$= \left\{\frac{u\left[\frac{dy_0}{ds_0}(z-z_0) - \cdots\right]}{r(BM)} + \frac{u^2}{V}\frac{d}{ds_0}\frac{\frac{dy_0}{ds_0}(z-z_0) - \frac{dz_0}{ds_0}(y-y_0)}{BM}\right\}\frac{\rho\,ds}{B\,M} .$$

In order to get α, one must integrate the preceding expression along the complete circuit while recalling that s_0 is a function of s that is defined by (19). However, it is more convenient to change the variable by taking s_0 instead of s.

When (19) is differentiated, it will become:

$$\frac{V^2}{u^2}(s - s_0)\,(ds - ds_0) = -\sum \frac{dx_0}{ds_0}(x-x_0)\,ds_0 ,$$

or since $(V / u)\,(s - s_0) = P_0\,M = r$:

$$r\,(ds - ds_0) = -\frac{u}{V}\sum \frac{dx_0}{ds_0}(x-x_0)\,ds_0 ,$$

or

$$\frac{ds}{BM} = \frac{ds_0}{r} .$$

The domain of integration with respect to s_0 is obviously the same as it is for s ; therefore:

$$\alpha = \int \left\{ \frac{u\left[\dfrac{dy_0}{ds_0}(z-z_0)-\cdots\right]}{r^2(BM)} + \frac{u^2}{V\,r}\frac{d}{ds_0}\frac{\dfrac{dy_0}{ds_0}(z-z_0)-\dfrac{dz_0}{ds_0}(y-y_0)}{BM} \right\} \rho\, ds_0\ .$$

Moreover:

$$\frac{dr}{ds_0} = -\frac{1}{r}\sum \frac{dx_0}{ds_0}(x-x_0)\ ,$$

and the second terms in the parentheses can be written:

$$\frac{u^2}{V}\frac{d}{ds_0}\frac{\dfrac{dy_0}{ds_0}(z-z_0)}{r\,(BM)} - \frac{u^2\sum\dfrac{dx_0}{ds_0}(x-x_0)}{V\,r^3(BM)}\left[\frac{dy_0}{ds_0}(z-z_0)-\cdots\right],$$

and α will become:

$$\alpha = \int_C \left\{ \frac{u\left[\dfrac{dy_0}{ds_0}(z-z_0)-\cdots\right]}{r^3(BM)}\left[r-\frac{u}{V}\sum\frac{dx_0}{ds_0}(x-x_0)\right]+\frac{u^2}{V}\frac{d}{ds_0}\frac{\cdots}{r\,(BM)} \right\} \rho\, ds_0$$

$$= \int_C \frac{\dfrac{dy_0}{ds_0}(z-z_0)-\dfrac{dz_0}{ds_0}(y-y_0)}{r^3}\,u\,\rho\, ds_0 + \int_C d\left\{\frac{\rho u^2}{V}\frac{\dfrac{dy_0}{ds_0}(z-z_0)-\dfrac{dz_0}{ds_0}(y-y_0)}{r\,(BM)}\right\}\ .$$

The second integral is zero identically, and the magnetic field is the same as the one that is produced by a conducting current of intensity $u\,\rho$.

Similarly, one will find the following value for $4\pi V^2 f$:

$$4\pi V^2 f = \int_C V^2\frac{x-x_0}{r^2}\rho\, ds_0 + \int_C d\,\frac{\rho u V(AM)}{r\,(BM)}\ .$$

Here again, the second integral is zero, and the electric field is the same as when the circuit is at rest.

II. – Let us now look for the energy lost by radiation. In order to do that, we shall evaluate the energy flux during the time dt across the sphere whose center is P_0 and radius is $P_0\,M$, which we shall call the *position* of the wave that is emitted by P_0 at time t, to simplify.

If D_t is the component of D in the tangent plane to the wave then the flux across the element dS will be equal to $V^2\,\overline{\overline{D_t}\,\overline{H}}\,dS\,dt$. However, D_t is normal to H and equal to $H / 4\pi V$, so that expression will be further equal to:

$$V\frac{1}{4\pi}H^2\,dS\,dt = V4\pi V D_t^2\,dS\,dt = \left(\frac{1}{8\pi}H^2 + 2\pi V^2 D_t^2\right)dS\,V\,dt\,.$$

Upon integrating over dS, one will get the total flux that passes from the interior of the wave surface to its exterior.

During the time dt, the radius of the wave surface will have increased by $V\,dt$, and the space that is swept out by the surface will contain an energy that is equal to:

$$Vdt\int\left[\frac{1}{8\pi}H^2 + 2\pi V^2 D^2\right]dS = Vdt\int\left[\frac{1}{8\pi}H^2 + 2\pi V^2 D_t^2\right]dS + V\,dt\int 2\pi V^2 D_n^2\,dS\,.$$

Finally, the quantity of energy that traverses the wave surface, which is considered to be moving, is equal to:

$$-\,V\,dt\int 2\pi V^2 D_n^2\,dS\,.$$

However, D_n is infinitely small of order two when r is infinitely large of order one, i.e., when t increases indefinitely, θ will vary at the same time in order for $t - \theta$ to remain constant and the point P_0 to correspondingly remain the same. As a result, the preceding integral will have order $1/r^2$ and will tend to zero. Therefore, if we consider two wave surfaces S and S' that correspond to the same value of t, but different values of θ, and for t increasing indefinitely then the energy found between those two wave surfaces will tend to a constant value, and since all of that energy will go out to infinity, it will be found to have been lost by radiation.

In order to perform the calculation, I shall first suppose that S and S' correspond to values of θ that differ by infinitely little, say, θ and $\theta + d\theta$. The surface S' will have a radius that is equal to $V(\theta + d\theta)$, and its center will not be at P_0, but at P_0', such that $\overline{P_0' P_0} = u\,d\theta$.

As a result, a volume element that is found between those two surfaces will be equal to $dS\,[V\,d\theta - u\cos\varphi\,d\theta]$ and the energy that is found between S and S' will be equal to:

$$dE = d\theta\int\left[\frac{1}{8\pi}H^2 + 2\pi V^2 D^2\right](V - u\cos\varphi)\,dS\,.$$

Since we seek only the limiting value of dE for r infinitely large, we must keep only the terms in $1/r^2$ in the parentheses, i.e.:

$$\frac{1}{8\pi}H^2 + 2\pi V^2 D_t^2 = \frac{H^2}{4\pi} = \frac{\alpha^2+\beta^2+\gamma^2}{4\pi} \ .$$

To simplify, I will take the x-axis to be parallel to $P_0\,A$, i.e., the direction of the velocity at P_0, and I will take Ox to be the axis of a system of polar coordinates r, φ, ψ. As a result, upon taking only the terms in $1/r$ in α, β, γ, one will have:

$$\frac{\alpha}{e}\frac{r}{V} = \frac{w_y\sin\varphi\sin\psi - w_z\sin\varphi\cos\psi}{[V-u\cos\varphi]^2},$$

$$\frac{\beta}{e}\frac{r}{V} = \frac{-[w_x\cos\varphi + w_y\sin\varphi\cos\psi + w_z\sin\varphi\sin\psi]u\sin\varphi\sin\psi + [V-u\cos\varphi][w_z\cos\varphi - w_x\sin\varphi\sin\psi]}{[V-u\cos\varphi]^3},$$

$$\frac{\gamma}{e}\frac{r}{V} = \frac{[w_x\cos\varphi+\cdots]u\sin\varphi\cos\psi + [V-u\cos\varphi][w_x\sin\varphi\cos\psi - w_y\cos\varphi]}{[V-u\cos\varphi]^3}.$$

Hence:

$$\frac{H^2}{4\pi}[V-u\cos\varphi]\,dS = \frac{\alpha^2+\beta^2+\gamma^2}{4\pi}[V-u\cos\varphi]\,r^2 d\varphi\sin\varphi\,d\psi$$

$$= \frac{e^2V^2\sin\varphi\,d\varphi\,d\psi}{4\pi[V-u\cos\varphi]^3}\left\{\begin{array}{l} [w_x\cos\varphi + w_y\sin\varphi\cos\psi + w_z\sin\varphi\sin\psi]^2u^2\sin^2\varphi \\ +2u[w_x\cos\varphi+\cdots][V-u\cos\varphi]\{w_x - \cos\varphi[w_x\cos\varphi+\cdots]\} \\ +\ [V-u\cos\varphi]^2\{w_x^2+w_y^2+w_z^2 - [w_x\cos\varphi+\cdots]^2\} \end{array}\right\}$$

$$= \frac{e^2V^2\sin\varphi\,d\varphi\,d\psi}{4\pi[V-u\cos\varphi]^3}\left\{\begin{array}{l} [w_x\cos\varphi + w_y\sin\varphi\cos\psi + w_z\sin\varphi\sin\psi]^2u^2\sin^2\varphi \\ +2u[w_x\cos\varphi+\cdots][V-u\cos\varphi]\{w_x - \cos\varphi[w_x\cos\varphi+\cdots]\} \\ +\ [V-u\cos\varphi]^2\{w_x^2+w_y^2+w_z^2 - [w_x\cos\varphi+\cdots]^2\} \end{array}\right\}.$$

One must integrate over ψ from 0 to 2π and over φ from 0 to π. In the first integration, the terms that contain $\cos\psi$, $\sin\psi$, or $\sin\psi\cos\psi$ disappear, while the ones that contain $\cos^2\psi$ or $\sin^2\psi$ will be found to be multiplied by π, in one case, and 2π, in the other. As a result:

$$dE = d\theta\int_0^\pi \frac{e^2V^2\sin\varphi\,d\varphi}{4[V-u\cos\varphi]^3}\left\{\begin{array}{l} -(V^2-u^2)[2w_x^2\cos^2\varphi + w_y^2\sin^2\varphi + w_z^2\sin^2\varphi] \\ +\,4u\,w_x^2[V-u\cos\varphi]\cos\varphi \\ +\,2[V-u\cos\varphi]^2(w_x^2+w_y^2+w_z^2) \end{array}\right\}.$$

To integrate this, I make the change of variables:

$$V - u \cos \varphi = x \,,$$

so

$$u \sin \varphi \, d\varphi = dx$$

and

$$\cos \varphi = \frac{V - x}{u} \,.$$

The new limits of integration will be $V - u$ and $V + u$.

$$dE = d\theta \frac{e^2 V^2}{4u} \int_{V-u}^{V+u} \frac{dx}{x^3} \{-(V^2 - u^2)[(2w_x^2 - w_y^2 - w_z^2)\frac{(V-x)^2}{u^2} + w_y^2 + w_z^2] + 4V\, w_x^2\, x + 2x^2 (w_x^2 + w_y^2 + w_z^2)\}$$

$$= d\theta \frac{e^2 V^2}{4u} \int_{V-u}^{V+u} \frac{dx}{x^3} \{[2w_x^2 V^2 - (V^2 - u^2)(w_y^2 + w_z^2)]\left[\frac{-(V-x)^2}{x^3} + \frac{2V}{x^4}\right] + \frac{(V^2 + u^2)(w_y^2 + w_z^2) - 2V^2 w_x^2}{x^2}\Bigg\}. \tag{20}$$

Now:

$$\int_{V-x}^{V+x} \frac{dx}{x^5} = \frac{1}{4}\left[\frac{1}{(V-u)^4} - \frac{1}{(V+u)^4}\right] = \frac{2uV(V^2 + u^2)}{(V^2 - u^2)^4} \,,$$

$$\int_{V-x}^{V+x} \frac{dx}{x^4} = \frac{2}{3} \frac{u(3V^2 + u^2)}{(V^2 - u^2)^3} \,,$$

$$\int_{V-x}^{V+x} \frac{dx}{x^3} = \frac{2uV}{(V^2 - u^2)^2} \,,$$

and after all reductions have been made, the preceding expression will become:

$$dE = d\theta \frac{2}{3} \frac{e^2}{V} \left[w_x^2 \frac{V^6}{(V^2 - u^2)^3} + (w_y^2 + w_z^2) \frac{V^4}{(V^2 - u^2)^2} \right]$$

$$= \frac{2}{3} \frac{e^2}{V} d\theta \left[w^2 \frac{V^4}{(V^2 - u^2)^3} + w^2 \cos^2(w, u) \frac{V^4 u^2}{(V^2 - u^2)^3} \right]. \tag{21}$$

When u^2 is negligible compared to V^2, one will have more simply:

$$dE = \frac{2}{3}\frac{e^2\,w^2}{V}\,d\theta\,,$$

which is an expression that was given by Larmor [Phil. Mag. **44** (1897), pp. 503]. Larmor calculated it simply by evaluating the energy that traversed a wave surface of infinite radius. That procedure, which is permissible when u is negligible, will be incorrect under the more general assumption, because the waves that the charge e emits are not concentric, and a sphere that is the wave surface at one moment will no longer be such a thing at the following instant.

Suppose that the charge is originally at rest and the field is invariable. One sets the charge into motion, only to bring it back to rest after a time τ. If the velocity u is less than V during all of the motion (as we have supposed implicitly up to now, moreover) then at a time t after the motion ends, the perturbation will be concentrated between two spheres, one of which has a radius of $V(\tau + t)$ and its center at the initial position, and the other of which has a radius of Vt and its center at the final position. Outside of that region, the field will be the same as it was originally, and inside of it, the field will be the same as if the charge had always been at rest in its present position. From the preceding calculation, the energy that is carried off to infinite by the perturbation will be equal to $\int_0^\tau dE$ and essentially positive. That expression then represents the total work that is done on the charge during the motion. However, it would not be true to say that the work done at an arbitrary instant is equal to dE, because the work done is infinitely large when one supposes that the charge is concentrated at a point, but the difference must be an exact differential of an (infinite) function of u and w that will disappear when one starts at rest and returns to rest.

Let us also seek the total impulse of the force that is necessary to produce the motion. We have established (*Écl. Él.*, t. XIV, pp. 45) that the projections of the forces developed by the field on the charges onto the impulsion axes during an arbitrary time are equal to the variations of the integrals:

$$-\int (g\,\gamma - h\,\beta)\,d\omega\,,\quad -\int (h\,\alpha - f\,\gamma)\,d\omega\,,\quad -\int (f\,\beta - g\,\alpha)\,d\omega\;.$$

Here, the desired quantity will be equal and of opposite sign. Now, at the initial instant, α, β, γ are zero and the integrals are zero. It will then suffice to have their values at the end of the motion. However, since the charge is no longer subject to any force, because it is isolated in the field, then the integrals will no longer vary, and we can seek their values for t infinite. We shall operate as before by first seeking the values of the integrals for the space that is found between the two wave surfaces S and S'.

$g\,\gamma - h\,\beta$, …, represent the components of the vectorial product $\overline{\bar{D}\,\bar{H}}$, which is equal to the resultant of the following two $\overline{\bar{D}_t\,\bar{H}}$ and $\overline{\bar{D}_n\,\bar{H}}$. For the same reason as before, the product containing D_n will be negligible, and as for the first one, as we have seen, it will be equal to $H^2/4\pi V$ and point in the direction of $P_0\,M$. Therefore, if dI_x, dI_y, dI_z represent the desired integrals then their expression will differ from those of dE only by suppressing a factor of V and introducing the factors $\cos\varphi$, $\sin\varphi\cos\psi$, $\sin\varphi\sin\psi$ under the $\int$ sign, which represent the direction cosines of the line $P_0\,M$.

We first address dI_x . $(1 / V) \cos \varphi$ is equal to $\frac{1}{u}\left(1-\frac{x}{V}\right)$.

Let dE' denote the value that dE takes when one multiplies the quantity under the $\int$ sign by x, which amounts to changing the integrals:

$$\int \frac{dx}{x^5}, \quad \int \frac{dx}{x^4}, \quad \int \frac{dx}{x^3}$$

in equation (20) into the following ones:

$$\int \frac{dx}{x^4}, \quad \int \frac{dx}{x^3}, \quad \int_{V-u}^{V+u} \frac{dx}{x^2} = \frac{2u}{V^2-u^2} .$$

One easily finds that $dE' = (V^2-u^2)/V\, dE$ and as a result:

$$dI_x = \frac{1}{u}\left[dE - \frac{1}{V}\, dE'\right] = \frac{u}{V^2}\, dE = \tfrac{2}{3}\, e^2\, u\, w^2 \left[1+\frac{u^2 \cos^2(w,u)}{V^2-u^2}\right] \frac{V\, d\theta}{(V^2-u^2)^2} .$$

As for I_y and I_z , I say that they are zero. Here, since we introduce a factor $\sin \varphi \cos \psi$ that contains ψ, it is necessary to perform the integration over ψ. In that integration, the terms in $\cos\varphi$, $\cos^3 \psi$, $\sin^2 \psi \cos \psi$, $\cos^2 \psi \sin \psi$, $\sin \psi \cos \psi$ will disappear, and all that will remain are the terms that contain $\cos^2 \psi$, i.e., upon dropping the constant factors:

$$\frac{\sin^2 \varphi\, d\varphi}{[V-u\cos\varphi]^5}\{-2(V^2-u^2)\, w_x\, w_y \cos\varphi \sin\varphi + 2u\, w_x\, w_y\, [V-u\cos\varphi \sin\varphi]\} ,$$

and upon introducing a factor of $\frac{-u^4}{2V\, w_x\, w_y}$, the identity to be verified will become:

$$0 = \int_u^{\pi} \frac{(V u \cos\varphi - u^2)\, u^2 \sin^3 \varphi\, d\varphi}{[V-u\cos\varphi]^5}$$

$$= \int_{V-u}^{V+u} \frac{(V^2-u^2-V\,x)[-(V^2-u^2)+2V\,x-x^2]\, dx}{x^5}$$

$$= \int_{V-u}^{V+u} \left[\frac{-(V^2-u^2)^2}{x^5} + \frac{3V\,(V^2-u^2)}{x^4} + \frac{u^2-3V^2}{x^3} + \frac{V}{x^2}\right] dx ,$$

which one does immediately upon replacing the integrals with their given values.

If one considers dI_x, dI_y, dI_z to be the components of a vector $\overline{dI}$ then one will see that $\overline{dI}$ is equal to $\overline{dI_x}$, and one will have the direction of O_x, i.e., of $\overline{u}$, and as a result:

$$\overline{dI} = \tfrac{2}{3} e^2 \, \overline{u} \, w^2 \left[1 + \frac{u^2 \cos^2(w,u)}{V^2 - u^2}\right] \frac{V\,d\theta}{(V^2 - u^2)^2}, \tag{22}$$

which is an expression that will be true independently of the particular choice of the axes that were adopted in order to simplify the calculation, and upon taking the geometric integral of $\overline{dI}$ over the duration of motion, one will get the total impulse during the time that the force produced the motion. One sees that it will not be zero, in general.

As we did before in regard to dE, we must point out that $\overline{dI}$ does not represent the elementary impulse of the force that acts upon the charge e, but the difference (which is infinite) will be an exact differential of a function of u and v that will disappear when one starts out at rest and returns to it.

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(continuation, pp. 53-59) [1]

III. – *Force exerted by a charged body of very small dimensions on itself when it is animated by a translatory motion.*

Let u be the velocity of the body at the instant t. I will take Ox to be parallel to that direction. Let $P(\xi, \eta, \zeta)$ and $M(x, y, z)$ be two points of the body (Fig. 4). I shall first look for the action of a charge de that surrounds P on a charge de' that surrounds M.

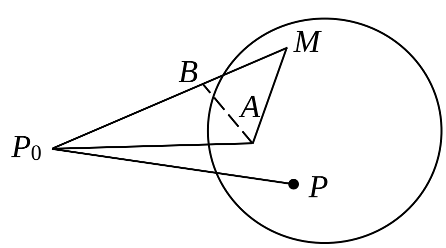

Figure 4.

At a previous time θ, the point P was at P_0, whose coordinates were:

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[1] See *L'Éclairage Électrique* on 2 July, pp. 5.

$$x_0 = \xi - u\,\theta + \tfrac{1}{2} w_x\,\theta^2 + \ldots, \qquad y_0 = \eta + \tfrac{1}{2} w_y\,\theta^2 + \ldots, \qquad z_0 = \zeta + \tfrac{1}{2} w_x\,\theta^2 + \ldots, \tag{23}$$

and at that moment, the components of the velocity were:

$$u_{0x} = u - w_x\,\theta + \ldots, \qquad u_{0y} = - w_y\,\theta + \ldots, \qquad u_{0z} = - w_z\,\theta + \ldots \tag{24}$$

We determine θ from the condition that $P_0\,M$ is equal to $V\theta$, and we will then get the equation:

$$\begin{aligned} V^2\,\theta^2 &= (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 + 2\theta\,u\,(x-\xi) + \theta^2 u^2 \\ &\quad - \theta^2\,[w_x\,(x-\xi) + w_y\,(y-\eta) + w_z\,(z-\zeta)] - \theta^3\,u\,w_x + \ldots \end{aligned} \tag{25}$$

If one considers PM to be infinitely small then θ will have the same order as PM, and the terms in the right-hand side of (25) to be neglected will be of order higher than three.

Since the velocity of M is parallel to Ox, the component dX along that direction of the force that de exerts upon de' will reduce to $4\pi V^2\,f\,de$ ([1]), or upon replacing f with its value (17):

$$\frac{dX}{de\,de'} = \frac{[V^2 - u_0^2 + \sum w_{0x}\,(x - x_0)][x - x_0 - \theta u_{0x}] - w_x(P_0M)(BM)}{(BM)^3} \,.$$

We seek only the principal value of dX. We will first be led to neglect terms in the expression for dX for which w has order higher than the ones in $V^2 - u_0^2$, and in those terms, we take only the principal value by setting $u_0 = u$, and equating points A and P in the calculation of BM when their distance is equal to $\tfrac{1}{2} w\theta^2 + \ldots$ However, to that degree of approximation, the value of dX' relative to the action of M on P will be equal and opposite to that of dX, and those two terms will cancel in the integration. It will then be necessary to calculate dX to a higher degree of approximation, and in order to do that, we must keep the terms in w (although we can calculate them to the first degree of approximation) and calculate the term $\dfrac{V^2 - u_0^2}{(BM)^3}$ while taking into account the difference between u_0 and u and between the points A and P.

From (24), u_0^2 is equal to $u^2 - 2\theta\,u\,w_x + \ldots$, and one can write:

$$\frac{(V^2 - u_0^2)(x - x_0 - \theta u_{0x})}{(BM)^3} = \frac{(V^2 - u^2)(x - x_0 - \theta u_{0x})}{(BM)^3} + \frac{2\theta u\,w_x(x - x_0 - \theta u_{0x})}{(BM)^3} \,,$$

or rather, upon replacing x_0 and u_{0x} with their values in the first term, the preceding will become:

([1]) See *L'Éclairage Électrique*, t. XIV, pp. 456.

$$\frac{(V^2-u^2)(x-\xi)}{(BM)^3}+\frac{\frac{1}{2}\theta^2 w_x\,(V^2-u^2)+2\theta\,u\,w_x(x-x_0-\theta u_{0x})}{(BM)^3}\ .$$

In the latter term, the numerator has order two, while the first term has order one. The second term can then be calculated by taking only the principal values. All that remains is to calculate $1/(BM)^3$ to the second degree of approximation. By definition:

$$BM = r-\frac{1}{V}\sum u_{0x}(x-x_0)\ = V\,\theta-\frac{(u-\theta\,w_x)(x-\xi+\theta u-\frac{1}{2}w_x\,\theta^2)-\theta\,w_y(y-\eta)-\theta\,w_z(z-\zeta)}{V},$$

or

$$V\,(BM) =\ (V^2-u^2)\,\theta-u\,(x-\xi)+\theta\Big[w_x\,(x-\xi)+w_y(y-\eta)+\cdots\Big]+\tfrac{3}{2}\theta^2 u\,w_x+\cdots$$

Taking the square will give:

$$\begin{aligned}V^2(BM)^2 &=\ (V^2-u^2)[(V^2-u^2)\,\theta^2-2\theta\,u\,(x-\xi)]+u^2(x-\xi)^2\\ &\quad+[\,(V^2-u^2)\ \ \theta-u\,(x-\xi)]\,\{2\theta\,[w_x\,(x-\xi)+\ldots]+\,3\theta^2\,u\,w_x\}\,+\,\ldots,\end{aligned}$$

in which we always confine ourselves to the second degree of approximation, i.e., we neglect fourth-order terms since (BM) has order two.

I replace the first parenthesis with its value:

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2+\ \theta^2\,[w_x\,(x-\xi)+\ldots]-\ \theta^3\,u\,w_x\ ,$$

which is inferred from equation (25), and get:

$$\begin{aligned}V^2(BM)^2 &=\ (V^2-u^2)[(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]+u^2(x-\xi)^2\\ &\quad+[\,(V^2-u^2)\,\theta^2-2\ \theta\,u\,(x-\xi)]\,[w_x\,(x-\xi)+\ldots]+[2(V^2-u^2)\,\theta^2-3\theta u\,(x-\xi)]\theta u\,w_x\,.\end{aligned}$$

Up to the coefficient V^2, the terms in the first row represent the value of $(BM)^2$ to the first degree of approximation. As a result, if one lets ρ denote the distance PM and lets χ denote the angle between PM and O then that quantity will be equal to $\rho^2\,[V^2-u^2\sin^2\chi]$.

In the last two products, I again replace $(V^2-u^2)\,\theta^2$ with its value (25), but while pointing out that here I am confining myself to the first degree of approximation, since $(V^2-u^2)\,\theta^2$ is found to be multiplied by an infinitesimal. Finally:

$$\begin{aligned}&V^2(BM)^2\\ &\quad=\ \rho^2\,[(V^2-u^2\sin^2\chi]+\rho^2[w_x\,(x-\xi)+w_y(y-\eta)+w_z(z-\zeta)]+\theta\,u\,w_x\,[2\rho^2+\theta u\,(x-\xi)]\ .\end{aligned}$$

Let ψ be the angle between the plane of PM and O_x and the xy-plane.
One will easily find that:

$$(x-\xi) = \rho\cos\chi\,, \qquad (y-\eta) = \rho\sin\chi\cos\psi\,, \qquad (z-\zeta) = \rho\sin\chi\sin\chi\,,$$

and to the first degree of approximation:

$$P_0\,M = V\,\frac{\sqrt{V^2-u^2\sin^2\chi}+u\cos\chi}{V^2-u^2}\,\rho\,,$$

$$\theta = \frac{\sqrt{V^2-u^2\sin^2\chi}+u\cos\chi}{V^2-u^2}\,\rho\,.$$

As a result:

$$V^2(BM)^2 = \rho^2[V^2-u^2\sin^2\chi]$$

$$+\rho^2\left\{[w_x\cos\chi+w_y\sin\chi\cos\psi+\cdots]-w_x\frac{u^4\cos^2\chi}{(V^2-u^2)^2}+u\,w_x\left[V^2-u^2\sin^2\chi\right]\frac{3u\cos\chi+2\sqrt{}}{(V^2-u^2)^2}\right\},$$

and finally:

$$\frac{1}{V^3(BM)^3} = \frac{1}{\rho^3\left[V^2-u^2\sin^2\chi\right]^{3/2}}$$

$$\left[1-\tfrac{3}{2}\rho\,\frac{[w_x\cos\chi+\cdots]-u\,w_x\dfrac{u^3\cos^3\chi-[V^2-u^2\sin^2\chi](3u\cos\chi+2\sqrt{})}{(V^2-u^2)^2}}{V^2-u^2\sin^2\chi}\right].$$

When one combines the various calculated terms, one will then have, to the assumed degree of approximation:

$$\frac{dX}{de\,de'} = V^3\frac{(V^2-u^2)\cos\chi}{\rho^3[V^2-u^2\sin^2\chi]^{3/2}}-\tfrac{3}{2}V^3\frac{[w_x\cos\chi+\cdots](V^2-u^2)-\dfrac{u^4w_x\cos^3\chi}{(V^2-u^2)}}{\rho[V^2-u^2\sin^2\chi]^{5/2}}\cos\chi$$

$$-\tfrac{3}{2}V^3\frac{u\,w_x(3u\cos\chi+2\sqrt{})\cos\chi}{\rho[V^2-u^2\sin^2\chi]^{3/2}(V^2-u^2)}+\frac{V^3}{\rho[V^2-u^2\sin^2\chi]^{3/2}}\left\{\tfrac{1}{2}w_x\frac{V^2+u^2(\cos^2\chi-\sin^2\chi)+2u\sqrt{}\cos\chi}{V^2-u^2}\right.$$

$$+\ 3u\,w_x\,\frac{\sqrt{\ }+u\cos\chi}{V^2-u^2}\cos\chi+[w_x\cos\chi+\cdots]\cos\chi-w_x\,\frac{V^2-u^2\sin^2\chi+u\sqrt{\ }\cos\chi}{V^2-u^2}\Bigg\}\ .$$

In the preceding expression, one can suppress all terms that change sign when one inverts the roles of the points P and M, which amounts to changing χ into $\pi-\chi$ and ψ into $\pi+\psi$, or rather, to changing the signs of cos χ, sin χ cos ψ, and sin χ sin ψ. As we have said, the terms in $1/\rho^2$ will disappear, and among the other ones, the only ones that will remain are the ones that do not contain the radical in the numerator: Moreover, the latter have a sum that is identically zero. As a result:

$$\frac{dX}{de\,de'}=-\frac{V^2[w_x\cos\chi+\cdots]\cos\chi}{2\rho[V^2-u^2\sin^2\chi]^{5/2}}[V^2-u^2\sin^2\chi-3u^2\cos^2\chi]$$

$$-\ \frac{V^3\,w_x}{2\rho[V^2-u^2\sin^2\chi]^{5/2}}\{V^2-u^2\sin^2\chi-3u^2\cos^2\chi\}\ .$$

Now, χ is the angle between the two directions ρ and u, and the expression w_x cos χ + … is equal to χ cos (w, ρ), in such a way that one can write:

$$\frac{dX}{de\,de'}=-\frac{V^3\,w[\cos(w,\rho)\cos(u,\rho)]}{2\rho[V^2-u^2\sin^2\chi]^{3/2}}+\frac{3V^3wu^2\cos^2(u,\rho)[\cos(w,\rho)\cos(u,\rho)-\cos(w,u)]}{2\rho[V^2-u^2\sin^2\chi]^{5/2}}\ .$$

We can now suppose that the axes are arbitrary. We need only replace dX with dF_u, since the component of the calculated force is the one that is parallel to u. If we multiply by $u\,dt$ and remark that the direction ρ remains fixed, since the motion is one of translation, then we can write:

$$\frac{d}{dt}\,u\cos(u,\rho)=w\cos(w,\rho)\ ,$$

$$\frac{d}{dt}u^2=2\,u\,w\cos(w,u)\ ,$$

which will give:

$$\frac{dF_u\,u\,dt}{de\,de'}=-\ d\,\frac{V^2-u^2\sin^2(u,\rho)+u^2\cos^2(u,\rho)}{2\rho[V^2-u^2\sin^2(u,\rho)]^{3/2}}\,V^3.$$

Return to the particular axes and operate with g as we did for f. That will give:

$$\frac{4\pi V^2\,g\,de'}{de\,de'}$$

$$= -\tfrac{3}{2}V^3 \frac{[w_x \cos\chi + \cdots](V^2 - u^2) - \dfrac{u^4 w_x \cos^3\chi}{(V^2 - u^2)}}{\rho[V^2 - u^2 \sin^2\chi]^{5/2}} \sin\chi\cos\psi - \tfrac{3}{2}V^3 \frac{u\, w_x \cos\chi}{\rho[\cdots]^{3/2}(V^2 - u^2)} \sin\chi\cos\psi$$

$$+ \frac{V^3}{\rho[\cdots]^{3/2}}\left\{\tfrac{1}{2} w_y \frac{V^2 + u^2(\cos^2 u - \sin^2 u)}{V^2 - u^2} + 3u\, w_x \frac{u\cos\chi}{V^2 - u^2}\sin\chi\cos\psi\right.$$

$$\left. + [w_x \cos\chi + \ldots]\sin\chi\cos\psi - w_y \frac{V^2 - u^2\sin^2\chi}{V^2 - u^2}\right\} .$$

The calculation for γ is analogous:

$$\frac{\gamma\, de'}{de\, de'} = \frac{[V^2 - u_0^2 + \sum w_{0x}(x - x_0)[u_{0x}(y - y_0) - u_{0y}(x - x_0)] + V(BM)[w_x(y - y_0) - w_y(x - x_0)]}{V^2(BM)^3}$$

$$= \frac{(V^2 - u^2)[u(y - y_0) - \theta\, w_x(y - y_0) + \theta\, w_y(x - x_0)]}{V^2(BM)^3} + \ldots$$

$$= \frac{(V^2 - u^2)u(y - y_0)}{V^2(BM)^3} + \frac{-(V^2 - u^2)u\,\tfrac{1}{2}\theta^2 w_y}{V^2(BM)^3} - \frac{\theta(V^2 - u^2)w_y(y - y_0) - w_y(x - x_0)}{V^2(BM)^3} .$$

The first term is the only one for which the numerator has order one and in which one must replace (BM) with its exact value. In the other two and all of the unwritten ones, the numerator has order two. It will suffice to take the principal value of each factor.

As a result, upon combining the terms in $w_x (y - y_0) - w_y (x - x_0)$, one will have (upon immediately suppressing all terms that vanish):

$$\frac{\gamma\, de'}{de\, de'}$$

$$= -\tfrac{3}{2}V u \frac{[w_x \cos\chi + \cdots](V^2 - u^2) - \dfrac{u^4 w_x \cos^3\chi}{(V^2 - u^2)}}{\rho[V^2 - u^2 \sin^2\chi]^{5/2}} \sin\chi\cos\psi - \tfrac{3}{2}V u \frac{u\, w_x(3u\cos\chi)}{\rho[\cdots]^{3/2}(V^2 - u^2)} \sin\chi\cos\psi$$

$$+ \frac{V^3}{\rho[\cdots]^{3/2}}\left\{-\tfrac{1}{2} u\, w_y \frac{V^2 + u^2(\cos^2 u - \sin^2 u)}{V^2 - u^2} - u\cos\chi\left[w_x \sin\chi\cos\psi - w_y\left(\cos\chi + \frac{u^2\cos\chi}{V^2 - u^2}\right)\right]\right.$$

$$\left. + 3u^2 w_x \sin\chi\cos\psi \frac{u\cos\chi}{V^2 - u^2} + [w_x\cos\chi + \cdots]u\sin\chi\cos\psi\right\} .$$

Finally:

$$\frac{dY}{de\,de'} = \frac{(4\pi V^2 g - u\,\gamma)\,de'}{de\,de'}$$

$$= -\frac{V\,(V^2-u^2)[w_x\cos\chi+\cdots][V^2-u^2\sin^2\chi-3u^2\cos^2\chi]}{2\rho[\cdots]^{5/2}}\sin\chi\cos\psi$$

$$-\frac{V\,u^2 w_x\cos\chi\sin\chi\cos\psi}{2\rho[\cdots]^{5/2}}[V^2-u^2\sin^2\chi-3u^2\cos^2\chi]-\frac{V\,w_x}{2\rho[\cdots]^{3/2}}[V^2-u^2\sin^2\chi+u^2\cos^2\chi]\ .$$

Upon observing that cos (u, y) is identically zero, one can further write:

$$\frac{dY}{de\,de'}$$

$$= \frac{-V\,w\{(V^2-u^2)\cos(w,\rho)+u^2\cos(w,u)\cos(u,\rho)\}}{2\rho[V^2-u^2\sin^2(\rho,u)]^{5/2}}[V^2-u^2\sin^2(u,\rho)-3u^2\cos^2(u,\rho)]\cos(\rho,y)$$

$$-\frac{V\,w[V^2-u^2\sin^2(u,\rho)+u^2\cos^2(u,\rho)]}{2\rho[\cdots]^{3/2}}\cos(w,y)$$

$$-\frac{V\,u^2\,w\cos(w,\rho)\cos(u,\rho)[V^2-u^2\sin^2(u,\rho)-3u^2\cos^2(u,\rho)]}{2\rho[\cdots]^{5/2}}\cos(u,y)$$

$$-\frac{V\,u^2\,w\cos(w,u)[V^2-u^2\sin^2(u,\rho)+3u^2\cos^2(u,\rho)]}{2\rho[\cdots]^{5/2}}\cos(u,y)\ .$$

If one changes the *y* in that expression into *x* then one can confirm (after some reductions) that one will return to the value of $\frac{dX}{de\,de'}$. Moreover, in their new forms, the expressions will be preserved unchanged under a rotation of the coordinate axes. The new form is then valid without one needing to suppose that the *x*-axis is parallel to *u*. We can then suppose that the axes are fixed in space.

Since the angle (ρ, y) is fixed, the preceding expression can be written:

$$\frac{dY}{de\,de'} = -V\,u\,\frac{d}{dt}\,\frac{(V^2-u^2)\cos(u,\rho)\cos(\rho,y)+[V^2-u^2\sin^2(u,\rho)+u^2\cos^2(u,\rho)]\cos(u,y)}{2\rho[\cdots]^{3/2}}\ .$$

Then again, geometrically, the elementary impulse is equal to:

$$- V\,de\,de'\,d\,\frac{(V^2 - u^2)\,\bar{u}_\rho + [V^2 - u^2\cos^2(u,\rho)]\,\bar{u}}{2\rho[V^2 - u^2\sin^2(u,\rho)]^{3/2}},$$

in which $\bar{u}_\rho$ represents the component of $\bar{u}$ along ρ.

Suppose, in particular, that the body is spherical and confine oneself to the first approximation, while supposing that u^2 is negligible compared to V^2. The value of impulse of the force that acts on the body will be:

$$- \iint (\bar{u} + \bar{u}_\rho)\,\frac{de\,de'}{2\rho} = - \,\bar{u}\iint \frac{de\,de'}{2\rho} - \iint \frac{\bar{u}_\rho\,de\,de'}{2\rho}.$$

Now, in electrostatics, $\iint \frac{de\,de'}{2\rho}$ represents the potential energy of the sphere with respect to itself – namely, $\frac{3}{5}\frac{e^2}{a}$ – if a is the radius of the sphere and the charge is distributed uniformly throughout the entire volume of the sphere. As for the second integral, by reason of symmetry, it will represent a vector that has the same direction as $\bar{u}$, and as a result, in order to evaluate it, one can replace $u\,\rho$ with its projection onto u, namely, $u\cos^2(\rho, u)$. The second integral is then equal to the factor $\bar{u}$, up to:

$$\iint \frac{\cos^2(\rho,u)\,de\,de'}{2\rho} = \iint \frac{\cos^2(\rho,u')\,de\,de'}{2\rho} = \iint \frac{\cos^2(\rho,u'')\,de\,de'}{2\rho}$$

$$= \tfrac{1}{3}\iint \frac{\cos^2(\rho,u) + \cos^2(\rho,u') + \cos^2(\rho,u'')}{2\rho}\,de\,de'$$

$$= \tfrac{1}{3}\iint \frac{de\,de'}{2\rho} = \frac{1}{5}\frac{e^2}{a},$$

by reason of symmetry, upon letting u' and u'' denote two directions that form a tri-rectangular trihedron with u. Finally, the impulse is equal to $-\frac{4}{5}\frac{e^2}{a}\bar{u}$, and its derivative is $-\frac{4}{5}\frac{e^2}{a}\bar{w}$, which is the resultant of the forces that are developed by the sphere on itself, constitutes what one can call the *electric inertia* of the sphere.

The expression that is obtained by the work done by electric forces will lead to the same result. Indeed, that work is equal (to the same degree of approximation) to:

$$\iint \frac{[V^2 - \tfrac{1}{2}u^2\sin^2(u,\rho) - u^2\cos^2(u,\rho)]\,de\,de'}{2\rho}.$$

One can drop the constant term $V^2 \iint \frac{de\,de'}{2\rho}$, and all that will remain is simply:

$$-\frac{u^2}{2}\iint \frac{1+\cos^2(u,\rho)}{2\rho}\,de\,de' = -\frac{2}{5}\frac{e^2}{a}u^2,$$

whose differential, namely, $-\frac{4}{5}\frac{e^2}{a}u\,w\cos(u,w)\,dt$, indeed represents the work done by the electric force of inertia.

If one supposes that the charge is only on the surface then the integral $\iint \frac{de\,de'}{2\rho}$ will be equal to $\frac{e^2}{2a}$, and as a result, the electric inertia will have the value $-\frac{2}{3}\frac{e^2}{a}\bar{w}$, which is the expression that Larmor gave before.

IV. – *We examine the particular case in which the velocity u becomes greater than or equal to the velocity V of the radiation.*

Up to now, we have always supposed that u was much smaller than V. Indeed, in order for the expressions that were found to not be infinite, it was necessary that the quantity BM, or $(P_0M / V)(V - u\cos\varphi)$, that appears in the denominator could not go to zero, and in order for that to be true, u had to be much smaller than V. It is appropriate to examine what happens in the opposite case, and in particular, to see if it is true, as Larmor ([1]) and Searle ([2]) asserted, that it is impossible to give an electrified body a velocity that is greater than or equal to V.

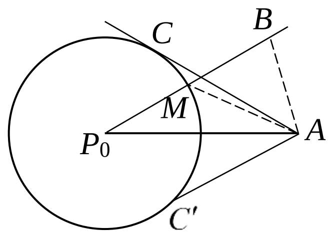

Figure 5.

When u is greater than V, our point A will be exterior to the sphere, which we have called a *wave*, with its center at P_0 and a radius of P_0M (Fig. 5). If we consider the cone whose summit is A and which circumscribes the sphere then the contact circle CC' will divide the sphere into two regions.

In the one that is furthest from A, $V - u\cos\varphi$ will be positive, and the preceding calculations will be valid without modification.

([1]) LARMOR, "On a dynamical theory of the electric and luminiferous medium," Phil. Trans. A (1894), pp. 809.
([2]) SEARLE, "On the steady motion of an Electrified Ellipsoid," Phil. Mag. **44** (1897), pp. 341.

In the other one, which closest to A, $V - u \cos \varphi$ will be negative, but finite. Under those conditions, I say that one must change all of the signs in the expression that were found for D and H or their components.

Indeed, refer to the determination of the function ψ that is defined by the integral (12). $d\varpi'$ is essentially positive, as well as $d\varpi$, which represents the volume that is swept out by the surface AB with respect to Ω, while $d\varpi'$ represents the volume that is actually swept out. We have seen that:

$$d\varpi' = \frac{d\varpi}{1 - \frac{u}{V}\cos(u, r)} .$$

Since $d\varpi'$ and $d\varpi$ are necessarily positive, one must change the relation into the following one:

$$d\varpi' = - \frac{d\varpi}{1 - \frac{u}{V}\cos(u, r)}$$

when the denominator is negative. Hence, there must be a change of sign for ψ and for all of the expressions that enter into the rest of the calculations. Moreover, that is the only modification that that must be made, as one easily convinces oneself by reviewing the calculations.

Finally, BM is zero on CC', and everything is infinite.

We should point out that as long as one supposes that u is less than V, each position of the point M will correspond to one and only one value of θ and a unique position for the point P_0. Indeed, when θ increases from 0 to infinity, the sphere of center M and radius $V\theta$ will dilate with a velocity V that is greater than that of the charge, and as a result it will necessarily be attained and can no longer meet itself again.

On the contrary, if u takes values that are greater than V then the number of positive real roots θ of equation (14) can be zero or even greater than 1. In that case, one must consider the various corresponding positions for the point P_0 and add (geometrically) the values of D and H that correspond to each other them.

Consider the state of the field at an instant that is determined when the charge is at P. Each value of θ then corresponds to a sphere and a circle CC'. The locus of those circles when θ varies will be the envelope E of the spheres of the wave S. That envelope will divide the space into two (or a much greater number of) regions. For the points of one of them, the values of θ will be imaginary, i.e., the perturbation will not have arrived at it yet, and the field will be zero there. For the points of the other ones, θ will have several real values, and the field can be determined in the manner that was just described. Finally, on the envelope itself, the equation in θ will have a double root for which BM will be zero, and the field will have an infinite intensity. We seek the order of magnitude of D and H for points that are infinitely close to the envelope.

Develop equation (14) into a Taylor series:

$$2\sum(x-x_0)(\delta x-\delta x_0)+\sum(\delta x-\delta x_0)^2 = 2V^2\,\theta\,\delta\theta+V^2\,\delta\theta^2.$$

Moreover:

$$\delta x_0 = -\,u_x\,\delta\theta+\tfrac{1}{2}\,w_x\,\delta\theta^2+\ldots,$$

so upon making that replacement, one will get:

$$2\sum(x-x_0)(\delta x+u_x\,\delta\theta-\tfrac{1}{2}\,w_x\,\delta\theta^2)+\sum(\delta x+u_x\,\delta\theta-\tfrac{1}{2}\,w_x\,\delta\theta^2)^2 = 2V^2\,\theta\,\delta\theta+V^2\,\delta\theta^2,$$

or rather, since $BM = \dfrac{1}{V}[V^2\theta-\sum u_x(x-x_0)]$ is zero:

$$2\sum(x-x_0)(\delta x-\tfrac{1}{2}\,w_x\,\delta\theta^2)+\sum(\delta x+\cdots)^2 = V^2\,\delta\theta^2.$$

That shows that $\delta\theta^2$ has the same order as δ_x, δ_y, δ_z, or rather that $\delta\theta$ is infinitely small of order 1/2 for a point that its located at a distance from the envelope that is infinitely small of order one. Moreover, one easily sees that *BM*, which is zero on the envelope, has the same order as $\delta\theta$ in its neighborhood. Therefore, *D* and *H*, whose expressions contain $(BM)^3$ in their denominators, will be infinitely large of order 3/2. As a result, the integrals:

$$2\pi V^2\int(f^2+g^2+h^2)\,d\omega, \qquad \frac{1}{8\pi}\int(\alpha^2+\beta^2+\gamma^2)\,d\omega,$$

which represent the electric and magnetic energy, resp., will be infinite, and one must expend an infinite amount of work in order to give the charge a velocity that is greater than or equal to *V*.

However, it should be noted that the energy is already infinite due to another reason, even when the velocity is less than *V*. Indeed, in the neighborhood of the charge (which we have supposed to be reduced to a point), *D* and *H* are infinitely large of order two. We must then examine the case of a charge that is distributed throughout a certain volume and see if the energy will again be infinite for a velocity that is greater than *V* if we are to decide whether or not it is possible to give a charged body a velocity that is greater or equal to *V*.

One can do that in two ways: Either one treats the problem directly by means of equations (5) and (6) or one decomposes the volume Ω of the body into infinitely-small elements $d\omega$ and calculates the field intensities that are produced by the charges in each of those elements $d\omega$, which are supposed to be reduced to a point, and then take the sum. That is what we shall examine in a later article.

(to be continued)

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(continuation, pp. 106-112)([1])

We shall examine the case of a charge that is distributed in a certain volume and see whether the energy will again be infinite for a value that is greater than V.

We begin with the second of the indicated processes.

1. Let P (ξ, η, ζ) be an arbitrary point of the body, around which we consider an element $d\omega$, and as always, let M be the point where we would like to evaluate the field. Each position of the point P in the interior of Ω will correspond to an envelope E with the equation:

$$f(x, y, z, \xi, \eta, \zeta) = 0 . \tag{26}$$

There is good reason to introduce the coordinates ξ, η, ζ of the point P in the equation for the envelope, since each position of P will correspond to a particular envelope. If one considers all of the possible positions of P inside of the electrified body Ω then the set of corresponding envelopes will occupy a region C of space. If the point M is outside of C then the values of D and H at that point will be represented by integrals such as $\int \bar{A}\,\rho\,d\omega$, where ρ is the electrical density, and the function $\bar{A}$, which represents (up to sign) the factor of e in the right-hand sides of equations (17′) or (18′), will never be infinite. The integral itself will then be finite.

On the contrary, let the point M (x, y, z) be in the region C. There will exist positions of the point P inside of Ω for which (BM) is zero, and those points will obviously be on the surface (26), when one now considers x, y, z to be given and ξ, η, ζ to be running coordinates. Let Σ be that surface. An argument that is analogous to the one that was made above will show that for the positions of P that are close to Σ, (BM) will be infinitely small of order 1/2, and as a result, $\bar{A}$ will be infinitely large of order 3/2, i.e., greater than 1. The integral $\int \bar{A}\,\rho\,d\omega$ is then infinite, and it will seem that the field has an infinite intensity in the entire region C.

Instead of seeking the values of D and H, we seek those of ψ and F, G, H. Here, the functions to be integrated are 1 / (BM), u_x / (BM), etc., which will only be infinitely large of order 1/2. As a result, the integrals will be finite, and the functions ψ, F, G, H will be finite and well-defined in all of space ([2]). However, f, g, h, α, β, γ, as they are determined from equations (9) and (10), will be themselves finite and well-defined.

The apparent contradiction between the two results is analogous to the peculiarity that is presented by the determination of the force inside of a magnet.

Let a magnet have a moment $M = (M_x, M_y, M_z)$. At a point at a distance r, the magnetic potential will have the value:

$$V = \frac{x\,M_x + y\,M_y + z\,M_z}{r^3},$$

([1]) See *L'Éclairage Électrique* on 2 and 9 July, pp. 5 and 53.

([2]) Unless, however, there is a point of Σ that corresponds to an infinite value of θ, because BM will then have order one. (Cf. *infra*) and the components of the magnetic force will be:

$$\alpha = M_x \frac{3x^2 - r^2}{r^5} + \cdots = -\frac{dV}{dx}, \qquad \beta = \ldots, \qquad \gamma = \ldots,$$

upon supposing that the magnet is zero-dimensional and is situated at the coordinate origin.

Now let a magnet be finite-dimensional. In order to calculate the magnetic potential and the magnetic force at a point, one takes the integral of the elementary values that are due to each part of the magnet. One knows that the potential will always have a well-defined value for an interior point of the magnet, although the function to be integrated will become infinite in the domain of integration and admit derivatives. However, the magnetic force will be indeterminate, and will no longer be equal to the derivative of a potential, as a result.

The same thing is true here. The values of f, g, h, α, β, γ that are calculated by integration are infinite or indeterminate, whereas the functions ψ, F, G, H will remain finite and admit derivatives. However, from the manner by which they were obtained, equations (9) and (10) are certainly applicable in all cases, even interior to the electrified body, whereas formulas (18) and (19) are the results of transformations of the calculations that were made by always supposing that (MB) is non-zero, so it is no wonder that those formulas will lead to incorrect results when (BM) becomes zero in the domain of integration.

Finally, we see that $\bar{D}$ and $\bar{H}$ remain finite and that, as a result, nothing stands in the way of assigning a velocity greater than V to an electrified body.

Before passing on to the second method, we further point out that BM can also be infinitely small in the domain of integration in another case, namely, when the point M is interior to the body Ω and one takes a point P that is infinitely close to M. However, ($A\,M$) and ($P_0\,M$) are also infinitely small, in such a way that the function A considered before will have order of magnitude two, and since it is only in the neighborhood of a point, and not a surface, the integral will be no less finite.

2. The other process will permit us to go deeper into the question a little more completely.

Consider equation (7). Upon traversing the surface of the electrified body, ρ will pass sharply from the value 0 to a finite value. Consequently, ρ is a discontinuous function of x, y, z, t. Now, ψ enters into (7) by way of its derivatives $\frac{d^2\psi}{dx^2}$, $\frac{d^2\psi}{dy^2}$, $\frac{d^2\psi}{dz^2}$, $\frac{d^2\psi}{dt^2}$. Those second-order derivatives can then be discontinuous, but those of first order $\frac{d\psi}{dx}$, $\frac{d\psi}{dy}$, $\frac{d\psi}{dz}$, $\frac{d\psi}{dt}$ must be well-defined and continuous, while those of second order can be infinite. Similar statements are true for F, G, H. Therefore, f, g, h, α, β, γ, whose expressions contain only the first-order derivatives of ψ, F, G, H are finite and continuous functions of x, y, z, t.

We should not be content with that proof, *a priori*, but study the continuity of ψ using the solved equation, i.e., (12). The results obtained will extend immediately to the functions F, G, H.

We then see that the preceding result is not always exact, due to the fact that the expression $\left(V^2\Delta - \frac{d^2}{dt^2}\right)\psi$ can remain finite, even for infinite values of $\frac{d^2\psi}{dx^2}$, etc.

We saw that the domain of integration in the expression (12) was determined in the following manner: *M* (Fig. 6) is the point where one can evaluate the value of ψ at the instant *t*, so we can describe *M* as the center of a sphere *S* of radius $V\,\theta$, and we seek the part *AB* of that sphere that is situated inside of the electrified body when it is taken in its position at the instant $t - \theta$. The region swept out by *AB* when θ varies from 0 to ∞ will constitute the domain of integration.

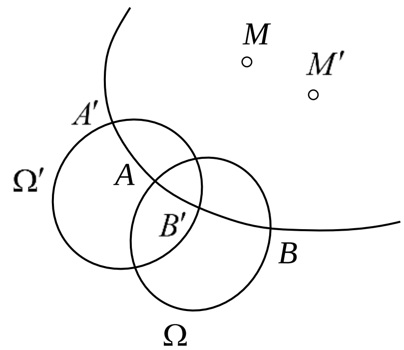

Figure 6.

If that domain of integration is finite then ψ will be finite. Indeed, the right-hand side of equation (12) represents the ordinary electrostatic potential of a certain electric distribution of finite density that is spread over the domain of integration, and the potential is always finite and well-defined, even when the point *M* is inside of the domain.

On the contrary, if the domain of integration extends to infinity then since ρ remains finite, the potential will be infinite.

We must then find only the cases in which the domain of integration extends up to infinity.

If the velocities of each point of the electrified charge are different from *V* for $t = -\infty$ (in one sense or another), while pointing in well-defined directions, then as θ increases indefinitely, all points of Ω will conclude by being found in either the interior or exterior of the sphere of radius $V\,\theta$, and the domain of integration will not extend out to infinity. ψ will be finite and well-defined for all points in space.

On the contrary, suppose that the velocities for $t = -\infty$ are equal to *V* and have a well-defined direction. In the limit, the velocity of dilatation of the sphere will then be equal to that of Ω, and for convenient positions of the point *M* the intersection *AB* will remain real for infinite θ, and ψ will be infinite. In particular, if one supposes that the body is animated with a permanent, uniform motion of velocity *V* then ψ will be infinite for all points of space that are found between the extreme tangent planes to the position of Ω at the instant *t*, which are planes that are drawn normal to *V*.

Larmor and Searle confined themselves to the study of permanent motions. In that case, one must reach the conclusion that the domain will become infinite for a velocity of the body that is equal to *V*, so it would be impossible for it to have that velocity. However, the result pertains solely to the consideration of permanent motions, so that will oblige one to suppose that the body is animated with the velocity V after an infinite time.

On the contrary, in order to conform to reality, one must suppose that the motion has a beginning, and it will result from the foregoing that under those conditions ψ will be finite and well-defined for all points of space.

It remains to be seen whether ψ is continuous in time, as it is in space, and admits well-defined derivatives.

Let M' be another point in space. In order to get the corresponding value of ψ', we proceed as before. For each value of θ, the two spheres S and S' that have M and M', resp., for their centers will have the same radius. We displace S' by a quantity that is equal and parallel to $M'M$ in such a manner as to make it coincide with S, and apply the same displacement to Ω, which will take it to Ω', where Ω' plays the same role with respect to M that Ω does with respect to M', and we will get the value of ψ' by proceeding with Ω' as we did with Ω.

That value will differ from ψ for two reasons: On the one hand, the domain of integration will be different, since it will be the space that swept out by, not only AB, but also $A'B'$, where $A'B'$ is the potion of S that is interior to Ω'. On the other hand, the density will have varied at each point.

If MM' is infinitely small and equal to ds then the variation of the density at a point will be equal to $(d\rho / ds)\, ds$, and the corresponding variation of ψ will be:

$$ds\int \frac{d\rho / ds}{r}\, d\omega \ .$$

That variation is infinitely small of the same order as ds, and its quotient by ds is finite and well-defined ([1]).

In order for ψ to be continuous and admit a derivative, it will then suffice that the difference between the two domains of integration is also infinitely small of the same order as ds and that its quotient by ds is finite and well-defined.

First consider a value of θ for which S is not tangent to the surfaces Σ and Σ' of Ω and Ω', resp., i.e., it cuts those surfaces at finite angles. The zone of S between AB and $A'B'$ will be infinitely small and proportional to ds, and if the preceding condition is satisfied between the values θ_1 and θ_2 of θ then the corresponding difference between the domains of integration will be itself proportional to ds.

Now let a sphere S cut Σ and Σ' at infinitely-small angles, and as a result, they will be essentially tangent to those two surfaces. Draw two spheres T, T' that are concentrated at S and tangent to the surfaces Σ and Σ' at the points Q and Q', resp. If h and h' are the differences between the radii of T and T', resp., and S then h and h' will represent the distances from Σ and Σ', resp., to S.

([1]) The argument supposes that $d\rho / ds$ is finite, and as a result, that ρ is continuous in all of the interior of Ω. If one has discontinuity surfaces then those surfaces will divide Ω into several regions, and one makes the same study for each of them that one did for Ω.

Two cases can present themselves:

1. The spheres T and T' touch the surfaces Σ and Σ', resp., without intersecting them at real points.

2. The intersections will be real and present a double point at Q or Q'.

Under the second hypothesis, the intersections of S with Σ and Σ' will also be real, and the spherical area of S that is found between the two will have the order of magnitude $h - h'$, which itself has the same order as ds. There is once more nothing special about this case.

On the contrary, under the first hypothesis, the intersections of S with Σ and Σ' can be real or imaginary in the vicinities of the points Q and Q'.

1. The two intersections are imaginary: The surface S will play no role in the determination of the domain of integration.

2. The two intersections are real and differ slightly by small spherical ellipses: If R_1 and R_2 are the radii of curvature of Σ at Q, and R is that of S then the ellipse AB will have the value:

$$2\pi\, h\, R \sqrt{\frac{R_1 R_2}{(R_1 - R)(R_2 - R)}}\ ,$$

while that of $A'B'$ will be:

$$2\pi\, h'\, R \sqrt{\frac{R_1' R_2'}{(R_1' - R)(R_2' - R)}}\ .$$

R_1' and R_2' are relative to Σ' and the point Q', and the difference will have the same order of magnitude as $h - h'$. The volume that is swept out by the difference between the areas AB and $A'B'$ will then be once more infinitely small and proportional to ds, which will neither introduce a discontinuity in ψ nor any indeterminacy in its derivative.

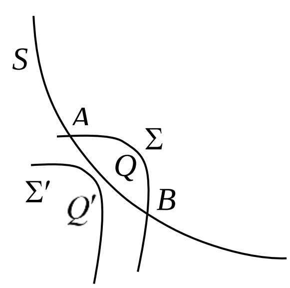

Figure 7.

3. One of the two intersections (the one with S, for example) is real and the other is imaginary (Fig. 7). h and h' will then have opposite signs. The area $A'B'$ will not exist, and the area AB will have the preceding value that is proportional to h. Now h and h' have opposite signs, so h will be

smaller in absolute values than $h - h'$, which is proportional to *ds*. The area *AB* will then be infinitely small at the same time as *ds*, but *it will no longer be proportional to it*, and the same thing will be true for the volume that is generated by *AB*.

If the normal velocities to the points Q and Q' are different from *V* then the values of θ between which *S* can cut Σ without cutting Σ' (or conversely) will be infinitely close, and the corresponding volume that is swept out by *AB* will be infinitely small of order two, and one can neglect it.

However, if those normal velocities are equal to *V* during a finite time, in such a way that the peculiarity been studied exists in a finite interval of the variation of θ then the volume generated by *AB* will not be negligible, and since it is infinitely small at the same time as *ds* without being proportional to it, the derivative of *V* will be indeterminate.

Let M'' be taken along MM' in such a manner that the corresponding position of Σ is tangent to *S*. Up to higher-order infinitesimals, one will obviously have the relation:

$$\frac{MM''}{h} = \frac{M''M'}{-h'}.$$

For two points that are located between *M* and M'', the intersections that correspond to *AB* will be real, and the differences between the values of the potential at those points will be proportional to the distance between them. The same thing will be true for two points that are located on the other side of M''. On the contrary, if the points considered are on one side and the other of M'' then there will no longer be any proportionality. The derivative of *V* will then be discontinuous at the point M''.

In summary, one sees that the function *V* is always continuous. Its derivatives are also well-defined, in general, but can be discontinuous for the points such that a sphere of radius $V\theta$ that describes one of those points as its center will remain tangent to the boundary surface of the electric charge without intersecting it during a finite time interval when that charge is taken for each value of θ in its position at the instant $t - \theta$.

In particular, there will be a discontinuity in the derivative upon traversing the boundary surface of the charge if that surface remains animated with a normal velocity that is equal to *V* during a finite time interval.

However, if the velocity just passes through that value without maintaining it then no discontinuity will result.

In order to study the continuity of the function ψ at a point with respect to time, instead of considering two spheres *S* and S' with the same radius and different centers, on the contrary, one considers two concentric spheres with radii that differ by *V dt*, which amounts to comparing the values of the potential at two instants that differ by *dt*.

The results will be the same as before, and the points of discontinuity of the derivatives with respect to space will also be ones for the derivatives with respect to time. Moreover, a moment of reflection will suffice for one to see that this is true.

The results that we just obtained for the continuity of the function ψ and its derivatives are immediately applicable to the functions *F*, *G*, *H*, as we have pointed out already.

Now, the quantities α, β, $V\,\gamma$, f, g, h are expressed linearly by means of the derivatives of those four functions. Therefore, the magnetic force and the electric displacement will always be finite and generally continuous, except at the points that were defined above. The energy of the field is not infinite then, and it will be possible to assign a velocity that is greater than or equal to V to the charge.

Remark I. – In the foregoing, we considered the functions y, F, G, H to be sufficiently defined by the equations (7) and (8) without taking into account the condition (11). We have that right, because that equation (11) expresses only the principle of the conservation of electricity, and is necessarily satisfied as a result. Indeed, we can write:

$$\left(V^2\Delta - \frac{d^2}{dt^2}\right)\left[\frac{d\psi}{dt} + \frac{dF}{dx} + \frac{dG}{dy} + \frac{dH}{dz}\right] = -\,4\pi V^2\left[\frac{d\rho}{dt} + \frac{d(\rho u_x)}{dx} + \frac{d(\rho u_y)}{dy} + \frac{d(\rho u_z)}{dz}\right],$$

and the bracketed expression on the right-hand side is zero, from the principle of the conservation of electricity.

In the first question that was treated, we can therefore also dispense with verifying that the condition (11) is indeed satisfied.

Remark II. – In the preceding analysis, we omitted one case, namely, the one in which the surfaces Σ and Σ' differ very slightly over a noticeable portion of their extent by spheres that have their centers at the point M, and the sphere S is included between them. In that case, one of the two areas AB or $A'B'$ will have to be zero and the other one will have to be finite, in such a way that if such conditions had been satisfied for a series of values of θ that form a finite time interval then the difference between the potential at the points M and M' would have been finite, and the derivative infinite. However, that is a very special case that demands, moreover, that the volume of Ω must be deformable in order for a portion of its surface to appreciably preserve the point M as the center of curvature during a finite time interval.

Remark III. – If one studies the case of a charge that is spread over a surface Σ, instead of a volume, then the results will be different. The functions ψ, F, G, H will then be continuous or discontinuous in the case where their derivatives can become infinite, and the same will be true for the magnetic force and displacement. In that case, it will be impossible to assign a normal velocity that is equal to V to an electrified surface, and the normal velocity must always remain either less than or greater than V.

Remark IV. – If the magnetic force and the displacement are generally continuous upon traversing a boundary surface of an electrified body then the same thing will *not* be true of their derivatives, which are always discontinuous, as was shown by equations (1) to (4).

Let M be a point of the surface Σ whose inward-pointing normal we take to be the x-axis, while the y and z-axes are in the tangent plane that the surface Σ agrees with in the neighborhood of the point M.

Only the derivatives with respect to x can then be discontinuous, and if I let $\Delta \frac{df}{dx}$ represent the discontinuity in the derivative $\frac{df}{dx}$ then equations (3) will immediately give:

$$\Delta \frac{df}{dx} = \rho , \qquad \Delta \frac{d\alpha}{dx} = 0 . \tag{27}$$

$\Delta\rho$ is indeed equal to ρ, and the Δ of $\frac{dg}{dy}$, $\frac{dh}{dz}$, $\frac{d\beta}{dy}$, $\frac{d\gamma}{dz}$, are zero.

Let M' be a point of Mx whose abscissa is $u_x\,dt$. At a time dt later, it will be found on the surface Σ since it will be displaced in the direction Mx by the quantity MM'. The difference between the values of f at M' and M at time t will be:

$$\left(\frac{df}{dx} + \Delta \frac{df}{dx}\right) u_x \, dt \ .$$

df / dx refers to the point M, but on the exterior face of Σ.

During dt, f will increase by $\frac{df}{dt}\,dt$ at M and by $\left(\frac{df}{dx} + \Delta \frac{df}{dx}\right) dt$ at M', in such a way that the original difference will become:

$$\left(u_x \frac{df}{dx} + u_x \Delta \frac{df}{dx} + \Delta \frac{df}{dt}\right) dt \ .$$

On the other hand, at the end of the time dt, MM' will be found to be completely exterior to Σ, so the difference between the values of f must become:

$$\left(\frac{df}{dx} + \frac{d^2 f}{dx\,dt}\,dt\right) u_x \, dt \ .$$

Equating those two values while neglecting the higher-order infinitesimals will give:

$$\Delta \frac{df}{dt} = - \, u_x \, \Delta \frac{df}{dx} . \tag{28}$$

One will have five identical relations for g, h, α, β, γ.

Upon taking the Δ of each term in equations (1) and (2) and neglecting the ones that are zero, one will easily find that:

$$\left.\begin{aligned} 0 &= \rho u_x + \Delta\frac{df}{dt}, \\ -\frac{1}{4\pi}\Delta\frac{d\gamma}{dx} &= \rho u_y + \Delta\frac{dg}{dt}, \\ -\frac{1}{4\pi}\Delta\frac{d\beta}{dx} &= \rho u_z + \Delta\frac{dh}{dt}, \end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} 0 &= - \quad \Delta\frac{d\alpha}{dt}, \\ -V^2\Delta\frac{d\gamma}{dx} &= -\frac{1}{4\pi}\Delta\frac{d\beta}{dt}, \\ +V^2\Delta\frac{d\beta}{dx} &= -\frac{1}{4\pi}\Delta\frac{d\gamma}{dt}. \end{aligned}\right\} \tag{30}$$

If one takes equations (27), (28) into account, along with the analogues to the latter, then one will see that the first equations in (29) and (30) are satisfied identically, and the other ones will become:

$$-\frac{1}{4\pi}\Delta\frac{d\gamma}{dx} = \rho u_y - u_x\,\Delta\frac{dg}{dx},$$

$$\frac{1}{4\pi}\Delta\frac{d\beta}{dt} = \rho u_z - u_x\,\Delta\frac{dh}{dx},$$

$$-V^2\Delta\frac{dh}{dx} = \frac{u_x}{4\pi}\Delta\frac{d\beta}{dx}, \qquad V^2\Delta\frac{dg}{dx} = \frac{u_x}{4\pi}\Delta\frac{d\gamma}{dx},$$

so one will infer that:

$$\left.\begin{aligned} \Delta\frac{dg}{dx} &= \frac{-\rho u_y u_x}{V^2 - u_x^2} = \frac{u_x}{4\pi V^2}\Delta\frac{d\gamma}{dx}, \\ \Delta\frac{dh}{dx} &= \frac{-\rho u_z u_x}{V^2 - u_x^2} = \frac{-u_x}{4\pi V^2}\Delta\frac{d\beta}{dx}. \end{aligned}\right\} \tag{31}$$

In order for u_x^2 to be equal to V^2, the Δ of $\frac{dg}{dx}$, $\frac{dh}{dx}$, $\frac{d\beta}{dx}$, $\frac{d\gamma}{dx}$, will become infinite, and *g*, *h*, β, γ will become discontinuous, but not *f* and α.

The normal components of the magnetic force and displacement will therefore always remain continuous, and only the tangential components will be discontinuous in the case that was previously studied.

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