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New equations of motion for material systems in MINKOWSKI space

Nowe równania ruchu materialnych w przestrzent MINKOWSKIEGO

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The mechanical equations of material systems that were given previously by M. Mathisson are derived with the help of retarded potentials for a system that is found in MINKOWSKI space and is characterized by its mass and angular impulse to a sufficient degree of approximation. That will permit an intuitive interpretation of the gravitational multipoles that were introduced by MATHISSON and which replace the system. If one can restrict the development of the gravitational potential outside of the world-tube to the monopole and dipole terms then the pole strength will give the energy-impulse four-vector, and the dipole strength will give the angular impulse. If one demands that the gravitational equations must be satisfied then one will obtain the law of the conservation of mass and the equations of motion of the system.

In a paper that appeared recently [**1**], MATHISSON found equations of motion for a material system from the general EINSTEIN gravitational equations. MATHISSON obtained these equations of motion, which differ from the classical ones in general, by a variational method that he had previously developed.

The purpose of the present article is to derive these equations (but only in a special case) from a more immediate method that makes it possible to give a more intuitive physical interpretation.

1. Matter shall fill up a fixed region in space; i.e., the energy-impulse tensor $T^{\alpha\beta}$ (α, β = 1, …, 4) shall vanish outside of a fixed world-tube (we thus exclude the presence of an electromagnetic field; we will then show that we can lift that restriction). At a sufficiently large distance, we can set:

$$g_{\alpha\beta} = \overset{0}{g}_{\alpha\beta} + \gamma_{\alpha\beta}\,, \tag{1}$$

in which $\overset{0}{g}_{\alpha\beta}$ are the components of the metric for MINKOWSKI space (viz., $-\overset{0}{g}_{ii}$ = $\overset{0}{g}_{44}$ = 1, i, k … = 1, 2, 3, while the other $\overset{0}{g}_{\alpha\beta}$ = 0), and the $\gamma_{\alpha\beta}$ are very small quantities.

We will consider the $\gamma_{\alpha\beta}$ to be a tensor field *against the background* of MINKOWSKI space. If we neglect the higher powers of $\gamma_{\alpha\beta}$ and their derivatives in the statement of the curvature tensor $R_{\alpha\beta\gamma\delta}$ then we will get the EINSTEIN equations of gravitation, viz.:

$$R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R = -\kappa T_{\alpha\beta} \tag{2}$$

(κ = gravitational constant) in the approximation of linear gravitation:

$$\tfrac{1}{2}\Box\varphi_{\alpha\beta} = -\kappa T_{\alpha\beta} \qquad \left(\Box \equiv g^{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\right), \tag{3}$$

$$\frac{\partial \varphi_\alpha^\beta}{\partial x^\beta} = 0, \tag{4}$$

in which we have:

$$\varphi_\alpha^\beta = \gamma_\alpha^\beta - \tfrac{1}{2}\gamma_\nu^\nu\,\delta_\alpha^\beta . \tag{5}$$

Later on, we will take the quantities $\varphi_{\alpha\beta}$ to be simply gravitational potentials. Everywhere outside of matter, the gravitational potentials will satisfy the system of equations:

$$\Box\varphi_{\alpha\beta} = 0, \tag{6}$$

$$\frac{\partial \varphi_\alpha^\beta}{\partial x^\beta} = 0. \tag{4}$$

In order to not go any deeper into the structure of matter, we can extend the exterior solutions to the interior of the tube in a completely formal way that points to the fact that the solutions will possess singularities along a fixed world-line, just as the NEWTONIAN potential, which satisfies the LAPLACE equation $\Delta\varphi = 0$ outside of the matter, is represented by a function that possesses singularities in the interior of matter. If the region V is likewise filled with matter of density ρ then the NEWTONIAN potential will be determined by the equation:

$$\varphi_p = \int \frac{\rho}{r_{PQ}} dV .$$

If we take an arbitrary reduction point O in the region V then at a sufficiently large distance away from it we can develop $1 / r$ into the following series:

$$\frac{1}{r} = \frac{1}{r_0} + \left(\frac{\partial(1/r)}{\partial x^i}\right)_0 y^i + \dots$$

in which y^i mean the coordinates of the point relative to O; we further get:

$$\varphi_p = \frac{1}{r_0}\int \rho\, dV + \left(\frac{\partial(1/r)}{\partial x^i}\right)_0 \int \rho\, y^i dV + \dots$$

The successive terms in this series represent the potential by the poles, dipoles, and multipoles of higher order that are found at this reduction point. We see that these potential have singularities of the type $\frac{1}{r}$, $\left(\frac{1}{r}\right)^2$, etc. At larger distances, one will already find a good approximation for the potential of a body by using just the first term of the series.

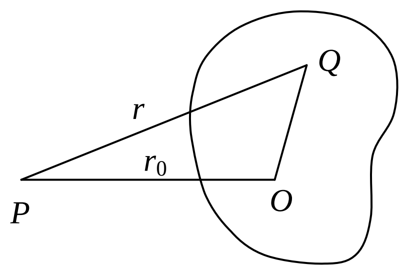

Figure 1

We will use this analogy to look for solutions to the system of equations (6), (4) that have the type of a pole, dipole, etc.; i.e., we will look for solutions that have singularities along the line L, which is temporarily taken to be inside the matter tube. We assume that the solutions to the system (3), (4) can be represented outside the tube by a series of multipoles, and that at large distances one can realize a good approximation by using the first term of this series. Equations (4) place one constraint on the reduction line L. We will associate the line L with the system very intimately; we will find mechanical laws for the material system from this.

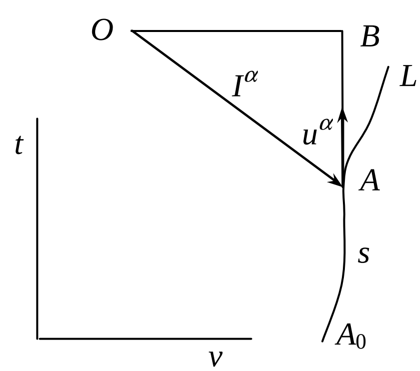

Figure 2.

2. In order to give an expression for the potential of a multipole, we must introduce a definition. Let there be a time-like world-line L (Fig. 2). We will measure out an interval s from the arbitrary time point A_0 along this line. (s is the proper time for a material point whose motion is represented by L.) Since we are measuring the parameter s along the line L, we can construct a scalar field over all of space in the following way: We define the past light cone at each point O, and assign the value of s to the point O, which represents the point A where the line L cuts the light cone. At the same time, we introduce the vector field l^α, where l^α is the vector that connects O with A. Since I^α lies on the light cone, we will have:

$$l^\alpha l_\alpha = 0. \tag{8}$$

We further denote a four-dimensional tangent to L by u^α; i.e., when the line L is given by the equations $x^\alpha = x^\alpha(s)$, its tangent will be given by:

$$u^\alpha = \frac{dx^\alpha}{ds}.$$

If we use u^α and l^α then we can introduce the scalar n by means of the equation:

$$n = l^\alpha u_\alpha. \tag{9}$$

In the proper system – i.e., in the system in which u^α is parallel to the time axis – n will mean the (space-like) distance between the points O and A (*). In the proper system $u^i = 0$, $u^4 = 1$, so $n = l^4 = t_A - t_0$. However, due to (7), $(l^4)^2 = r^2$ and $l^4 = -r$. (l^4 is always negative, since it was chosen to be in the past.) We will then have $n = -r$ in the proper system.

MATHISSON gave the rules for the differentiation of the quantities that were introduced (with respect to the coordinates of the world-point) in the previously-cited paper [**2**]. We shall state them here without proof (**).

$$\frac{\partial s}{\partial x^\alpha} = \frac{l_\alpha}{n}, \tag{10}$$

$$\frac{\partial l_\alpha}{\partial x^\beta} = -g_{\alpha\beta} + \frac{u_\alpha l_\beta}{n}. \tag{11}$$

We can now imagine that every function f that measures arcs s along L is a function that is defined over all space. It will follow from (10) that its gradient will be:

(*) In the figure, the segment OB – i.e., the projection of la onto the normal to u_A^α – is the measure of the separation OA of the points in the proper system.

(**) From now on, we shall write $g_{\alpha\beta}$, instead of $\overset{0}{g}_{\alpha\beta}$.

$$\frac{\partial f(s)}{\partial x^{\alpha}} = \frac{df}{ds}\frac{l_{\alpha}}{n} = \dot{f}\,\frac{l_{\alpha}}{n}, \tag{12}$$

in which a dot means differentiation with respect to s. We will find the derivatives of n with the help of the given formula:

$$\frac{\partial n}{\partial x^{\alpha}} = -\,u_{\alpha} + \frac{l_{\alpha}}{n}(1 + l^{\alpha}\dot{u}_{\alpha})\,. \tag{13}$$

3. By considering these relations, we can establish, with no further assumptions, that the identity:

$$\Box\frac{f}{n} = 0 \tag{14}$$

is true everywhere, except along the line L, for any arbitrary function $f(s)$. We will thus choose the desired solution to equations (6), (4) to have the form:

$$\varphi^{\alpha\beta} = \frac{m^{\alpha\beta}}{n}\,. \tag{15}$$

The relation above represents the potential of a pole; namely, it has a singularity of type $1 / r$ along the line L. If one chooses the desired equation of equations (6) to satisfy the relation (15) then (6) will be fulfilled, and the conditions (4) will still remain, which will impose some restrictions on $m^{\alpha\beta}(s)$ and the line L.

Namely, one has:

$$\frac{\partial \varphi^{\alpha\beta}}{\partial x^{\beta}} = \frac{1}{n^2}\cdot m^{\alpha\beta}\left(u_{\beta} - \frac{l_{\beta}}{n}\right) + \frac{1}{n}\left(m^{\alpha\beta}\frac{l_{\beta}}{n} - m^{\alpha\beta}\frac{l_{\beta}}{n}\frac{l_{\nu}\dot{u}^{\nu}}{n}\right). \tag{16}$$

This equation should be valid for any arbitrary l^{α}. However, since l^{α} / n is always a finite quantity, and $1 / n$ grows without bound along the line L with this approximation, any term that we find for the coefficients of $1 / n$ and $1 / n^2$ must vanish. The arbitrariness in l^{α} will then lead to the following conditions (MATHISSON [**2**]):

$$\begin{aligned} &m^{\alpha\beta} = m\, u^{\alpha}\, u^{\beta}, \\ &m = 0, \qquad m\dot{u}^{\alpha} = 0. \end{aligned} \tag{17}$$

In order to interpret the result this obtained, we remark that:

$$\varphi^{\alpha\beta} = -\,\frac{\kappa}{2\pi}\int \frac{T^{\alpha\beta}_{(t-r)}}{r}\,dV \tag{18}$$

represents the solution to (3), (4).

We now compare this solution with (15), and ideally in the proper system while neglecting the acceleration, since we will be dealing with only the principal meanings of the quantities m that were introduced into (17). We then let $T^{\alpha\beta}$ denote the energy-impulse tensor of incoherent matter:

$$T^{\alpha\beta} = \sum \mu\, v^\alpha v^\beta \,, \tag{19}$$

in which v^α is the velocity relative to the point of reduction, which we will assume to the center of mass of the system. Moreover, we assume that the velocity is sufficiently small that we can neglect its square. (Speed of light c = 1.) One will therefore have:

$$T^{44} = \sum \mu \,, \tag{20}$$

$$\varphi^{44} = -\frac{\kappa}{2\pi}\sum\frac{\mu}{r} = -\frac{\kappa}{2\pi}\frac{1}{r}\sum\mu - \frac{\kappa}{2\pi}\left(\frac{\partial(1/r)}{\partial x^i}\right)_0 \sum \mu y^i - \dots \tag{21}$$

If y^i are the coordinates of the center of mass then the second term of the series development will vanish. If one compare the values:

$$\varphi^{44} = \frac{m}{n} = -\frac{m}{r}$$

that follow from (15) (in the proper system) with (21) then one will see that m means the mass of the material system in equation (17), up to the constant coefficient $\kappa / 2\pi$. In the absence of forces (we have assumed the MINKOWSKI background and the absence of an electromagnetic field), one can derive the law of conservation of mass for the material system that is represented by our pole to a sufficient degree of approximation from the conditions (17), along with the uniform motion of the center of mass of the system along a line.

4. Up to now, we have obtained results that agree with the classical ones. Considering the dipole term will yield something new. By considering the fact that if $f(s)$ fulfills the equation $\Box f = 0$ then $\partial f / \partial x^\alpha$ will also fulfill that condition, we would like to look for a solution of equations (6) that takes the form:

$$\varphi^{\alpha\beta} = \frac{\partial}{\partial x^\lambda}\left(\frac{m^{\lambda\alpha\beta}}{n}\right) + \frac{m^{\alpha\beta}}{n}\,. \tag{22}$$

The pole term m^{ab} / n is considered in this expression, in addition to the proper dipole potential $\dfrac{\partial}{\partial x^\lambda}\left(\dfrac{m^{\lambda\alpha\beta}}{n}\right)$.

Since equations (6) will already be fulfilled by this assumption, we must still satisfy the conditions (4). The expression $\partial\varphi^{\alpha\beta} / \partial x^\beta$ will assume the form:

$$\frac{\partial\varphi^{\alpha\beta}}{\partial x^\beta} = \frac{1}{n^8} A^\alpha + \frac{1}{n^4} B^\alpha + \frac{1}{n} C^\alpha , \tag{23}$$

in which (*):

$$A^\alpha = m^{\lambda\alpha\beta}\left(g_{\lambda\beta} + 2u_\lambda u_\beta - 6\frac{u_{(\lambda}u_{\beta)}}{n} + 3\frac{l_\lambda l_\beta}{n^2}\right), \tag{23a}$$

$$\begin{aligned} B^\alpha = {} & \dot{m}^{\lambda\alpha\beta}\left(-g_{\lambda\beta} + 4\frac{u_{(\lambda}l_{\beta)}}{n} - 3\frac{l_\lambda l_\beta}{n^2}\right) + m^{\lambda\alpha\beta}\left(g_{\lambda\beta}\frac{l_\nu \dot{u}^\nu}{n} - 6\frac{u_{(\lambda}l_{\beta)}}{n}\frac{l_\nu \dot{u}^\nu}{n} \right. \\ & \left. + \ 6\frac{l_\lambda l_\beta}{n^2}\frac{l_\nu \dot{u}^\nu}{n} + 2\frac{\dot{u}_{(\lambda}l_{\beta)}}{n}\right) + m^{\alpha\beta}\left(u_\beta - \frac{l_\beta}{n}\right), \end{aligned} \tag{23b}$$

$$\begin{aligned} C^\alpha = {} & \ddot{m}^{\lambda\alpha\beta}\frac{l_\lambda l_\beta}{n^2} - 3\dot{m}^{\lambda\alpha\beta}\frac{l_\lambda l_\beta}{n^2}\frac{l_\nu \dot{u}^\nu}{n} - m^{\lambda\alpha\beta}\frac{l_\lambda l_\beta}{n^2}\left[\frac{l_\nu \ddot{u}^\nu}{n} - 3\left(\frac{l_\nu \dot{u}^\nu}{n}\right)^2\right] \\ & + \ \dot{m}^{\alpha\beta}\frac{l_\beta}{n} - m^{\alpha\beta}\frac{l_\beta}{n}\frac{l_\nu \dot{u}^\nu}{n} . \end{aligned} \tag{23c}$$

As in (16), the expressions of equal order in $1 / n$ are combined. Also, just as in (16), the vanishing of the factors of $\frac{1}{n}$, $\left(\frac{1}{n}\right)^2$, and $\left(\frac{1}{n}\right)^3$ will follow from the vanishing of the entire expression (23) for an arbitrary l^α:

$$A^\alpha = 0, \tag{24a}$$
$$B^\alpha = 0, \tag{24b}$$
$$C^\alpha = 0. \tag{24c}$$

In order to be able to bring these equations to a conclusion effortlessly, we will make use of a method of decomposition. One can represent any tensor that is symmetric in $\alpha\beta$ ($m_{\lambda\beta\alpha}$ must be symmetric in $\alpha\beta$ due to the symmetry of the gravitational potentials) as follows:

$$m^{\lambda\alpha\beta} = {}^*m^{\lambda\alpha\beta} + n^{\alpha\lambda}\, u^\beta + n^{\beta\lambda}\, u^\alpha + q^{\alpha\beta}\, u^\lambda + p^\lambda\, u^\alpha\, u^\beta + w^\beta\, u^\lambda\, u^\alpha + w^\beta\, u^\lambda\, u^\alpha + w\, u^\alpha\, u^\beta\, u^\lambda$$

$$m^{\alpha\beta} = {}^*m^{\alpha\beta} + n^\alpha\, u^\beta + n^\beta\, u^\alpha + m\, u^\alpha\, u^\beta,$$

(*) $u_{(\lambda}\, l_{\beta)} = \frac{1}{2}\,(u_\lambda\, l_\beta + u_\beta\, l_\lambda)$.

in which all tensors on the right-hand side that are not the u^α vector are orthogonal to u^α:

$$^*m^{\lambda\alpha\beta}\,u_\beta = 0, \quad n^{\alpha\beta}\,u_\beta = 0, \qquad p^\lambda\,u_\lambda = 0, \qquad ^*m^{\alpha\beta}\,u_\beta = 0, \quad n^\alpha\,u_\alpha = 0, \qquad \text{etc.} \tag{25}$$

This decomposition is always possible and unique. However, it also shows that one must not consider $m^{\lambda\alpha\beta}$ in the most general form. It follows from the easily-proved identity:

$$\frac{\partial}{\partial x^\lambda}\left(\frac{f\,u^\lambda}{n}\right) = \frac{\dot{f}}{n} \tag{26}$$

that all of the terms that have u^λ as a coefficient will contribute an expression of the form:

$$\frac{\partial}{\partial x^\lambda}\left(\frac{Q^{\alpha\beta}\,u^\lambda}{n}\right) = \frac{\dot{Q}^{\alpha\beta}}{n}$$

to the potential. We can then regard these terms as the ones that were already considered in the monopole potential (viz., the second term in (22)). With no loss of generality, we can then pose:

$$m^{\lambda\alpha\beta} = {}^*m^{\lambda\alpha\beta} + n^{\alpha\lambda}\,u^\beta + n^{\beta\lambda}\,u^\alpha + p^\lambda\,u^\alpha\,u^\beta. \tag{27}$$

Substituting this in (24a) above will yield:

$$m^{\lambda\alpha\beta}\left(g_{\lambda\beta} + 2u_\lambda u_\beta - 6\frac{u_{(\lambda}l_{\beta)}}{n} + 3\frac{l_\lambda l_\beta}{n^2}\right) = 0.$$

However, since:

$$u^\beta\left(g_{\lambda\beta} + 2u_\lambda u_\beta - 6\frac{u_{(\lambda}l_{\beta)}}{n} + 3\frac{l_\lambda l_\beta}{n^2}\right) = 0,$$

all that will remain is:

$$({}^*m^{\lambda\alpha\beta} + n^{\beta\lambda}u^\alpha)\left(g_{\lambda\beta} + 2u_\lambda u_\beta - 6\frac{u_{(\lambda}l_{\beta)}}{n} + 3\frac{l_\lambda l_\beta}{n^2}\right) = 0,$$

and if we consider (25):

$$^*m^{\lambda\alpha\beta}\,u_\beta = {}^*m^{\lambda\alpha\beta}\,u_\lambda = n^{\beta\lambda}\,u_\lambda = n^{\beta\lambda}\,u_\beta = 0$$

then we will get:

$$({}^*m^{\lambda\alpha\beta} + n^{\beta\lambda}u^\alpha)\left(g_{\lambda\beta} + 3\frac{l_\lambda l_\beta}{n^2}\right) = 0. \tag{28}$$

We would like to consider this condition in the proper system. The orthogonality conditions for $^*m^{\lambda\alpha\beta}$ and $n^{\beta\lambda}$ with respect to u_α can be expressed as follows:

$$^{*}m^{\lambda\alpha 4} = {}^{*}m^{4\alpha\beta} = 0, \qquad\qquad n^{4\alpha} = n^{\alpha 4} = 0.$$

Furthermore, $l_i / n = \gamma_i$ are the direction cosines for the guiding ray. These direction cosines fulfill only the condition that:

$$g^{ik}\,\gamma_i\,\gamma_k + 1 = 0, \tag{29}$$

but otherwise they are completely arbitrary, and in particular, linearly-independent. Since $u^i = 0$, equation (28) will decompose into two parts:

$$^{*}m^{\lambda\alpha\beta}\left(g_{\lambda\beta} + 3\frac{l_\lambda l_\beta}{n^2}\right) = 0 \qquad\qquad (\alpha \neq 4), \tag{28a}$$

$$n^{\beta\lambda}\left(g_{\lambda\beta} + 3\frac{l_\lambda l_\beta}{n^2}\right) = 0 \qquad\qquad (\alpha = 4). \tag{28b}$$

In the proper system, it will then follow that (†):

$$^{*}m^{(i\,|\,\alpha\,|\,j)} = {}^{*}m^{\alpha}\,g^{ij} \qquad\qquad ({}^{*}m^{\alpha}\,u_\alpha = 0), \tag{30a}$$

$$n^{(ij)} = \rho\, g^{ij}. \tag{30b}$$

If we bring the orthogonality of $^{*}m^{\lambda\alpha\beta}$ and $n^{\alpha\beta}$ to u_α under consideration – i.e., the vanishing of the components $^{*}m^{4\alpha\beta}$, $^{*}m^{\lambda\alpha 4}$, and $n^{4\alpha}$ – then we will get:

$$^{*}m^{(\lambda\,|\,\alpha\,|\,\beta)} = {}^{*}m^{\alpha}\,(g^{\lambda\beta} - u^\lambda\,u^\beta), \tag{31a}$$

$$n^{(\lambda\beta)} = \rho\,(g^{\lambda\beta} - u^\lambda\,u^\beta). \tag{31b}$$

The relations (31) are presented in tensor form, so they will be independent of the coordinate system. Since $m^{\lambda\alpha\beta}$ is symmetric in $\alpha\beta$, we will get the following expressions for $^{*}m^{\lambda\alpha\beta}$ and $n^{\alpha\beta}$ (††):

$$^{*}m^{\lambda\alpha\beta} = {}^{*}m^{\alpha}\,(g^{\lambda\beta} - u^\lambda\,u^\beta) + {}^{*}m^{\beta}\,(g^{\lambda\alpha} - u^\lambda\,u^\alpha) - {}^{*}m^{\lambda}\,(g^{\alpha\beta} - u^\alpha\,u^\beta), \tag{32}$$

$$n^{\lambda\beta} = \rho\,(g^{\lambda\beta} - u^\lambda\,u^\beta) + n^{[\lambda\beta]}. \tag{33}$$

We now consider what sort of contribution the expression $^{*}m^{\lambda\alpha\beta} + n^{(\lambda\beta)}\,u^\alpha + n^{(\lambda\alpha)}\,u^\beta$ makes to the potential more closely. The potential that arises from it has the form:

(†) $^{*}m^{(\lambda\,|\,\alpha\,|\,\beta)} = \frac{1}{2}\,({}^{*}m^{\lambda\alpha\beta} + {}^{*}m^{\beta\,\alpha\lambda})$.

(††) $n^{[\lambda\beta]} = \frac{1}{2}\,(n^{\lambda\beta} - n^{\beta\lambda})$.

$$\varphi^{\alpha\beta} = \frac{\partial}{\partial x^\lambda}\left(\frac{{}^*m^{\lambda\alpha\beta} + n^{(\lambda\alpha)}u^\beta + n^{(\lambda\beta)}u^\alpha}{n}\right)$$

$$= g^{\alpha\nu}\frac{\partial}{\partial x^\nu}\left(\frac{{}^*m^\beta + \rho u^\beta}{n}\right) + \frac{\partial}{\partial x^\nu}\left(\frac{{}^*m^\alpha + \rho u^\beta}{n}\right) - g^{\alpha\beta}\frac{\partial}{\partial x^\nu}\left(\frac{{}^*m^\nu + \rho u^\nu}{n}\right) + \frac{\partial}{\partial x^\lambda}\left(\frac{{}^*m^\lambda + \rho u^\lambda}{n}\right)$$

$$+ \frac{1}{n}\left[\dot{\rho}g^{\alpha\beta} - 2\frac{d}{ds}({}^*m^{(\alpha}u^{\beta)} + \rho u^\alpha u^\beta)\right]; \tag{34}$$

the identity (26) was employ in the proof of this.

The first three terms in the relation (34) can be (with the use of the covariant and contravariant derivatives, which are converted into ordinary derivatives for the MINKOWSKI background) represented as follows:

$$\psi^{\alpha\beta} = \nabla^\alpha \xi^\beta + \nabla^\beta \xi^\alpha - g^{\alpha\beta}\nabla_\nu \xi^\nu, \tag{35}$$

in which ξ^ν fulfills the condition $\Box\xi^\nu = 0$. As one can easily prove, the identity:

$$\frac{\partial \psi^{\alpha\beta}}{\partial x^\beta} = 0$$

will follow from this.

This part then imposes no restriction on the remaining terms of the potential. It will then define a physically inessential term (in electrodynamics, this part has its analogue in the four-dimensional gradients that one can add to the electromagnetic potential), and one can omit it with no loss of generality. The next term in (34) be linked quite simply to the term $p^\lambda u^\alpha u^\beta$ without affecting the orthogonality, and further with the monopole term. Therefore, it suffices to choose $m^{\lambda\alpha\beta}$ in the form:

$$m^{\lambda\alpha\beta} = n^{\alpha\lambda}u^\beta + n^{\beta\lambda}u^\alpha + p^\lambda u^\alpha u^\beta, \tag{36}$$

in which $n^{\alpha\beta}$ is already antisymmetric.

Before we go on to the consequences of the vanishing of the factors of the higher powers of $1 / n$, we must speak about the physical sense of the $n^{\lambda\beta}$ and p^λ. As in no. **3**, we will compare the potential $\varphi^{\alpha\beta} = \frac{\partial}{\partial x^\lambda}\left(\frac{m^{\lambda\alpha\beta}}{n}\right)$ with the known relation (18) in the proper system, in which we will substitute the energy-impulse tensor of incoherent matter for $T^{\alpha\beta}$ while neglecting the accelerations and the squares of the velocities. From a comparison of the φ^{44} components, we will get:

$$\varphi^{44} = -\frac{\kappa}{2\pi}\sum\frac{\mu}{r} = -\frac{\kappa}{2\pi}\frac{1}{r_0}\sum\mu - \frac{\kappa}{2\pi}\left(\frac{\partial(1/r)}{\partial x^i}\right)\sum\mu\, y^i + \ldots$$

$$= \frac{m^{44}}{n} + p^i \frac{\partial}{\partial x^i}\left(\frac{1}{n}\right) = -p^i \frac{\partial}{\partial x^i}\left(\frac{1}{r}\right) - \frac{m^{44}}{r}.$$

We see from this that p^λ *fundamentally* represents a static moment relative to the reduction point. Previously, L was assumed to be entirely arbitrary, except that it had to be inside the tube; however, if one lets it go through the center of mass of the section of the tube that is orthogonal to it then the moment above will vanish.

For that reason, from now on, we will assume that L is the "center of mass line," so we will then have:

$$p^\lambda = 0. \tag{37}$$

For the sake of interpreting the $n^{\lambda\beta}$, we must compare the expressions for the components φ^{4i}.

$$\varphi^{4i} = -\frac{\kappa}{2\pi}\sum \frac{\mu v^i}{r} = -\frac{\kappa}{2\pi}\frac{1}{r_0}\sum \mu v^i - \frac{\kappa}{2\pi}\left(\frac{\partial(1/r)}{\partial x^k}\right)_0 \sum \mu y^k v^i + \ldots$$
$$= \frac{m^{4i}}{n} + \frac{\partial}{\partial x^\lambda}\left(\frac{n^{i\lambda}}{n}\right) = \frac{m^{4i}}{n} + n^{ik}\frac{\partial}{\partial x^k}\left(\frac{1}{n}\right) = -\frac{m^{4i}}{r} - n^{ik}\frac{\partial}{\partial x^k}\left(\frac{1}{r}\right).$$

From the comparison, one sees that:

$$n^{ik} = \sum \mu\, y^k v^i, \tag{38}$$

up to some constant coefficients.

The equation above demands antisymmetry on the right-hand side. Its validity is exhibited by, e.g., systems that move like rigid bodies whose ellipsoid of inertia is a sphere. The dipole will then be a satisfactory approximation; the quadrupole term must be considered in the case of a triaxial ellipsoid of inertia.

Since we would like to truncate with the dipole term, we assume antisymmetry on the right-hand side, and for that reason, we will have:

$$n^{ik} = \tfrac{1}{2}\sum \mu(y^k v^i - y^i v^k) = \Omega^{ik}, \tag{39}$$

in which $\Omega^{\alpha\beta}$ is the moment of impulse tensor. As we will see, $n^{\alpha\beta}$ will be *fundamentally* the moment of impulse.

We now consider the terms in $\left(\frac{1}{n^2}\right)$ in (23). With consideration given to (36), we will obtain:

$$(\dot{n}^{\alpha\beta} - {}^*m^{\alpha\beta} - n^\beta u^\alpha)\left(u_\beta - \frac{l_\beta}{n}\right) = 0. \tag{40}$$

It follows from this (e.g., in the proper system) that:

$$\dot{n}^{\alpha\beta} - {}^{*}m^{\alpha\beta} - n^{\beta}u^{\alpha} = Q^{\alpha}\,u^{\beta}. \tag{41}$$

Multiplication by u^{β} yields:

$$Q^{\alpha} = \dot{n}^{\alpha\beta}u_{\beta}. \tag{42}$$

Since $n^{\alpha\beta}$ is antisymmetric, Q^{α} must be orthogonal to u^{α}:

$$Q^{\alpha}\,u_{\alpha} = \dot{n}^{\alpha\beta}u_{\alpha}u_{\beta} = 0. \tag{43}$$

If one multiplies (41) by u^{α} then it will follow that:

$$n^{\beta} = \dot{n}^{\alpha\beta}u_{\alpha} = -\,\dot{n}^{\beta\alpha}u_{\alpha}. \tag{44}$$

Substituting the relation above in (41) yields:

$${}^{*}m_{\alpha\beta} = \dot{n}^{\alpha\beta} + n^{\beta\nu}u_{\nu}u^{\alpha} - \dot{n}^{\alpha\nu}u_{\nu}u^{\beta}. \tag{45}$$

Two equations follow from the symmetry of the right-hand side and the antisymmetry of the left-hand side of the relation above:

$${}^{*}m^{\alpha\beta} = 0, \tag{46}$$

$$\dot{n}^{\alpha\beta} + n^{\beta\nu}u_{\nu}u^{\alpha} - \dot{n}^{\alpha\nu}u_{\nu}u^{\beta} = 0. \tag{47}$$

Differentiating the equation $n^{\alpha\beta}\,u_{\beta} = 0$ with respect to s yields:

$$\dot{n}^{\alpha\beta}u_{\beta} = -\,n^{\alpha\beta}\dot{u}_{\beta}. \tag{48}$$

For that reason, we can replace (47) with:

$$\boxed{\dot{n}^{\alpha\beta} - n^{\beta\nu}\dot{u}_{\nu}u^{\alpha} + n^{\alpha\nu}\dot{u}_{\nu}u^{\beta} = 0.} \tag{49}$$

With consideration given to the antisymmetry of $n^{\alpha\beta}$, (49) will yield the relations (48) and (47). Equation (49) is a relativistic generalization of the law of conservation of angular momentum.

Finally, we consider the terms in $1 / n$ in (23). It follows from them that:

$$\dot{n}^{\alpha\lambda}\frac{l_{\lambda}}{n} - n^{\alpha\lambda}\frac{l_{\lambda}}{n}\cdot\frac{l_{\nu}\dot{u}^{\nu}}{n} + \dot{n}^{\alpha} + \dot{n}^{\beta}\frac{l_{\beta}}{n}u^{\alpha} + n^{\beta}\frac{l_{\beta}}{n}\dot{u}^{\alpha} - n^{\beta}\frac{l_{\beta}}{n}u^{\alpha}\frac{l_{\nu}\dot{u}^{\nu}}{n} + \dot{m}u^{\alpha} + m\dot{u}^{\alpha} = 0. \tag{50}$$

With the use of the relations (49) and (44), we get:

$$\dot{m}u^{\alpha} + m\dot{u}^{\alpha} + 2\dot{n}^{\alpha} = 0. \tag{51}$$

Differentiating (44) with respect to *s* yields:

$$\dot{n}^{\alpha} = -\ddot{n}^{\alpha\beta}u_{\beta} - \dot{n}^{\alpha\beta}\dot{u}_{\beta}\,. \tag{51}$$

However, it follows from (49) that $\dot{n}^{\alpha\beta}\dot{u}_{\beta}$, so $\dot{n}^{\alpha} = -\ddot{n}^{\alpha\beta}u_{\beta}$. Moreover, the vanishing of the quantity $\dot{n}^{\alpha\beta}\dot{u}_{\beta}$ is linked with the relation:

$$\ddot{n}^{\alpha\beta}u_{\beta} = -n^{\alpha\beta}\ddot{u}_{\beta}\,, \tag{52}$$

and for that reason we will get:

$$\dot{m}u^{\alpha} + m\dot{u}^{\alpha} + 2n^{\alpha\beta}\ddot{u}_{\beta} = 0. \tag{53}$$

After multiplying this equation by u_{α}, we will see that:

$$\boxed{\dot{m} = 0,} \tag{54}$$

so it will follow from (53) that:

$$\boxed{m\dot{u}^{\alpha} + 2n^{\alpha\beta}\ddot{u}_{\beta} = 0.} \tag{55}$$

The center of mass of a separate (i.e., isolated) system that is endowed with a moment of impulse no longer moves uniformly along a line.

The motion of the center of mass is still coupled with the moment of impulse in the absence of forces, as in this case. MATHISSON found equations (49), (55) in the previously-cited paper by using a variational method in the general case of a RIEMANNIAN background. Independent terms appeared there that were due to the curvature of the background and would vanish for the MINKOWSKI background. The consideration of the MINKOWSKI background as a special case has the advantage that it allows one to compare the potential of the multipole with the successive terms in a known expression for the gravitational potential $-\frac{\kappa}{2\pi}\int \frac{T^{\alpha\beta}_{(t-r)}}{r}\,dV$, which will yield an intuitive, physical interpretation of the quantities that were introduced.

Up to now, we have assumed that the energy-impulse tensor $T^{\alpha\beta}$ vanishes outside of matter. We thus excluded the electromagnetic field. From the linearity of the basic equations (3) and (4), we see, with no further assumptions, that in order for one to have $T^{\alpha\beta} \neq 0$ outside of matter, one will need to find only one integral of equation (3) and add it to the series of multipoles that represents the solution to the linear system. The integral was given by MATHISSON [**3**] in the case where the material system was coupled to an electric pole.

In conclusion, I would like to express my thanks to Herrn Doz. Dr. M. MATHISSON for suggesting this topic and for his valuable remarks while we were discussing it

References

1. M. MATHISSON, Acta Phys. Pol. **6** (1937), 176.
2. M. MATHISSON, Zeit Phys. **67** (1931), 270.
3. M. MATHISSON, Zeit Phys. **67** (1931), 826.