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The motion of a rigid body

(An exercise in the theory of extensions)

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§ 1.

In the following paper, I will employ the point calculations that were taught in the *Ausdehnungslehre* of 1862 (*Grassmann's* book, volume one, part two, especially) (*). I will then make no use of the regressive multiplication, because I do not regard its introduction as advantageous from a pedagogical standpoint. Only experience will teach us whether one can manage without that product. This is known to be true in mechanics; kinematical considerations, as in the present treatise, suggest the use of a special regressive product, which one can, however, introduce independently, as *Peano* did in the last-cited paper below.

One can as little do without the extension of a vector or product of two vectors – viz., a *bivector*, which will be denoted by a line | – in geometry as one can do without the concept of perpendicularity. (By contrast, one does not need the extension of a product of points.) I therefore understand the extension of the vector a – which will be denoted by $|\, a$ – to mean the bivector bc whose factors are perpendicular to a, and are so arranged that the area of the surfaces of the parallelogram that is defined by b and c is equal to the length of a. Therefore, the sense shall be such that when one looks along a, a right rotation through an angle < 180° of the vector c will bring it into the direction of c. One understands the extension of the bivector de – which will be denoted by $|\, de$ – to be the vector f for which $|\, f = de$. The law of distribution is then true for extensions, so one will have $a \mid b = b \mid a$, $(|\, ab) \mid c = abc$ is then equal to a number, namely, the volume of the parallelepiped that the three vectors a, b, c define, and:

$$|\, (c \mid ab) = (c \mid b)\, a - (a \mid c)\, b,$$

(*) One finds a brief presentation by an original method in *Peano: Calcolo geometrico secondo l'Ausdehnungslehre di H. Grassmann*, Torino, 1888. In German, under the title: *The Gründzüge des geometrischen Kalküls*, German version by Schepp, Leipzig, 1891. *Carvallo:* "La méthode de Grassmann," Nouv. Annales, 3me Série, t. 11 (1892), pp. 8. *Peano:* "Saggio di calcolo geometrico," Accad. d. Scienza di Torino, 1895/96.

$$(ab) \mid cd = (a \mid c)\,(b \mid d) - (a \mid d)\,(b \mid c) = cd \mid ab,$$

which are two formulas that essentially agree with the ones in nos. 180 and 176 of the second *Ausdehnungslehre* (*Grassmann's* book, v. 1, pt. 2, pp. 136).

The operation ω that *Peano* introduced relates to second and third-degree forms. One can always write a second-degree form F:

$$F = Pa + bc,$$

where P is an arbitrary point, and a, b, c are vectors. Therefore, the vector a does not depend upon the choice of P (*Grassmann*, *loc. cit.*, no. 347, pp. 222); let that vector be denoted by ωF. If $a = 0$ then F will be equal to a bivector, one must set $\omega F = 0$.

A third-degree form F can be represented as either a product of three vectors $F = abc$ or as the product of a point with a bivector $F = Pab$ (*Carvallo*, page 26, no. 21). In the former case, one understands ωF to be zero, while in the latter case, where the bivector ab does not depend upon the choice of the point P, one understands ωF to mean just that bivector.

The operation ω is distributive in both cases.

In order to minimize the number of brackets, I, with *Carvallo* and *Peano*, have omitted the brackets around a product of points that *Grassmann* had applied in order to distinguish that product from the other ones. The effect of | and ω shall always extend up to the next operation symbol, such that, e.g., one writes $\omega(ab)$ briefly as ωab, $\omega\mathfrak{A}(b - c)$ for $\omega[\mathfrak{A}\,(b - c)]$, and $\mid (a - b)(c - d)$ for $\mid [(a - b)(c - d)]$, when no misunderstanding can arise.

§ 2.

Let AB, A_1B_1 be two pairs of points in such a position that the separation $\overline{AB}$ is equal to $\overline{A_1B_1}$. This can be expressed by the equation:

$$(A_1 - B_1)^2 = (A - B)^2,$$

or by:

$$(B_1 - A_1 - B + A) \mid (B_1 - A_1 + B - A) = 0.$$

If one denotes the midpoints of the lines AA_1 and BB_1 – and thus, the points, $(A + A_1) / 2$ and $(B + B_1) / 2$ – by $\mathfrak{A}$ and $\mathfrak{B}$, resp., then one can write this equation as:

$$[B_1 - B - (A_1 - A)] \mid (\mathfrak{B} - \mathfrak{A}) = 0. \tag{1}$$

We next assume that $\mathfrak{B} - \mathfrak{A}$ is not equal to zero. One can then replace this equation with:

$$B_1 - B - (A_1 - A) = \mid (\mathfrak{A} - \mathfrak{B})\, a, \tag{2}$$

where a is a vector that is not determined completely. It would be convenient if one could separate the symbols A and B in this equation, so to speak. However, this is not immediately possible, because one cannot decompose $\mid(\mathfrak{B} - \mathfrak{A})\, a$ into $\mid \mathfrak{B}a - \mid \mathfrak{A}a$ on the right, since – at least, for us here – a form like $\mid \mathfrak{A}a$ has no meaning. However, with the use of an arbitrary point P, one can write:

$$(\mathfrak{A} - \mathfrak{B})\, a = \omega P\, (\mathfrak{A} - \mathfrak{B})\, a = \omega(\mathfrak{A} - \mathfrak{B})\, Pa$$
$$= \omega \mathfrak{B}\, Pa - \omega \mathfrak{A}\, Pa,$$

and then decompose the latter equation into:

$$B_1 - B - \mid \omega \mathfrak{B}\, Pa = A_1 - A \mid \omega \mathfrak{A}\, Pa,$$

in which the symbols are separated. If one sets the vector that both sides are equal to equal to b then one will have:

$$A_1 - A = \mid \omega \mathfrak{A}\, Pa + b = \mid (\omega \mathfrak{A}\, Pa + \mid b)$$
$$B_1 - B = \mid \omega \mathfrak{B}\, Pa + b = \mid (\omega \mathfrak{B}\, Pa + \mid b),$$

or, since $\omega(\mathfrak{A} \mid b) = \omega(\mathfrak{B} \mid b) = \mid b$, if one sets:

$$Pa + \mid b = \Gamma$$

then one will have:

(3) $$\begin{cases} A_1 - A = \mid \omega \mathfrak{A}\, \Gamma, \\ B_1 - B = \mid \omega \mathfrak{A}\, \Gamma. \end{cases}$$

These formulas are then also true when one has $A_1 - A = B_1 - B$. Namely, one will then have:

$$a = 0, \quad \Gamma = \mid b, \qquad \mid \omega \mathfrak{A}\, \Gamma = \mid \omega \mathfrak{B}\, \Gamma = b,$$

so $A_1 - A = B_1 - B = b$, as it must be.

§ 3.

In order to recognize the geometric meaning of these formulas, we would first like to assume that a point P has the following relationship with another one P_1 :

(4) $$P_1 - P = \tfrac{1}{2} \mid \omega P\, Q\, R,$$

where Q, R are given points. If one writes:

$$PQR = P\,(Q - P)(R - P)$$

then one will see that $\omega PQR = (Q - P)(R - P)$, so it is equal to a bivector whose factors lie in the plane PQR, and that $| \, \omega PQR$ is correspondingly a vector that is perpendicular to the plane PQR. The length of this vector is equal to the surface area of the parallelogram whose sides are PQ and PR or equal to twice the area of the triangle PQR. As far as its direction is concerned, when one looks outward from it, one will see $R - P$ to the right of $Q - P$; i.e., if one places P perpendicular to the plane PQR, such that one must make a proper rotation around it through an angle $< 180^{\circ}$ in order to see the direction PQ in the direction PR then the direction of the vector will go from the foot to the head. Instead of that, one can also say that if one stands on QR with one's feet at R and one's head at Q and looks towards P then the direction of the vector will go from right to left.

If one draws PS perpendicular to QR then the magnitude of PP_1 will be equal to $\frac{1}{2}\overline{PS}\cdot\overline{QR}$. Therefore, if one determines an acute angle φ from the equation $\tan \varphi / 2 = \overline{QR}$ then one can say that the plane P_1QR defines an angle of $\varphi / 2$ with PQR, and indeed with a rotation that goes right to left when seen from QR.

If we secondly assume that the two points PP_1 are connected with each other and their midpoint $\mathfrak{P}$ by the equation:

$$P_1 - P = | \, \omega \mathfrak{P} \, QR. \tag{5}$$

Since $\mathfrak{P} = (P + P_1) / 2$, one will have:

$$P_1 - P = 2\,(P_1 - \mathfrak{P}) = 2\,(\mathfrak{P} - P),$$

such that one will have:

$$\begin{aligned} P_1 - \mathfrak{P} &= \tfrac{1}{2} \, | \, \omega \mathfrak{P} \, QR, \\ P - \mathfrak{P} &= -\tfrac{1}{2} \, | \, \omega \mathfrak{P} \, QR \, . \end{aligned}$$

P, $\mathfrak{P}$, P_1 then lie on a straight line that is perpendicular to the plane $\mathfrak{P}QR$ at P, the plane P_1QR makes an angle of $\varphi / 2$ to the left with the plane $\mathfrak{P}QR$, while PQR makes an angle of $\varphi / 2$ to the right. P and P_1 will then have the same distance from a point T on QR. One can then set $QR = UT$ if one determines the point U on QR suitably, and then writes equation (5) as:

$$P_1 - T + T - P = | \, \omega \mathfrak{P} \, UT \, .$$

It will follow from this that:

$$\begin{aligned} (P_1 - T) \, | \, (P_1 - T) + (P_1 - T) \, | \, (T - P) &= (P_1 - T) \; \omega \mathfrak{P} \, UT, \\ (T - P) \, | \, (P_1 - T) + (T - P) \, | \, (T - P) &= (T - P) \; \omega \mathfrak{P} \, UT, \end{aligned}$$

and by subtraction, that:

$$\begin{aligned} (P_1 - T)^2 - (P - T)^2 &= (P_1 + P - 2T) \; \omega \mathfrak{P} \, UT, \\ &= 2\,(\mathfrak{P} - T) \; \omega \mathfrak{P} \, UT. \end{aligned}$$

However, one has $\omega \mathfrak{P}\, UT = (U - \mathfrak{P})\,(T - \mathfrak{P})$, and as a result, the right-hand side is zero, so $\overline{P_1T} = \overline{PT}$.

Therefore, P_1 emerges from P by a rotation around the axis QR through the angle φ, and in fact, a rotation that proceeds to the left when seen from QR. The magnitude and sense of the rotation, and its axis are determined completely by the line segment QR.

§ 4.

One can alter the second-degree form Γ by altering the point P. One can then write:

$$\Gamma = Qa + (P - Q)\,a + |\,b.$$

Here, $(P - Q)\,a + |\,b$ is a bivector that one can set equal to $|\,b'$. One will obtain a distinguished form for Γ when b' is parallel to a, so one will get:

$$(P - Q)\,a + |\,b = \lambda\,|\,a, \tag{6}$$

if one understands λ to mean a number (*).

The multiplication by a will give $a\,|\,b = \lambda\, a\,|\,a$, and then:

$$(P - Q)\,a = \frac{(a\,|\,b)\,|\,a - (a\,|\,a)\,|\,b}{a\,|\,a} = \frac{1}{a\,|\,a}\,a\,(|\,ab),$$

$$\left(P - Q + \frac{|\,ba}{a\,|\,a}\right) a = 0.$$

The first factor on the left must then be a vector that is parallel to a that one can set equal to $\mu\, a$ if one understands μ to mean a number. That will yield:

$$Q = P + \frac{|\,ba}{a\,|\,a} - \mu\, a.$$

By substituting this into equation (6), one sees that m remains completely arbitrary, such that Q can be chosen arbitrarily in a certain line that is parallel to a. One finally has:

$$\Gamma = Qa + \frac{a\,|\,b}{a\,|\,a} \cdot |\,a,$$

as the normal form of Γ. Formula (3) then yields:

(*) The following reduction is identical with the search for the central axis of a system of forces, which can also be represented by a second-degree form.

$$A_1 - A = |\ \omega \mathfrak{A}\ Q\ a + \lambda\ a.$$

Now, if ρ is an arbitrary number, and one sets:

$$\begin{aligned} A_1 - (\lambda - \rho)\ a &= A_1', \\ A + \rho\ a &= A', \end{aligned}$$

then A' and A_1' will be two points, the first of which emerges from A by a displacement ρa parallel to a, while one must displace the second one by $(\lambda - \rho)\ a$ in order to obtain A_1. With that, one will have:

$$\begin{aligned} \mathfrak{A} &= \frac{A' + A_1'}{2} + \tfrac{1}{2}(\lambda - 2\rho)\ a, \\ \mathfrak{A}\ Q\ a &= \frac{A' + A_1'}{2} - Qa, \\ A_1' - A' &= |\ \omega\ \frac{A' + A_1'}{2} - Qa. \end{aligned}$$

The last formula shows that A_1' emerges from A' by a rotation around the axis that goes through Q and is parallel to a, which will be denoted by (Q, a). A_1 then arises from A by a translation parallel to a and a rotation around an axis parallel to a.

The second-degree form Γ thus represents a screwing motion, or a *wrench* (*).

Now, if Γ' is a second wrench that, like Γ, must take the points A, B to A_1, B_1, resp., in any case, then one must have:

$$\omega \mathfrak{A}\ \Gamma = \omega \mathfrak{A}\ \Gamma', \qquad \omega \mathfrak{B}\ \Gamma = \omega \mathfrak{B}\ \Gamma'. \tag{7}$$

However, if one has:

$$\omega \mathfrak{A}\ \mathsf{T} = 0$$

for a second-degree form T then one will set:

$$\mathsf{T} = \mathfrak{A}\ d + ef,$$

where d, e, f are vectors. One will then arrives at the equation:

$$\omega \mathfrak{A}\ ef = ef = 0,$$

so $\mathsf{T} = \mathfrak{A}\ d$, or, when one sets the point $\mathfrak{A} + d = R$, one will have $\mathsf{T} = \mathfrak{A}\ R$.

Equations (7) will then be fulfilled when:

$$\Gamma' - \Gamma = \mathfrak{A}\ R = \mathfrak{B}\ S.$$

(*) *Grassmann*, *loc. cit.*, pp. 223. Remark.

However, this relation between four points $\mathfrak{A}$, $\mathfrak{B}$, R, S says that they lie upon a straight line, so two vectors $R - \mathfrak{A}$ and $S - \mathfrak{B}$ will be equal. If one then sets:

$$R = \mathfrak{A} + \rho\,(\mathfrak{B} - \mathfrak{A}),$$
$$S = \mathrm{A} + \sigma\,(\mathfrak{B} - \mathfrak{A})$$

then it will follow that $\rho = \sigma - 1$, so:

$$\Gamma' - \Gamma = \rho\,\mathfrak{A}\mathfrak{B},$$

where ρ is an arbitrary number. Any wrench of the form:

$$\Gamma + \rho\,\mathfrak{A}\mathfrak{B}$$

will then take AB to A_1B_1, such that there is an entire pencil of wrenches that will accomplish that objective.

§ 5.

Should there be a wrench among this pencil that is a pure rotation then one must be able to determined ρ in such a way that $\Gamma' = \Gamma + \rho\,\mathfrak{A}\mathfrak{B}$ is equal to a line segment (*).

However, one will then have $\Gamma'\,\Gamma' = 0 = \Gamma\Gamma + 2\rho\,(\Gamma\,\mathfrak{A}\mathfrak{B})$, so:

$$\rho = -\frac{\Gamma\Gamma}{2(\Gamma\mathfrak{A}\mathfrak{B})}.$$

Let ρ be so determined. If one then brings Γ' into the form:

$$\Gamma' = Pa' + |\,b$$

with the help of a point P, in which a' and b are vectors, then it will follow from:

$$0 = \Gamma'\,\Gamma' = 2\,P\,(a'\,|\,b)$$

that b will be perpendicular to a' when $a' \neq 0$. One can then set:

$$b = |\,a'c,$$

where c is again a vector, and one will get:

$$\Gamma' = Pa' + a'c = (P - c)\,a',$$

(*) *Loc. cit.*, no. 286.

or with $P - c = Q$, $a' = R - Q$, where Q and R are points, so $\Gamma' = QR$ will, in fact, be a line segment. However, if $a' = 0$ then one will have $\Gamma' = | \, b$, and that will represent a pure translation. There will then be a rotation that takes AB to A_1B_1 when one has $a' = \omega\Gamma' \neq 0$. However, one will have:

$$\omega\Gamma' = \omega\Gamma + \rho\omega\mathfrak{A}\mathfrak{B} = \omega\Gamma + \rho\,(\mathfrak{B} - \mathfrak{A}),$$

so if one had $\omega\Gamma' = 0$ then one would have:

$$\omega\Gamma = \rho\,(\mathfrak{B} - \mathfrak{A}).$$

However, the normal form of Γ is:

$$\Gamma = Q\omega\Gamma + \lambda \,|\, \omega\Gamma,$$

so here it will be equal to $Q\rho\,(\mathfrak{A} - \mathfrak{B}) + \rho\lambda \,|\, (\mathfrak{A} - \mathfrak{B})$, and that will imply:

$$\begin{aligned}
\mathfrak{A}\,\Gamma &= -\rho\, Q\, \mathfrak{A}\mathfrak{B} + \rho\lambda\, \mathfrak{A} \,|\, (\mathfrak{A} - \mathfrak{B}),\\
\mathfrak{B}\,\Gamma &= -\rho\, Q\, \mathfrak{A}\mathfrak{B} + \rho\lambda\, \mathfrak{B} \,|\, (\mathfrak{A} - \mathfrak{B}),\\
\omega\mathfrak{A}\,\Gamma &= -\rho\omega Q\, \mathfrak{A}\mathfrak{B} + \rho\lambda \,|\, (\mathfrak{A} - \mathfrak{B}),\\
\omega\mathfrak{A}\,\Gamma &= -\rho\,\omega Q\, \mathfrak{A}\mathfrak{B} + \rho\lambda \,|\, (\mathfrak{A} - \mathfrak{B}),
\end{aligned}$$

such that $A_1 - A = B_1 - B$, so only a displacement will be necessary.

§ 6.

The derivation of formula (2) from (1) is justified only when $\mathfrak{A} - \mathfrak{B}$ is not equal to zero. However, if the two points $\mathfrak{A}$ and $\mathfrak{B}$ coincide then one will determine a unit vector a such that one has:

$$(A_1 - A) \,|\, a = (B_1 - B) \,|\, a\,.$$

If ρ is the common value, and one sets:

$$A_1 - A = \rho\, a + a_1, \qquad B_1 - B = \rho\, a + b_1$$

then it will follow that $a \,|\, a_1 = a \,|\, b_1 = 0$. Therefore, if one defines the points A' and B' by the equations:

$$A' = A + \rho\, a, \qquad B' = B + \rho\, a$$

then one will have $A_1 - A' = a_1$, $B_1 - B' = b_1$, and these vectors will be perpendicular to a when they are not zero. Furthermore, one has:

$$A_1 + A' = 2\mathfrak{A} + \rho\, a, \quad B_1 + B' = 2\mathfrak{B} + \rho\, a\,.$$

The lines A_1A' and B_1B' will then be met by the line $(\mathfrak{A}, a)$ in their midpoints, when they do not vanish. As a result, one can take AB to A_1B_1 by just a displacement through ρa, or in connection with a rotation through 180° around the axis $(\mathfrak{A}, a)$.

§ 7.

We now go on to the consideration of three points, and let ABC and $A'B'C'$ be two congruent triangles. Let the midpoints:

$$\tfrac{1}{2}(A + A_1), \qquad \tfrac{1}{2}(B + B_1), \qquad \tfrac{1}{2}(C + C_1)$$

of the three lines AA_1, BB_1, CC_1 be denoted by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$. It will first be assumed that these three points do not lie on a straight line and that no two of them will coincide, either. One can then apply the considerations of § 2 to the three pairs of points AB, BC, CA, and thus find three second-degree forms A, B, Γ such that

$$\text{(8)} \qquad \begin{cases} A_1 - A = | \, \omega \mathfrak{A} \Gamma, & B_1 - B = | \, \omega \mathfrak{B} \mathsf{A}, \quad C_1 - C = | \, \omega \mathfrak{C} \mathsf{B}, \\ B_1 - B = | \, \omega \mathfrak{B} \Gamma, & C_1 - C = | \, \omega \mathfrak{C} \mathsf{A}, \quad A_1 - A = | \, \omega \mathfrak{A} \mathsf{B}. \end{cases}$$

One must then also have:

$$\omega \mathfrak{A} (\Gamma - \mathsf{B}) = 0, \qquad \omega \mathfrak{B} (\mathsf{A} - \Gamma) = 0, \qquad \omega \mathfrak{C} (\mathsf{B} - \mathsf{A}) = 0,$$

so, from § 4, three points P, Q, R must then exist that imply:

$$\text{(9)} \qquad \Gamma - \mathsf{B} = \mathfrak{A} P, \quad \mathsf{A} - \Gamma = \mathfrak{B} Q, \quad \mathsf{B} - \mathsf{A} = \mathfrak{C} R,$$

and must then fulfill the equation:

$$\text{(10)} \qquad \mathfrak{A} P + \mathfrak{B} Q + \mathfrak{C} R = 0.$$

If one multiplies this by $\mathfrak{B}\mathfrak{C}$, $\mathfrak{C}\mathfrak{A}$, and $\mathfrak{A}\mathfrak{B}$ then it will follow that:

$$\mathfrak{A}\mathfrak{B}\mathfrak{C}P = \mathfrak{A}\mathfrak{B}\mathfrak{C}Q = \mathfrak{A}\mathfrak{B}\mathfrak{C}R = 0,$$

and thus, since $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ should not lie on a straight line, PQ and R must lie in the plane $\mathfrak{A}\mathfrak{B}\mathfrak{C}$. One can thus set:

$$P = \alpha \mathfrak{A} + \beta'' \mathfrak{B} + \gamma \mathfrak{C},$$

where α, β'', γ are numbers that fulfill the condition $\alpha + \beta'' + \gamma = 1$, and obtain:

$$\mathfrak{A}P = \beta'' \mathfrak{A}\mathfrak{B} - \gamma \mathfrak{C}\mathfrak{A}.$$

Similarly, there are numbers β, β', γ', γ'' for which it follows that:

$$\mathfrak{B}Q = \beta \mathfrak{B}\mathfrak{C} - \gamma' \mathfrak{A}\mathfrak{B},$$
$$\mathfrak{C}R = \beta' \mathfrak{C}\mathfrak{A} - \gamma'' \mathfrak{B}\mathfrak{C}.$$

Equation (10) then demands that:

$$(\beta'' - \gamma')\, \mathfrak{A}\mathfrak{B} + (\beta - \gamma'')\, \mathfrak{B}\mathfrak{C} + (\beta' - \gamma)\, \mathfrak{C}\mathfrak{A} = 0,$$

and upon multiplying this equation by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, in turn, one will get the three equations:

$$\beta'' - \gamma' = 0, \qquad \beta - \gamma'' = 0, \qquad \beta' - \gamma = 0,$$

resp., which will yield:

$$\Gamma - \mathrm{B} = \beta'' \mathfrak{A}\mathfrak{B} - \beta' \mathfrak{C}\mathfrak{A},$$
$$\mathrm{A} - \Gamma = \beta\ \mathfrak{B}\mathfrak{C} - \beta'' \mathfrak{A}\mathfrak{B},$$
$$\mathrm{B} - \mathrm{A} = \beta'\ \mathfrak{C}\mathfrak{A} - \beta\ \mathfrak{B}\mathfrak{C},$$

and furthermore:

$$\mathrm{A} - \beta \mathfrak{B}\mathfrak{C} = \mathrm{B} - \beta' \mathfrak{C}\mathfrak{A} = \Gamma - \beta'' \mathfrak{A}\mathfrak{B}.$$

Finally, if one denotes the common value of these three second-degree forms by Σ then it will follow that:

$$\mathrm{A} = \Sigma + \beta \mathfrak{B}\mathfrak{C}, \qquad \mathrm{B} = \Sigma + \beta' \mathfrak{C}\mathfrak{A}, \qquad \Gamma = \Sigma + \beta'' \mathfrak{A}\mathfrak{B},$$

and

(11) $$A_1 - A = |\ \omega \mathfrak{A}\Sigma, \qquad B_1 - B = |\ \omega \mathfrak{B}\Sigma, \qquad C_1 - C = |\ \omega \mathfrak{C}\Sigma.$$

There is then a wrench that takes the triangle ABC to the congruent one $A_1B_1C_1$. If there is yet a second wrench Σ' that accomplishes the same thing then one must have:

$$\omega \mathfrak{A}\, (\Sigma' - \Sigma) = \omega \mathfrak{B}\, (\Sigma' - \Sigma) = \omega \mathfrak{C}\, (\Sigma' - \Sigma) = 0.$$

There must then exist three points U, V, W, such that one has:

$$\Sigma' - \Sigma = \mathfrak{A}\, U = \mathfrak{B}\, V = \mathfrak{C}\, W.$$

However, it follows from these equations that:

$$\mathfrak{A}\mathfrak{B}\, U = 0, \qquad \mathfrak{A}\mathfrak{C}\, U = 0;$$

i.e., U is necessarily identical with $\mathfrak{A}$, and therefore:

$$\Sigma' - \Sigma = 0.$$

There is then only one wrench that satisfies equations (11).

§ 8.

If the three points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ lie on a straight line then one will set:

$$\begin{aligned} A &= \mathfrak{A} - a, & A_1 &= \mathfrak{A} + a, \\ B &= \mathfrak{B} - b, & B_1 &= \mathfrak{B} + b, \\ C &= \mathfrak{C} - c, & C_1 &= \mathfrak{C} + c, \end{aligned}$$

where a, b, c are vectors. The following equations must then be true:

$$(12) \qquad \left\{ \begin{aligned} (\mathfrak{B} - \mathfrak{A}) \,|\, (b - a) &= 0, \\ (\mathfrak{C} - \mathfrak{B}) \,|\, (c - b) &= 0, \\ (\mathfrak{A} - \mathfrak{C}) \,|\, (a - c) &= 0. \end{aligned} \right.$$

The three vectors $\mathfrak{B} - \mathfrak{A}$, $\mathfrak{C} - \mathfrak{B}$, $\mathfrak{A} - \mathfrak{C}$ will be parallel, since $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ lie on a straight line. If q is a unit vector that is parallel to them then it will follow from the last three equations that:

$$(13) \qquad a \,|\, q = b \,|\, q = c \,|\, q = \rho .$$

Therefore, if one sets:

$$(14) \qquad \left\{ \begin{aligned} a &= \rho q + a_1, \\ b &= \rho q + b_1, \\ c &= \rho q + c_1 \end{aligned} \right.$$

then one will have:

$$a_1 \,|\, q = b_1 \,|\, q = c_1 \,|\, q = \rho .$$

If one now defines three points by the equations:

$$A' = A + 2\rho\, q, \qquad B' = B + 2\rho\, q, \qquad C' = C + 2\rho\, q$$

then one will get:

$$\begin{aligned} \frac{A_1 + A'}{2} &= \mathfrak{A} + \rho\, q, & A_1 - A' &= 2a_1, \\ \frac{B_1 + B'}{2} &= \mathfrak{B} + \rho\, q, & B_1 - B' &= 2b_1, \\ \frac{C_1 + C'}{2} &= \mathfrak{C} + \rho\, q, & C_1 - C' &= 2c_1 . \end{aligned}$$

The lines A_1A', B_1B', C_1C' will then be met perpendicularly by the line $(\mathfrak{A}, q)$ at their midpoints, when they are not all equal to zero, and therefore $A_1B_1C_1$ will emerge from $A'B'C'$ by rotating through 180° around the axis $(\mathfrak{A}, q)$, and $A_1B_1C_1$ will arise from ABC by a displacement through $2\rho q$ and the rotation that was just referred to. If the three vectors $a_1b_1c_1$ are zero then one will arrive at a displacement; one will thus also have a screwing motion here. However, this is contained in formulas (11) of the previous paragraphs only as a limiting case that corresponds to approximating $\varphi / 2$ by 90°.

The above conclusion is also possible when two of the points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ coincide, since two of equations (12) will then become non-illusory, and the vector q will then be determinate.

However, if $\mathfrak{A} = \mathfrak{B} = \mathfrak{C}$ then one will determine q such that it will be perpendicular to the vector $b - a$ and $c - a$. Since one will then have:

$$(b - a) \mid q = 0, \qquad (c - a) \mid q = 0,$$

equations (13) will be true, along with their further consequences.

q can be indeterminate when the three vectors a, b, c are coplanar. If one takes q to be perpendicular to the plane abc in that case then it will follow that $\rho = 0$, while everything else will remain as before.

Finally, the plane abc can be indeterminate when a, b, c are parallel. If one then determines q such that one has $a \mid q = 0$ then one will have $\rho = 0$, and the previous results will be true.

§ 9.

The congruent triangles ABC and $A_1B_1C_1$ will now have yet a fourth point D and D_1, resp., added to them, which will lie in such a way that:

$$\overline{AD} = \overline{A_1D_1}, \qquad \overline{BD} = \overline{B_1D_1}, \qquad \overline{CD} = \overline{C_1D_1}.$$

Let the middle of DD_1 – i.e., the point $\frac{1}{2}(D + D_1)$ – be denoted by $\mathfrak{D}$. We will next assume that no three of the four points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ lie on a straight line. One can then apply the considerations of § 7 to each of the four triangles ABC, ABD, ACD, BCD, and thus obtain the equations:

$$(15) \qquad \left\{ \begin{array}{lll} A_1 - A = \mid \omega \mathfrak{A}\,\Gamma, & A_1 - A = \mid \omega \mathfrak{A}\,\mathrm{B}, & B_1 - B = \mid \omega \mathfrak{B}\,\mathrm{A}, \\ B_1 - B = \mid \omega \mathfrak{B}\,\Gamma, & C_1 - C = \mid \omega \mathfrak{C}\,\mathrm{B}, & C_1 - C = \mid \omega \mathfrak{C}\,\mathrm{A}, \\ D_1 - D = \mid \omega \mathfrak{D}\,\Gamma, & D_1 - D = \mid \omega \mathfrak{D}\,\mathrm{B}, & D_1 - D = \mid \omega \mathfrak{D}\,\mathrm{A}, \end{array} \right.$$

$$(16) \qquad A_1 - A = \mid \omega \mathfrak{A}\,\Sigma, \qquad B_1 - B = \mid \omega \mathfrak{B}\,\Sigma, \qquad C_1 - C = \mid \omega \mathfrak{C}\,\Sigma,$$

where A, B, Γ, Σ are four second-degree forms.

Comparing the two values of $A_1 - A$ and $B_1 - B$ gives:

$$\omega \mathfrak{A} (\Sigma - \Gamma) = 0, \qquad \omega \mathfrak{B} (\Sigma - \Gamma) = 0,$$

from which, it will follow that:

$$\Sigma - \Gamma = - \nu \mathfrak{A}\mathfrak{B},$$

where ν is a number. Comparing $B_1 - B$ and $C_1 - C$ likewise yields:

$$\Sigma - \mathrm{B} = - \mu \mathfrak{C}\mathfrak{A},$$

and the comparison of $B_1 - B$ and $C_1 - C$ yields:

$$\Sigma - \mathrm{A} = - \lambda \mathfrak{B}\mathfrak{C} .$$

Therefore:

$$\begin{aligned} D_1 - D &= | \omega \mathfrak{D} \Sigma + \lambda | \omega \mathfrak{D}\mathfrak{B}\mathfrak{C}, \\ &= | \omega \mathfrak{D} \Sigma + \mu | \omega \mathfrak{D}\mathfrak{C}\mathfrak{A}, \\ &= | \omega \mathfrak{D} \Sigma + \nu | \omega \mathfrak{D}\mathfrak{A}\mathfrak{B} . \end{aligned}$$

Since $\omega \mathfrak{D}\mathfrak{B}\mathfrak{C} = (\mathfrak{B} - \mathfrak{C}) (\mathfrak{C} - \mathfrak{D}) = \mathfrak{B}\mathfrak{C} + \mathfrak{C}\mathfrak{D} + \mathfrak{D}\mathfrak{B}$, in order for the three expressions above to be equal, one must have:

$$\begin{aligned} \lambda (\mathfrak{D}\mathfrak{B} + \mathfrak{B}\mathfrak{C} + \mathfrak{C}\mathfrak{D}) &= \mu (\mathfrak{D}\mathfrak{C} + \mathfrak{C}\mathfrak{A} + \mathfrak{A}\mathfrak{D}) \\ &= \nu (\mathfrak{D}\mathfrak{A} + \mathfrak{A}\mathfrak{B} + \mathfrak{B}\mathfrak{D}). \end{aligned}$$

If one then multiplies this by $\mathfrak{D}$ then it will follow that:

$$\lambda \mathfrak{B}\mathfrak{C}\mathfrak{D} = \mu \mathfrak{C}\mathfrak{A}\mathfrak{D} = \nu \mathfrak{A}\mathfrak{B}\mathfrak{D}, \tag{17}$$

and the further multiplication by $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ will give:

$$\lambda \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D} = \mu \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D} = \nu \mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D} = 0.$$

These equations can be fulfilled in two ways: In one case, $\mathfrak{A}\mathfrak{B}\mathfrak{C}\mathfrak{D} \neq 0$, and $\lambda = \mu = \nu = 0$, and one must add to equations (16):

$$D_1 - D = | \omega \mathfrak{D} \Sigma. \tag{16*}$$

The tetrahedron *ABCD* will thus be taken to $A_1B_1C_1D$ by the wrench Σ.

In the other case, $\mathfrak{ABCD} = 0$, and the four points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ lie in a plane. One can then determine the number ρ such that $\rho\,\mathfrak{ABC}$ is equal to the three products in (17) that are set equal to each other, and one will then get:

$$D_1 - D = |\ \omega\mathfrak{D}\ \Sigma + \rho\ |\ \omega\mathfrak{ABC}\ . \tag{16**}$$

§ 10.

The tetrahedron $A_1B_1C_1D_1$ was assumed to be such that the six separations of the four edges were equal to the corresponding lengths of the edges of the tetrahedron $ABCD$. Therefore, the two figures are either congruent or symmetric. Since taking equations (16) to equations (16*) can be performed by a wrench Σ, the two tetrahedra will be congruent, and therefore if they are symmetric then equations (16) and (16**) must be true. However, they assume that the four points $\mathfrak{ABCD}$ lie in a plane.

We thus have the theorem:

If two tetrahedral are symmetrically equal then the midpoints of the connecting lines of their corresponding edges will lie in one and the same plane, which might be called the middle plane.

One can show that the two tetrahedra are congruent in the first case when one proves the equation:

$$ABCD = A_1B_1C_1D_1\ .$$

In fact, if we set, as in § 8:

$$\begin{aligned} A_1 &= \mathfrak{A} + a, & A &= \mathfrak{A} - a,\\ B_1 &= \mathfrak{B} + b, & B &= \mathfrak{B} - b,\\ C_1 &= \mathfrak{C} + c, & C &= \mathfrak{C} - c,\\ D_1 &= \mathfrak{D} + d, & D &= \mathfrak{D} - d \end{aligned}$$

then it will follow that:

$$\left\{ \begin{aligned} A_1B_1C_1D_1 - ABCD &= 2[\mathfrak{ABCD}d - \mathfrak{ABD}c + \mathfrak{ACD}b - \mathfrak{BCD}a] \\ &\quad + 2[\mathfrak{A}bcd - \mathfrak{B}acd + \mathfrak{C}abd - \mathfrak{D}abc]. \end{aligned} \right. \tag{18}$$

Since a product of four vectors is zero, one will have:

$$\mathfrak{B}acd = \mathfrak{B}acd + (\mathfrak{A} - \mathfrak{B})\ acd = \mathfrak{A}acd, \qquad \text{etc.,}$$

and therefore the second bracket will be:

$$\begin{aligned} &= \mathfrak{A}\ (bcd - acd + abd - abc) \\ &= \mathfrak{A}\ (b - a)\ (c - a)\ (d - a). \end{aligned}$$

Now, if u, v, w, t are four vectors then one will have:

$$(|\, ut)\,(|\, vt)\,(|\, wt) = 0,$$

since the three factors will be vectors that are perpendicular to t, and thus coplanar.

On the other hand, in order to give a different proof, let $|\, vt = v'$, $|\, wt = w'$, so one has:

$$v'\, w' \,|\, u\, t = (v' \,|\, u)\,(w' \,|\, t) - (v' \,|\, t)(w' \,|\, u).$$

However, $v' \,|\, t = t \,|\, v' = tvt$ is equal to 0, and likewise $w' \,|\, t$, so one will have:

$$v'\, w' \,|\, ut = 0,$$

and this is the equation to be proved.

However, one has:

$$2\,(b - a) = |\, \omega(\mathfrak{B} - \mathfrak{A})\, \Sigma.$$

Therefore, if one brings the wrench Σ into the form $Pe + fg$, where e, f, g are vectors, with the help of an arbitrary point then it will follow that:

$$\omega(\mathfrak{B} - \mathfrak{A})\, \Sigma = e\,(\mathfrak{B} - \mathfrak{A}),$$

so

$$b - a = \tfrac{1}{2} |\, (\mathfrak{A} - \mathfrak{B})\, e,$$
$$c - a = \tfrac{1}{2} |\, (\mathfrak{A} - \mathfrak{C})\, e,$$
$$d - a = \tfrac{1}{2} |\, (\mathfrak{A} - \mathfrak{D})\, e,$$

such that from the formula that was just proved, one will have:

$$(b - a)(c - a)(d - a) = 0.$$

One can write:

$$\mathfrak{ABC}\,(d - a) - \mathfrak{ABD}\,(c - a) + \mathfrak{ACD}\,(b - a) + a\,[\mathfrak{ABC} - \mathfrak{ABD} + \mathfrak{ACD} - \mathfrak{BCD}]$$

for the first bracket in equation (18).

The coefficient of a is equal to $(\mathfrak{A} - \mathfrak{D})(\mathfrak{B} - \mathfrak{D})(\mathfrak{C} - \mathfrak{D})$, so the second summand will be equal to zero, since it is the product of four vectors.

However, the first summand will be equal to:

$$\tfrac{1}{2}[\mathfrak{ABC} \,|\, (\mathfrak{A} - \mathfrak{D})\, e - \mathfrak{ABD} \,|\, (\mathfrak{A} - \mathfrak{C})\, e + \mathfrak{ACD} \,|\, (\mathfrak{A} - \mathfrak{B})\, e],$$

and when one introduces $\mathfrak{ABC} = \mathfrak{A}\,(\mathfrak{B} - \mathfrak{A})(\mathfrak{C} - \mathfrak{A})$, etc., this will become:

$$= \frac{\mathfrak{A}}{2}\,[(\mathfrak{B}-\mathfrak{A})(\mathfrak{C}-\mathfrak{A}) \mid (\mathfrak{A}-\mathfrak{D})\,e - (\mathfrak{B}-\mathfrak{A})\,(\mathfrak{A}-\mathfrak{D}) \mid (\mathfrak{A}-\mathfrak{C})\,e$$
$$+ (\mathfrak{C}-\mathfrak{A})(\mathfrak{D}-\mathfrak{A}) \mid (\mathfrak{A}-\mathfrak{B})\,e]\,.$$

Now, if u, v, w, t are once more four arbitrary vectors then one will have:

$$\begin{aligned} uv \mid wt &= (u \mid w)(v \mid t) - (u \mid t)(v \mid w), \\ vw \mid ut &= (v \mid u)(w \mid t) - (v \mid t)(w \mid u), \\ wu \mid vt &= (w \mid v)(u \mid t) - (w \mid t)(u \mid v), \end{aligned}$$

so by addition, one will get (*):

$$uv \mid wt + vw \mid ut + wu \mid vt = 0.$$

As a result of this equation, the coefficient of $\frac{1}{2}\mathfrak{A}$ will be equal to zero, and therefore $ABCD = A_1B_1C_1D_1$.

§ 11.

One can also prove the theorem about symmetric tetrahedra in yet another way. We imagine that a perpendicular has been dropped from D to the plane ABC, whose base point in this plane is the point:

$$A + \beta(B-A) + \gamma(C-A),$$

where β and γ are numbers. Since $\mid \omega ABC$ is a vector that is perpendicular to the plane ABC, one can set:

(19) $$D = A + \beta(B-A) + \gamma(C-A) + \lambda \mid \omega ABC\,.$$

One will then have:

$$D_1 = A_1 + \beta(B_1-A_1) + \gamma(C_1-A_1) + \lambda \mid \omega A_1B_1C_1$$

for the congruent tetrahedron.

Therefore, one will have:

(19*) $$D_1 = A_1 + \beta(B_1-A_1) + \gamma(C_1-A_1) + \lambda \mid \mathfrak{a}\, A_1B_1C_1\,,$$

for the symmetric case, so:

(20) $$\mathfrak{D} = \mathfrak{A} + \beta(\mathfrak{B}-\mathfrak{A}) + \gamma(\mathfrak{C}-\mathfrak{A}) + \lambda \mid \omega(A_1B_1C_1 - ABC).$$

However, with the notations of the previous paragraph, one has:

$$A_1B_1C_1 - ABC = 2\,[\mathfrak{A}\mathfrak{B}c + \mathfrak{B}\mathfrak{C}a + \mathfrak{C}\mathfrak{A}b + abc],$$

(*) *Grassmann*, *loc. cit.*, no. 185.

so, since $\omega abc = 0$, one will have:

$$\begin{aligned}\omega(A_1B_1C_1 - ABC) &= 2\omega[\mathfrak{A}\mathfrak{B}c + \mathfrak{B}\mathfrak{C}a + \mathfrak{C}\mathfrak{A}b] \\ &= 2\,[(\mathfrak{B} - \mathfrak{A})\,c + (\mathfrak{C} - \mathfrak{B})\,a + (\mathfrak{A} - \mathfrak{C})\,b] \\ &= 2\,[\mathfrak{A}\,(b - c) + \mathfrak{B}\,(c - a) + \mathfrak{C}\,(a - b)] \\ &= \mathfrak{A} \mid (\mathfrak{C} - \mathfrak{B})\,e + \mathfrak{B} \mid (\mathfrak{A} - \mathfrak{C})\,e + \mathfrak{C} \mid (\mathfrak{B} - \mathfrak{A})\,e.\end{aligned}$$

In order to be able to extend this, one must also represent the right-hand side, which is a sum of bivectors, as such. To that end, we convert it into:

$$(\mathfrak{A} - \mathfrak{C}) \mid (\mathfrak{C} - \mathfrak{B})\,e + (\mathfrak{B} - \mathfrak{C}) \mid (\mathfrak{A} - \mathfrak{C})\,e + \mathfrak{C}\,[\mid(\mathfrak{C} - \mathfrak{B})\,e + \mid (\mathfrak{A} - \mathfrak{C})\,e + \mid (\mathfrak{B} - \mathfrak{A})\,e],$$

which is equal to the first sum, since the bracket vanishes. It then follows from this that:

$$\begin{aligned}\mid \omega(A_1B_1C_1 - ABC) &= \mid [(\mathfrak{A} - \mathfrak{C}) \mid (\mathfrak{C} - \mathfrak{B})\,e] + \mid [(\mathfrak{B} - \mathfrak{C}) \mid (\mathfrak{A} - \mathfrak{C})\,e] \\ &= [(\mathfrak{A} - \mathfrak{C}) \mid e]\,(\mathfrak{C} - \mathfrak{B}) - [(\mathfrak{A} - \mathfrak{C}) \mid (\mathfrak{C} - \mathfrak{B})]\,e \\ &\quad + [[(\mathfrak{B} - \mathfrak{C}) \mid e]\,(\mathfrak{A} - \mathfrak{C}) - [(\mathfrak{A} - \mathfrak{C}) \mid (\mathfrak{B} - \mathfrak{C})]\,e \\ &= \mathfrak{A}\,[(\mathfrak{B} - \mathfrak{C}) \mid e] + \mathfrak{B}\,[(\mathfrak{C} - \mathfrak{A}) \mid e] + \mathfrak{C}\,[(\mathfrak{A} - \mathfrak{B}) \mid e].\end{aligned}$$

However, this expression represents a vector that is contained in the plane $\mathfrak{A}\mathfrak{B}\mathfrak{C}$; if one adds – 1 / 2 times it to the point:

$$\mathfrak{A} + \beta(\mathfrak{B} - \mathfrak{A}) + \gamma(\mathfrak{C} - \mathfrak{A})$$

in the plane $\mathfrak{A}\mathfrak{B}\mathfrak{C}$, using the prescription of equation (20), then a point in that plane will again arise in $\mathfrak{D}$.

§ 12.

If the two symmetric tetrahedra $ABCD$ and $A_1B_1C_1D_1$ are given then the wrench will be determined completely, so the numbers λ, μ, ν and ρ in (17) and (16**) will be, as well. However, if one assumes that Σ and ρ are arbitrary and then calculates $A_1B_1C_1D_1$ from equations (16) and (16**) then the triangle $A_1B_1C_1$ will be congruent to ABC. Furthermore, one will have:

$$D_1 - D - (A_1 - A) = \mid \omega(\mathfrak{D} - \mathfrak{A})\,\Sigma + \rho \mid \omega\mathfrak{A}\mathfrak{B}\mathfrak{C},$$

so

$$\begin{aligned}(D_1 - D)^2 - (D - A)^2 &= 2\,(\mathfrak{D} - \mathfrak{A}) \mid (D_1 - D - A_1 + A) \\ &= 2\rho\,(\mathfrak{D} - \mathfrak{A})\;\omega\mathfrak{A}\mathfrak{B}\mathfrak{C}, \\ &= 2\rho\,(\mathfrak{D} - \mathfrak{A})\,(\mathfrak{B} - \mathfrak{A})\,(\mathfrak{C} - \mathfrak{A}).\end{aligned}$$

Therefore, one will have:

$$\overline{D_1A_1} = \overline{DA} \quad \text{and likewise} \quad \overline{D_1B_1} = \overline{DB}, \quad \overline{D_1C_1} = \overline{DC},$$

only when either $\rho = 0$ or the four points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ lie in a plane. Only then will one get a tetrahedron that is congruent or symmetric to $\mathfrak{ABCD}$ as a result.

Now, in the latter case, this will yield:

$$A_1B_1C_1D_1 + ABCD = 2\,[\mathfrak{ABCD} + abcd + F],$$

where

$$F = \mathfrak{AB}cd - \mathfrak{AC}bd + \mathfrak{AD}bc + \mathfrak{BC}ad - \mathfrak{B}\mathfrak{d}ac + \mathfrak{CD}ab\,.$$

However, since $abcd$ vanishes (as the product of four vectors) and $\mathfrak{ABCD}$ vanishes because the four points lie in a plane, only F needs be calculated.

It is equal to:

$$\begin{aligned}
&\mathfrak{AB}\,(c-a)(d-a) - \mathfrak{AB}\,(b-a)(d-a) + \mathfrak{AD}\,(b-a)(c-a)\\
&\qquad + \mathfrak{AB}\,(ad-ac) - \mathfrak{AC}\,(ad-ab) + \mathfrak{AD}\,(ac-ab)\\
&\qquad + \mathfrak{BC}\,ad - \mathfrak{BD}\,ac + \mathfrak{CD}\,ab.
\end{aligned}$$

The last six summands, when combined differently, yield:

$$\begin{aligned}
&(\mathfrak{AB} + \mathfrak{BC} + \mathfrak{CA})\,ad + (\mathfrak{BA} + \mathfrak{AD} + \mathfrak{DB})\,ac + (\mathfrak{AC} + \mathfrak{CD} + \mathfrak{DA})\,ab\\
&= (\mathfrak{A}-\mathfrak{C})(\mathfrak{B}-\mathfrak{C})\,ad + (\mathfrak{B}-\mathfrak{D})(\mathfrak{A}-\mathfrak{D})\,ac + (\mathfrak{A}-\mathfrak{D})(\mathfrak{C}-\mathfrak{D})\,ab = 0,
\end{aligned}$$

since each summand will vanish by itself as the product of four vectors.

If one denotes the three vectors $\mathfrak{A}-\mathfrak{B}$, $\mathfrak{A}-\mathfrak{C}$, and $\mathfrak{A}-\mathfrak{D}$ by u, v, w then the first part of F will be equal:

$$\tfrac{1}{4}\,\mathfrak{A}\,\{u \mid ve\,(\mid we + \rho \mid uv) - v \mid ue\,(\mid we + \rho \mid uv) + w \mid ue \mid ve\}.$$

The content of the bracket is $\rho T_1 + T_2$, where:

$$\begin{aligned}
T_1 &= u \mid ve \mid uv - v \mid ue \mid uv,\\
T_2 &= u \mid ve \mid we - v \mid ue \mid we + w \mid ue \mid ve\,.
\end{aligned}$$

However, one has:

$$\begin{aligned}
u \mid ve \mid wt &= wt \mid (u \mid ve)\\
&= wt\,[v\,(u \mid e) - e\,(u \mid v)],
\end{aligned}$$

so

$$T_1 = -\,uve\,(u \mid v) - vue\,(v \mid u) = 0.$$

When one applies the same conversion to T_2, one will find that:

$$T_2 = e\,[vw\,(u \mid e) + wu\,(v \mid e) + uv\,(w \mid e)].$$

The bracket does not vanish for arbitrary vectors u, v, w. However, here, these three vectors are coplanar, so one can set:

$$w = \mu\, u + \nu\, v,$$

which yields:

$$T_2 = e\,[\mu\, vu\,(u \mid e) + \nu\, vu\,(v \mid e) + \mu\, uv\,(u \mid e) + \nu\, uv\,(v \mid e)] = 0.$$

With that, one has shown that:

$$A_1B_1C_1D_1 = -\,ABCD \tag{21}$$

will be true for any arbitrary value of ρ, only when the four points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ lie in a plane, such that the tetrahedra will then be symmetric.

§ 13.

If, as was assumed, $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ lie in a plane then one can set:

$$\mathfrak{D} = \mathfrak{A} + \mu\,(\mathfrak{B} - \mathfrak{A}) + \nu\,(\mathfrak{C} - \mathfrak{A})\;; \tag{22}$$

equations (16) and (16**) will then yield:

$$D_1 - D = (1 - \mu - \nu)\,(A_1 - A) + \mu\,(B_1 - B) + \nu\,(C_1 - C) + \rho \mid \omega\mathfrak{A}\mathfrak{B}\mathfrak{C},$$

so the vector:

$$D_1 - (1 - \mu - \nu)\,A_1 - \mu\,B_1 - \nu\,C_1$$

will be equal to:

$$D_1 - (1 - \mu - \nu)\,A - \mu\,B - \nu\,C + \rho \mid \omega\mathfrak{A}\mathfrak{B}\mathfrak{C}\;.$$

If this equals x then equation (22) will yield:

$$x = \frac{\rho}{2}\,\omega\mathfrak{A}\mathfrak{B}\mathfrak{C},$$

and it will then follow that:

$$\left\{\begin{aligned} D &= (1-\mu-\nu)A + \mu B + \nu C - \frac{\rho}{2}\,\omega\mathfrak{A}\mathfrak{B}\mathfrak{C}, \\ D_1 &= (1-\mu-\nu)A_1 + \mu B_1 + \nu C_1 + \frac{\rho}{2}\,\omega\mathfrak{A}\mathfrak{B}\mathfrak{C}, \end{aligned}\right. \tag{23}$$

If $\rho = 0$ then D will be a point of the plane ABC, D_1 will be a point of the plane $A_1B_1C_1$ that corresponds to it under a congruent transformation, and the two tetrahedra $ABCD$, $A_1B_1C_1D_1$ will then be zero.

Equations (23) imply the theorem:

The perpendiculars that one drops from the vertices D and D_1 to the middle plane are equal in length and meet the triangles ABC and $A_1B_1C_1$, resp., at corresponding points.

If one substitutes the value of:

$$|\,\omega(A_1B_1C_1 - ABC)$$

that was found in § 11 in formula (20) then it will follow that:

$$\mathfrak{D} = \mathfrak{A} + \beta(\mathfrak{B} - \mathfrak{A}) + \gamma(\mathfrak{C} - \mathfrak{A}) - \tfrac{1}{2}[\mathfrak{A}\,(\mathfrak{B} - \mathfrak{C} \mid e) + \mathfrak{B}\,(\mathfrak{C} - \mathfrak{A} \mid e) + \mathfrak{C}\,(\mathfrak{A} - \mathfrak{B} \mid e)].$$

Comparing this with (22) then yields:

$$\begin{aligned} 1 - \mu - \nu &= 1 - \beta - \gamma = \tfrac{1}{2}(\mathfrak{B} - \mathfrak{C} \mid e), \\ \mu &= \beta - \tfrac{1}{2}(\mathfrak{C} - \mathfrak{A} \mid e), \\ \nu &= \gamma - \tfrac{1}{2}(\mathfrak{A} - \mathfrak{B} \mid e), \end{aligned}$$

and thus, from (19) and (19*):

$$D = A + \left(\mu + \tfrac{1}{2}(\mathfrak{C} - \mathfrak{A} \mid e)\right)(B - A) + \left(\nu + \tfrac{1}{2}(\mathfrak{A} - \mathfrak{B} \mid e)\right)(C - A) + \lambda \mid \omega ABC.$$

One gets a similar expression for D_1; it arises from the foregoing one when one replaces A, B, C with A_1, B_1, C_1, resp., and $\lambda \mid \omega ABC$ with $-\lambda \mid \omega ABC$. We have similar expressions in (23); we shall now prove that the two agree. If one writes the expression above as:

$$(24)\qquad \left\{ \begin{aligned} D &= (1-\mu-\nu)A + \mu B + \nu C + X, \\ D_1 &= (1-\mu-\nu)A_1 + \mu B_1 + \nu C_1 + Y, \end{aligned} \right.$$

then one will get:

$$\begin{aligned} X + Y &= \lambda \mid \omega(ABC - A_1B_1C_1) + \lambda\,(\mathfrak{C} - \mathfrak{A} \mid e)(\mathfrak{B} - \mathfrak{A}) + \lambda\,(\mathfrak{A} - \mathfrak{B} \mid e)(\mathfrak{C} - \mathfrak{A}) \\ &= \lambda\,[\mid \omega(ABC - A_1B_1C_1) + (\mathfrak{B} - \mathfrak{C} \mid e)\,\mathfrak{A} + (\mathfrak{C} - \mathfrak{A} \mid e)\,\mathfrak{B} + (\mathfrak{A} - \mathfrak{B} \mid e)(\mathfrak{C}] = 0, \end{aligned}$$

as was shown above in § 11.

By contrast, one has:

$$\begin{aligned} X - Y &= -\lambda\,(\mathfrak{C} - \mathfrak{A} \mid e)(b - a) - \lambda\,(\mathfrak{A} - \mathfrak{B} \mid e)(c - a) + \lambda \mid \omega(ABC + A_1B_1C_1) \\ &= \lambda \mid \omega(ABC + A_1B_1C_1) - \tfrac{1}{2}\lambda\,(\mathfrak{C} - \mathfrak{A} \mid e) \mid (\mathfrak{A} - \mathfrak{B})\, e - \tfrac{1}{2}\lambda\,(\mathfrak{C} - \mathfrak{A} \mid e) \mid (\mathfrak{A} - \mathfrak{C})\, e. \end{aligned}$$

It follows further that:

$$\begin{aligned} A_1B_1C_1 + ABC &= 2\mathfrak{A}\mathfrak{B}\mathfrak{C} + 2\mathfrak{A}bc + 2\mathfrak{B}ca + 2\mathfrak{C}ab, \\ \omega(A_1B_1C_1 + ABC) &= 2\,\omega\mathfrak{A}\mathfrak{B}\mathfrak{C} + 2\,(bc + ca + ab), \\ &= 2\,\omega\mathfrak{A}\mathfrak{B}\mathfrak{C} + 2\,(b - a)\,(c - a), \end{aligned}$$

so

$$X - Y = 2\lambda \mid \omega\mathfrak{A}\mathfrak{B}\mathfrak{C} + \tfrac{1}{2} \mid [\mid (\mathfrak{A} - \mathfrak{B})\, e \cdot \mid (\mathfrak{A} - \mathfrak{C})\, e] - \tfrac{1}{2} \mid (\mathfrak{C} - \mathfrak{A} \mid e) \mid (\mathfrak{A} - \mathfrak{B})\, e$$
$$- \tfrac{1}{2} (\mathfrak{A} - \mathfrak{B} \mid e) \mid (\mathfrak{A} - \mathfrak{C})\, e\,.$$

If one sets:

$$\mathfrak{B} - \mathfrak{A} = u, \qquad \mathfrak{C} - \mathfrak{A} = v$$

then when one multiplies the right-hand side above by $\lambda / 2$, it will become:

$$\mid (\mid ue \cdot \mid ve) + (v \mid e) \mid ue - (u \mid e) \mid ve = U.$$

When one replaces $\mid ue$ with w in:

$$w \mid ve = (w \mid e) \mid v - (w \mid v) \mid e,$$

it will follow that:

$$\mid ue \cdot \mid ve = -(vue) \mid e = (euv) \mid e.$$

On the other hand, it follows from the fact that $\mid (e \mid uv) = u\,(v \mid e) - v\,(u \mid e)$ that:

$$- \mid e\, [\mid e\, (\mid uv)] = (v \mid e) \mid ue - (u \mid e) \mid ve.$$

However, the left-hand side is equal to $e\,(euv) + \mid uv \cdot (e \mid e)$. Therefore:

$$U = (euv)\, e - (euv)\, e + (e \mid e) \mid uv = (e \mid e) \mid uv,$$

so

$$X - Y = 2\lambda \mid \omega\mathfrak{A}\mathfrak{B}\mathfrak{C} + \tfrac{1}{2}\, (e \mid e) \mid (\mathfrak{B} - \mathfrak{A})(\mathfrak{C} - \mathfrak{A}),$$

or, since $\omega\mathfrak{A}\mathfrak{B}\mathfrak{C} = (\mathfrak{B} - \mathfrak{A})(\mathfrak{C} - \mathfrak{A})$, one will have:

$$X - Y = \frac{\lambda}{2}(4 + \varepsilon^2) \mid \omega\mathfrak{A}\mathfrak{B}\mathfrak{C},$$

if one denotes the length of e by ε. One will then have:

$$X = -Y = \frac{\lambda(4+\varepsilon^2)}{4} \mid \omega\mathfrak{A}\mathfrak{B}\mathfrak{C},$$

and comparing (23) with (24) will give:

$$\rho = -\frac{\lambda(4+\varepsilon^2)}{4}\,.$$

If the aforementioned midpoints of two congruent tetrahedra lie in a plane, so one has:

$$\mathfrak{D} = \mathfrak{A} + \mu\,(\mathfrak{B} - \mathfrak{A}) + \nu\,(\mathfrak{C} - \mathfrak{A}),$$

then equations (16) and (16*) will yield:

$$D_1 - D = A_1 - A + \mu\,(B_1 - B - A_1 + A) + \nu\,(C_1 - C - A_1 + A),$$

and both of them together will imply that:

$$\begin{aligned} D\;\; &= A\;\; + \mu\,(B\; - \;A) + \nu\,(C\; - \;A), \\ D_1 &= A_1 + \mu\,(B_1 - A_1) + \nu\,(C_1 - A_1), \end{aligned}$$

which say that A, B, C, D lie in a plane.

§ 14.

The considerations of §§ 3 and 4 show that when Σ is a second-degree form, the formula:

(25) $$P_1 - P = |\; \omega\,\frac{P_1 + P}{2}\,\Sigma$$

will represent a congruent transformation of space. In order to derive an expression for P_1 in terms of P from it, we assume that Σ has been put into the normal form:

$$\Sigma = Ua + \lambda \mid e,$$

where e is a unit vector that is parallel to the vector a, and λ is a number. Since:

$$\omega\left(\frac{P_1 + P}{2} \mid e\right) = \mid e,$$

one will then have:

$$P_1 - P = \lambda e + \tfrac{1}{2} \mid \omega PUa + \tfrac{1}{2} \mid \omega P_1 Ua$$

or

(26) $$P_1 - P = P - U + \lambda e + \tfrac{1}{2} \mid (P - U) + \tfrac{1}{2} \mid (P_1 - U)\, a.$$

It will then follow from this that:

$$\begin{aligned} a \mid (P_1 - U) &= a \mid (P - U) + \lambda a \mid e, \\ \mid a\,(P_1 - U) &= \mid a\,(P - U) - \tfrac{1}{2}\{(P - U)\,(a \mid a) - a\,[a \mid (P - U)]\} \\ &\quad - \tfrac{1}{2}\{(P_1 - U)\,(a \mid a) - a\,[a \mid (P_1 - U)]\}. \end{aligned}$$

If one substitutes this into (26) then one will get:

$$\begin{aligned} P_1 - U = P - U + \lambda e\; &- \mid (P - U)\, a - \tfrac{1}{4}(P - U)\,(a \mid a) + \tfrac{1}{4} a\,[a \mid (P - U)] \\ &- \tfrac{1}{4}(P_1 - U)(a \mid a) + \tfrac{1}{4} a\,\{[a \mid (P - U)] + \lambda\,(a \mid e)\}. \end{aligned}$$

If one sets the length of the vector a equal to 2 tan φ / 2, such that one has:

$$a = 2\tan\frac{\varphi}{2}\cdot e,$$

then one will get:

$$\begin{aligned}P_1 - U = P - U + \lambda e \; - 2\tan\frac{\varphi}{2}\,|\,(P-U)\,e - \tan^2\frac{\varphi}{2}(P-U)\\ + \tan^2\frac{\varphi}{2}\,e\,(e\,|\,P-U) - \tan^2\frac{\varphi}{2}(P_1-U)\\ + \tan^2\frac{\varphi}{2}\,e\,\{[e\,|\,(P-U)] + \lambda\},\end{aligned}$$

from which, it will follow that:

$$P_1 - P = (P-U)\cos\varphi + \lambda e + 2\sin^2\frac{\varphi}{2}\,e\,(e\,|\,P-U) - \sin\varphi\,|\,(P-U)\,e. \tag{27}$$

If one sets tan φ / 2 = μ in this equation and goes over to orthogonal coordinates then one will obtain the well-known *Euler* formulas for the transformation of these coordinates from them.

While formula (25) loses its validity for φ = 180°, this is not the case for (27). Moreover, it gives:

$$\frac{P+P_1}{2} = U + \tfrac{1}{2}e + e\,(e\,|\,P-U)$$

which is a point in the axis (U, e), and since:

$$(P_1 - P)\,|\,e = -\,2\,(P-U)\,|\,e + \lambda\,(e\,|\,e) + (e\,|\,e)\,[e\,|\,P-U] = \frac{\lambda}{2},$$

the projection of $P_1 - P$ onto the axis will be constant, as it must be for a rotation through 180°.

In formula (27), one can write:

$$(P-U)\,e = \omega UPe,$$

$$U + (P-U)\cos\varphi + 2\sin^2 e\,(e\,|\,P-U)$$

$$\begin{aligned}&= P\left(1-2\sin^2\frac{\varphi}{2}\right) + 2\sin^2\frac{\varphi}{2}\,U + 2\sin^2\frac{\varphi}{2}\,e(e\,|\,P-U)\\ &= P + 2\sin^2\frac{\varphi}{2}\,[e\,(e\,|\,P-U) - (e\,|\,e)(P-U)]\end{aligned}$$

$$= P + 2\sin^2\frac{\varphi}{2}\,[e\,(e \mid P - U)]\,.$$

It then follows from this that:

$$P_1 = P - 2\sin^2\frac{\varphi}{2} \mid e\,(\mid \omega UP) - \sin\varphi \mid \omega UP\, e + \lambda e, \tag{28}$$

or, when one denotes the second-degree form Ue by Π, since $e = \omega\Pi$, that:

$$P_1 = P + 2\sin^2\frac{\varphi}{2} \mid (\mid \omega\Pi \cdot \mid \omega P\Pi) + \sin\varphi \mid \omega P\Pi + \lambda\omega\Pi\,.$$

If one now considers the general formula:

$$P_1' = P + \mid (\omega\Sigma' \cdot \mid \omega P\,\Sigma') + \tau \mid \omega P\,\Sigma', \tag{29}$$

where τ is a numerical coefficient, and Σ' is a second-degree form whose normal form is:

$$\xi\, U\, e + \eta \mid e,$$

where ξ and η are numbers, such that one can assume that ξ is positive, then one will get:

$$\begin{aligned} P_1' &= P + \mid (\xi e\,(\xi\omega PUe + \eta e) + \tau\,(\xi\omega PUe + \eta \mid e) \\ &= P + \xi^2 \mid e\,(\mid \omega PUe) + \tau\xi \mid \omega PUe + \tau\eta e\,. \end{aligned}$$

The right-hand side will be identical with (28) when:

$$\xi^2 = 2\sin^2\frac{\varphi}{2}, \qquad \tau\xi = \sin\varphi, \qquad \tau\eta = \lambda.$$

One can thus determine the wrench Σ' and the number τ such that the same congruent transformation is represented by formula (29) as the one that is represented by formula (28). It will follow that:

$$\xi = \sqrt{2}\cdot\sin\frac{\varphi}{2}, \qquad \tau = \sqrt{2}\cdot\cos\frac{\varphi}{2}, \qquad \eta = \frac{1}{\sqrt{2}}\sec\frac{\varphi}{2}\,.$$

(29) will then represent a screwing motion when $\xi^2 + \tau^2 = 2$.

§ 15.

If we understand Σ to mean an arbitrary wrench then the problem suggests itself of considering the space transformation that is given by:

$$P_1 = P + \rho \mid (\omega\Sigma \cdot \mid \omega P\Sigma) + \sigma \mid \omega P\Sigma, \tag{30}$$

where ρ and σ are numbers.

If we replace the point P and the one P_1 that corresponds to it with another pair of associated points QQ_1 then it will follow that:

$$Q_1 - P_1 = Q - P + \rho \mid \omega\Sigma \cdot \mid \omega(Q - P) + \sigma \mid \omega(Q - P)\,\Sigma.$$

Now let Σ have the normal form:

$$\Sigma = \xi\, U\, e + \eta \mid e,$$

as was assumed above, so one will get:

$$\omega(Q - P)\,\Sigma = \xi\,(P - Q)\,e, \qquad \omega\Sigma = \xi\, e;$$

thus:

$$\begin{aligned} Q_1 - P_1 &= Q - P + \xi^2\,\rho \mid e\,[\mid (P - Q)\,e] + \sigma\xi \mid (P - Q)\,e \\ &= Q - P + \xi^2\rho\,[P - Q - e\,(P - Q) \mid e] + \sigma\xi \mid (P - Q)\,e \\ \mid (Q_1 - P_1) &= \mid (Q - P)(1 - \xi^2\rho) - \xi^2\rho\,(P - Q \mid e) \mid e + \sigma\xi\,(P - Q)\,e. \end{aligned}$$

The multiplication of both expressions then gives:

$$\begin{aligned} (Q_1 - P_1) \mid (Q_1 - P_1) &= (Q_1 - P_1)^2 \\ &= (1 - \xi^2\rho)(Q - P)^2 + \xi^2\rho\,(1 - \xi^2\rho)(P - Q \mid e)^2 + \xi^2\rho\,(1 - \xi^2\rho)(P - Q \mid e)^2 \\ &\qquad + \xi^4\rho^2\,(P - Q \mid e)^2 - \sigma^2\xi^2\,[(P - Q)^2 - (P - Q \mid e)^2] \\ &= [(1 - \xi^2\rho)^2 + \xi^2\sigma^2]\,(Q - P)^2 + [2\,\xi^2\rho - \sigma^2\xi^2 - \xi^4\rho^2]\,(P - Q) \mid e)^2. \end{aligned}$$

If one sets:

$$1 - \xi^2\rho = \lambda \cos\kappa; \qquad \xi\sigma = \lambda \sin\kappa$$

then one can write this as:

$$(Q_1 - P_1)^2 = (Q - P)^2 \cdot \lambda^2 + (1 - \lambda^2)(P - Q \mid e)^2.$$

Formula (30) then represents a congruent transformation only when $\lambda = 1$. However, since:

$$e \mid (Q_1 - P_1) = (1 - \xi^2\rho)\,(e \mid Q - P) - \xi^2\rho\,(P - Q \mid e)$$

or

$$(Q_1 - P_1) \mid e = (P - Q) \mid e, \tag{31}$$

the last equation can be written as:

$$(Q_1 - P_1)^2 - (P_1 - Q_1 \mid e)^2 = \lambda^2\,[(Q - P)^2 - (Q - P \mid e)^2]. \tag{32}$$

$(Q_1 - P_1)^2 - (P_1 - Q_1 \mid e)$ is the square of the projection of the segment PQ onto a plane that is perpendicular to the vector e. Therefore, equation (32) says that the component of P_1Q_1 that is perpendicular to e is λ times as long as the component of PQ that is perpendicular to e, while equation (31) shows that the components of these segments that are parallel to the e are equal. One can then say that equation (30) represents an affine space transformation, under which the dimensions that are parallel to e remain unchanged, but the ones that are perpendicular to e will be extended by a factor of λ, while a screw around an axis that is parallel to e will be present.

§ 16.

In § 9, we assumed that no three of the four points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ were on a straight line.

However, if the three points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ do lie on a straight line, without two of them coinciding, then one must add the three equations:

$$\begin{aligned}(\mathfrak{D} - \mathfrak{A}) \mid d &= (\mathfrak{D} - \mathfrak{A}) \mid a,\\ (\mathfrak{D} - \mathfrak{B}) \mid d &= (\mathfrak{D} - \mathfrak{B}) \mid b,\\ (\mathfrak{D} - \mathfrak{C}) \mid d &= (\mathfrak{D} - \mathfrak{C}) \mid c,\end{aligned}$$

to the three equations (12) of § 8, which say that the point D_1 is just as far from the three points A_1, B_1, C_1 as D is from A, B, C.

Two cases will now be possible: In one case, $\mathfrak{D}$ likewise lies on the line $\mathfrak{ABC}$. If q is a unit vector on that line then the three equations above will say that:

$$q \mid a = q \mid b = q \mid c = q \mid d = \rho,$$

with the notation of (13). If one then sets:

$$d = \rho\, q + d$$

then one must have $q \mid d_1 = 0$. If one then defines the point D' by the equation:

$$D' = D + 2\rho\, q$$

then $\dfrac{D_1 + D'}{2} = \mathfrak{D} + \rho\, q$ will be a point on the axis $(\mathfrak{A}, q)$, and:

$$q \mid (D_1 - D') = 2q \mid d = 0.$$

From § 8, the displacement $2\rho\, q$, and possibly the rotation through 180° around the axis ($\mathfrak{A}$, q) that takes ABC to $A_1B_1C_1$, will thus make D overlap with D_1 .

However, if the four points $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ do not lie on the same line then (in agreement with § 9) one will assume that $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ lie on a line that does not, however, contain $\mathfrak{C}$. The last two systems (15) and (16) will then be valid, and a comparison of the different expressions for the vectors in question will give the relations:

$$\mathsf{A} = \Sigma + \alpha\,\mathfrak{B}\mathfrak{C},$$
$$\mathsf{B} = \Sigma + \beta\,\mathfrak{A}\mathfrak{C},$$

which agree with formulas (16) for $A_1 - A$, $B_1 - B$, $C_1 - C$, but for $D_1 - D$ they will yield:

$$D_1 - D = |\ \omega\mathfrak{D}\ \Sigma + \alpha|\ \omega\mathfrak{D}\mathfrak{B}\mathfrak{C},$$
$$= |\ \omega\mathfrak{D}\ \Sigma + \beta|\ \omega\mathfrak{D}\mathfrak{A}\mathfrak{C}\ .$$

Since one must have $\alpha\,\mathfrak{D}\mathfrak{B}\mathfrak{C} = \beta\,\mathfrak{D}\mathfrak{A}\mathfrak{C}$, one can set both equal to $\rho\,\mathfrak{A}\mathfrak{B}\mathfrak{C}$ and set:

$$D_1 - D = |\ \omega\mathfrak{D}\ \Sigma + \rho\,|\ \omega\mathfrak{A}\mathfrak{B}\mathfrak{C},$$

as was found in (16^{**}).

The considerations of §§ 12 and 13 are then unchanged, except that one must set ν = 0 in the latter.

In the first of the two cases that were treated in those paragraphs, the two tetrahedra $ABCD$ and $A_1B_1C_1D_1$ were congruent, since there was a screw that changed the one into the other. Therefore, the figures were symmetric in the second case. One can therefore state the theorem:

If three of the four midpoints of the connecting lines of corresponding corners of two congruent tetrahedra lie on a straight line then the fourth one will also lie on that line.

Such a theorem is not true for symmetric tetrahedra. In order to prove it directly, we employ the Ansatz of § 11, which will yield:

$$\mathfrak{D} = \mathfrak{A} + \beta(\mathfrak{B} - \mathfrak{A}) + \gamma(\mathfrak{C} - \mathfrak{A}) + \tfrac{1}{2}|\ \omega(A_1B_1C_1 + ABC).$$

From § 13, however, one has:

$$\omega(A_1B_1C_1 + ABC) = 2\,[\omega\mathfrak{A}\mathfrak{B}\mathfrak{C} + (b - a)(c - a)].$$

Since $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ lie on a straight line, one will have $\omega\,\mathfrak{A}\mathfrak{B}\mathfrak{C} = 0$, and from equations (14), one will have $(b - a)(c - a) = (b_1 - a_1)(c_1 - a_1)$. It will then follow that:

$$\mathfrak{D} = \mathfrak{A} + \beta(\mathfrak{B} - \mathfrak{A}) + \gamma(\mathfrak{C} - \mathfrak{A}) + \pi\lambda q,$$

which proves the theorem.

§ 17.

The formulas of the previous paragraph will also still be true when the point $\mathfrak{D}$ coincides with $\mathfrak{A}$ or $\mathfrak{B}$. However, if three of the four points coincide in $\mathfrak{A}$ and the fourth one is $\mathfrak{D}$ then the three equations at the beginning of the previous paragraph will become:

$$(33)\qquad \begin{cases} (\mathfrak{D}-\mathfrak{A}) \mid (d-a)=0, \\ (\mathfrak{D}-\mathfrak{A}) \mid (d-b)=0, \\ (\mathfrak{D}-\mathfrak{A}) \mid (d-c)=0. \end{cases}$$

Either $\mathfrak{D} = \mathfrak{A}$, or the three vectors $d - a$, $d - b$, $d - c$ are perpendicular $\mathfrak{D} - \mathfrak{A}$, and thus coplanar. In the second case, there will then be three numbers α, β, γ such that:

$$\alpha (d - a) + \beta (d - b) + \gamma (d - c) = 0.$$

If $\alpha + \beta + \gamma \neq 0$ then it will follow from this equation that:

$$D_1 - \frac{\alpha A_1 + \beta B_1 + \gamma C_1}{\alpha+\beta+\gamma} = D - \frac{\alpha A + \beta B + \gamma C}{\alpha+\beta+\gamma}.$$

Let these two equal vectors be called m. If one denotes the two points:

$$\frac{\alpha A + \beta B + \gamma C}{\alpha+\beta+\gamma} \text{ by } E \text{ and } \frac{\alpha A_1 + \beta B_1 + \gamma C_1}{\alpha+\beta+\gamma} \text{ by } E_1$$

then E_1 and E will be corresponding points on the two planes $A_1B_1C_1$ and ABC that go to each other under congruent transformations.

One will then have:

$$\frac{E+E_1}{2} = \mathfrak{A}, \quad D_1 = E_1 + m, \quad D = E + m, \qquad \mathfrak{D} = \mathfrak{A} + m.$$

Since one must have:

$$m \mid (d - a) = m \mid (d - b) = m \mid (d - c) = 0,$$

one must also have:

$$m \mid (b - a) = 0, \qquad m \mid (c - a) = 0$$

such that one will have $m \parallel q$. If one further sets:

$$D' = \frac{\alpha A' + \beta B' + \gamma C'}{\alpha+\beta+\gamma} + m = \frac{\alpha A + \beta B + \gamma C}{\alpha+\beta+\gamma} + 2\rho\, q + m$$

$$= E + 2r\,q + m,$$

then it will follow, on the one hand, that:

$$D_1 - D' = \frac{\alpha(A_1 - A') + \beta(B_1 - B') + \gamma(C_1 - C')}{\alpha + \beta + \gamma}$$
$$= 2\,\frac{\alpha a_1 + \beta b_1 + \gamma c_1}{\alpha + \beta + \gamma}$$

with

$$q \mid (D_1 - D') = 0,$$

and on the other hand:

$$\frac{D_1 + D'}{2} = \mathfrak{A} + \rho\,q + m.$$

The point $\frac{D_1 + D'}{2}$ then lies on the axis $(\mathfrak{A}, q)$ that will be met perpendicularly by the line $D'D_1$ when it is not zero, and the motion that was depicted in § 8 will take the tetrahedron $ABCD$ to $A_1B_1C_1D_1$, such that the case that was treated above will be that of congruence. Therefore, if $\mathfrak{D}$ coincides with $\mathfrak{A}$ then the two tetrahedra must be symmetric. This can also be inferred from a consideration of the difference $A_1B_1C_1D_1 - ABCD$, as in § 10, which, since:

$$\mathfrak{ABC} = \mathfrak{ABD} = \mathfrak{ACD} = \mathfrak{BCD} = 0$$

here, will reduce to:

$$2\mathfrak{A}\,(bcd - acd + acd - dbc).$$

However, since one has:

$$d = \frac{\alpha a + \beta b + \gamma c}{\alpha + \beta + \gamma},$$

it will follow that:

$$bcd = \frac{\alpha abc}{\alpha + \beta + \gamma}, \qquad acd = -\frac{\beta abc}{\alpha + \beta + \gamma}, \qquad abd = -\frac{\gamma abc}{\alpha + \beta + \gamma},$$

and this will imply that:

$$A_1B_1C_1D_1 - ABCD = 0.$$

If $\alpha + \beta + \gamma = 0$ then $\alpha a + \beta b + \gamma c = 0$, so the three vectors a, b, c will be coplanar. The vector q will then be perpendicular to the plane of those three. It follows from equations (33) that:

$$(\mathfrak{D} - \mathfrak{A}) \mid (b - a) = 0, \qquad (\mathfrak{D} - \mathfrak{A}) \mid (c - a) = 0,$$

which show that $\mathfrak{D} - \mathfrak{A}$ is parallel to q. Equations (33) then imply:

$$q \mid d = q \mid a = q \mid b = q \mid c = 0.$$

Thus, a rotation through 180° around the axis ($\mathfrak{A}$, q) will succeed in taking $ABCD$ to $A_1B_1C_1D_1$.

§ 18.

The theorems of §§ 10, 16, 16 on the midpoints of segments that connect corresponding vertices of two congruent or symmetric tetrahedra can be deduced from the known properties of orthogonal transformations with real coefficients, since each symmetric or congruent space transformation is indeed orthogonal.

If:

$$\left\{\begin{aligned} x_1 &= A + a_{11}x + a_{12}y + a_{13}z, \\ y_1 &= B + a_{21}x + a_{22}y + a_{23}z, \\ z_1 &= C + a_{31}x + a_{32}y + a_{33}z \end{aligned}\right. \tag{34}$$

are the transformation formulas, and ξ, η, ζ are the coordinates of the midpoint of the segment that connects the point (x, y, z) with its corresponding point (x_1, y_1, z_1) then one will have:

$$\left\{\begin{aligned} 2\xi &= A + (a_{11}+1)x + a_{12}y + a_{13}z, \\ 2\eta &= B + a_{21}x + (a_{22}+1)y + a_{23}z, \\ 2\zeta &= C + a_{31}x + a_{32}y + (a_{33}+1)z. \end{aligned}\right.$$

The determinant:

$$\begin{vmatrix} a_{11}-s & a_{12} & a_{13} \\ a_{21} & a_{22}-s & a_{23} \\ a_{31} & a_{32} & a_{33}-s \end{vmatrix} = \varphi(s)$$

is the so-called *characteristic* function of the substitution (34), and it is known that:

$$\varphi(0) = \varepsilon = \pm 1,$$

according to whether the transformation is congruent or symmetric.

Equation $\varphi(s)$ can have $s = +1$ or $s = -1$, along with complex values, for its roots. If $s = +1$ is a μ-fold root and $s = -1$ is an ν-fold root then one will have:

$$\mu + \nu = 1 \text{ or } 3,$$
$$(-1)^{\nu} = \varepsilon,$$

and for an m-fold root all sub-determinants of degree $(4 - m)$ will vanish in the determinant $\varphi(s)$, while those of degree $(3 - m)$ will not all be zero (*).

Therefore, the following cases are possible:

(*) *Stickelberger:* "Über reelle orthogonale Transformationen," Beilage zum Programm des Polytechnikums, Zurich, 1877, page VII.

$$\begin{array}{ll} \varepsilon = +1, & \nu = 0 \text{ or } 2, \\ \quad -1, & \quad\;\; 1 \quad\;\; 3. \end{array}$$

However, $\varphi(-1)$ is the determinant of equations (35). Thus, if $\nu = 1$ then it will vanish. A linear equation will then exist between the quantities:

$$2\xi - A, \quad 2\eta - B, \quad 2\zeta - C,$$

or the midpoints of the segments considered will lie on a plane under a symmetric transformation.

The transformation is likewise symmetric for $\nu = 3$, but all first-degree determinants will vanish for $s = -1$; i.e., one will have:

$$a_{11} = a_{22} = a_{33} = -1,$$

or the midpoints will coincide at the same point.

One is dealing with congruent transformations for $\nu = 2$. The second-degree sub-determinants vanish for $s = -1$, so *two* linear equations will exist between the quantities $2\xi - A$, $2\eta - B$, $2\zeta - C$, and the midpoints of the segments will lie on a line.

For congruent transformations the midpoints will either fill up all of space or they will lie on a line, while for symmetric transformations they will lie on a plane or coincide in a point.

These theorems can be deduced with no difficulty from the general theorems that *H. Wiener* gave on the transformation of a spatial system into an equal one [Berichte d. math.-phys. Klasse d. kgl. Sächs. Gesellsch. d Wissensch. (1891), pp. 659.]