"Eine Übertragung der Diracschen Theorie des Elektrons in gewohnte Formen," Zeit. f. Phys. **54** (1929), 303-306.

A transcription of Dirac's theory of the electron into a customary form.

By **E. Madelung** at Frankfurt am Main

(Received on 1 March 1929)

Translated by D. H. Delphenich

It will be shown that Dirac's theory leads to equations in conventional vector notation that are very similar to those of Maxwell's theory. The method of solution will be given for a special problem.

The form in which Dirac has represented his wonderful theory of the electron [*] deviates so much from the customary forms that any insight into their actual essence is very difficult. The following representation, by contrast, relates to only customary notions.

I assume the following system of equations:

$$\left.\begin{aligned}
&\frac{1}{ic}\frac{\partial\sigma}{\partial t}-\operatorname{div}\mathfrak{S}+\frac{mc^2}{K}\sigma=\frac{e\varphi}{K}\sigma+\frac{ie}{K}(\mathfrak{A}\mathfrak{S}), \quad \left(K=\frac{hc}{2\pi}\right),\\
&\frac{1}{ic}\frac{\partial\mathfrak{S}}{\partial t}+\operatorname{grad}\sigma+\operatorname{rot}\mathfrak{T}-\frac{mc^2}{K}\mathfrak{S}=\frac{e\varphi}{K}\mathfrak{S}-\frac{ie}{K}(\mathfrak{A}\sigma+[\mathfrak{A}\mathfrak{S}]),\\
&\frac{1}{ic}\frac{\partial\tau}{\partial t}-\operatorname{div}\mathfrak{T}-\frac{mc^2}{K}\tau=\frac{e\varphi}{K}\tau+\frac{ie}{K}(\mathfrak{A}\mathfrak{T}),\\
&\frac{1}{ic}\frac{\partial\mathfrak{T}}{\partial t}+\operatorname{grad}\tau+\operatorname{rot}\mathfrak{S}+\frac{mc^2}{K}\mathfrak{T}=\frac{e\varphi}{K}\mathfrak{T}-\frac{ie}{K}(\mathfrak{A}\tau+[\mathfrak{A}\mathfrak{S}]).
\end{aligned}\right\} \tag{1}$$

One easily confirms by elimination that for $\varphi = 0$ and $\mathfrak{A} = 0$ – i.e., in the absence of an external field (φ and $\mathfrak{A}$ are the electric potentials) – any of the quantities σ, τ, $\mathfrak{S}$, $\mathfrak{T}$ (their components, resp.) fulfill the wave equation:

$$\Delta-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\left(\frac{mc^2}{H}\right)^2=0.$$

[*] Proc. Roy. Soc. (A) **117** (1928), 610.

The right-hand sides are so defined that any $\frac{1}{ic}\frac{\partial}{\partial t}$ corresponds to $\frac{e\varphi}{K}$ on the left and any $\frac{\partial}{\partial x_i}$ on the left corresponds to $-\frac{ie}{K}\mathfrak{A}_{x_i}$ on the right.

By the presentation of the system of equations, Dirac's requirement is fulfilled that the wave equation of the electron must represents a simultaneous system of first order that lead to the relativistic wave equation. In contrast to it, we have, however, eight, instead of four, component equations, and therefore complete symmetry.

The similarity with the Maxwell equations is likewise made very remarkable here.

The case that Dirac treated of a centrally-symmetric static force field $\mathfrak{A} = 0$, $\varphi = \varphi(r)$ can be disposed of by means of our equations as follows: As usual, one first sets $\frac{1}{ic}\frac{\partial}{\partial t} = -\frac{E}{K}$ and obtains the special system:

$$\left.\begin{aligned}
\frac{1}{K}(mc^2 - E - e\varphi)\sigma &= \operatorname{div}\mathfrak{S}, \\
\frac{1}{K}(mc^2 + E + e\varphi)\mathfrak{S} &= \operatorname{grad}\sigma + \operatorname{rot}\mathfrak{T}, \\
\frac{1}{K}(mc^2 + E + e\varphi)\tau &= -\operatorname{div}\mathfrak{T}, \\
\frac{1}{K}(mc^2 - E - e\varphi)\mathfrak{T} &= -\operatorname{grad}\tau - \operatorname{rot}\mathfrak{S}.
\end{aligned}\right\} \tag{2}$$

If $f(r)$ is a function of only r then one has the following identities:

$$\begin{aligned}
(\mathfrak{r}\operatorname{rot} f(r)\,\mathfrak{a}) &= f(r)\,(\mathfrak{r}\operatorname{rot}\mathfrak{a}), \\
[\mathfrak{r}\operatorname{grad} f(r)\,\alpha) &= f(r)\,[\mathfrak{r}\operatorname{rot}\alpha], \\
\operatorname{rot}[\mathfrak{r}\,f(r)\,\mathfrak{a}] - [\mathfrak{r}\operatorname{rot} f(r)\,\mathfrak{a}] &= f(r)\,(\operatorname{rot}[\mathfrak{r}\mathfrak{a}] - [\mathfrak{r}\operatorname{rot}\mathfrak{a}]), \\
\mathfrak{r}\operatorname{div} f(r)\,\mathfrak{a} - \operatorname{grad}(\mathfrak{r}\,f(r)\,\mathfrak{a}) &= f(r)\,(\mathfrak{r}\operatorname{div}\mathfrak{a} - \operatorname{grad}(\mathfrak{r}\,\mathfrak{a})).
\end{aligned}$$

One further observes that:

$$\begin{aligned}
(\mathfrak{r}\operatorname{rot}\mathfrak{a}) &= -\operatorname{div}[\mathfrak{r}\,\mathfrak{a}], \\
[\mathfrak{r}\operatorname{grad}\mathfrak{a}] &= -\operatorname{rot}\mathfrak{r}\,\alpha, \\
\operatorname{rot}[\mathfrak{r}\,\mathfrak{a}] - [\mathfrak{r}\operatorname{rot}\mathfrak{a}] &= \mathfrak{r}\operatorname{div}\mathfrak{a} - \operatorname{grad}(\mathfrak{r}\,\mathfrak{a}) - \mathfrak{a}.
\end{aligned}$$

One thus confirms that the same equations (2) are true for the following quantities:

$$\begin{aligned}
\sigma' &= -(\mathfrak{r}\operatorname{rot}\mathfrak{T}) + \sigma, \\
\mathfrak{S}' &= \operatorname{rot}[\mathfrak{r}\,\mathfrak{S}] - [\mathfrak{r}\operatorname{rot}\mathfrak{S}] - [\mathfrak{r}\operatorname{grad}\tau] + \mathfrak{S},
\end{aligned}$$

$$\tau' = -(\mathfrak{r}\ \text{rot}\ \mathfrak{S}) - \tau,$$
$$\mathfrak{T}' = -\text{rot}\ [\mathfrak{r}\ \mathfrak{T}] + [\mathfrak{r}\ \text{rot}\ \mathfrak{T}] + [\mathfrak{r}\ \text{grad}\ \sigma] - \mathfrak{T}$$

as for the unprimed ones. It is therefore permissible to make the Ansatz:

$$\sigma' = k\sigma, \qquad \mathfrak{S}' = k\mathfrak{S}, \qquad \tau' = k\tau, \qquad \mathfrak{T}' = k\mathfrak{T}.$$

This yields, *inter alia:*

$$\frac{1}{K}(mc^2 + E + e\varphi)\ (\mathfrak{r}\ \mathfrak{S}) = \frac{\partial}{\partial r}(r\sigma) - \frac{k}{r}(r\sigma),$$
$$\frac{1}{K}(mc^2 - E - e\varphi)\ (\mathfrak{r}\ \sigma) = \frac{\partial}{\partial r}(r\,\mathfrak{S}) + \frac{k}{r}(r\,\mathfrak{S}).$$

These are precisely the Dirac equations (*loc. cit.*, pp. 622, *infra*) when one replaces $r\sigma = \psi_\alpha$, $(\mathfrak{r}\ \mathfrak{S}) = \psi_\beta$, $k = -j$. The problem is then easily solved with no use made of the Dirac operators.

We shall not suggest that the operator method is undesirable. On the contrary, its formal character often eases the necessary conversions and admits very concise notations. For the sake of illustration, we translate our basic equations into a form that is very close to the Dirac equations.

First, I sketch out the following schema:

	α_0	α_1	α_2	α_3	$\beta_0 = \alpha_1\alpha_2\alpha_3$	$\beta_1 = \alpha_2\alpha_3$	$\beta_2 = \alpha_3\alpha_1$	$\beta_3 = \alpha_1\alpha_2$
σ …	σ	$-S_x$	$-S_y$	$-S_z$	$-\tau$	$-T_x$	$-T_y$	$-T_z$
S_x …	$-S_x$	$-\sigma$	$-T_z$	T_y	T_x	τ	S_z	$-S_y$
S_y …	$-S_y$	T_z	$-\sigma$	$-T_x$	T_y	$-S_z$	τ	S_x
S_z …	$-S_z$	$-T_y$	T_x	$-\sigma$	T_z	S_y	$-S_x$	τ
τ …	$-\tau$	T_x	T_y	T_z	σ	$-S_x$	$-S_y$	$-S_z$
T_x …	T_x	τ	S_z	$-S_y$	$-S_x$	σ	T_z	$-T_y$
T_y …	T_y	$-S_x$	τ	S_x	$-S_y$	$-T_z$	σ	T_x
T_z …	T_z	S_y	$-S_x$	τ	$-S_z$	T_y	$-T_x$	σ

The schema is understood to mean that, e.g., $T_z = -\alpha_2 S_x$; i.e., the symbols α_i (β_i , resp.) are operators that take quantities on the left to corresponding quantities in the column of the operator. As one sees, one is dealing with permutation operators with sign rules.

Our system of equations can thus be written symbolically *:

* The half parenthesis should be interpreted as saying that in each case the variable – e.g., σ – is to be placed behind the operator. Perhaps this is superfluous, since, e.g., $\frac{\partial}{\partial t}\alpha_0$ makes no sense by itself. The notation $\frac{\partial}{\partial t}\alpha_0$ seems dangerous to me!

$$\frac{1}{ic}\frac{\partial}{\partial t}(\alpha_0 + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\alpha_i + \frac{mc^2}{K} = \frac{e}{K}\left(\varphi(\alpha_0 - i\sum_{i=1}^{3}\mathfrak{A}_i(\alpha_i\right),$$

when it is applied to all of the quantities from σ to T_z .

One easily confirms that $(\alpha_i)^2 = 1$ and $\alpha_i\,\alpha_k = -\,\alpha_k\,\alpha_i$, as is also true for Dirac.

One can further combine the α_1 , α_2 , α_3 into an operator $\mathfrak{s}$ (and likewise β_1, β_2, β_3 into $\mathfrak{t}$) and write $\mathfrak{t} = \frac{1}{2}[\mathfrak{s}\ \mathfrak{s}]$, and:

$$\frac{1}{ic}\frac{\partial}{\partial t}(\alpha_0 + \mathrm{div}(\mathfrak{s} + \frac{mc^2}{K} = \frac{e}{K}\left(\varphi(\alpha_0 - i(\mathfrak{A}(\mathfrak{s})\right).$$

One can then write all of Dirac’s calculations in this form, and translate each of them into conventional vector equations.

One clearly sees the similarity between our formulas and those of electrodynamics by the following comparison:

If one sets [*]:

$$\mu\,\mathfrak{H} = \mathrm{rot}\; i\,\mathfrak{F}; \qquad \mathfrak{E} = -\,\frac{i}{c}\frac{\partial\mathfrak{F}}{\partial t} + \frac{1}{\varepsilon}\,\mathrm{grad}\,\mathcal{F},$$

$$\varepsilon\,\mathfrak{E} = -\,\mathrm{rot}\;\mathfrak{G}, \qquad \mathfrak{H} = -\,\frac{i}{c}\frac{\partial\mathfrak{G}}{\partial t} - \frac{i}{\mu}\,\mathrm{grad}\,\gamma$$

then Maxwell’s equations are true for $\mathfrak{E}$ and $\mathfrak{G}$ when one has

$$\left.\begin{aligned}
&\frac{\varepsilon}{ic}\frac{\partial\mathfrak{F}}{\partial t} + \mathrm{grad}\,\mathcal{F} + \mathrm{rot}\,\mathfrak{G} = 0,\\
&\frac{\mu}{ic}\frac{\partial\mathfrak{G}}{\partial t} + \mathrm{grad}\,\gamma + \mathrm{rot}\,\mathfrak{F} = 0,\\
&\frac{\mu}{ic}\frac{\partial\mathcal{F}}{\partial t} - \mathrm{div}\,\mathfrak{F} = 0,\\
&\frac{\varepsilon}{ic}\frac{\partial\gamma}{\partial t} - \mathrm{div}\,\mathfrak{G} = 0.
\end{aligned}\right\} \tag{3}$$

If one now sets:

$$\varepsilon = 1 + \frac{mc^2}{E}, \qquad \mu = 1 - \frac{mc^2}{E}$$

and

$$\frac{\varepsilon}{ic}\frac{\partial\mathfrak{F}}{\partial t} = \frac{1}{ic}\frac{\partial\mathfrak{F}}{\partial t} + \frac{mc^2}{E}\left(-\frac{E}{K}\mathfrak{F}\right)$$

---

[*] $\mathcal{F}$ is read as: digamma = Greek *w*.

and analogously for $\mathfrak{G}$, $\mathcal{F}$, γ, then these equations will formally read the same as our system (1).

This analogy seems meaningful to me, because it makes it clear that, by the addition of terms in σ, τ, $\mathfrak{S}$, $\mathfrak{T}$ to the right-hand side of system (3), one can arrive at a rational theory of the interaction of matter with the field – i.e., to a theory of radiation – that extends to the present Ansätze in a manner that is similar to the one that led Dirac from scalar wave mechanics.

____