"Sugli spostamenti elastici discontinui," Rend. Reale Accad. dei Lincei (5) **17** (1908), 571-576.

Mechanics – *On discontinuous, elastic displacements*, Note by the correspondent GIAN ANTONIO MAGGI.

The teaching of mathematical physics in the current year has given us an occasion to return to the topic of discontinuous, elastic displacements that we pointed out in a preceding communication ([1]) as a concrete interpretation of Volterra's polydromic displacements, and on the question of the double representation of physico-mathematical elements by means of discontinuous functions and polydromic functions, which are mutually related to each other, permit me to announce the principal points of my exposition in this brief communication.

1. Instead of the continuous displacements that are, as a rule, considered exclusively in the theory of elastic equilibrium, in our study we will address discontinuous displacements, of which, experiment provides obvious examples, and the simplest type of them are displacements that present discontinuities of the first kind at a certain surface, but which correspond entirely to dilatation parameters ([2]) and consequently to pressure parameters ([2]) that are regular in such a way that despite the tear that is represented by the discontinuous displacement at the surface in question, the elements of dilatation, deformation, or internal tension will remain distributed continuously in the body considered. That is the kind of elastic displacement that first attracted the attention of Weingarten ([3]) as the principal object of his geometric research. I shall begin with that concept in order to first of all treat the question of the possibility of such displacements.

2. I caution you that I intend that the position of elastic equilibrium for the body in question that will be considered is determined by means of a corresponding displacement of its natural position under which the displacement of any infinitesimal particle can be composed in a known way from a certain rigid displacement and a certain dilatational displacement that has its origin at the natural position of the particle. Given that, imagine that dilatation parameters are associated with any point (x, y, z) of the domain that represents the natural – or reference – position of the elastic body considered, and that they are regular ([4]) functions of that point. Note that the so-called Saint Venant equations

([1]) See, these Rendiconti, fasc. of 5 Nov. 1905.

([2]) What we call dilatation parameters and pressure parameters are the six quantities that others call the components, characteristics, etc., of dilatations, or deformation, pressure, or tension, respectively. The term "parameters" does not seem opportune to me, because these quantities, which might very well vary from point to point, behave like constants in the examination of an infinitesimal particle around the point, where the first kind exhibits with a kinematical aspect (*strain* or deformation) and the second kinds exhibits a dynamical aspect (*stress* or tension).

([3]) See, these Rendiconti, fasc. on 3 February 1901.

([4]) That is, uniform, continuous, finite, and endowed with derivatives of a similar kind up to the order that happens to be considered; in the present case, that would be the second.

must be satisfied ([1]): By virtue of them, by means of their first derivatives, one can associate the general point (x, y, z) in question with the total (i.e., exact) differentials:

$$\text{(1)} \qquad \begin{cases} dp = p_1 dx + p_2 dy + p_3 dz, \\ dq = q_1 dx + q_2 dy + q_3 dz, \\ dr = r_1 dx + r_2 dy + r_3 dz \end{cases}$$

of the components p, q, r of the rotation that relates to that point (x, y, z). It then follows that if the p, q, r are thought of as formed with the aid of them and the dilatation parameters above then one can further associate the point (x, y, z) considered with the total (exact) differentials:

$$\text{(2)} \qquad \begin{cases} d\xi = \xi_1 dx + \xi_2 dy + \xi_3 dz, \\ d\eta = \eta_1 dx + \eta_2 dy + \eta_3 dz, \\ d\zeta = \zeta_1 dx + \zeta_2 dy + \zeta_3 dz \end{cases}$$

of the components ξ, η, ζ of the displacement that relates to that point.

Given that, in the first place, let the domain that represents the body be simply connected. For any closed path C, (1) then gives:

$$\int_C (p_1\, dx + p_2\, dy + p_3\, dz) = 0, \text{ etc.;}$$

with that, the p, q, r become regular functions of the relevant point in that domain. One imagines introducing the functions in (2) that that will give, in turn:

$$\int_C (\xi_1\, dx + \xi_2\, dy + \xi_3\, dz) = 0, \quad \text{etc.;}$$

with that, the ξ, η, ζ also become regular functions. One concludes that if the body is represented by a simply-connected domain with regular dilatation parameters then it cannot correspond to anything but regular displacements, and consequently, displacements of the kind considered would not be possible. That proposition, which was emphasized by Weingarten ([2]), reduces to the analogous theorem of Volterra for polydromic displacements ([3]).

Suppose instead that the domain is multiply-connected, so for any system of simple paths that do not belong to circuits that define the complete contour of a cap that is included in the domain, but which pair-wise form the contour of a zone that is included in the domain (i.e., paths that are reducible to each other), (1) give:

$$\int_C (p_1\, dx + p_2\, dy + p_3\, dz) = \Delta p,$$

([1]) Cf., for example, Marcolongo, *Teoria dell'equilibrio dei corpi elastici*, Milano, Hoepli, 1904, Chap. III, § 6.

([2]) *loc. cit.*

([3]) See, these Rendiconti, fasc. on 5 February, 1905.

$$(3) \qquad \begin{aligned} \int_C (q_1\,dx + q_2\,dy + q_3\,dz) &= \Delta q, \\ \int_C (r_1\,dx + r_2\,dy + r_3\,dz) &= \Delta r, \end{aligned}$$

in which Δp, Δq, Δr denote constants that depend upon the system of paths and are generally assumed to be non-zero. The p, q, r can then be formally represented as polydromic functions that have one or more fundamental moduli of peridocity, according to the order of multiplicity of the connection in the domain. However, the mechanical significance of the p, q, r – namely, that they are the components of the rotation that relates to the point (x, y, z) – does not seem to allow for any point of the domain that lies upon various paths that begin at a point at which one assumes a certain value as initial value to attain various limits, but does allow for that situation to be verified by certain points that define the points of discontinuity of the rotations. On the other hand, one imagines that the domain has been rendered simply-connected by means of diaphragms that intersect the individual fundamental systems of circuits that are mutually reducible, so the p, q, r will be defined as uniform and discontinuous functions in the new domain that present a discontinuity of the first kind at any diaphragm, which is, conforming to (3), characterized by the differences Δp, Δq, Δr of the limits that they tend to at any point of the diaphragm from opposite sides of it. It then results that the location of any tear can be conceived of as being freely mobile, together with the corresponding diaphragm, along the relevant circuits. However, we would not like to say that one can ignore their existence for the mechanical interpretation under discussion. We would like to say that the dilatation parameters above allow an infinitude of problems that can be deduced from each other by moving the location of the discontinuity in the indicated fashion.

In the domain that has been reduced to a simply-connected one, in which one can distinguish the two sheets of the diaphragm as distinct parts of the contour, one can operate on the p, q, r as if they were regular functions. With that, for any system of simple paths that have their extremes on the corresponding points of two opposite sheets of the same diaphragm, (2) will give:

$$\begin{aligned} \int_C (\xi_1\,dx + \xi_2\,dy + \xi_3\,dz) &= \Delta \xi, \\ \int_C (\eta_1\,dx + \eta_2\,dy + \eta_3\,dz) &= \Delta \eta, \\ \int_C (\zeta_1\,dx + \zeta_2\,dy + \zeta_3\,dz) &= \Delta \zeta, \end{aligned}$$

in which the $\Delta\xi$, $\Delta\eta$, $\Delta\zeta$ denote three constants that relate to the diaphragm.

Since the continuity of the dilatation parameters requires that one have:

$$\Delta x_x = 0, \ \ldots, \ \Delta y_x = 0, \ \ldots,$$

one will get:

$$\begin{aligned} \Delta\xi &= \Delta a + (z - z_0)\,\Delta q - (y - y_0)\,\Delta r, \\ \Delta\eta &= \Delta b + (x - x_0)\,\Delta r - (z - z_0)\,\Delta q, \\ \Delta\zeta &= \Delta c + (y - y_0)\,\Delta p - (x - x_0)\,\Delta p, \end{aligned}$$

in which Δp, Δq, Δr have the preceding significance, and Δa, Δb, Δc are new constants.

One then finds a discontinuity in the displacement at the same location as the first one, and that discontinuity will be represented by a rigid displacement ([1]). Moreover, with that peculiarity, one can prove the possibility of a displacement of the type considered in a body that is represented by a multiply-connected domain: It will then once more reduces to the analogous theorem of Volterra for polydromic displacements ([2]).

One can then immediately define corresponding polydromic and continuous functions that satisfy (2) from the uniform and discontinuous functions, and thus revert to Volterra's polydromic displacements. These functions have the quality of not involving the diaphragms. However, as far as their ability to represent displacements with the indicated position of the problem is concerned, as well as for the mobile position of the diaphragms – i.e., the location of the discontinuities of the uniform displacement – the considerations that were made previously *a propos* of rotations are still valid.

3. Permit me to add the following observations in regard to the discussion of the relationships between polydromy and discontinuity.

Set:

$$V = -\int_\sigma \frac{\partial(1/r)}{\partial n}\,d\sigma\,,$$

in which σ denotes a regular surface that has a certain contour, n denotes the normal at its generic point, which points in a certain sense, and r is the distance from that point to a certain point that is taken on the surface P and has coordinates x, y¸ z, so:

$$dL = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

will represent the differential of the work that corresponds to a motion of a magnetic pole of unit intensity that is placed at P, as well as the force that is exerted on a magnetic layer that has its location on the surface σ and a specific magnetic moment of unit magnitude that is oriented like n, and also the force that is exerted on an electric current of unit intensity that circulates in the line that is represented by the contour of σ in the positive sense with respect to the the normal n. For any closed, simple path C that belongs to a circuit that is the concatenation of such lines, whose senses agree with that of the normal n, one will have:

$$\int_C \left(\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz\right) = 4\pi\,.$$

On the basis of that result, L – viz., the work that corresponds to a path that goes from one fixed point P_0 to the point P – can be defined as either a uniform, discontinuous function of the surface σ or as a polydromic, continuous function of the multiply-

([1]) See the cited notes of Weingarten and Volterra.

([2]) *Loc. cit.*

connected domain that one obtains from the space, minus the contour of σ, in which an infinitude of values of P are represented by:

$$\bar{L} + 4\pi\nu,$$

in which $\bar{L}$ denotes the value of the preceding uniform, discontinuous function at P, and ν denotes the difference between the number of turns in the sense of n and in the opposite sense, concatenated with the contour of σ, that one makes along the path considered. Now, *just as one did for the physical significance*, one can make the first or second hypothesis according to whether one is dealing with the magnetic layer or the electric current.

In other cases, the two hypotheses can be assumed indifferently. Thus, Volterra, in his celebrated memoir "Sur les vibrations lumineuses dans les milieux birefringents" ([1]), has invalidated the results of Sofia Kovalevskij and concluded that Lamé's hypothesis that luminous vibrations that emanate from a center in a birefringent medium and propagate as wave surfaces will lead to expressions for the vibrations in terms of polydromic functions that have parallels to the optical axes that are described at the center for their critical lines. He reaches that conclusion by proceeding with uniform, discontinuous functions on the plane of the aforementioned parallel, which lend themselves to analogous conclusions. Finally, let me note that this discontinuity can be seen in the results of Lamé by observing that the vibration at any point on each stratum of the wave surface will have non-zero amplitudes at any point in the directions of the tangents to the spherical intersections that pass through the point, and at any instant, in order to have identity of the phase, their senses must agree for all points of that spherical intersection. Thus, each of the two halves that the spherical intersection is divided into by the plane of the parallels to the optical axes that go through the center and tend to that place will tend to an arc of the circle that is placed in that plane, so one sees that the vibrations that belong to the points that are situated in the opposite part of the aforementioned plane and tend to the same point of the plane will compete for limits that are equal and of opposite sense.

([1]) Acta Mathematica, Tome 16.