"Posizione e soluzione di alcune questioni attinenti alla teoria delle distorsioni elastiche," Rend. R. Acc. dei Lincei (5) **26** (1917), 350-357.

Mathematics. – *Statement and solution of some questions related to the theory of elastic distortions*. Note by the member GIAN ANTONIO MAGGI.

Translated by D. H. Delphenich

The theory of elastic distortions, which was initiated by Weingarten, to whom is due the basic concept ([1]), was developed from the beginning and enriched by copious interesting results of Volterra ([2]) and then cultivated with success in various aspects, and I believe that I am not mistaken in asserting that it took on its most definitive formulation only in the recent papers of Somigliana ([3]).

As a matter of fact, in proposing to study the deformations in elastic equilibrium that are provoked by discontinuities in the displacements of a given internal surface, Weingarten established the conditions that the parameters of dilatation (viz., components of the deformation, characteristics of the deformations, etc.)
:

$$
(1)\qquad
\begin{gathered}
x_x = \frac{\partial \xi}{\partial x}, \qquad y_y = \frac{\partial \eta}{\partial y}, \qquad z_z = \frac{\partial \zeta}{\partial z}, \\
y_z = z_y = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z}, \qquad z_x = x_z = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x}, \qquad x_y = y_x = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}
\end{gathered}
$$

must be continuous on that surface, in order that what one calls *stresses* should be continuous. It then appears that for them one must intend that the pressure parameters (viz., pressure components, pressure characteristics, etc.):

$$
(2)\qquad X_x, \quad Y_y, \quad Z_z, \quad Y_z = Z_y, \quad Z_x = X_z, \quad X_y = Y_x,
$$

should be homogeneous, linear functions of the preceding ones that, in turn, prove to be homogeneous, linear functions of the latter.

Volterra defined regular deformations to be ones for which (1) are finite, continuous, monodromic functions in the entire domain that the elastic body represents, and whose

([1]) "Sulle superficie di discontinuità nella teoria della elasticità dei corpi solidi," these Rendiconti (5) **10** (1901), 1st sem.

([2]) A series of notes in these Rendiconti (5) **14** (1905), 1st sem; "Sull'equilibrio dei corpi elastici molteplicemente connessi," in Nuovo Cimento (5) **10** and **11** (1905-06); "Sur l'équilibre des corps élastiques multiplement connexes," in Annales de l'École Normale (3) **24** (1907).

([3]) "Sulla teoria delle distorsioni elastiche," two notes in these Rendiconti (5) **23** (1914), 1st sem. and Nuovo Cimento (6) **11** (1916); "Sulle discontinuità dei potenziali elastici," in Atti della R. Accademia delle Scienze di Torino **51** (1915-16).

first and second order derivatives are also finite, continuous, and monodromic ([1]), and it was that kind of deformation that formed the object of his research. Under these conditions, the elastic displacement, leaving aside the case in which it degenerates into rigid displacements – is necessarily continuous in a simply-connected region. The monodromy is always assumed, unless stated to the contrary. Therefore, discontinuities can arise in a multiply-connected region, and any surface that represents a "diaphragm" will tend to diminish the degree of connectivity of the region, and the resulting discontinuity will be defined by a rigid infinitesimal displacement of the two edges of a slit that is imagined to have been made in the diaphragm with respect to each other. The fact that one can assign an infinitude of diverse positions to the surface of discontinuity in relation to the assignment of the diaphragm without changing the distribution of the dilatation parameters translates into the possibility of representing that displacement of the elastic body in the form of continuous, polydromic functions of the coordinates is attributed to Volterra ([2]).

Therefore, we adopt Somigliana's terminology of "Weingarten displacements" and "Volterra displacements" ([3]), which are a special case of them that confers special interest upon the nature of the discontinuity and the indifference, if only between certain limits, of the position of the surface upon which that discontinuity is verified.

Now, the discussion that Weingarten gave to the condition that we recalled seems to affirm its necessity if one is to maintain the integrity of the body considered ([4]). Whereas in order for this to be true, it is also necessary to have continuity of the displacement and the specific pressure across the assumed discontinuity surface at any point of that surface relative to the ray that has the normal direction at that point and a pre-established sense ([5]). Namely, the continuity of (2) is not necessary for this purpose, but only that of:

$$
\tag{3}
\left\{
\begin{aligned}
X_n &= X_x \cos\widehat{nx} + X_y \cos\widehat{ny} + X_z \cos\widehat{nz},\\
Y_n &= Y_x \cos\widehat{nx} + Y_y \cos\widehat{ny} + Y_z \cos\widehat{nz},\\
Z_n &= Z_x \cos\widehat{nx} + Z_y \cos\widehat{ny} + Z_z \cos\widehat{nz},
\end{aligned}
\right.
$$

which are the components of the specific pressure above.

([1]) "Un teorema sulla teoria della elasticità," these Rendiconti (5) **10** (1905), 1st sem. and chap I, art. I of the cited paper in Nuovo Cimento.

([2]) See my note, "Sugli spostamenti elastici discontinui" in these Rendiconti (5) **17** (1908), 1st sem. There, I endeavored to deduce the aforementioned results of Volterra in a way that I believe is more direct. I take this opportunity to point out a correction on page 574, line 17 of the constants in the functions, which should be noted in the rest of the text, moreover. Whoever would seek the reason for the difference in the behavior of the p, q, r and ξ, η, ζ is cautioned that the circuits relate to the former pre-existing application of the diaphragm, while it is only after interruption the corresponding circuits with them that one can proceed with the calculation of the latter.

([3]) "If the stresses that exist internally are continuous all of the space that is occupied by the body then it will have the character of a single, unique body, but if the stresses become discontinuous or the displacement are discontinuous then the body can be considered to have the character of several distinct bodies." *loc. cit.*

([4]) "Sulle deformazione elastiche non regolari," Atti del IV Congresso Internazionale dei Mathematici, Roma, 1908.

([5]) See, e.g., my *Principii della teoria matematica del movimento dei corpo*, § 405.

Therefore, the same Weingarten displacements do not represent a very special case among the possibilities.

The substantial progress that Somigliana made in the theory of elastic distortions consists of the introduction of the necessary condition above in place of the Weingarten condition.

In that way, if let D denote the difference between the limits that are approached as a point considered approaches a point of the surface from the side for which the sense of normal is taken to be positive and from the opposite side (viz., $D = \lim_{n>0} - \lim_{n<0}$) for a well-defined form of the discontinuity of the displacement then one will have the six equations:

$$D\xi = \xi_\sigma, \qquad D\eta = \eta_\sigma, \qquad D\zeta = \zeta_\sigma, \tag{4}$$

$$DX_z = 0, \qquad DY_z = 0, \qquad DZ_z = 0 \tag{5}$$

on any surface σ of discontinuity, in which the limit point is taken to be the origin and z-axis is defined by the normal n, when taken in the positive sense. With that, the x-axis and y-axis will prove to be tangent to the surface at the limit point and ξ_σ, η_σ, ζ_σ will represent any given functions that one desires of either the curvilinear coordinates that belong to the surface or x, y, which are, in that case, deduced from known functions of x, y, z that make $z = 0$.

Somigliana proved for the first time (we recall only the results that suit our purposes) that the six equations (4), (5) serve to determine the discontinuity D in each of the six dilatation parameters ([1]).

From the expressions for these discontinuities, he then deduced the discontinuities in the first derivatives with respect to the coordinates of the components of the displacement ξ, η¸ ζ and then, with the aid of the equations of equilibrium, when the limiting force (i.e., the volume force) is assumed to be zero (or simply continuous on the surface σ), the discontinuities in the second derivatives of those functions. In that way, Somigliana arrived at the result upon which he insisted especially that the discontinuities in the dilatation parameters are determined by equations (4), taking into account (5) and the equations of elastic equilibrium, as well as those of the first and second derivatives of the components of the displacement ([2]).

One recognizes immediately how, with the aid of the equations that one obtains, differentiating both sides of the equations of equilibrium with respect to the single coordinates will result in the determination of the discontinuities in the derivatives of successive order of the components of the displacement in the same way. A result of all that we propose to examine here is that the previously-enumerated conditions, such as the ones that define a regular deformation (with the Volterra terminology), represent a surplus. Therefore, to the extent that it is permissible, we shall demand the continuity of the dilatation parameters and the first and second derivatives of the dilatation parameters, and they seem to belong to a displacement that is provoked by a discontinuity of a pre-established form.

The form that is representable by means of a rigid displacement of the two edges of a slit that is made in the discontinuity surface with respect to each other falls within the

---

([1]) These Rendiconti and Nuovo Cimento, note 1.
([2]) These Rendiconti and Nuovo Cimento, note 2.

category that relates to Weingarten displacements ([1]). For them, one can also assert the continuity of the dilatation parameters. It then remains to examine the first and second derivatives, and we shall now prove their continuity: That is, we prove the continuity of the derivatives of the components of the displacements up to those of third order, at which point, it jumps, of course. I believe that the conclusions that Volterra made relative to the possibility of the deformation of elastic equilibrium that we speak of, which point to existence theorems when they are applied to auxiliary deformations ([2]) (by a calculation that is not simple), remove the opportunity to formulate the mathematical theory of a type of elastic equilibrium that experiment has brought to our attention by this direct verification of an indispensible property.

Preserve the significance that was attributed to x, y, z and ξ, η, ζ in (4) and (5), by which the x and y axes are tangent to the surface σ at the point considered, which is assumed to be the origin. Then imagine, in the known way, a grid of orthogonal coordinate lines that are adjacent to that surface and let u and v denote the variable parameters along the lines of the two families that determine the points of that surface by their intersection. With that, as usual:

$$E\,du^2 + G\,dv^2$$

represents the square of the differential length of an arc that goes through the point (u, v).

If one understands that the x and y axes are tangent to the first and second of the indicated line coordinates, respectively (v = const. and u = const., resp.), then one will verify the relations ([3]):

$$\frac{\partial}{\partial u} = \sqrt{E}\,\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial v} = \sqrt{G}\,\frac{\partial}{\partial y}, \tag{6}$$

$$\left\{\begin{aligned} \frac{\partial^2}{\partial u^2} &= E\,\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial}{\partial y}\frac{\partial^2 y}{\partial u^2} + \frac{\partial}{\partial z}\frac{\partial^2 z}{\partial u^2}, \\ \frac{\partial^2}{\partial v^2} &= G\,\frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial x}\frac{\partial^2 x}{\partial v^2} + \frac{\partial}{\partial y}\frac{\partial^2 y}{\partial v^2} + \frac{\partial}{\partial z}\frac{\partial^2 z}{\partial v^2}, \\ \frac{\partial^2}{\partial u\,\partial v} &= \sqrt{EG}\,\frac{\partial^2}{\partial x\,\partial y} + \frac{\partial}{\partial x}\frac{\partial^2 x}{\partial u^2} + \frac{\partial}{\partial y}\frac{\partial^2 y}{\partial u\,\partial v} + \frac{\partial}{\partial z}\frac{\partial^2 z}{\partial u\,\partial v}. \end{aligned}\right. \tag{7}$$

Now, let φ denote a function of x, y, z or u, v, n, as are the ξ, η, ζ, to which, one attributes the properties that these functions possess (at least, in the vicinity of σ), and set:

$$\lim_{n>0} - \lim_{n<0} = D, \qquad D\varphi = \varphi_\sigma, \tag{8}$$

([1]) Cited note.

([2]) "Sull'equilibrio dei corpi elastici più volte connessi," these Rendiconti (5) **14** (1905), 1st sem. and chap. II, art. II of the cited paper in Nuovo Cimento.

([3]) Cf., my note, "Sopra una formola commutative e alcune sue applicazioni" on page 189 of the present volume of these Renidconti.

with which, φ_σ will represent a function of u, v or x, y that is deduced from a function of x, y, z that makes $z = 0$.

Recall the commutation formula, where the function φ is intended to satisfy the conditions that are required for its validity ([1]):

$$D\,\frac{\partial^{\mu+\nu}\varphi}{\partial u^\mu\,\partial u^\nu} = \frac{\partial^{\mu+\nu} D\varphi}{\partial u^\mu\,\partial u^\nu}. \tag{9}$$

With that, one gets from (6) that:

$$D\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi_\sigma}{\partial x}, \qquad D\frac{\partial\varphi}{\partial y} = \frac{\partial\varphi_\sigma}{\partial y}, \tag{10}$$

$$D\frac{\partial^2\varphi}{\partial x\,\partial z} = \frac{\partial}{\partial x}D\frac{\partial\varphi}{\partial z}, \qquad D\frac{\partial^2\varphi}{\partial y\,\partial z} = \frac{\partial}{\partial y}D\frac{\partial\varphi}{\partial z}, \tag{11}$$

and from (7):

$$\left\{\begin{aligned} D\frac{\partial^2\varphi}{\partial x^2} &= \frac{\partial^2\varphi_\sigma}{\partial x^2} + \frac{1}{E}\frac{\partial^2 z}{\partial x^2}\delta, \\ D\frac{\partial^2\varphi}{\partial y^2} &= \frac{\partial^2\varphi_\sigma}{\partial y^2} + \frac{1}{G}\frac{\partial^2 z}{\partial v^2}\delta, \qquad \delta = \frac{\partial\varphi_\sigma}{\partial z} - D\frac{\partial\varphi}{\partial z}, \\ D\frac{\partial^2\varphi}{\partial x\,\partial y} &= \frac{\partial^2\varphi_\sigma}{\partial x\,\partial y} + \frac{1}{\sqrt{EG}}\frac{\partial^2 z}{\partial u\,\partial v}\delta. \end{aligned}\right. \tag{12}$$

Take φ to be the ξ, η, ζ. Using (10), form the jump in their first derivatives with respect to the tangential coordinates x and y.

If the body is understood to be isotropic, i.e.:

$$X_x = -\lambda\kappa - 2\mu\, x_x, \qquad Y_z = -\mu\, y_z, \qquad \kappa = x_x + y_y + z_z,$$

and analogous formulas, in which λ, μ denote the "elastic constants," with the aid of (5), and invoking (1), one will find immediately that ([2]):

$$D\frac{\partial\xi}{\partial z} = -\frac{\partial\zeta_\sigma}{\partial x}, \qquad D\frac{\partial\eta}{\partial z} = -\frac{\partial\zeta_\sigma}{\partial y}, \qquad D\frac{\partial\zeta}{\partial z} = \frac{\lambda}{\lambda+2\mu}\left(\frac{\partial\xi_\sigma}{\partial x} + \frac{\partial\eta_\sigma}{\partial y}\right). \tag{13}$$

Now, introduce the special form of the discontinuity in the displacement that refers to our present discussion, and then set:

$$\xi_\sigma = a + qz - ry, \qquad \eta_\sigma = b + rx - pz, \qquad \zeta_\sigma = c + py - qx \qquad (z = 0), \tag{14}$$

([1]) See my note above.
([2]) Somigliana: the first of the cited notes "Sulle teoria delle distorsioni elastiche."

in which a, b, c, p, q, r represent arbitrary constants when one fixes the limit point.

From (10) and (13), one has:

$$\text{(15)} \qquad \left\{ \begin{array}{lll} D\dfrac{\partial \xi}{\partial x} = 0, & D\dfrac{\partial \xi}{\partial y} = -r, & D\dfrac{\partial \xi}{\partial z} = q, \\ D\dfrac{\partial \eta}{\partial x} = 0, & D\dfrac{\partial \eta}{\partial y} = 0, & D\dfrac{\partial \eta}{\partial z} = -p, \\ D\dfrac{\partial \zeta}{\partial x} = -q, & D\dfrac{\partial \zeta}{\partial y} = p, & D\dfrac{\partial \zeta}{\partial z} = 0. \end{array} \right.$$

Therefore, the jumps in each of the dilatation parameters [cf., (1)] are then zero, which agrees with the result that was recalled already. From (14), all of the second derivatives of ξ_σ, η_σ, ζ_σ with respect to x and y will then be zero. In addition, from (14) and (15), one verifies that:

$$\text{(16)} \qquad \delta = \frac{\partial \varphi_\sigma}{\partial z} - D\frac{\partial \varphi}{\partial z} = 0,$$

no matter which of ξ, η, ζ is represented by φ.

It follows from (11) and (12) that all of the jumps in the second derivatives of ξ, η, ζ with respect to the coordinates x, y, z are zero, except (for the moment) $\dfrac{\partial^2 \xi}{\partial z^2}$, $\dfrac{\partial^2 \eta}{\partial z^2}$, $\dfrac{\partial^2 \zeta}{\partial z^2}$.

For them, we recall the equations of equilibrium:

$$\text{(17)} \qquad \left\{ \begin{array}{l} (\lambda+\mu)\dfrac{\partial \kappa}{\partial x} + \mu\,\Delta_2 \xi + X = 0, \\ (\lambda+\mu)\dfrac{\partial \kappa}{\partial y} + \mu\,\Delta_2 \eta + Y = 0, \\ (\lambda+\mu)\dfrac{\partial \kappa}{\partial z} + \mu\,\Delta_2 \zeta + Z = 0, \\ \kappa = \dfrac{\partial \xi}{\partial x} + \dfrac{\partial \eta}{\partial y} + \dfrac{\partial \zeta}{\partial z}, \qquad \Delta_2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}, \end{array} \right.$$

in which X, Y, Z are meant to be continuous on the surface σ.

Applying D to these equations will give three equations that give $D\dfrac{\partial^2 \xi}{\partial z^2}$, $D\dfrac{\partial^2 \eta}{\partial z^2}$, $D\dfrac{\partial^2 \zeta}{\partial z^2}$, respectively, as homogeneous, linear functions of D of the remaining second derivatives of the ξ, η, ζ. They will all be zero, in such a way that it will also follow that:

(18) $$D\frac{\partial^2\xi}{\partial z^2} = 0, \qquad D\frac{\partial^2\eta}{\partial z^2} = 0, \qquad D\frac{\partial^2\zeta}{\partial z^2} = 0.$$

One concludes that all of the jumps in the second derivatives of ξ, η, ζ.are zero, and consequently, conforming to (1) the jumps in the first derivatives of the dilatation parameters will be zero.

In order to prove that the jumps in the second derivatives of those dilatation parameters are zero, one proves, in an analogous way, that all of the jumps in the third derivatives of the ξ, η, ζ.with respect to the coordinates x, y, z are zero.

We therefore make use of series of formulas that are established by continuing the procedure that was followed in order to define (10), (11), (12), into which, of course, we introduce the preceding results on the jumps in the second derivatives.

With the same significance for φ, while always applying (9), one gets from (6), in the first place:

$$D\frac{\partial^2\varphi}{\partial x\,\partial z^2} = \frac{\partial}{\partial x}D\frac{\partial^2\varphi}{\partial z^2}, \qquad D\frac{\partial^2\varphi}{\partial y\,\partial z^2} = \frac{\partial}{\partial y}D\frac{\partial^2\varphi}{\partial z^2},$$

from which:

$$D\frac{\partial^2\varphi}{\partial x\,\partial z^2} = 0, \quad D\frac{\partial^2\varphi}{\partial y\,\partial z^2} = 0\ ;$$

in the second place, one gets from (7) that:

$$D\frac{\partial^2\varphi}{\partial x^2\,\partial z} = \frac{\partial^2}{\partial x^2}D\frac{\partial\varphi}{\partial z} + \frac{1}{E}\frac{\partial^2 z}{\partial u^2}\delta',$$

$$D\frac{\partial^2\varphi}{\partial y^2\,\partial z} = \frac{\partial^2}{\partial y^2}D\frac{\partial\varphi}{\partial z} + \frac{1}{G}\frac{\partial^2 z}{\partial v^2}\delta', \qquad \delta' = \frac{\partial}{\partial z}D\frac{\partial\varphi}{\partial z} - D\frac{\partial^2\varphi}{\partial z^2},$$

$$D\frac{\partial^2\varphi}{\partial x\,\partial y\,\partial z} = \frac{\partial^2}{\partial x\,\partial y}D\frac{\partial\varphi}{\partial z} + \frac{1}{\sqrt{EG}}\frac{\partial^2 z}{\partial u\,\partial v}\delta',$$

by which:

$$D\frac{\partial^2\varphi}{\partial x^2\,\partial z} = 0, \qquad D\frac{\partial^2\varphi}{\partial y^2\,\partial z} = 0, \qquad D\frac{\partial^2\varphi}{\partial x\,\partial y\,\partial z} = 0.$$

As for the remaining three derivatives, we begin by finding the formulas that result from (6) and (7). For example, we obtain:

$$\frac{\partial^3}{\partial u^3} = E\sqrt{E}\,\frac{\partial^3}{\partial x^3} + 3\sqrt{E}\left(\frac{\partial^2}{\partial x^2}\cdot\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2}{\partial x\,\partial y}\cdot\frac{\partial^2 y}{\partial u^2} + \frac{\partial^2}{\partial x\,\partial z}\cdot\frac{\partial^2 z}{\partial u^2}\right)$$

$$+\frac{\partial}{\partial x}\cdot\frac{\partial^3 x}{\partial u^3}+\frac{\partial}{\partial y}\cdot\frac{\partial^3 y}{\partial u^3}+\frac{\partial}{\partial z}\cdot\frac{\partial^3 z}{\partial u^3}.$$

It then follows, making use of (9) and taking (10), (11), and (12) into account, that:

$$D\frac{\partial^3\varphi}{\partial x^3}=\frac{\partial^3\varphi_\sigma}{\partial x^3}+\frac{3}{E\sqrt{E}}\left(\frac{1}{\sqrt{E}}\frac{\partial^2 x}{\partial u^2}\frac{\partial^2 z}{\partial u^2}+\frac{1}{\sqrt{G}}\frac{\partial^2 y}{\partial u^2}\frac{\partial^2 z}{\partial u\,\partial v}\right)\delta,$$

from which, conforming to the hypotheses and the preceding results:

$$D\frac{\partial^3\varphi}{\partial x^3}=0.$$

One finds, in the same way, that:

$$D\frac{\partial^3\varphi}{\partial y^3}=0,\qquad D\frac{\partial^3\varphi}{\partial x^2\,\partial y}=0,\qquad D\frac{\partial^3\varphi}{\partial x\,\partial y^2}=0.$$

What remain are:

(19) $$D\frac{\partial^3\xi}{\partial z^3},\qquad D\frac{\partial^3\eta}{\partial z^3},\qquad D\frac{\partial^3\zeta}{\partial z^3}.$$

In order to find them, take the ordinary derivatives of the three equilibrium equations (17) with respect to x, y, and z, and suppose that $\partial X / \partial x$, $\partial Y / \partial y$, $\partial Z / \partial z$ are continuous on the surface σ, and then apply D to the two sides of the equations thus formed. That will give (19) as homogeneous, linear functions of the D of the remaining third derivatives of ξ, η, ζ, which all prove to be zero. Finally, it then results from this that:

$$D\frac{\partial^3\xi}{\partial z^3}=0,\qquad D\frac{\partial^3\eta}{\partial z^3}=0,\qquad D\frac{\partial^3\zeta}{\partial z^3}=0.$$