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ON THE SINGULARITIES OF DIFFERENTIAL FORMS

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TABLE OF CONTENTS

Introduction

Let M be a differentiable manifold and let ω be an exterior differential form on M; numerous examples justify the value of a general study of the pair (M, ω); such as:

1) A field of contact elements of codimension p on M, i.e., a sub-bundle of codimension p of the tangent bundle, is defined, at least locally, by a non-null decomposable p-form.

2) A Pfaff form ω without zeros such that $\omega \wedge d\omega = 0$ defines a foliation of codimension 1 on M.

3) At the other extreme, if M is of odd dimension $2p + 1$ and $\omega \wedge d\omega^p \neq 0$ at any point then the Pfaff form ω defines a classical structure called a "contact structure."

4) A 2-form ω (closed 2-form, resp.) of maximum rank on a manifold of even dimension $2p$ (i.e., $\omega^p \neq 0$ at any point) defines an almost-symplectic (symplectic, resp.) structure.

5) If M is a Riemannian manifold, then the *minimal* submanifolds of M are solutions of differential systems that are defined by canonical p-forms on the tangent bundle TM.

The preceding examples bring into play forms that are sufficiently *regular:* constant rank in 1), class equal to 1 or 2 in 2), etc. By contrast, a general study of the pair (M, ω) will be essentially a theory of singularities of the differential forms; this is the theory that is sketched in this work.

The notions of singularities of differentiable maps and vector fields are now classical; I begin the study of forms at an analogous viewpoint.

One so often hears the remark that the singularities of Pfaff forms must correspond to the singularities of vector fields (via a Riemannian metric, for example) that I must first deduce this little myth: any *non-null* vector field has the expression $\partial / \partial x_1$ in a convenient local coordinate system; by contrast, it is practically impossible to classify the germs of *non-null* Pfaff forms up to isomorphism; the rank of the exterior differential $d\omega$ will intervene in this classification, and, more precisely, the *class* (in the sense of E. Cartan) of ω; therefore, the points where the class changes will be the singular points of ω.

In the theory of differentiable maps (R. Thom [15], H. Levine [9]), the rank plays a central role.

In this work, we will be concerned with singularities of three types of objects:

1) Exterior differential forms.

2) Closed exterior differential forms.

3) Pfaff equations, i.e., fields of tangent hyperplanes on a manifold (defined locally by an equation $\omega = 0$, where ω is a Pfaff form with no zeroes).

The role of the rank for differentiable maps will be played by the *rank of an alternating multilinear form* in the case of closed forms, and the *class* in the cases of forms that are not closed and Pfaff equations.

Chapter I is dedicated to *rank* in the exterior algebra of a vector space; there, one very closely studies the stratifications that are defined by sets of forms of a given rank in $\Lambda^p E^*$. Due to the very elementary nature of this chapter, I believe it is quite original; in any case, I have found hardly a trace of the preoccupations that are found here in the classical works of Goursat and E. Cartan. In the course of this study, I have become convinced that the search for invariants for forms of "intermediate" degrees that are more definitive than the rank will be undoubtedly interesting and amusing.

In Chapter II, with the aid of transversality theorems, I then study the generic nature of sets points where the rank or the class of a differential form (or a Pfaff equation) on a given differentiable manifold decreases. Here is an example of a result in that direction:

Let M be a differentiable manifold of dimensions n and let ω be a Pfaff form on M; let $\zeta_d(\omega)$ be the set of points where the class of ω is $n - d$; generically, $\zeta_d(\omega)$ is a regular sub-manifold of M of codimension $\frac{d(d+1)}{2}$; in particular, the class is generally very large everywhere, and minorized by a quantity that has the approximate value $n-\sqrt{2n}$ (cf. II.4.3.3).

One may then study singularities of higher order, as in the case of differentiable maps; I have realized this objective only in the case of closed 2-forms in dimension 4, and Pfaff equations in dimension 3, where one obtains a classification that is already satisfied by the generic singularities (II.3 and II.6) in orders 1 and 2.

Chapter III is essentially dedicated to the search for *models* of a given singularity; this amounts to classifying the germs of forms that present the singularity being considered, up to isomorphism.

One knows the classical models in a certain number of regular situations: volume form, closed 2-form of maximal rank, Pfaff form of maximal class (Darboux's theorem). Starting with these results, I may very easily show that in a very large number of cases the simplest singularities admit a model; for example, the simplest singularities of a Pfaff form in even dimension admit:

$$\omega = (1 \pm x_1^2)dy_1 + x_2 dy_2 + \cdots + x_p dy_p ,$$

for a model. The tools that are used here are the implicit function theorem, the theorem of existence and uniqueness of solutions to differential equations, and the divisibility properties of differentiable functions. Meanwhile, given the simplicity of the situations envisioned, I have not had to use the differentiable preparation theorem.

On the other hand, I have included several remarks in this chapter (III.B) that relate to the notions of *stability* and *infinitesimal stability* for a germ of a differential form; in this context, one may pose a problem that is analogous to the one that was recently solved by J. Mather in the case of differentiable maps, but a different order of difficulty; it essentially amounts to a linear problem for the maps. By contrast, a differential operator (of order 1) intervenes in the case of forms.

We do not begin to discuss the global problems of the theory here. Nonetheless, we mention that one may make a homological study of the singular set of a differential form that is analogous to the case of differentiable maps (cf. [8]). On the other hand, one may cite several results that relate to the global stability of forms or Pfaff equations (in particular, see [7] and [14]).

Finally I have judged it useful to give a very brief summary of the theory of transversality in an Appendix; it was difficult for me to indicate the references that led directly to the results that I found necessary. Above all, I insist on the transversality theorem that relates sections of a vector bundle, which is technically easier; for the sections of an arbitrary fiber bundle, one comes back to the preceding by a simple linearization procedure.

In conclusion, there are no new techniques in this work; one only applies the methods that are now available to a situation that has not been studied up till now and is certainly very rich in interest. On the other hand, I regret that I have not given any applications of the theory that is sketched out here; it is possible that it provides a means of approach to the problem of the existence of structures that defined on a given manifold by differential forms *without singularities* (for example, the existence of a contact structure on a compact orientable manifold of dimension 3; cf., S.S. Chern [4]).

A part of the results in this work has been announced in two notes to the Comptes Rendus ([11], [12]).

It remains for me to point out that Professor E. Calabi has independently obtained certain results that appear here (as well as others that do not), and that he has made remarks to me that allowed me to ameliorate certain points.

This article constitutes the essence of the work that I have presented as a doctoral thesis to the Faculté des Sciences de Grenoble. I would like to heartily thank G. Reeb, who suggested this study to me, C. Chabauty, who graciously presided over the jury, and O. Galvani and R. Thom, for the interest that they have shown in my work and their participation in the jury.

CHAPTER I

RANK IN EXTERIOR ALGEBRA

All of the vectors spaces envisioned in this chapter are of finite dimension over **R**.

1. Notations – Definitions – Summary.

1.1 Let E be a vector space of dimension n; for any p ($1 \leq p \leq n$), one notates the p^{th} exterior product of E by $\Lambda^p E$.

If h: $E \to F$ is a linear map, then one notates the p^{th} exterior product of h by:

$$h^p: \Lambda^p E \to \Lambda^p F.$$

Let F be vector subspace of E and let i: $F \to E$ be the canonical injection. i^p: $\Lambda^p F \to \Lambda^p E$ is an injection, and in the sequel one will always identify $\Lambda^p F$ and its image in $\Lambda^p E$ by i^p.

By means of this identification, the following relations are verified ([2]):

$$\Lambda^p(F_1 \cap F_2) = \Lambda^p F_1 \cap \Lambda^p F_2 \tag{1}$$

for any subspaces F_1 and F_2 of E.

$$h^p(\Lambda^p E) = \Lambda^p[h(E)] \tag{2}$$

where h: $E \to F$ is a linear map.

1.2. DEFINITION ([2], pp. 72). – *Let E be a vector space and let $\omega \in \Lambda^p E$ be a p-vector of E. The support of ω is the smallest subspace $S_\omega \subset E$ such that $\omega \in \Lambda^p S_\omega$; its dimension is the rank of ω; the corank of ω is the codimension of S_ω in E.*

This definition is justified by relation (1).

PROPOSITION. – *Let h: $E \to F$ be an injective linear map and let $\omega \in \Lambda^p E$; one has*:

$$h(S_\omega) = S_{h^p(\omega)}.$$

This is an immediate consequence of relations (1) and (2). This result, when applied to an automorphism of E, shows that rank is an *invariant* of the canonical action of the linear group, $Gl(E)$, in $\Lambda^p E$.

The rank of a non-null p-vector of E is obviously between p and $n = \dim E$; the preceding proposition, when applied to the injection i: $S_\omega \to E$, permits us to consider ω as a p-vector of maximum rank in S_ω.

1.3. Let E be a vector space; E^* will denote the dual of E and $\Lambda^p E^*$, the space of p-vectors of E^*, which is identified with the space of alternating p-forms on E.

One will notate the *interior product* of $\omega \in \Lambda^p E^*$ with $x \in E$ by $x \rfloor \omega$.

DEFINITION. – ([3], [5], [6]). *Suppose* $\omega \in \Lambda^p E^*$; *one calls the subspace* $A_\omega \subset E$ *that is defined by*:

$$A_\omega = \{x \in E : x \rfloor \omega = 0\},$$

the associated space *of* ω.

PROPOSITION. – *Suppose* $\omega \in \Lambda^p E^*$; *the subspace* $A_\omega \subset E$ *that is associated with* ω *and the subspace* $S_\omega \subset E^*$, *which is the support of* ω, *are orthogonal.*

The rank of a p-form on E is therefore likewise the codimension of its associated space in E. In this context, the support of a p-form ω will also be called the *associated system* to ω.

Remark. – Suppose $\omega \in \Lambda^p E^*$; let $i_\omega : E \to \Lambda^{(p-1)} E^*$ be the linear map defined by $i_\omega(x) = x \rfloor \omega$, let $j_\omega : \Lambda^{(p-1)} E^* \to E^*$ be the linear map defined by $j_\omega(X) = (-1)^{p-1} X \rfloor \omega$. It is immediate that j_ω is the transpose of i_ω; the image of j_ω is thus the orthogonal to the kernel of i_ω; i.e., the support of ω. In other words, if ω is a p-form then the forms:

$$\omega(x_1, \ldots, x_{p-1}), \qquad (x_1, \ldots, x_{p-1} \in E)$$

generate the support of ω in E^*.

1.4 Immediate properties of rank.

1.4.1. Let $\omega = \alpha_1 \wedge \ldots \wedge \alpha_p$ be a non-null decomposable p-form; it is immediate that S_ω is the subspace of E^* that is generated by the independent k-forms $\alpha_1, \ldots, \alpha_p$; the rank of ω is therefore equal to p.

Conversely, if $\omega \in \Lambda^p E^*$ has rank p, like $\omega \in S_\omega$, where $\dim S_\omega = p$, then one has $\omega = \lambda\alpha_1 \wedge \ldots \wedge \lambda\alpha_p$, where $(\alpha_1, \ldots, \alpha_p)$ is a basis for S_ω and λ is a convenient scalar; ω is therefore decomposable.

1.4.2. Let $\omega \in \Lambda^{(n-1)} E^*$, $\omega \neq 0$, with $n = \dim E$; let $\Omega \in \Lambda^n E^*$, $\Omega \neq 0$. The linear map $i_\Omega : E \to \Lambda^{(n-1)} E^*$ that is defined by $i_\Omega(x) = x \rfloor \Omega$, is an injection; it is therefore an isomorphism since the dimensions of the source and target are equal. There thus exists a vector $x \neq 0$ such that $\omega = x \rfloor \Omega$; then $x \rfloor \omega = 0$, i.e., $x \in A_\omega$; therefore the rank of ω is strictly less than n; it is therefore equal to $n - 1$, and ω is decomposable.

1.4.3. One immediately deduces from 1.4.2 that a p-form may not be of rank $(p + 1)$.

1.4.4. Proposition. – *Let* ω *and* ω' *be two p-forms on E* ($p \geq 2$); *if* $\dim(S_\omega \cap S_{\omega'}) \leq p - 2$, *then* $S_{\omega+\omega'} = S_\omega + S_{\omega'}$; *in particular, if* $S_\omega \cap S_{\omega'} = \{0\}$ *then* $rank(\omega + \omega') = rank(\omega) + rank(\omega')$.

Proof. – One considers the linear equation:

$$x \rfloor (\omega + \omega') = x \rfloor \omega + x \rfloor \omega' = 0 \qquad (x \in E).$$

It is clear that $x \rfloor \omega \in \Lambda^{(p-1)} S_\omega$ and $x \rfloor \omega' \in \Lambda^{(p-1)} S_{\omega'}$; however:

$$\Lambda^{(p-1)} S_\omega \cap \Lambda^{(p-1)} S_{\omega'} = \{0\},$$

by hypothesis; one therefore has $x \rfloor \omega = 0$ and $x \rfloor \omega' = 9$, namely:

$$A_{\omega+\omega'} = A_\omega \cap A_{\omega'},$$

and, by passing to orthogonal complements, $S_{\omega+\omega'} = S_\omega + S_{\omega'}$.

1.4.5. PROPOSITION. – *Let* $\omega \in \Lambda^p E^*$ *and* $\omega' \in \Lambda^q E^*$, *with:*

$$S_\omega \cap S_{\omega'} = \{0\};$$

then $S_{\omega \wedge \omega'} = S_\omega \oplus S_{\omega'}$ *and* $rank(\omega \wedge \omega') = rank(\omega) + rank(\omega')$.

The proof is likewise very easy; we shall not give it.

2. General study of the sets $\Sigma_{n,p}^p$.

In all of what follows, we will let $\Sigma_{n,r}^p$ denote the set of elements of $\Lambda^p E^*$ that have rank r (where $n = \dim E$).

We propose to study the figure that is formed in $\Lambda^p E^*$ by the sets $\Sigma_{n,r}^p$ in a very detailed manner, and to compare these sets, which are invariant under the action of $Gl(E)$, with the orbits of that group.

Before we begin this study for different values of p, we state a general result.

PROPOSITION. – *Suppose* $\Sigma_{n,r}^p \neq \varnothing$; *then*:

a) $\Sigma_{n,r}^p$ *is a regular submanifold of* $\Lambda^p E^*$ *of dimension* $C_r^p + r(n-r)$.

b) $\overline{\Sigma_{n,r}^p}$ *is an algebraic sub-variety of* $\Lambda^p E^*$, *and:*

$$\overline{\Sigma_{n,r}^p} = \bigcup_{r' \leq r} \Sigma_{n,r'}^p .$$

Proof. – We first remark that once one has eliminated the trivial case $r = 0$ one necessarily has $r \geq p$.

Let $G_{r,n-r}$ be the Grassmannian of r-planes of E^*. Let $F_r^p \to G_{r,n-r}$ be the fiber bundle defined by the pairs (ν, α) where $\nu \in G_{r,n-r}$, and $\alpha \in \Lambda^p \nu$ (the fiber over ν is therefore the vector space $\Lambda^p \nu$).

Set $h(\nu, \alpha) = \alpha \in \Lambda^p E^*$; h is a continuous algebraic map.

From proposition 1.2, it is clear that $h(F_r^p) = \bigcup_{r' \leq r} \Sigma_{n,r'}^p$.

The set $\bigcup_{r' \leq r} \Sigma_{n,r'}^p$ is an algebraic sub-variety of $\Lambda^p E^*$ (since it is the set of $\omega \in \Lambda^p E^*$ such that the linear map i_ω: $E \to \Lambda^{p-1} E^*$ is of rank less than or equal to r).

On the other hand, the set of pairs $(\nu, \alpha) \in F_r^p$ such that $\alpha \in \Lambda^p \nu$ is of maximum rank r (i.e., its support is ν) is an open set Ω, which non-vacuous, by hypothesis, therefore it is dense in F_r^p; since $h(\Omega) = \Sigma_{n,r}^p$, part *b*) is proved.

It is almost obvious that h defines a homeomorphism of Ω on $\Sigma_{n,r}^p$. It remains to show that h is an immersion upon restriction to Ω.

Let $(\nu, \alpha) \in \Omega$; let $(e_1, \ldots, e_n)$ be a base of E, such that in the dual basis $(e_1^*, \cdots, e_n^*)$ of E^* the forms $(e_1^*, \cdots, e_r^*)$ constitute a basis for ν; one lets w notate the supplement to v that is generated by $(e_{r+1}^*, \cdots, e_n^*)$.

We can associate a trivialization of F_r^p:

$$\mathrm{Hom}(v, w) \times \Lambda^p \nu \to F_r^p,$$

with the basis $(e_1^*, \cdots, e_n^*)$ in a neighborhood of (v, α); to the point:

$$(a_i^j, \mu_\sigma),$$

where $i = 1, \ldots, r$, $j = r + 1, \ldots, n$, $\sigma = (\sigma_1, \ldots, \sigma_p)$, $1 \leq \sigma_1 < \ldots < \sigma_p \leq r$, we can associate the pair (v', α') such that:

1) v' has $(e_1^* + a_1, \cdots, e_r^* + a_r)$ for a basis, where $a_i \in w$, such that:

$$a_i = \sum_{j=r+1}^{n} a_i^j e_j^*.$$

2) $\alpha' = \sum_\sigma \mu_\sigma (e^* + a)_\sigma$ where:

$$(e^* + a)_\sigma = (e_{\sigma_1}^* + a_{\sigma_1}) \wedge \cdots \wedge (e_{\sigma_p}^* + a_{\sigma_p}).$$

Formula 2) represents the expression for h in the chart considered for F_r^p. An immediate calculation shows that the expression for the derivative h' of h at the point (v, α) with the coordinates $\{a_i^j = 0; \mu_\sigma = \lambda_\sigma\}$ is:

$$\begin{aligned} h'(\varepsilon_i^j;\eta_\sigma) &= \sum_\sigma \eta_\sigma e_\sigma^* + \sum_\sigma \lambda_\sigma \left(\sum_{i=1}^p e_{\sigma_1}^* \wedge \cdots \wedge \varepsilon_{\sigma_i} \wedge \cdots \wedge e_{\sigma_p}^* \right) \\ &= \sum_\sigma \eta_\sigma e_\sigma^* + \sum_{\substack{\sigma \\ i=1,\cdots,r}} \lambda_\sigma \varepsilon_i \wedge (e_i \lrcorner e_\sigma^*) \\ &= \sum_\sigma \eta_\sigma e_\sigma^* + \sum_{j=r+1}^n e_\sigma^* \wedge \left[\left(\sum_{i=1}^r \varepsilon_i^j e_j \right) \lrcorner \alpha \right]. \end{aligned}$$

Therefore, $h'(\varepsilon_i^j;\eta_\sigma) = 0$ is equivalent to $\eta_\sigma = 0$ for all α, and:

$$\left(\sum_{i=1}^r \varepsilon_i^j e_j \right) \lrcorner \alpha = 0,$$

for all i. However, by hypothesis, the associated space to α in E has (e_{r+1}, …, e_n) for a basis; one therefore has $\varepsilon_i^j = 0$ for all i and j, and h' is injective. Q.E.D.

Remarks.

1) If n and p are given, and r has its maximum value then $\Sigma_{n,r}^p$ is a dense open set in $\Lambda^p E^*$.
2) It is clear that $\overline{\Sigma_{n,r}^p}$ is always a closed set in $\Lambda^p E^*$.
3) The set $\Sigma_{n,r}^p$ of p-forms of rank p (i.e., decomposable ones) is always non-vacuous, and is an orbit of $Gl(E)$ identically.

3. Trivial case.

We always set dim $E = n$.

3.1. A form in E^* is of rank 0 or 1 depending on whether it is null or not; the sets $\Sigma_{n,0}^1 = \{0\}$ and $\Sigma_{n,1}^1 = E^* - \{0\}$ are therefore the orbits of $Gl(E)$ in E^*.

3.2. A non-null n-form is of rank n, and written $\alpha_1 \wedge \ldots \wedge \alpha_n$ for a convenient basis for E^*; $\Sigma_{n,0}^n = \{0\}$ and $\Sigma_{n,n}^n = \Lambda^n E^* - \{0\}$ are the orbits of $Gl(E)$.

3.3. A non-null $(n-1)$-form is of rank $(n-1)$, and may be written $\alpha_1 \wedge \ldots \wedge \alpha_{n-1}$ ($\alpha_1, \ldots, \alpha_n$ are independent in E^*); $\Sigma_{n,0}^{n-1} = \{0\}$ and $\Sigma_{n,n-1}^{n-1} = \Lambda^{n-1} E^* - \{0\}$ are the orbits of $Gl(E)$.

4. Case of 2-forms.

4.1. Recall that:

PROPOSITION. – ([3], pp. 12, [5], pp. 31). *Any* 2*-form* $\omega \in \Lambda^2 E^*$ *is of even rank, and the following conditions are equivalent*:

1) ω *is of rank* $2k$.
2) $\omega^k \neq 0$ *and* $\omega^{(k+1)} = 0$ ($\omega^k = \omega \wedge \ldots \wedge \omega$, k times).
3) *There exist independent forms* $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k, \in E^*$ *such that*:

$$\omega = \sum_{i=1}^{k} \alpha_i \wedge \beta_i \ .$$

Remark. – From proposition 1.4.4, it is clear that if $\omega = \sum_{i=1}^{k} \alpha_i \wedge \beta_i$ then since the forms α_i and β_i are independent, they constitute a basis for the support of ω.

4.2. PROPOSITION. – *In* $\Lambda^2 E^*$, *the sets* $\Sigma_c = \Sigma^2_{n.n-c}$ *are identical to orbits of* $Gl(E)$; *for any* c ($n - c$ even, $0 \leq n - c \leq n$), Σ_c *is a regular submanifold of* $\Lambda^2 E^*$, *and:*

$$\operatorname{codim} \Sigma_c = \frac{c(c-1)}{2} \ .$$

From 4.1 (condition 3) and 2., the proof is obvious. It is remarkable that the codimensions of these manifolds depend only on the corank; this phenomenon is reminiscent of the result concerning sets of matrices of given rank (for which the codimension is the product of the coranks of the source and target).

Meanwhile, the situation is different according to the parity of the dimension n of E.

If n is even then the admissible values of c are 0, 2, 4, 6, …, and the corresponding codimensions are 0, 1, 6, 15, …

If n is odd then the admissible values of c are 1, 3, 5, 7, …, and the corresponding codimensions are 0, 3, 10, 21, …

Remark 1. – For any $\omega \in \Lambda^2 E^*$ one may indicate a local system of equations for the manifold Σ_c that passes through ω; one proceeds in the following manner:

One identifies the exterior forms of degree 2 with the anti-symmetric linear maps of E into E^*.

One then lets $(\alpha_1, \ldots, \alpha_n)$ be a basis for E^* such that:

$$\omega = \alpha_1 \wedge \alpha_2 + \ldots + \alpha_{2k-1} \wedge \alpha_{2k} ,$$

where $2k = n - c = \operatorname{rank}(\omega)$.

The matrix $\overline{\omega}$ that corresponds to ω is written:

$$\overline{\omega} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where I is the $2k \times 2k$ *anti-symmetric* matrix:

$$\begin{pmatrix} 0 & -1 & 0 & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We then let Ω denote the open set of $\Lambda^2 E^*$ that is comprised of forms $\lambda = \sum_{1 \le i < j \le n} \lambda_{i,j} \alpha_i \wedge \alpha_j$ such that $\sum_{1 \le i < j \le 2k} \lambda_{i,j} \alpha_i \wedge \alpha_j$ is of (maximum) rank $2k$. For any $\lambda \in \Omega$, the corresponding anti-symmetric matrix $\overline{\lambda} = (\overline{\lambda}_{i,j})$ is defined by for $i < j$; set:

$$\overline{\lambda} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

in which A is an anti-symmetric $2k \times 2k$ matrix and $C = -$ (transpose of B); as in ([9], Prop. 2, pp. 6), one shows that $\lambda \in \Omega \cap \Sigma_c$ ($c = n - 2k$) if and only if $D - CA^{-1}B = 0$; this shows that $\Omega \cap \Sigma_c$ is defined as the set of common zeroes of $\frac{c(c-1)}{2}$ independent functions since D is an anti-symmetric $c \times c$ matrix.

Remark 2. – From proposition 2, $\overline{\Sigma_c} = \bigcup_{c' \ge c} \Sigma_{c'}$ is an algebraic sub-variety of $\Lambda^2 E^*$.

If $\overline{\Sigma_{c+2}}$ is non-vacuous then we just showed that for any $\omega \in \overline{\Sigma_{c+2}}$ the tangent at $\omega \in$ contains a basis for $\Lambda^2 E^*$; we will then have proved that *the set of singular points of* $\overline{\Sigma_c}$ *is equal to* $\overline{\Sigma_{c+2}}$ (if c does not have its minimum value).

Let $(\alpha_1, \ldots, \alpha_k)$ be a basis for E^*; for any i, j ($1 \le i < j \le n$), the curve, $t \to \omega + t\, \alpha_i \wedge \alpha_j$ ($t \in [0, 1]$) is traced in $\overline{\Sigma_c}$, and its velocity at the origin of ω is $\alpha_i \wedge \alpha_j$. Q.E.D.

Remark 3. – One knows that the bilinear forms of corank c in the space $\otimes^2 E^*$ define a regular submanifold S_c of codimension c^2 ([9], pp. 5).

On the other hand, $\Sigma_c = \Lambda^2 E^* \cap S_c$.

If the manifold S_c intersects the subspace $\Lambda^2 E^*$ of $\otimes^2 E^*$ in general position then the codimension of S_c in $\otimes^2 E^*$ will be equal to the codimension of S_c in $\Lambda^2 E^*$; one sees that there is nothing else; the loss of codimension in the intersection is likewise increasing with c.

5. Case of forms of degree (n – 2).

5.1. In $\Lambda^{n-2}E^*$, rank may take only the values 0, $n - 2$, n (cf. 1.4.3). A priori, one does not know if there always exist (n – 2)-forms of maximal rank n.

However, $\Sigma_{n,n-2}^{n-2}$, the set of forms of rank $n - 2$ – i.e., decomposable ones – is a submanifold of dimension:

$$C_{n-2}^{n-2} + 2(n-2) = 2n - 3;$$

when $n > 3$ one has $2n - 3 < \dim \Lambda^{n-2}E^*$, and there necessarily exist (n – 2)-forms of rank n.

5.2. Orbits of $Gl(E)$ in $\Lambda^{n-2}E^*$. The results of this section will not be used in the sequel.

5.2.1. Let $\Omega \in \Lambda^n E^*$ be a volume form on E, and let:

$$h_\Omega : \Lambda^{n-2}E^* \to \Lambda^2 E^*$$

be the isomorphism that associates every $\omega \in \Lambda^{n-2}E^*$ with the unique bivector $X = h_\Omega(\omega)$ such that $\omega = X \rfloor \Omega$; if Ω and Ω' are two volume forms then $h_\Omega(\omega)$ and $h_{\Omega'}(\omega)$ are proportional, and one likewise has rank $2k$; the integer k will be called the *length* of the (n – 2)-form ω. *One will denote the set of forms of length k by* S_k.

5.2.2. With the preceding notations, let:

$$\omega = X \rfloor \Omega, \qquad \text{in which } X = h_\Omega(\omega).$$

If g is an automorphism of E then one has:

$$g \cdot \omega = (g^{-1} \cdot X) \rfloor (g \cdot \Omega).$$

However, $g \cdot \Omega = \Delta(g)\Omega$, where $\Delta(g)$ is the determinant of g. If ω and ω' are (n – 2)-forms then upon setting $X = h_\Omega(\omega)$ and $X' = h_\Omega(\omega')$, it is clear that:

$$\omega' = g \cdot \omega \qquad (g \in Gl(E))$$

is equivalent to:

$$g \cdot X' = \Delta(g) X .$$

This shows that *length is an invariant of the action of $Gl(E)$*.

5.2.3. Let $\omega = X \rfloor \Omega$ be an $(n - 2)$-form of length k; from proposition 4.1, one easily deduces the existence of a basis $(e_1, \ldots, e_n)$ of E such that if $(\alpha_1, \ldots, \alpha_n)$ denotes the dual basis for E^* then one has:

$$X = e_1 \wedge e_2 + \ldots + e_{2k-1} \wedge e_{2k},$$
$$\Omega = \lambda\alpha_1 \wedge \ldots \wedge \alpha_n \qquad \text{where } \lambda = \pm 1,$$

so:

$$\omega = \lambda \cdot \sum_{i=1}^{k} \alpha_1 \wedge \ldots \wedge \alpha_{2i-2} \wedge \alpha_{2i+1} \wedge \ldots \wedge \alpha_n .$$

This shows that the set S_k is composed of at most two orbits of Gl(E).

a) $2k < n$; upon possibly changing the sign of e_n and α_n, one reduces ω to the "canonical form":

$$\omega = \sum_{i=1}^{k} \alpha_1 \wedge \ldots \wedge \alpha_{2i-2} \wedge \alpha_{2i+1} \wedge \ldots \wedge \alpha_n .$$

S_k is therefore an orbit of $Gl(E)$ in $\Lambda^{n-2}E^*$, which is bijectively related to $\Sigma_k \in \Lambda^2 E^*$ by h_Ω.

b) $2k = n$; from 5.2.1, $-\omega = g \cdot \omega$ is equivalent to:

$$g \cdot X = -\Delta(g) \cdot X.$$

However, in the present case X^k is a non-null n-vector of E, and one necessarily has:

$$g \cdot X^k = \Delta(g) \cdot X^k = (-1)^k \Delta(g)^k \cdot X^k,$$

namely:

$$(-1)^k = \Delta(g)^{k-1}.$$

b_1) If k is odd then this equality is impossible. Any $(n - 2)$-form of length k therefore reduces to one of the two following "canonical expressions," which characterize the open orbits of $Gl(E)$ in $\Lambda^{n-2}E^*$:

$$\omega_1 = \sum_{i=1}^{k} \alpha_1 \wedge \ldots \wedge \alpha_{2i-2} \wedge \alpha_{2i+1} \wedge \ldots \wedge \alpha_n$$

$$\omega_2 = -\omega_1.$$

By means of h_Ω, these orbits divide $\Sigma_k \subset \Lambda^2 E$ into two half-cones.

b_2) If k is even then one necessarily has $\Delta(g) = 1$, and the automorphism g that is defined by:

$$g(e_i) = e_{i+1} \qquad \text{if } i \text{ is odd,}$$
$$g(e_i) = e_{i-1} \qquad \text{if } i \text{ is even}$$

answers the question.

S_k is therefore the open orbit of $Gl(E)$ in $\Lambda^{n-2}E^*$.

5.2.4. *Remarks.*

1) It is clear that a form ω is decomposable if and only if it is of length 1; therefore, the open set of forms of maximal rank n is decomposed into the (disjoint) union:

$$\Sigma_{n,n}^{n-2} = \bigcup_{k\geq 2} S_k .$$

2) The *length* of an $(n-2)$-form ω may be also defined as the *minimal number k* of *decomposable* forms ω_i that are necessary in order to write $\omega = \sum_{i=1}^{k} \omega_i$; this definition does not use duality and obviously extends to forms of arbitrary degree.

6. Intermediate case $(3 \leq p \leq n-3)$.

6.1. PROPOSITION. – $\Sigma_{n,r}^{p}$ *is non-vacuous for* $3 \leq p \leq n-3$ *if and only if* $r = 0, p, p+2, p+3, \ldots, n$.

Proof.

a) *If* $r \leq r < n$ *and* $\Sigma_{n,r}^{p}$ *is non-vacuous then* $\Sigma_{n,r+1}^{p+1}$ *is non-vacuous*; indeed, let $\omega \in \Lambda^p E^*$ be a p-form of rank r; since r is strictly less than n there exists a form $\alpha \in E^*$ that does not belong to S_ω; from proposition 1.4.5, the form $\omega \wedge \alpha$ is a $(p+1)$-form of rank $(r+1)$.

b) *It therefore suffices to establish the proposition for* $p = 3$ (the rest is deduced by recurrence upon using *a*).

b_1) It is obvious that $\Sigma_{n,3}^{3}$ is non-vacuous.

b_2) If $3 \leq r \leq n-2$ and if $\Sigma_{n,r}^{3}$ is non-vacuous then $\Sigma_{n,r+2}^{3}$ is non-vacuous; let $\alpha_1, \alpha_2 \in E^*$, such that the plane that is generated by α_1 and α_2 is in general position with respect to S_ω; let $\alpha_3 \in S_\omega$ be non-null; from proposition 1.4.4, the form $\omega + \alpha_1 \wedge \alpha_2 \wedge \alpha_3$ is of rank $r+2$.

b_3) If $3 \leq r \leq n-3$ and if $\Sigma_{n,r}^{3}$ is non-vacuous then $\Sigma_{n,r+3}^{3}$ is non-vacuous; the proof is analogous to the proof for 2 upon using $\alpha_1, \alpha_2, \alpha_3 \in E^*$ to generate a 3-plane in general position with respect to S_ω .

For the case $p = 3$, the proposition is deduced immediately from these remarks by recurrence on r.

6.2. PROPOSITION. – *If* $\Sigma^p_{n,r}$ *is non-vacuous then the algebraic variety* $\overline{\Sigma^p_{n,r}}$ *admits the set* $\bigcup_{r'\le r}\Sigma^p_{n,r'}$ *as its set of singular points.*

The proof may be made in analogous manner to the one in remark 1) of 4.2.

Remark. – I am ignoring the question of whether the stratification Σ^p_n of the space $\Lambda^p E^*$ that is defined by the varieties $\Sigma^p_{n,r}$ is "*coherent*" (cf. Appendix, sec. 2.2).

6.3. PROPOSITION. – *For* $n \le 9$ *and* $3 \le p \le n - 3$, *the dimension of* $Gl(E)$ *is strictly less than that of* $\Lambda^p E^*$; *so the group* $Gl(E)$ *admits a continuous infinitude of orbits in* $\Lambda^p E^*$.

For sufficiently large n, rank is therefore a very coarse invariant of the forms of intermediate degree.

6.4. *Remarks.* – Let ω be a p-form on E; we call any decomposition $\omega = \sum_{i=1}^{k}\omega_i$ of ω into a sum of *decomposable* forms ω_i, where k is the *length* of ω (cf. 5.2.3, remark 2)), a *minimal expression* for ω; if we are given such a minimal expression for ω then we denote the support of ω_i by F_i; for any $i = 1, \ldots, k$, F_i is a subspace of dimension p of E^* and the F_i are all distinct (the dimensions of the pair-wise intersections are likewise necessarily less than $p - 2$); one denotes the collection of subspace F_i by F.

Let $\omega = \sum_{i=1}^{k}\omega_i$ and $\omega = \sum_{j=1}^{k}\omega'_j$ be two minimal expressions for ω; let F and F' be the corresponding collections of subspaces.

CONJECTURE. – *The collections* F *and* F' *are equal in* E^*; *i.e., there exists an automorphism g of E such that for any i there exists a j with* $g(F_i) = F'_j$.

One may associate a numerical symbol to each class of equal collections (for example, the number k of subspaces, the dimensions of the sums 1 to 1, 2 to 2, …, k to k); the solution of the preceding conjecture will therefore permit us to attach a numerical symbol to any p-form that will be an invariant under the action of $Gl(E)$ (furthermore, the *rank* will appear in the symbol; it will be the dimension of the sum of all of the spaces F_i).

One must then study the "admissible" symbols for each p; one may hope that the set of forms that admit a given symbol is a subvariety of $\Lambda^p E^*$ in which the orbits of $Gl(E)$ have a constant dimension.

7. The representation of $Gl(E)$ in $E^* \oplus \Lambda^2 E^*$.

In what follows, one sets $F = E^* \oplus \Lambda^2 E^*$ and $g(\alpha, \beta) = (g\alpha, g\beta)$ for $g \in Gl(E)$ and $(\alpha, \beta) \in F$.

7.1. Let $(\alpha, \beta) \in F$; one calls the subspace $S_{(\alpha,\beta)}$ of E^* that is the sum of the supports of α and β the *support* of (α, β); the *rank* of (α, β) will be the dimension of $S_{(\alpha,\beta)}$. It is obvious that the rank is an invariant under the action of $Gl(E)$ on F.

(*Remark*: At the beginning of this chapter we directly defined the support and rank of an arbitrary, but possibly non-homogenous, element of the exterior algebra of a space E.)

For an element of the form $(0, \beta)$ the support and rank are those of the 2-form β that was studied in 4; *let* Ω *be the open set of F that consists of the set of* (α, β) *such that* $\alpha \neq 0$.

One sets:

$$\gamma_p(\alpha, \beta) = \begin{cases} \beta^k & \text{for} \quad p = 2k, \\ \alpha \wedge \beta^k & \text{for} \quad p = 2k+1. \end{cases}$$

PROPOSITION. – *For any pair* $(\alpha, \beta) \in \Omega$, *the following conditions are equivalent*:

a) $\operatorname{rank}(\alpha, \beta) = r$,
b) $\gamma_r(\alpha, \beta) \neq 0$ *and* $\gamma_{r+1}(\alpha, \beta) = 0$,
c) $\gamma_p(\alpha, \beta) \neq 0$ *for* $p \leq r$ *and* $\gamma_p(\alpha, \beta) = 0$ *for* $p > r$,
d) *There exist independent forms* $\alpha_1, \ldots, \alpha_r \in E^*$ *such that*:
$\alpha = \alpha_1$, *and* $\beta = \alpha_1 \wedge \alpha_2 + \ldots + \alpha_{2k-1} \wedge \alpha_{2k}$ *if* $r = 2k$,
$\alpha = \alpha_1$, *and* $\beta = \alpha_2 \wedge \alpha_3 + \ldots + \alpha_{2k} \wedge \alpha_{2k+1}$ *if* $r = 2k + 1$.

The proof is very simple if one starts with proposition 4.1.

Remark. – The rank of (α, β) is $2k$ if and only if β is of rank $2k$ and if $\alpha \in S_\beta$; the rank of (α, β) is $2k + 1$ if and only if β is of rank $2k$ and $\alpha \notin S_\beta$.

7.2. In what follows, we will let Σ_c denote the set of $(0, \beta)$ with $\operatorname{rank}(\beta) = n - c$, and let S_d denote the set of $(\alpha, \beta) \in \Omega$ of rank $n - d$ (corank equal to d), where $n = \dim E$.

PROPOSITION. – *The sets* Σ_c *and* S_d *are orbits of* $Gl(E)$ *in* F; *they are regular submanifolds of F, whose adherences are algebraic varieties*; *finally*:

$$\operatorname{codim} \Sigma_c = n + \frac{c(c-1)}{2}$$

$$\operatorname{codim} S_d = \frac{d(d+1)}{2}$$

where $n - c$ *is even, with* $0 \leq n - c \leq n$ *and* $0 \leq d \leq n - 1$.

One denotes the stratification of F that is so defined by S.

Everything that relates to the sets Σ_c results from proposition 4.1. From condition *d*) of proposition 7.1, the sets S_d are precisely the orbits of $Gl(E)$ in Ω.

$n - d = 2k + 1$: From 7.1 b), S_d is the set of (α, β) such that $\beta^{k+1} = 0$ and $\alpha \wedge \beta^k \neq 0$; $\overline{S_d}$ is the algebraic set that is defined by the equation $\beta^{k+1} = 0$; S_d is a dense open subset of the "cylinder" $E^* \times \Sigma_{n-k}$ in an obvious way; from this, one deduces its codimension.

$n - d = 2k$: From 7.1. b), S_d is the set of (α, β) such that $\alpha \wedge \beta^k = 0$ and $\beta^{k+1} \neq 0$; $\overline{S_d}$ is the algebraic variety of the equations $\alpha \wedge \beta^k = 0$ and $\beta^{k+1} = 0$. The projection $\pi: F \to \Lambda^2 E^*$ takes S_d to Σ_{n-k} and, for $\beta \in \Sigma_{n-k}$, $\pi^{-1}(\beta) \cap S_d$ is the set of $(\alpha, \beta) \in \Omega$ such that $\alpha \in S_\beta$; one easily deduces that S_d is an open regular submanifold of F of codimension $d(d + 1)/2$.

7.3. If we have the ultimate goal of studying *the class of a Pfaff equation* (cf. II.5) then it is useful to study the following notion:

DEFINITION. – The *reduced rank* of an element $(\alpha, \beta) \in \Omega \subset E^* \oplus \Lambda^2 E^*$ (i.e., such that $\alpha \neq 0$) is the odd integer $2k + 1$ such that $\alpha \wedge \beta^k \neq 0$ and $\alpha \wedge \beta^{k+1} = 0$ (it is also the rank of the 3-form $\alpha \wedge \beta$).

One lets $S'_c \subset \Omega$ denote the elements of reduced rank $n - c$ (where $n = \dim E$), and one seeks to describe the partition of Ω that this defines.

Let H be the group of jets of order 1 of non-null numerical functions at the origin of E; the group H is composed of pairs $f = (\lambda, h)$, where λ is a non-null real number and $h \in E^*$. The formula:

$$d(f\omega) = f d\omega + df \wedge \omega,$$

in which ω is a Pfaff form, f is a function, and d is exterior differentiation, suggests that we make H act on $F = E^* \oplus \Lambda^2 E^*$ according to the rule:

$$(\lambda, h) \cdot (\alpha, \beta) = (\lambda\alpha, \lambda\beta + h \wedge \alpha),$$

so that the reduced rank becomes an *invariant of the action of* H; indeed, it suffices to remark that $\lambda\alpha \wedge (\lambda\beta + h \wedge \alpha)^k = \lambda^{k+1} \alpha \wedge \beta^k$.

Now let $(f, g) \in H \times Gl(E)$ and $(\alpha, \beta) \in F$; one sets:

$$(f, g) \cdot (\alpha, \beta) = f \cdot g(\alpha, \beta) = (\lambda \cdot g\alpha, \lambda \cdot g\beta + h \wedge g\alpha),$$

where $f = (\lambda, h)$.

It is clear that one defines a law of operation of $\overline{G} = H \times Gl(E)$ in this way, which is a group that is endowed with the natural structure of a semi-direct product in F. The reduced rank, which is invariant under the action of H and $G = Gl(E)$, is invariant under the action of $\overline{G}$.

PROPOSITION. – *The orbits of* $\overline{G}$ *in F are, on the one hand, the submanifolds* $\Sigma_c \subset \{0\} \times \Lambda^2 E^*$, *and, on the other hand, the sets* $S'_c \subset \Omega$ ($n - c$ *odd*); S'_c *is a regular submanifold of F with algebraic adherence and codimension* $\frac{c(c-1)}{2}$ *for any c such that* $n - c$ *is odd and* $1 \leq n - c \leq n$.

Proof. – One has $(f, g)(0, \beta) = (0, \lambda \cdot g\beta)$ where $(f, g) \in \overline{G}$ and $f = (\lambda, h)$. It is then clear that the orbits of $\overline{G}$ in $0 \times \Lambda^2 E^*$ are the sets Σ_c.

On the other hand, $S'_c = S_c \cup S_{c-1}$. Set $n - c = 2k + 1$; let $(\alpha, \beta) \in S_c$ and show that $\overline{G}(\alpha, \beta) = S'_c$. Let $f = (1, h)$, where $h \in E^*$ is such that $h \wedge \alpha \wedge \beta^k \neq 0$. The element $(f, 1) \cdot (\alpha, \beta) = (\alpha, \beta + h \wedge \alpha)$ has rank $2l + 2$. Therefore, $\overline{G}$ intersects S_c and S_{c-1}. It therefore contains their union since the action of $\overline{G}$ prolongs that of $Gl(E)$. On the other hand, $\overline{G}(\alpha, \beta) \subset S'_c$, since the reduced rank is an invariant, and one has $\overline{G}(\alpha, \beta) = S'_c$, precisely.

The set S'_c is then a submanifold, since it is the orbit of a Lie group; however, it is also the union of two regular submanifolds. It is therefore a regular submanifold whose dimension is equal to that of S_{c-1}. The rest of the proposition is immediate.

Remarks.

1) The stratification that is defined in F by the sets $S'_c \subset \Omega$ and $S_c \subset \{0\} \times \Lambda^2 E^*$ is therefore *coherent* (cf. Appendix, sec. 2.2).
2) The stratification that is defined in F by the sets Σ_c and $S_c \subset \Omega$, has the following very curious property: For any c, the set $S_c \cup S_{c-1}$ is a regular submanifold of F of codimension $\dfrac{c(c-1)}{2}$. For odd $(n - c)$, this results from the preceding proposition; for even $(n - c)$, one simply deals with the product $(E^* - \{0\}) \times \Sigma_{\frac{n-c}{2}}$.

CHAPTER II

SINGULARITIES OF THE RANK AND CLASS OF A DIFFERENTIAL FORM

1. Generalities; notations and definitions.

1.1. Let F_k^p ($\mathcal{F}_k^p$, resp.) ($k \geq 0$, $1 \leq p \leq n$) denote the vector space of jets of order k of exterior differential forms (*closed* exterior differential forms, resp.) of degree p at the origin of $\mathbf{R}^n$.

Denote the dual to $\mathbf{R}^n$ by $\mathbf{R}_n$; one therefore has $F_0^p = \mathcal{F}_0^p = \Lambda^p \mathbf{R}_n$.

For any k and $k' \geq k$, denote the *restriction* homomorphisms by $\rho: F_{k'}^p \to F_k^p$ or $\rho: \mathcal{F}_{k'}^p \to \mathcal{F}_k^p$, resp.

For any p and $k \geq 1$, exterior differentiation defines a homomorphism $d: F_k^p \to F_{k-1}^{p+1}$ whose image is $\mathcal{F}_{k-1}^{p+1}$.

Now, for any $k \geq 0$, let L_k be the Lie group of invertible $(k + 1)$-jets of $\mathbf{R}^n$ into $\mathbf{R}^n$, with source and target 0 (i.e., the jets of order $(k + 1)$ of the germs of diffeomorphisms that preserve the origin of $\mathbf{R}^n$). For $k' \geq k$, one further denotes the restriction homomorphism by $\rho: L_{k'} \to L_k$.

For any $k \geq 0$, the rule for changing the variables of a differential form defines a natural law of operation for L_k on F_k^p and $\mathcal{F}_k^p$.

These laws of operation commute with the restriction morphisms.

DEFINITION. – One calls any submanifold Σ of F_k^p ($\mathcal{F}_k^p$, *resp.*) that is regular and invariant under L_k a *p-form* (*closed p-form*) *singularity of order k.*

1.2. Let M be an n-dimensional manifold of class C^∞ with a denumerable neighborhood basis.

One denotes by:

TM and T^*M	the tangent and cotangent bundles of M,
$\Lambda^p T^*M$	the p^{th} exterior power of T^*M,
$\Lambda^p T_k^* M$	the vector bundle of k-jets of sections of $\Lambda^p T^*M$,
$\Lambda^p \mathcal{T}_k^* M$	the fiber bundle of k-jets of *closed* p-forms.

One remarks that $\Lambda^p T^*M = \Lambda^p \mathcal{T}_0^* M$.

The fiber bundle $\Lambda^p T_k^* M$ has fiber type F_k^p and structure group L_k. Similarly, $\Lambda^p \mathcal{T}_k^* M$ has fiber type $\mathcal{F}_k^p$ and structure group L_k.

For any k and $k' \geq k$, one denotes the *restriction* morphism by $\rho: \Lambda^p T_{k'}^* M \to \Lambda^p T_k^* M$, or $\rho: \Lambda^p \mathcal{T}_{k'}^* M \to \Lambda^p \mathcal{T}_k^* M$.

For any $k \geq 1$, one notates the bundle morphism that is defined by exterior differentiation by d: $\Lambda^p T_k^* M \to \Lambda^{p+1} \mathcal{T}_{k-1}^* M$.

One lets $D_k^p(M)$ denote the space of *differential* p-forms (closed p-forms, resp.) on M with k times continuously differentiable coefficients, which is endowed with the C^k-topology (Appendix, sec. 1.2). $D^p(M)$ ($\mathcal{D}^p(M)$, resp.) denotes the space of C^∞ p-forms (closed p-forms, resp.), which is endowed with the C^∞-topology.

Let $\omega \in D_{k'}^p(M)$, and $k \leq k'$. One lets $j_k\omega$ denote the section of $\Lambda^p T_k^* M$ that is defined by $j_k\omega(x) = k$-jet of ω at x.

1.3. Let $\Sigma \subset F_k^p$ be a p-form singularity of order k. Since this submanifold is invariant under the structure group L_k, it defines a submanifold of $\Lambda^p T_k^* M$ (fibered over M with fiber type Σ) in a natural way, which we notate $\Sigma(M)$.

Therefore let $\omega \in D_{k'}^p(M)$, where $k \leq k' \leq \infty$. The set $\Sigma(\omega)$ of points $x \in M$ such that $j_k\omega(x) \in \Sigma(\mathrm{M})$ will be called the *singular set of type* Σ *of* ω; the points of $\Sigma(\omega)$ will be called *singular points of type* Σ *of* ω.

One defines the singular points and singular set of a close form, for a given singularity, in a similar manner.

In this chapter, I propose to define a certain number of singularities that are related to the notions of *rank* and *class* of a differential form, and then to study the generic nature of the corresponding singular sets with the aid of transversality techniques.

More precisely, for any p and a given k, one seeks to define a *finite stratification* (cf. Appendix, sec. 2.2) *of* F_k^p ($\mathcal{F}_k^p$, *resp.*) *by its singularities,* such that one obtains a "maximum of information" about the behavior of a differential form ω on a manifold M at each point when $j_k\omega$ is *transverse* to this stratification.

There exists no actual *systematic method* that permits us to construct an "optimal natural stratification" of F_k^p.

Here, I use the *rank* as an invariant of order 0, the *class* as an invariant of order 1, and I employ the same strategy for the construction of the singularities of higher order as in the case of differentiable maps (only for the closed 2-forms in dimension 4 and Pfaff equations in dimension 3).

1.4. *Remarks.*

1) The sense given to the term "singularity" here is a little broader than its usual intuitive sense. Therefore, if ω is a Pfaff form then a point where ω is of maximum class may be considered to be a singular point of ω of a given type.

2) Let $\Sigma \subset F_k^p$ ($\mathcal{F}_k^p$, resp.) be a singularity of codimension c. Let M be a manifold of dimension n, and let $\omega \in D_{k'}^p(M)$ ($\mathcal{D}_{k'}^p(M)$, resp.), where $k' \geq k + 1$. One then knows (Appendix, sec. 6 and 7) $j_k\omega$ is transverse of $\Sigma(M)$ generically in $D_{k'}^p(M)$ ($\mathcal{D}_{k'}^p(M)$, resp.). The singular set $S(\omega)$, *if it is non*-vacuous, will then be regular submanifold of M

of codimension c. In what follows, this will always be summarized by the expression: "$S(\omega)$ is generically a regular submanifold of codimension c."

Moreover, we note that, from a theorem of Mather ([13], pp. 29), if $c \leq n$ then there always exists a p-form ω such that $j_k\omega$ is transverse to $\Sigma(\omega)$ and $\Sigma(\omega)$ is non-empty.

2. Rank and its singularities.

2.1. It is clear that $F_0^p = \mathcal{F}_0^p = \Lambda^p\mathbf{R}_n$, where $\mathbf{R}_n = (\mathbf{R}^n)^*$. On the other hand, L_0 is the group $Gl(n, \mathbf{R})$, and the law of operation defined in 1.1 is the natural law that was already considered in Chap. I. The study made in this chapter shows that the sets $\Sigma_{n,r}^p$ (recall that in $\Lambda^p\mathbf{R}_n$ this amounts to the set of forms of rank r) are singularities of order 0, and give us the codimensions of these singularities.

If M is a manifold of dimension n then for each integer p one considers the stratification of $\Sigma_n^p(M)$ that is comprised of the submanifolds $\Sigma_{n,r}^p(M)$. From the transversality theorems (Appendix, Th. 6), the subset of D_l^p ($\mathcal{D}_l^p$, resp.) that is comprised of forms (closed forms, resp.) that are transverse to this stratification is a residual set for $l \geq 1$. We now detail a few aspects of this generic situation.

2.2. THEOREM. – *Let M be a manifold of dimension $n \geq 7$. For $3 \leq p \leq n - 2$, the set of p-forms of D_1^p that are of maximum rank n at every point of M, is a dense C^1-open set. This result is also true for $\mathcal{D}_1^p$.*

Indeed, from I, prop. 2, the codimension of $\Sigma_{n,r}^p$ is strictly greater than n whenever $r < n$, provided that $n \geq 7$ and $3 \leq p \leq n - 2$. Thus, a form that is in general position with respect to the manifolds $\Sigma_{n,r}^p(M)$ may not be of rank less than n at any point. On the other hand, from the Appendix, sec. 6, these forms constitute a C^1-open set. For the second part of the theorem, one applies theorem 7 of the Appendix.

2.3. For $p = 1, n - 1, n$, the situation is very simple since the rank may take only two values then. Generically, the zeroes of an n-form constitute a compact submanifold of codimension 1 in M; the zeroes of a Pfaff form (closed or not) or an $(n-1)$-form are isolated.

2.4. Generic behavior of the rank of a 2-form. Taking I.4.2 into account, we set:

$$r(n) = \begin{cases} Max\left(c; c \text{ is even and } \dfrac{c(c-1)}{2} \leq n\right) & \text{for } n \text{ even}, \\ Max\left(c; c \text{ is odd and } \dfrac{c(c-1)}{2} \leq n\right) & \text{for } n \text{ odd}. \end{cases}$$

Let Σ be the stratification of $\Lambda^2\mathbf{R}_n$ that is defined by the sets Σ_c (the set of 2-forms of corank c).

2.4.1. – One then has:

PROPOSITION. – *Let M be a manifold of dimension n. The set T*(Σ) *of 2-forms* $\omega \in D_k^2(M)$ ($\mathcal{D}_k^2(M)$, resp.) *that are transverse to the stratification* Σ(*M*) ⊂ $\Lambda^2 T^*M$ *is a dense* C^k-*open set for* $k \geq 1$.

Such a 2-form has the following properties:

a) *The rank of ω is greater than or equal to n – r*(*n*) *at every point*; *an asymptotic expression of this minorant is* $n - \sqrt{2n}$.

b) *For* $c \leq r(n)$, the *set* $\Sigma_c(\omega)$ *of points of M where ω is of rank n – c is a regular submanifold of codimension* $c(c-1)/2$, *if it is non-vacuous.*

This result is an immediate consequence of Theorem 6 (remark 2) of the Appendix and Theorem 7 for closed forms.

Examples. – If $n = \dim M = 4$, then the corank is necessarily even. Generically, $\Sigma_4(\omega)$ is vacuous, i.e., ω has no zero, and $\Sigma_4(\omega)$ is a (closed) hypersurface of *M*. ω is of maximum rank 4 on the open set $\Sigma_0(\omega) = M - \Sigma_2(\omega)$.

If $n = \dim M = 6$, then $\Sigma_6(\omega)$ is generically vacuous (ω has no zero), $\Sigma_4(\omega)$ is composed of isolated points, and $\Sigma_2(\omega)$ is a hypersurface such that $\overline{\Sigma_2(\omega)} = \Sigma_2(\omega) \cup \Sigma_4(\omega)$.

For even *n*, $6 \leq n \leq 14$, the description of the generic behavior of the rank remains the same, i.e., only $\Sigma_2(\omega)$ and $\Sigma_4(\omega)$ (and obviously $\Sigma_0(\omega)$) may be non-vacuous. They have codimensions 1 and 6, respectively, and $\Sigma_4(\omega)$ is a locus of singular points for $\overline{\Sigma_2(\omega)}$.

If $n = 5$ then the corank is necessarily odd. Generically, $\Sigma_5(\omega)$ is vacuous (no zero) and $\Sigma_3(\omega)$ is a (closed) submanifold of codimension 3. ω has the maximum rank of 4 on the open set:

$$\Sigma_1(\omega) = \mathrm{M} - \Sigma_3(\omega);$$

the description for $n = 7, 9$, works the same way.

2.4.2. – PROPOSITION. – *For* $k \geq 2$, *the singular sets of the rank of a 2-form in T*(Σ) $\subset D_k^2(M)$ (*T*(Σ) ⊂ $\mathcal{D}_k^2(M)$, resp.) *are isotopically stable.*

Proof. – First, this proposition signifies that if $\omega \in D_k^2(M)$ is transverse to Σ(*M*) then there exists a neighborhood *V* of ω (for the C^k-topology) such that if c *V* (?) then the stratifications $\Sigma(\omega)$ and $\Sigma(\omega')$ of *M* that are formed of the singular sets $\Sigma_c(\omega)$ and $\Sigma_c(\omega')$ are isotopic. This result is an immediate consequence of theorem 4.2 of the Appendix since the stratification Σ(*M*) is coherent, like the stratification Σ of $\Lambda^2\mathbf{R}_n$ (recall that the strata $\Sigma_c \subset \Lambda^2\mathbf{R}_n$ are the orbits of the group $L_0 = Gl(n, \mathbf{R})$).

3. "Generic" singularities of a closed 2-form in dimension 4.

3.1. Description of the generic situation. Let M be manifold (compact, for the sake of specificity) of dimension 4. The situation that is described in the following paragraphs for a closed 2-form ω on M is generic in $\mathcal{D}_3^2(M)$, the space of closed 2-forms that are at least three times continuously differentiable (recall that this signifies that the set of closed 2-forms that have the properties specified above is residual in $\mathcal{D}_3^2(M)$); indeed, the same will be true for dense C^3-open set).

3.1.1. *The set* $\Sigma(\omega)$ *of points where* ω *has corank* 2 (hence, rank 2) *is a* (compact) *submanifold of codimension* 1 (if it is non-vacuous); ω is not zero at any point of M. (cf. II. 2.4.1, example 1)).

Remark. – $\Sigma_2(\omega)$ *is oriented in a canonical way*; indeed, let $x \in \Sigma_2(\omega)$, and let Ω be a volume form on a neighborhood of x in M. One has $\omega_2 = f \cdot \Omega$. By the definition of $\Sigma_2(\omega)$, $f(x) = 0$. In the generic situation $f'(x)$ is a non-null linear form. Its kernel is $T_x\Sigma_2(\omega)$, which inherits an orientation from the pair $(\Omega(x),\ f'(x))$. If one changes the sign of Ω then f and f' change sign, and the induced orientation does not change.

3.1.2. Let ω' be the restriction of ω to $\Sigma_2(\omega)$, and let $\Sigma_{2,2}(\omega)$ be *the set of points where* ω' *is null*. $\Sigma_{2,2}(\omega)$ *is a* (*compact*) *submanifold of codimension* 3 *in* M (or codimension 2 in $\Sigma_2(\omega)$). It is therefore a finite union of closed simple curves of M. On the other hand, one sets $\Sigma_{2,0}(\omega) = \Sigma_2(\omega) - \Sigma_{2,2}(\omega)$.

3.1.3. At each point, $x \in \Sigma_2(\omega)$, $\omega(x)$ is, by the definition of $\Sigma_2(\omega)$, a 2-form of corank 2 on T_xM. The associated space $A_\omega(x)$ is therefore a plane. It is clear that $\Sigma_{2,2}(\omega)$ is the set of points $x \in \Sigma_2(\omega)$, such that $A_\omega(x) \subset T_x\Sigma_2(\omega)$. If $x \in \Sigma_{2,0}(\omega)$, then the associated plane to ω is transverse to $T_x\Sigma_2(\omega)$; the intersection is a line. One thus defines a *field of directions* D *on* $\Sigma_{2,0}(\omega)$ *that is also the field that is associated to the induced form* ω'. *This field is canonically oriented.* It suffices to choose a volume form Ω on $\Sigma_2(\omega)$ that is positive with respect to the orientation of canonical orientation of $\Sigma_2(\omega)$ and consider the vector field X that is defined at each point by the linear equation, $\omega' = X \,\lrcorner\, \Omega$. The field X is defined up to a positive factor, and it is obviously supported by D.

Let $\Sigma_{2,2,1}(\omega)$ *be the set of points* $x \in \Sigma_{2,2}(\omega)$ *such that the line* $T_x\Sigma_{2,2}(\omega)$ *is included in the plane* $A_\omega(x)$. $\Sigma_{2,2,1}(\omega)$ *is a set of isolated points.*

One sets $\Sigma_{2,2,0}(\omega) = \Sigma_{2,2}(\omega) - \Sigma_{2,2,1}(\omega)$. At any point $x \in \Sigma_{2,2,0}(\omega)$, $T_x\Sigma_{2,2}$ and the plane $A_\omega(x)$, are transverse in $T_x\Sigma_2(\omega)$.

Consider a point $x \in \Sigma_{2,2}(\omega)$. Since the section:

$$\omega' : \Sigma_2(\omega) \to \Lambda^2 T^*\Sigma_2(\omega)$$

is *null* at this point, it *Jacobian* at x is a linear map $T_x\omega' : T_x\Sigma_2 \to \Lambda^2 T_x^*\Sigma_2$. By using a positive volume form on $T^*\Sigma_2$ and the corresponding duality isomorphism of $\Lambda^2 T_x^*\Sigma_2$ with $T_x\Sigma_2$, $T_x\omega'$ defines an endomorphism:

$$\Lambda_\omega(x): T_x\Sigma_2 \to T_x\Sigma_2 .$$

This endomorphism is defined up to a *positive* homothety (according to the choice of volume element).

One may also define $\Lambda_\omega(x)$ in the following fashion: let Ω be a positive volume form and let X be the vector field that was constructed in the preceding paragraph; the point x is a zero of X. The Jacobian of X at x is then the matrix $\Lambda_\omega(x)$ that corresponds to the volume $\Omega(x)$ on $T_x\Sigma_2(\omega)$. One may therefore say that $\Lambda_\omega(x)$ is *the Jacobian of the direction field D at x* ($x \in \Sigma_{2,2}$).

By construction, $\Lambda_\omega(x)$ has *a rank that is less than or equal to* 2. Indeed, since ω' is annulled at the points of $\Sigma_{2,2}(\omega)$, the line tangent to it is certainly contained in the kernel of $\Lambda_\omega(x)$.

On the other hand, $\Lambda_\omega(x)$ *always has a zero trace.* Indeed, since ω' is closed, the auxiliary vector field X that was constructed above verifies $\theta(X)\Omega = 0$. It is therefore *unimodular.*

The generic situation that was described in 3.1.3 may be then stated more precisely:

a) $\Lambda_\omega(x)$ *has rank* 2 *at every point* $x \in \Sigma_{2,2}(\omega)$. *The kernel of* $\Lambda_\omega(x)$ *is the tangent to* $\Sigma_{2,2}(\omega)$ *at* x; *the image of* $\Lambda_\omega(x)$ *is* $A_\omega(x)$.

b) $\Sigma_{2,2,0}(\omega)$ *is the set of points x such that* $\Lambda_\omega(x)$ *is not nilpotent.* This matrix has one null proper value, and the other two are non-null and opposite. They are thus pure imaginary or real. The first case defines the subset $\Sigma^e_{2,2,0}(\omega)$ of "*elliptical points,*" and the second set $\Sigma^h_{2,2,0}(\omega)$ defines the "*hyperbolic points.*"

c) *The points of* $\Sigma_{2,2,1}(\omega)$ *separate the elliptic and hyperbolic arcs.* One may call them "*parabolic.*" At such a point, the proper values of $\Lambda_\omega(x)$ are all null, and the image of $\Lambda_\omega(x)$, which is always $A_\omega(x)$, contains the tangent to $\Sigma_{2,2}(\omega)$.

3.2. To convince the reader, we give an example of each of the types of singular points that were described above.

3.2.1. At every point of $\Sigma_0(\omega)$ (i.e.: ω has maximum rank 4) there exist (from Darboux's theorem; see the following chapter) local coordinates (x, y, z, t) such that:

$$\omega = dx \wedge dy + dz \wedge dt.$$

3.2.2. At every point of $\Sigma_{2,0}(\omega)$ there exist local coordinates, (x, y, z, t), in which:

$$\omega = x\, dx \wedge dy + dz \wedge dt.$$

This fact will be proved in III.A, 4.2.2.

3.2.3.

a) Now here is an example of a form on $\mathbf{R}^4$ that presents a point of type $\Sigma^e_{2,2,0}(\omega)$ (pure elliptic) at the origin:

$$\omega = dx \wedge dy + z\, dz \wedge dt + d\left(xz + ty - \frac{z^3}{3}\right) \wedge dt .$$

In this case, $\Sigma_2(\omega)$ is defined by $x = 0$ (one has $\omega^2 = x\, dx \wedge dy \wedge dz \wedge dt$); a positive volume element on $\Sigma_2(\omega)$ is $dy \wedge dz \wedge dt$. On the other hand:

$$\omega' = z\, dy \wedge dz + t\, dy \wedge dt - z^2\, dz \wedge dt.$$

The associated vector field is defined by:

$$X = -z^2 \frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t} .$$

$\Sigma_{2,2}(\omega)$ is therefore the y-axis. The integral curves of the field X are "helices of the axis Oy."

Remark. – At an *elliptic* point x of $\Sigma_{2,2}(\omega)$ the tangent to is canonically oriented. Indeed, at such a point, $\Lambda_\omega(x)$ defines a "quadrant" in the image plane that is transverse to $\Sigma_{2,2}(\omega)$. This plane is therefore oriented, and since the tangent space to Σ_2 is also, one thus deduces an orientation for the line $T_x\Sigma_{2,2}(\omega)$.

b) One obtains an example of a *hyperbolic* point (type $\Sigma^h_{2,2,0}(\omega)$) by changing a sign in the preceding example:

$$\omega = dx \wedge dy + z\, dz \wedge dt + d\left(xz - ty - \frac{z^3}{3}\right) \wedge dt .$$

Remark. – I have ignored the issue of whether the germ of a closed 2-form that presents an elliptic or hyperbolic singular point is *isomorphic* (in the sense of III.A, 1.1.2) to the corresponding example above. Meanwhile, see III.B, 2.2, on this subject.

3.2.4. The remark made in 3.2.2 a) shows that there are at least *two types of parabolic points*, according to whether the elliptic arc "begins or ends there." Here are two examples that correspond to these two behaviors:

$$\omega = dx \wedge dy + z\, dz \wedge dt + d\left(xz + 2tx \pm \frac{y^2}{2} - \frac{z^3}{3} - tz^2\right) \wedge dt .$$

In the two cases, $\Sigma_{2,2}(\omega)$ is defined by $x = 0$. One has:

$$\omega' = z\, dy \wedge dz \pm y\, dy \wedge dt - z(z + 2t)\, dz \wedge dt.$$

The corresponding vector field X is:

$$X = -z(z+2t)\frac{\partial}{\partial y} \mp y\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}.$$

The manifold $\Sigma_{2,2}(\omega)$ is defined by $x = y = z = 0$. The proper value equation for the Jacobian Λ of X at the point $(0, 0, t_0)$ of (?) is $\lambda(\lambda^2 \pm 2\, t_0) = 0$. One verifies that the elliptic arc "starts" at the origin ($t > 0$) in the case $(-\,y^2/2)$ and ends there ($t < 0$) in the case $(+\,y^2/2)$.

I also ignore whether any germ of a 2-form that presents a parabolic point is isomorphic to one of the two examples above.

3.3. Justification of the genericity of the situation described in 3.1.

One works in the space $\mathcal{F}_1^2$ of jets of order 1 of closed 2-forms at the origin of $\mathbf{R}^4$. I will construct a sequence of singularities in $\mathcal{F}_1^2$ such that the transversal position of $j_2\omega$ with respect to these singularities (where ω is a closed 2-form) will imply the behavior that was specified in 3.1.

We write any jet in the form $\omega = \omega_0 + \omega_1$, where ω_0 is a 2-form with constant coefficients, and ω_1 is a *closed* 2-form with linear homogenous coefficients.

3.3.1. One first considers the stratification $\mathcal{F}_1^2 = \Sigma_0 \cup \Sigma_2' \cup \Sigma_4$ of $\mathcal{F}_1^2$, where Σ_i denotes the set of ω such that ω_0 is of *corank* i; Σ_0, Σ_2', Σ_4 are regular submanifolds of $\mathcal{F}_1^2$ of codimension 0, 1, 6 (from Proposition I, 4.2).

3.3.2. One now stratifies the submanifold Σ_2'. It is a (vector) bundle over the set of ω_0 *of rank* 2. One works in a fiber, upon remarking that for any $\omega \in \Sigma_2'$ there exists a basis for $\mathbf{R}^4$ in which $\omega_0 = dx \wedge dy$.

Then let $\omega_1 = h\, dz \wedge dt + \ldots$, where $h, \ldots$ are linear forms.

It is clear that "ω is transverse to $\Sigma_2 \subset \Sigma_2'$" is equivalent to "$h$ is a non-null linear form."

One then decomposes Σ_2' into $\Sigma_2 \cup \Sigma_2''$ where Σ_2'' is the set of ω such that $h \equiv 0$. Σ_2'' is a regular submanifold of *codimension* 4 *in* Σ_2 ; therefore, *it has codimension* 5 *in* $\mathcal{F}_1^2$.

3.3.3. Let $\omega = \omega_0 + \omega_1 = dx \wedge dy + h\, dz \wedge dt + \ldots \in \Sigma_2$. Let H notate the kernel hyperplane of h.

One defines $\Sigma_{2,2}' \subset \Sigma_2$ to be the set of ω such that the restriction of the form $dx \wedge dy$ to the hyperplane H is null. This is equivalent to $\frac{\partial h}{\partial z} = \frac{\partial h}{\partial t} = 0$. These conditions define a submanifold of codimension 2 in Σ_2 ; therefore it has *codimension* 3 *in* $\mathcal{F}_1^2$. One sets $\Sigma_{2,0} = \Sigma_2 - \Sigma_{2,2}'$.

3.3.4. Suppose we have an element $\omega \in \Sigma'_{2,2}$. One easily sees that it is possible to choose a basis for $\mathbf{R}^4$ such that $\omega_0 = dx \wedge dy$, and:

$$\omega_1 = x\, dz \wedge dt + \dots$$

One now works in the set of elements of $\Sigma'_{2,2}$ whose expression is as above, and with a fixed basis.

More precisely, one sets:

$$\omega_1 = x\, dz \wedge dt + k\, dy \wedge dz + l\, dy \wedge dt + \dots$$

where k, l, … are linear forms.

Let $\bar{\omega}_1$ be the restriction of ω_1 to H, which is the hyperplane $x = 0$ here. One has $\bar{\omega}_1 = \bar{k} dy \wedge dz + \bar{l} dy \wedge dt$, where $\bar{k}$ and $\bar{l}$ denote the restrictions of k and l to $x = 0$. To them, one associates the endomorphism Λ of H that is defined by the matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ -\dfrac{\partial l}{\partial y} & -\dfrac{\partial l}{\partial z} & -\dfrac{\partial l}{\partial t} \\ \dfrac{\partial k}{\partial y} & \dfrac{\partial k}{\partial z} & \dfrac{\partial k}{\partial t} \end{pmatrix}$$

(of course, this is what corresponds to the $\Lambda_\omega(x)$ we previously envisioned). *The only relation* between these coefficients, which expresses that ω_1 is closed, is: $-\dfrac{\partial l}{\partial t} + \dfrac{\partial k}{\partial t} = 0$ (i.e., Λ has null trace).

One sets $\Sigma_{2,2} = \Sigma'_{2,2} - \Sigma''_{2,2}$, where $\Sigma''_{2,2}$ denotes the set of ω such that Λ is of rank ≤ 1. $\Sigma''_{2,2}$ is an algebraic variety of codimension 2 in $\Sigma'_{2,2}$, and therefore of codimension 5 in $\mathcal{F}_1^2$.

Finally, $\Sigma_{2,2,1} \subset \Sigma_{2,2}$ will be defined by the equation:

$$-\frac{\partial l}{\partial z} \cdot \frac{\partial l}{\partial t} + \frac{\partial l}{\partial t} \cdot \frac{\partial l}{\partial z} = -\left(\frac{\partial k}{\partial t}\right)^2 + \frac{\partial l}{\partial t} \cdot \frac{\partial k}{\partial z} = 0 .$$

Since this has *rank* 1, the three coefficients $\dfrac{\partial k}{\partial t}$, $\dfrac{\partial l}{\partial t}, \dfrac{\partial k}{\partial z}$ may not all be null in $\Sigma_{2,2}$ (Λ will have rank 1). $\Sigma_{2,2,1}$ is therefore a regular submanifold of $\mathcal{F}_1^2$ of codimension 4. The submanifold:

$$\Sigma_{2,2,0} = \Sigma_{2,2} - \Sigma_{2,2,1}$$

is then the set of ω such that Λ has two (opposite) non-null proper values. The subset $\Sigma^{e}_{2,2,0}$ ($\Sigma^{h}_{2,2,0}$, resp.) corresponds to the case where the determinant $-\left(\frac{\partial k}{\partial t}\right)^2 + \frac{\partial l}{\partial t}\cdot\frac{\partial k}{\partial z}$ is positive (negative, resp.).

3.3.5. One thus stratifies the space of jets of order 1 of closed 2-forms into:

$$\mathcal{F}_1^2 = \Sigma_0 \cup \Sigma_{2,0} \cup \Sigma_{2,2,0} \cup \Sigma_{2,2,1} \cup \Sigma_2'' \cup \Sigma_{2,2}'' \cup \Sigma_4$$

(the strata are classified in order of increasing codimensions 0, 1, 3, 4, 5, 5, 6). By construction, this stratification is invariant under the group L_1 of 2-jets of automorphisms.

From theorem 7 of the Appendix, the set of $\omega \in \mathcal{D}_3^2(M)$ such that $j_1\omega$ is transverse to the stratification induced in $\Lambda^2\mathcal{T}_1^*(M)$ by the preceding is *residual*; it is easy to verify that such a form has the properties that are indicated in 3.1.

4. Class and its singularities.

4.1. Consider the vector space F_1^p of jets of order 1 of differential p-forms at the origin of $\mathbf{R}^n$ ($1 \le p \le n$); let

$$d\colon\ F_1^p \to F_0^{p+1} = \Lambda^{p+1}\mathbf{R}_n$$

be the linear map that is defined by exterior multiplication and let:

$$\rho\colon\ F_1^p \to F_0^p = \Lambda^p\mathbf{R}_n,$$

the linear map of restriction.

DEFINITION. – *Let* $\omega_1 \in F_1^p$. *The support of* ω_1 *in* $\mathbf{R}_n$ *is the subspace that is the sum of the supports* (*cf.* I, 1.2) *of* $\rho(\omega_1)$ *and* $d(\omega_1)$. *The class of* ω_1 *is the dimension of its support.*

The associated space to ω_1 is the intersection of the associated spaces to $\rho(\omega_1)$ and $d(\omega_1)$. It is orthogonal to the support of ω_1. The class of ω_1 is therefore the codimension of the associated space as well.

Class obviously invariant under the action of the group L_1.

If M is a manifold and $\omega \in F_1^p$ is a p-form on M then the *class of* ω *at* $x \in M$ will be the class of $j_1\omega(x)$. The associated space to $j_1\omega(x)$ is also called the *characteristic space of* ω *at x.*

Remark. – The class of a closed form is equal to its rank.

4.2. PROPOSITION. – ([12]) *Let M be a manifold of dimension,* $n \ge 7$. *For* $2 \le p \le n - 2$, *the set of p-forms in* D_1^p *that have maximal class n at every point is residual.*

Proof.

a) For $3 \leq p \leq n - 2$, this is an immediate consequence of Theorem II, 2.2, since a form of maximal rank is necessarily of maximal class.

b) For $p = 2$, consider the morphism $d: \Lambda^2 T_1^* M \to \Lambda^3 T^* M$ that is defined by exterior differentiation. Let S be the stratification of $\Lambda^2 T_1^* M$ that is the reciprocal image under d of the stratification $\Sigma^3(M)$ (cf. II, 2.1). The codimension of S is equal to that of $\Sigma^3(M)$, since d is a submersion. Therefore, for $n \geq 7$, codim $S > n$.

From Remark 1 of sec. 6 of the Appendix, the set of $\omega \in D_1^p(M)$ such that $j_1\omega(x)$ is transverse to S (i.e., intersects no non-open strata) is a dense open set. This set is also that set of ω such that $d\omega$ has maximum rank at every point. Its class is, a fortiori, maximal, and the proposition is proved.

4.3. Case of Pfaff forms. This section uses the notations and results of I.7.

4.3.1. Consider the projection $\pi: F_1^1 \to \mathbf{R}_n \oplus \Lambda^2\mathbf{R}_n = F$, defined by $\pi(\omega_1) = (\alpha, \beta) = (\rho(\omega_1), d(\omega_1))$. The *class* of ω_1 is the rank of $\pi(\omega_1)$, in the sense of I, 7.1.

Let ζ notate the stratification of F_1^1 that is the reciprocal image of S by π. One sets $\Sigma'_c = \pi^{-1}(\Sigma_c)$ and $\zeta_d = \pi^{-1}(S_d)$. Σ'_c is therefore the set of 1-jets ω_1 of Pfaff forms such that $\rho(\omega_1) = 0$ (i.e., ω_1 is a zero of order 0), and $d(\omega_1)$ is a 2-form of corank c. ζ_d is the set of jets ω_1 that are not zero of order 0, and have co-class d. The submanifolds Σ'_r constitute a stratification of the vector subspace that is the kernel of ρ in F_1^1. The manifolds ζ_r stratify the open set Ω' that is complementary to the kernel of ρ.

4.3.2. Let us compare the stratification ζ with the set of orbits of L_1 in F_1^1.

Since class is an invariant, the orbits of L_1 are contained in the strata of ζ.

On the other hand, the space F_1^1 is identified in a natural fashion with $\mathbf{R}_n \oplus (\otimes^2\mathbf{R}_n)$. By means of this identification, the map π is defined by:

$$\pi(\alpha, \overline{\beta}) = (\alpha, \beta),$$

where β denotes the *anti-symmetrization* of $\overline{\beta}$. The group L_1 is identified by the set of pairs (a, b), where $a \in Gl(n, \mathbf{R})$ and $b \in Hom_s^2(\mathbf{R}^n, \mathbf{R}^n)$ (the space of symmetric bilinear maps of $\mathbf{R}^n \times \mathbf{R}^n$ into $\mathbf{R}^n$). As a brief calculation shows, the action of L_1 on F_1^1 is then expressed by:

$$(a, b) \cdot (a, \overline{\beta}) = (\alpha \otimes a,\ \alpha \otimes a + \beta \otimes \otimes^2 a).$$

From this, one deduces:

1) In the kernel of ρ, i.e., the set of $(\alpha, \overline{\beta})$ such that $\alpha = 0$, the action of L_1 is identified with that of $Gl(n, \mathbf{R})$. There is therefore an infinitude of orbits.

2) Let $(\alpha, \overline{\beta}) \in \Omega'$ (i.e., $\alpha \neq 0$). There then exists a $b \in Hom_s^2(\mathbf{R}^n, \mathbf{R}^n)$ such that the bilinear form $\alpha \otimes \beta$ is equal to *the opposite of the symmetric part* of $\overline{\beta}$. From this, it results that $(1, b) \cdot (\alpha, \overline{\beta}) = (\alpha, \beta)$. This remark suffices to prove that $L_1(\alpha, \overline{\beta}) = \pi^{-1}[Gl(n, \mathbf{R}) \cdot (\alpha, \beta)]$. One has therefore proved the:

PROPOSITION. – *In the open set* Ω', *which is the set of jets* ω_1 *such that* $\rho(\omega_1) \neq 0$ (i.e., *they are no null of order* 0), *the orbits of* L_1 *are the submanifolds* ζ_d.

Remarks.

1) The fact that ζ_0 is an orbit of L_1 in the preceding proposition may be considered to be the version that Darboux's theorem takes for order 1 of (cf. III.A, 4.1.3 and 4.1.4).

2) Let M be a manifold of dimension n. One has an exact sequence:

$$0 \to \otimes^2 T^*M \to T_1^*M \xrightarrow{\rho} T^*M \to 0,$$

but there is no *canonical* decomposition of the bundle T_1^{*M} into a Whitney sum $T^*M \oplus (\otimes^2 T^*M)$.

4.3.3. From I, 7.2, the subsets Σ'_c and ζ_d of F_1^1, which define the stratification ζ are singularities of order 1 of the Pfaff forms.

Let M be manifold of dimension n. For any Pfaff form ω on M, one lets $Z(\omega)$ denote the set of zeroes of ω, and let $\zeta_d(\omega)$ denote the set of singular points of ω of type ζ_d (i.e., the set of points where ω is different from 0 and has co-class d).

Remark. – A singular point of type ζ_d of a Pfaff form ω is a point x where:

a) $\omega \wedge d\omega^k \neq 0$ and $d\omega^{k+1} = 0$ if $d = n - (2k + 1)$, i.e., if the class is odd.

b) $\omega \neq 0$, $d\omega^{k+1} \neq 0$, and $\omega \wedge d\omega^k = 0$ if $d = n - 2k$; *i.e., the class is even.*

The stratification $\zeta(M)$ of T_1^* that issues from ζ is of codimension 1 and coherent. From the Appendix (Th. 6 and I, 7.2), one has the

THEOREM. – ([11]) *In* $D_k^1(M)$, *for* $k \geq 2$ *the set* $T(\zeta)$ *of forms* ω *such that* $j_1\omega$ *is transverse to* $\zeta(M)$ *is a dense open set. Any form* $\omega \in T(\zeta)$ *has the following properties:*

a) It is transverse to the zero section of $T^*(M)$. $Z(\omega)$ *is a closed set of isolated points.* $d\omega$ *is of maximal rank at each point of* $Z(\omega)$,

b) *For any* d $(0 \le d \le n-1)$, *if the singular set* $\zeta_d(\omega)$ *is non-vacuous then it is a regular submanifold of M of codimension* $\frac{d(d+1)}{2}$.

c) *The class of* ω *is everywhere greater than or equal to* $n - f(n)$, *where* $f(n) = \mathrm{Max}\{p: p(p+1)/2 \le n\}$. *An asymptotic expression for* $n - f(n)$ *is* $n - \sqrt{2n}$.

Remarks.

1) For $\omega \in T(\zeta)$, $\zeta_d(\omega) \cup \zeta_{d-1}(\omega)$ is always a regular submanifold of M of codimension $\frac{d(d+1)}{2}$ if it is non-vacuous. Indeed, from I, 7.3, $\zeta_d(\omega) \cup \zeta_{d-1}(\omega)$ is a submanifold of T_1^*M and $j_1\omega$ is tranverse to it, since it is transverse to $\zeta_d(\omega)$ and $\zeta_d(\omega)$.

2) Let $x \in Z(\omega)$ be a zero of $\omega \in T(\zeta)$. x will be adherent to $\zeta_d(\omega)$ if and only if $d = 0$ (since ζ_0 is obviously a dense open set in M) or $d = 1$, when the ambient dimension n is *odd*.

Examples.

1) Dim $M = 3$: $\zeta_1(\omega)$ is the set of points where $\omega \wedge d\omega = 0$, with $\omega \neq 0$. Generically, this is a hypersurface. $\zeta_2(\omega)$ is the set of points where $d\omega = 0$ and $\omega \neq 0$. Its codimension is 3, so it amounts to a closed set of isolated point, in general. Finally, $\overline{\zeta_1(\omega)} = \zeta_1(\omega) \cup \zeta_2(\omega) \cup Z(\omega)$. $\zeta_1(\omega) \cup \zeta_2(\omega)$ is again a hypersurface of M, which will be *closed* if and only if ω has no zero.

2) Dim $M = 5$: Class is generically greater than or equal to 3. For a form $\omega \in T(\zeta)$ with no zero (if it exists), $\zeta_1(\omega) \cup \zeta_2(\omega)$ will be a closed hypersurface of M, since $\zeta_2(\omega)$ is a closed submanifold of codimension 3 in M.

3) Dim $M = 8$: Generically, class is everywhere greater than or equal to 5, i.e., for $\omega \in T(\zeta)$, one has $\omega \wedge d\omega^2 \neq 0$ at any point of $M - Z(\omega)$. $\zeta_1(\omega)$ is a hypersurface, $\zeta_2(\omega)$ is a submanifold of codimension 3, and $\zeta_3(\omega)$ is a submanifold of codimension 6. Moreover, one has:

$$\overline{\zeta_1(\omega)} = \zeta_1(\omega) \cup \zeta_2(\omega) \cup \zeta_3(\omega), \qquad \overline{\zeta_2(\omega)} = \zeta_2(\omega) \cup \zeta_3(\omega),$$

so $\zeta_3(\omega)$ is closed. $Z(\omega)$ is disjoint from $\zeta_1(\omega)$. $\zeta_1(\omega) \cup \zeta_2(\omega)$ and $\zeta_2(\omega) \cup \zeta_3(\omega)$ are submanifolds of codimensions 1 and 3, respectively.

4.3.4. PROPOSITION. – *There is isotopic stability of the singular sets of the class in* $T(\zeta) \subset D_k^1(M)$ *for* $k \ge 3$.

This signifies that any $\omega \in T(\zeta)$ admits a neighborhood U such that for any $\omega' \in U$, the stratifications of M that are defined the singular sets of the class of ω and ω', respectively, are exchanged by an isotopy of M. This proposition is an immediate corollary of the Appendix, Th. 4.2, since the stratification $\zeta(M)$ is coherent.

4.4. Case of $(n - 1)$-forms. The situation is very simple here. If ω is an $(n - 1)$-form on a manifold M of dimension n then we let $Z(\omega)$ denote the set of its zeroes, and $S(\omega)$, the set of zeroes for $d\omega$. Therefore, in $D_k^1(M)$, for $k \geq 2$, the following properties are generic and define a dense open set:

a) ω is transverse to the zero section of $\Lambda^{n-1}T^*M$, and $Z(\omega)$ is therefore a closed set of isolated points.

b) $d\omega$ is transverse to the zero section of $\Lambda^n T^*M$, and $S(\omega)$ is therefore a hypersurface in M.

c) $S(\omega)$ is closed and disjoint from $Z(\omega)$.

In this case, $S(\omega)$ is the set of points where the class of ω is equal to $(n - 1)$.

The situation for closed $(n - 1)$-forms is also quite simple: Generically, a closed $(n - 1)$-form is transverse to the zero section of the bundle $\Lambda^{n-1}T^*M$, and thus admits isolated zeroes.

5. The class of a Pfaff equation and its singularities.

5.1. Let M be a manifold. One lets $P \to M$ denote the projective bundle that is associated with the vector bundle $T^*M \to M$ (i.e., the set of lines in T^*M), and let T_0^* denote the open subset of T^* that is comprised of non-null forms, and let q: $T_0^* \to P$ be the canonical projection.

One calls any section of the projective bundle over M a *Pfaff equation.* In what follows, we consider only sections that are at least once continuously differentiable.

A Pfaff equation may also be interpreted as a sub-bundle of T^*M in the form of a line bundle, or again, by passing to the orthogonal complement, as a sub-bundle of codimension 1 in the tangent bundle to M (a field of tangent hyperplanes on M).

One denotes the space of k-times continuously differentiable sections of the bundle P by $\Gamma_k(M)$ $(1 \leq k \leq \infty)$ (cf. Appendix, sec. 1.2).

Let σ be a section of P and U, an open set of M. A Pfaff form ω that is defined and non-null at every point of U is called a *covering of σ on U* if $\sigma = q \otimes \omega$ on U. A Pfaff equation admits coverings locally, but not necessarily a global covering.

5.2. If ω and ω' are two coverings of a Pfaff equation σ over an open subset U then one has $\omega' = f \cdot \omega$, in which f is an everywhere non-null function on U. From this, one deduces:

$$\omega' \wedge d\omega' = f^{\,p+1} \cdot \omega \wedge d\omega^p$$

for any integer p.

This justifies the:

DEFINITION. – A Pfaff equation σ is said to have *class* $2p + 1$ at a point $x \in M$ if any covering ω of σ in a neighborhood of x is such that $\omega \wedge d\omega \neq 0$ and $\omega \wedge d\omega^{p+1} = 0$ at x.

Therefore, the class of a Pfaff equation at a point is an odd number. It may also be interpreted as the *reduced rank* (cf. I, 7.3) of the pair $(\omega, d\omega)$ in which ω is an arbitrary local covering of σ. It is an invariant of order 1 of the equation at each point, i.e., it may be defined on the bundle J^1P of jets of order 1 of the sections of P.

5.3. In this section, one uses the notations and results of I, 7 and II, 4.3.

Denote the projection deduced from $q: T_0^* \to P$, by $\tilde{q} = j_1 q: J^1 T_0^* \to J^1 P$.

The fiber type of $J^1 T_0^*$ is the open set Ω' of F_1^1, which is the set of jets ω_1 such that $\rho(\omega_1) \neq 0$ (see II, 4.3).

One denotes the fiber type of J^1P by P_1, and one further notates the projection deduced from $\tilde{q}$ by $\tilde{q}: \Omega' \to P_1$ (P_1 is the manifold of 1-jets of Pfaff equations at the origin of $\mathbf{R}_n$).

The Lie group L_1 of 2-jets of automorphisms at the origin of $\mathbf{R}_n$ acts canonically on P_1 (change of variables in a Pfaff equation), and the actions of L_1 on P_1 and are comparable with $\tilde{q}$.

Let $\sigma_1 \in P_1$. Let $\omega_1 \in \Omega'$ such that $\tilde{q}(\omega_1) = \sigma_1$. The *reduced rank* of $\pi(\omega_1) \in \Omega \subset \mathbf{R}_n \oplus \Lambda^2 \mathbf{R}_n$ (where $\pi: F_1^1 \to \mathbf{R}_n \oplus \Lambda^2\mathbf{R}_n$ is the surjection that was defined in II, 4.3.1) is independent of the choice of ω; it is the *class* of σ_1.

One denotes the set of $\sigma_1 \in P_1$ *that are of co-class* d *– i.e., of class* $n - d$ *– by* C_d $(0 \leq d \leq n - 1)$.

PROPOSITION. – *The orbits of the group* L_1 *in* P_1 *are the sets* C_d $(0 \leq d \leq n - d$ *odd*$)$. *For each* d, C_d *is a regular submanifold of* P_1 *of codimension* $\frac{d(d+1)}{2}$. *One has, moreover:*

$$\overline{C}_d = \bigcup_{d' \geq d} C_{d'} .$$

One lets C denote the stratification of P_1 comprised of the submanifolds C_d. It is a coherent stratification (since it is defined by the orbits of a Lie group).

Proof.

a) Let H be the Lie group of jets of order 1 of *non-null* numerical functions at the origin of $\mathbf{R}^n$. The multiplication of a form by a function defines a law of operation of H on Ω'. With the notations of II, 4.3.2, this law is given by the rule:

$$f \cdot \omega_1 = (\lambda\alpha, h \otimes \alpha + \lambda\overline{\beta}) \quad \text{where} \quad (\alpha, \overline{\beta}) \in \Omega' \subset \mathbf{R}_n \oplus \otimes^2 \mathbf{R}_n,$$
$$\textit{and} \quad f = (\lambda, h) \in (\mathbf{R} - \{0\}) \times \mathbf{R}_n = H.$$

It is clear that P_1 is the quotient of Ω' by the action of H.

b) If one sets, for $f \in H$, $g \in L_1$, $\omega_1 \in \Omega'$ then:

$$(f, g) \cdot \omega_1 = f \cdot (g \cdot \omega_1),$$

and one then remarks that this rule defines a law of operation on the semi-direct product $\overline{G}_1 = H \times L_1$ (L_1 operates in H in an obvious fashion) in Ω'. This law of operation is such that the orbits of L_1 in P_1 are the projections (by $\tilde{q}$) of orbits of $\overline{G}_1$ in Ω'.

c) The action of $\overline{G}_1$ in Ω' and the action of $\overline{G}$ in Ω (cf. I, 7.3.) are compatible with the surjection π: $\Omega' \to \Omega$. It then results from Propositions I, 7.3 and II, 4.3.2, that the orbits of $\overline{G}_1$ in Ω' are the sets $\pi^{-1}(S'_d)$. They are therefore regular submanifolds of Ω' with the same codimensions as the manifolds S'_d. One also has:

$$\overline{\pi^{-1}(S'_d)} = \bigcup_{d' \geq d} \pi^{-1}(S'_{d'}) .$$

From b), the orbits of L_1 in P_1 are therefore the sets C_d. Finally, since the manifold $\pi^{-1}(S'_d)$ is invariant under H, it results from a) that the set C_d is a regular submanifold of P_1 of the same codimension as $\pi^{-1}(S'_d)$, namely, $d(d - 1)/2$. On the other hand, the projection $\tilde{q} : \Omega' \to P_1$ obviously preserves the properties that relate to adherences.

Remarks.

1) P_1 is obviously an algebraic variety. One may easily verify that the sets C_d are algebraic submanifolds of P_1, and that the set of singular points of $\overline{C}_d$ is $\overline{C_{d+2}}$.

2) If the ambient dimension n is even then the class must be odd. The admissible values of d are then 1, 3, 5, …, and the corresponding codimensions of the strata C_d are 0, 3, 10, …, resp. The stratification C then has codimension 3 in this case.

If the ambient dimension n is odd then d must be even. The codimensions of the strata C_d are then 0, 1, 6, 15, …, and the codimension of C is 1.

5.4. Generic behavior of the class of a Pfaff equation.

If we are given a manifold M then the stratification C of P_1, which is invariant under the structure group L_1 of J^1P, induces a stratification $C(M)$ of J^1P for which the strata have the same codimensions as the ones in C, and which is coherent.

One then has, as a corollary to the Appendix, sec. 6.2, the:

THEOREM. – *Let M be a manifold of dimension n. In* Γ_k(M), $k \geq 2$, *the set* $T(C)$ *of Pfaff equations* σ *such that* $j_1 s$ *is transverse to* $C(M)$ *is a dense open set.*

Examples.

1) Dim $M = 3$: for $\sigma \in T(C)$, the set $C_2(\sigma)$ of points where the class is 1 is a closed surface of M (if it is non-vacuous). The class is 3 outside of this surface.

2) Dim $M = 5$: generically, the class is everywhere greater than or equal to 3. $C_2(\sigma)$ is a closed hypersurface for $\sigma \in T(C)$.

3) Dim $M = 4$: the class may be only 1 or 3. The set $C_2(\sigma)$, where the class drops to 1 is generically a closed submanifold of codimension 3.

Remarks.

1) For a "generic" Pfaff equation (in the sense of the preceding section), the class is very large everywhere. It is minorized by a quantity whose asymptotic expression is (as in the case of the rank of a 2-form or the class of a 1-form) $n - \sqrt{2n}$.
2) Since the stratification C is coherent, there is isotopic stability of singular sets of the class of a Pfaff equation in $\Gamma_k(M)$, whenever $k \geq 3$.

6. Generic singularities of a Pfaff equation in dimension 3.

6.1. Let M be a (compact) manifold of dimension 3. The following situation, which relates to a Pfaff equation σ on M, is generic (in $\Gamma_k(M)$ for k very large):

1) *The set $C_2(\sigma)$ of points of M where the class of σ is* 1 (*co-class* = 2) *is a* (*compact*) *surface.*

2) For any point $x \in M$, we denote the plane that is defined by the equation σ by $P_x \subset T_xM$.

Therefore let $C_{2,1}(\sigma) \subset C_2(\sigma)$ be the set of points x such that $P_x = T_xC_2(\sigma)$. $C_{2,1}(\sigma)$ is a set of isolated points.

Set $C_{2,0}(\sigma) = C_2(\sigma) - C_{2,1}(\sigma)$. P_x and $T_xC_2(\sigma)$ are in general position on $C_{2,0}(\sigma)$, so they have a line as their intersection. σ therefore induces a *field of directions D on* $C_{2,0}(\sigma)$, such that the points of $C_{2,1}(\sigma)$ are its singularities.

3) One may make the *generic* situation at the points of $C_{2,1}$ more precise in the following manner:

Let $x \in C_{2,1}(\sigma)$, and let ω be a (local) covering of σ over a neighborhood of x. Let ω' denote the restriction of ω to $C_2(\sigma)$; by definition, ω' is annulled at x. The Jacobian of ω' defines a linear map,

$$T_x\omega' : T_xC_2(\sigma) \to T_x^*C_2(\sigma);$$

i.e., a *bilinear form* on $T_xC_2(\sigma)$.

Since $C_2(\sigma)$ is the locus of zeroes of the product $\omega \wedge d\omega$, one easily shows that $T_x\omega'$ is a *symmetric* bilinear form. *Generically,* it will be *non-degenerate* at any point of $C_{2,1}(\sigma)$. This form may further interpreted as the Jacobian at the point x of the direction field D. From this viewpoint, the preceding remark signifies that the singular points of the field D in $C_2(\sigma)$ may be only *foci* (or centers) or *collars.*

6.2. The proof of the preceding assertions is not difficult. It suffices to define a convenient stratification of the manifold of 2-jets of Pfaff equations such that an equation σ that has the preceding properties of $j_2\sigma$ is transverse to this stratification. I confine myself to giving an example of each of the types of singularities that we just enumerated.

1) At a point where the class of σ is 3, from Darboux's theorem (cf. III, A, 4.1.3), there exist local coordinates (x, y, z) in which σ may thus be defined by the equation:

$$dx + y\,dx = 0.$$

2) At a point of $C_{2,0}(\sigma)$, there exist (cf. III, 4.3) local coordinates in which s may be defined by the equation:

$$x\,dx + (1 + y)\,dz = 0.$$

3) I ignore the question of whether there exists a *model* at a point of $C_{2,0}(\sigma)$. The following Pfaff equations present a singular point of type $C_{2,0}(\sigma)$ at the origin ($x = y = z = 0$), which are a collar and a focus (or center), respectively:

$$\omega_1 = d\left(xz + y^2 + \frac{x^3}{3}\right) + (1+y)dz = 0\,,$$

$$\omega_2 = d\left(xz + y^2 + \frac{x^3}{3}\right) + (1+y)dz = 0\,.$$

For example, in the case of ω_1, $d\omega_1 = dy \wedge dz$:

$$\omega_1 \wedge d\omega_1 = (z + x^2)\,dx \wedge dy \wedge dz.$$

The surface $C_2(\sigma)$ has the equation, $z + x^2 = 0$. The induced form is:

$$\omega' \quad = 2y\,dy - 2x(1 + y + x^2)\,dx.$$

The quadratic form $T_x\omega'$ is then $y^2 - x^2$.
I propose to ultimately return to this point.

CHAPTER III

LOCAL STUDY OF SINGULARITIES

A. LOCAL MODELS

1. Generalities.

1.1. One lets D_k^p, $k \geq 0$, ($\mathcal{D}_k^p$, $k \geq 1$, resp.) denote the vector space of *germs* of (closed, resp.) exterior differential forms of degree p at the origin of $\mathbf{R}^n$, with coefficients that are k-times continuously differentiable. One sets $D^p = D_\infty^p$ and $\mathcal{D}^p = \mathcal{D}_\infty^p$.

Let $\mathcal{L}_k$ be the group of germs of automorphisms at the origin of $\mathbf{R}^n$ that are k-times continuously differentiable. One sets $\mathcal{L} = \mathcal{L}_\infty$.

DEFINITION. – *Two germs* ω *and* ω'*, resp., are called* C^r*-isomorphisms if there exists a* $g \in \mathcal{L}_r$ (*with* $1 \leq r \leq k + 1$) *such that*:

$$\omega' = g^* \omega,$$

in which $g^*\omega$ *denotes the inverse image of* ω *by* g.

If $r = k = \infty$ then ω and ω' will simply be called *isomorphic.*

1.2. The notations that are concerned with the jet spaces are the same as in chapter II.

For any $\omega \in D_{k'}^p$, one lets $j_k : D_{k'}^p \to F_k^p$ ($k' \geq k$) notate the restriction homomorphism that is defined by $j_k(\omega)$ = jet of order k of ω at 0. In a similar fashion, one defines the restriction homomorphism:

$$j_k : \mathcal{D}_k^p \to \mathcal{F}_k^p$$

(the space of k-jets of closed p-forms).

Let $\Sigma \subset F_k^p$ be a singularity of order k of a p-form. An essential problem is the classification of the germs ω in $D_{k'}^p$ ($k' \geq k$) that present the singularity Σ (i.e., such that $j_k\omega \in \Sigma$) from the standpoint of the relation of C^r-isomorphism ($1 \leq r \leq k' + 1$).

DEFINITION. – A singularity $\Sigma \subset F_k^p$ ($\mathcal{F}_k^p$, *resp.*) *is called rigid if all of the germs* $\omega \in D^p$ ($\mathcal{D}^p$, *resp.*) *such that* $j_k\omega \in \Sigma$ *are isomorphic.*

Any element of Σ, which we interpret to mean a differential form with polynomial coefficients (of degree less than or equal to k), will then be called a (local) *model* for the singularity.

We remark that, in this case Σ is an orbit of L_k in F_k^p ($\mathcal{F}_k^p$, resp.) and for any $k' \geq k$, $\rho^{-1}(\Sigma) \subset$ is an orbit of $L_{k'}$.

The primary objective of this chapter is to commence the study of singularities from the standpoint just defined.

1.3. I now recall a result that concerns the Lie derivative.

If X is a vector field and ω is a differential form then we notate the Lie derivative of ω with respect to X by $\theta(X)\omega$. If X and ω are of class C^k then $\theta(X)\omega$ is of class C^{k-1}. One recalls that $\theta(X)\omega = d(X \rfloor \omega) + X \rfloor d\omega$.

Now let X_t be a "time-dependent" vector field that is defined by C^k ($k \geq 1$) on $U \times [0, 1]$, in which U is an open neighborhood of 0 in $\mathbf{R}^n$ such that $X(0, t) = X_t(0)$ for any t.

The differential equation $\dfrac{dx}{dt} = X(x, t)$ defines a map $\varphi: U' \times [0, 1] \to \mathbf{R}^n$ (where is an open neighborhood of 0), such that:

$$\frac{\partial \varphi(x,t)}{\partial t} = X(t, x) \qquad \text{and} \qquad \varphi(x, 0) = x,$$

for any $x \in U'$.

φ is C^k and for any $t \in [0, 1]$, the map φ_t defined by $\varphi_t(x) = \varphi(x, t)$ defines an element of $\mathcal{L}_k$, and φ_0 is the identity.

Let ω_t be a one-parameter family of differential forms that is defined and C^k on $U \times [0, 1]$. One notates the "velocity of deformation" at the time t by $\dot{\omega}_t = \dfrac{\partial \omega_t}{\partial t}$.

PROPOSITION. – *With the preceding givens and notations, the following conditions are equivalent*:

a) $\theta(X_t)\omega_t = \dot{\omega}_t$,

b) $\varphi_t^*(\omega_0) = \omega_t$.

Proof. – One sets $\alpha_t = (\varphi_t^{-1})^*(\omega_t)$. An elementary calculation shows that:

$$\frac{\partial \alpha_t}{\partial t} = \dot{\alpha}_t = (\varphi_t^{-1})^*[\dot{\omega}_t - \theta(X_t)\omega_t],$$

If one assumes a) then one has $\dot{\alpha}_t = 0$, but $\alpha_0 = (\varphi_t^{-1})^*(\omega_0) = \omega_0$, since φ_0 is the identity. Therefore, $\alpha_t = \omega_0$ for any t.

1.4. Before proceeding, I need to make the following two remarks. Let $\omega \in D_{k'}^p$ be a germ of a differential form at the origin of $\mathbf{R}^n$, whose natural coordinates are $(x_1, \ldots, x_n)$. One denotes the restriction of ω to the hyperplane $x_1 = 0$ by ω_0. One denotes the canonical projection of $\mathbf{R}^n$ on that hyperplane by π.

1.4.1. If $k \geq 1$ and if $\frac{\partial}{\partial x_1} \zeta\, \omega = 0$ and $\frac{\partial}{\partial x_1} \zeta\, d\omega = 0$, i.e., $\theta\left(\frac{\partial}{\partial x_1}\right)\omega = 0$ and $\omega = 0$, then one has:

$$\omega = \pi^*(\omega_0).$$

In other words, the form ω is uniquely expressed with the aid of the coordinates x_i $(2 \leq i \leq n)$ and their differentials.

1.4.2. If $k \geq 1$ and if $\frac{\partial}{\partial x_1} \zeta\, \omega = 0$ and $\frac{\partial}{\partial x_1} \zeta\, \omega = f \cdot \omega$ (in which f denotes a C^{k-1} function), i.e., if $\frac{\partial}{\partial x_1} \zeta\, \omega = 0$ and:

$$\theta\left(\frac{\partial}{\partial x_1}\right)\omega = f \cdot \omega$$

then one has:

$$\omega = h\pi^*(\omega_0),$$

in which h is a C^{k-1} function that is equal to 1 when $x_1 = 0$.

2. Case of n-forms.

2.1. First consider the simplest case, for which:

$$\Sigma = \Sigma_{n,n}^n \subset F_0^n = \Lambda^n \mathbf{R}_n ,$$

is the set of non-null n-forms. A germ $\omega \in D_k^n$ presents the "singularity" $\Sigma_{n,n}^n$ if it is non-null; it is therefore the germ of a volume form.

THEOREM. – *For $k \geq 0$, any germ of a volume form $\omega \in D_k^n$ is C^k-isomorphism to the germ that is defined by the expression*:

$$dx_1 \wedge \ldots \wedge dx_n$$

($x_1, \ldots, x_n$ *denote the coordinates in* $\mathbf{R}^n$).

This result is classical and trivial. If:

$$\omega = f(x_1, \ldots, x_n)\, dx_1 \wedge \ldots \wedge dx_n$$

is a volume germ – i.e., $f(0) \neq 0$ – then the change of variables:

$$X_1 = \int_0^{x_1} f(t, x_2, \cdots, x_n)dt\,,$$
$$X_i = x_i \qquad \text{for} \qquad i = 2, \ldots, n$$

brings us to the indicated "canonical" form and defines a C^k-automorphism.

2.2. One further considers the set $\Sigma_{n,0}^n \subset F_1^n$ composed of *non*-null jets of order 1 whose image under $\rho: F_1^n \to F_0^n$ is null. A germ $\omega \in D_k^n$ ($k \geq 1$) is such that $j_1\omega \in \Sigma_{n,0}^n$ if and only if ω is annulled at 0 *transversally* to the zero section of bundle of n-forms.

THEOREM. – ([12]) *For $k \geq 3$ any germ $\omega \in D_k^n$ such that $j_1\omega \in \Sigma_{n,0}^n$ is C^{k-2}-isomorphic to the germ that is defined by the expression*:

$$x_1\, dx_1 \wedge \ldots \wedge dx_n.$$

Proof. – Let $\omega = f(x_1, \ldots, x_n)\, dx_1 \wedge \ldots \wedge dx_n$ be the expression for ω in natural coordinates. By hypothesis, f is C^k, $f(0)$ and df is non-null at 0. The equation $f = 0$ determines a germ of a hypersurface S the set of zeroes of ω. We shall establish a result that is more precise than is necessary for the present situation because it will relate to what follows. The theorem is an obvious consequence of it.

LEMMA. – *Let $\omega \in D_k^n$ ($l \geq 3$) such that $j_1\omega \in \Sigma_{n,0}^n$. Let S be the germ of the hypersurface of zeroes of ω. If $u_1, \ldots, u_n$ are C^k functions ($3 \leq k \leq l$) that are null at and independent by restriction to S then there exists a function u_1 of class C^{k-2} that is null at 0 such that*:

a) $(u_1, \ldots, u_n)$ *is a system of local coordinates,*
b) $\omega = \pm\, u_1\, du_1 \wedge \ldots \wedge du_n$ (*the sign is determined by the given of ω, $u_2, \ldots, u_n$*).

Proof. – The independence of the functions $u_2, \ldots, u_n$ on S signifies that $df \wedge du_2 \wedge \ldots \wedge du_n \neq 0$ at the origin. The functions $(f, u_2, \ldots, u_n)$ therefore constitute a system of local C^k coordinates. In this system, one has:

$$\omega = h(f, u_2, \ldots, u_n)\, df \wedge du_2 \wedge \ldots \wedge du_n;$$

the function h is of class C^{k-1}. One obviously has $h(0, u_2, \ldots, u_n) = 0$, and dh is non-null at the origin; i.e., $\dfrac{\partial h}{\partial f}(0) \neq 0$. There thus exists one (and only one) function g such that:

$$h(f, u_2, \ldots, u_n) = f \cdot g(f, u_2, \ldots, u_n).$$

The function g is of class C^{k-2} and non-null at the origin. To prove the lemma it then suffices to establish the existence of a function u_1 that is of class C^{k-2} and verifies the equation:

$$\frac{u_1^2}{2} = \varepsilon \int_0^f t \cdot g(t, u_2, \cdots, u_n)dt\,,$$

in which ε = sign of $g(0)$.

(One will then have $\omega = \varepsilon\, u_1\, du_1 \wedge \ldots \wedge du_n$.)

Now:

$$\int_0^f t \cdot g(t, u_2, \cdots, u_n)\,dt = f^2 \cdot \int_0^1 x \cdot g(f \cdot x, u_2, \cdots, u_n)\,dx$$
$$= f^2 \cdot \varphi(f, u_2, \ldots, u_n),$$

and φ is a function of class C^{k-2} that is non-null at the origin and has the same sign as g. The function:

$$u_1 = \sqrt{2} \cdot f \cdot \sqrt{\varepsilon\varphi}$$

answers the question.

2.3. *Remark.* – The singularities $\Sigma^n_{n,n}$ and $\Sigma^n_{n,0}$ are the only singularities that are generically presented (II, 2.3). We just showed that they are rigid. The problem of classification is therefore completely solved in the case of n-forms.

3. Case of $(n - 1)$-forms.

3.1. Let $\Sigma^{n-1}_{n,n} \subset F^{n-1}_1$ be the set of jets ω_1 of order 1 of $(n - 1)$-forms such that $j_0(\omega_1) \neq 0$ and $d\omega_1 \neq 0$. A germ $\omega \in D^{n-1}_k$ $(k \geq 1)$ is such that $j_1\omega \in \Sigma^{n-1}_{n,n}$ if and only if ω and $d\omega$ are non-null at 0. $\Sigma^{n-1}_{n,n}$ is therefore an open singularity in F^{n-1}_1.

THEOREM. – *For $k \geq 2$, any germ $\omega \in D^{n-1}_k$ such that $j_1\omega \in \Sigma^{n-1}_{n,n}$ is C^{k-1}-isomorphic to the germ that is defined by the expression:*

$$(1 + x_1)\, dx_2 \wedge \ldots \wedge dx_n.$$

Proof. – Let D be the *direction field associated to* ω (which defined at each point by the line associated to ω). Let $u_2, \ldots, u_n$ be $n - 1$ independent C^k first integrals of the differential equation D that are null at 0. One then has:

$$\omega = h\, du_2 \wedge \ldots \wedge du_n,$$

where h is a function of class C^{k-1} that is non-null at the origin. One may choose functions $u_2, \ldots, u_n$ in such a way that $h(0) = 1$. However, $d\omega = dh \wedge du_2 \wedge \ldots \wedge du_n \neq 0$ at the origin. The functions $u_1 = 1 - h, u_2, \ldots, u_n$ thus define a C^{k-1} system of local coordinates at the origin in which ω has the required expression.

3.2. Because of the remarks of II, 4.4, it is natural to define that singularity $\Sigma^{n-1}_{n,n-1} \subset F^{n-1}_2$, which is the set of jets ω_2 of order 2 of $(n - 1)$-forms such that $d\omega_2 \in \Sigma^n_{n,0}$ (cf. III, 2.2) and $j_0(\omega_2) \neq 0$. If ω is a germ in D^{n-1}_k $(k \geq 2)$ then one will have $j_2(\omega) \in \Sigma^{n-1}_{n,n-1}$ if and only if ω is non-null and if $d\omega$ is annulled *transversally* to the origin (ω is therefore of class $n - 1$ at the origin). It is obvious that $\Sigma_{n,n-1}^{n-1}$ is a singularity of codimension 1 in F_2^{n-1}.

Let $\omega \in D_k^{n-1}$ $(k \geq 2)$ such that $j_2(\omega) \in \Sigma_{n,n-1}^{n-1}$. One may consider the direction field D associated with ω and the hypersurface S of zeroes of $d\omega$. It is clear that the *order of contact* of the field D and the surface S at the origin is an invariant of the (C^∞) isomorphism class of the germ ω. Therefore, if one considers – in $\mathbf{R}^3$ (x, y, z coordinates), for example – the germs that are defined by the expressions:

$$\omega_1 = (1 + x^2)\, dy \wedge dz,$$
$$\omega_2 = (1 + xy - x^3)\, dy \wedge dz,$$
$$\omega_3 = (1 + xz - x^2y - x^4)\, dy \wedge dz$$

then the associated direction field D is the field parallel to the x-axis in all three cases. On the other hand:

$$d\omega_1 = 2x^2\, dx \wedge dy \wedge dz,$$
$$d\omega_2 = (y - 3x^2)\, dx \wedge dy \wedge dz,$$
$$d\omega_3 = (z - 2xy - 4x^3)\, dx \wedge dy \wedge dz,$$

and the equations of the surfaces defined by the zeroes are (S_1) $x = 0$, (S_2) $y - 3x^2 = 0$, (S_3) $z - 2xy - 4x^3 = 0$, respectively. In the first case, the field D is transverse to S_1, i.e., the restriction of ω_1 to S_1 is non-null. In the second case, the projection that is defined by the integral curves of D (i.e., the projection parallel to Ox) defines a *fold* (in the sense of Whitney [17]) of the surface S_2 in the plane yOz. The restriction of ω_2 to S_2 presents the singularity $\Sigma_{n-1,0}^{n-1}$ at the origin. In the latter case, the projection of S_3 in yOz parallel to Ox defines a *cusp* (in the sense of Whitney [17]). Here again, the restriction of ω_3 to S_3 represents the singularity $\Sigma_{n-1,0}^{n-1}$ at the origin.

These remarks show that the germs of ω_1, ω_2, ω_3 at the origin are not C^k-isomorphic $(k \geq 4)$.

One defines the singularity $\Sigma_{n,n-1,0}^{n-1} \subset F_2^{n-1}$ *to be the set of jets* ω_2 *such that*:

1) $d\omega_2 \in \Sigma_{n,0}^{n}$,
2) *The restriction of* ω_2 *to the hypersurface of zeroes of* $d\omega_2$ *is non-null.*

These conditions define an open set in $\Sigma_{n,n-1}^{n-1}$, and therefore a singularity of codimension 1.

THEOREM. – ([12]) *For* $k \geq 5$, *any germ* $\omega \in D_k^{n-1}$ *such that* $j_2\omega \in \Sigma_{n,n-1,0}^{n-1}$ *is* C^{k-4}-*isomorphic to one of the germs defined by the expressions*:

$$\omega_1 = \left(1 + \frac{x_1^2}{2}\right) dx_2 \wedge \cdots \wedge dx_n,$$
$$\omega_2 = \left(1 - \frac{x_1^2}{2}\right) dx_2 \wedge \cdots \wedge dx_n.$$

Proof. – Let S be the hypersurface of zeroes of $d\omega$ and let D be the direction field associated with ω. S is transverse to D since the restriction of ω to S is non-null. We notate this restriction by ω'.

The manifold S is of class C^{k-1}, and ω' is a volume germ of class C^{k-2} at the origin in S. From Th. 2.1, there exist C^{k-2} local coordinates $\bar{u}_2, \ldots, \bar{u}_n$ in S such that:

$$\omega' = d\bar{u}_2 \wedge \ldots \wedge d\bar{u}_n . \tag{1}$$

One then *prolongs* the functions $\bar{u}_2, \ldots, \bar{u}_n$ to a neighborhood of 0 in $\mathbf{R}^n$ and keeps them constant on the integral curves of D. One thus obtains functions $\bar{u}_2, \ldots, \bar{u}_n$ that are always of class C^{k-2} and, by construction, they are independent upon restriction to S.

Upon applying the lemma of paragraph 2.2 to $(d\omega, u_2, \ldots, u_n)$, one finds a function u_1 of class C^{k-4} such that $(u_1, \ldots, u_n)$ is a system of local coordinates at the origin, and:

$$d\omega = \pm\, u_1\, du_1 \wedge \ldots \wedge du_n . \tag{2}$$

We study the expression for ω in the coordinates $(u_1, \ldots, u_n)$. One obviously has:

$$\omega = h(u_1, \ldots, u_n)\, du_1 \wedge \ldots \wedge du_n ,$$

in which, by construction, the functions $u_2, \ldots, u_n$ comprise a system of first integrals of D. However, (1) is equivalent to $h(0, u_2, \ldots, u_n) = 1$ and (2) is equivalent to $\frac{\partial h}{\partial u_1} = \pm u_1$. One therefore has:

$$h = 1 \pm \frac{u_1^2}{2} \qquad \text{Q.E.D.}$$

Remarks.

1) Let ω be a germ such that $j_2(\omega) \in \Sigma^{n-1}_{n,n-1,0}$. At any point that does not belong to the surface S the linear equation $X \rfloor d\omega = \omega$ defines a non-null vector in the line that is associated to ω. The vector field that is thus associated to the form ω_1 (ω_2, resp.) of the preceding theorem is $X_1 = \left(\frac{1}{x_1} + \frac{x_1}{2}\right)\frac{\partial}{\partial x_1}$:

$$\left(X_2 = \left(-\frac{1}{x_1} + \frac{x_1}{2}\right)\frac{\partial}{\partial x_1}, resp.\right).$$

This field is *directed towards S* in the case of ω_2 and *directed towards the exterior* in the case of ω_1. This observation illustrates the fact that ω_1 and ω_2 may not be isomorphic.

2) I will ignore the issue of whether the higher-order singularities that correspond to orders of contact higher than 1 for the associated field D and the hypersurface S are rigid.

3.3. In II, 4.4, we saw that an $(n – 1)$-form ω may generically present isolated zeroes that are *disjoint* from the set of zeroes of $d\omega$. We simply mention that at such a point one may canonically associate ω with the germ of vector fields X that are defined by linear equation $X \,\lrcorner\, d\omega = \omega$. Like ω, X is annulled at the point considered. All of the invariants of the field X are invariants of ω.

3.4. Case of closed (n – 1)-forms. I recall only the following classical result: *Any germ of a closed* $(n – 1)$-*form* $\omega \in \mathcal{D}_k^{n-1}$ $(k \geq 1)$, *which is non-null at the origin is* C^k-*isomorphic to the germ* $dx_2 \wedge \ldots \wedge dx_n$.

On the other hand, we remark that if Ω denotes an n-form without zeroes (volume form), and if X denotes a vector field then the form $\omega = X \,\lrcorner\, \Omega$ is closed if and only if the Lie derivative $\theta(X)\Omega$ is *null*, since $\theta(X)\Omega = X \,\lrcorner\, d\Omega + d(X \,\lrcorner\, \Omega)$.

The study of a closed $(n – 1)$-form is therefore very close to the study of a *unimodular vector field* (i.e., one that preserves a volume).

4. Cases of closed 2-forms and Pfaff forms.

4.1. Darboux's theorem. This theorem establishes the existence of "canonical forms" for the germ of a closed 2-form of *maximum rank* (in either even or odd dimension) and the germ of a Pfaff form of *maximum class.* I shall not give a complete of it here, since it is found in numerous places in the literature ([3], [5], [6], [10], [18]). The central idea of the proof that one finds in the school of J. Moser ([14]), and seems most natural to me, takes into account considerations regarding the stability and infinitesimal stability of differential forms.

First recall that Σ_0 (Σ_1, resp.) denotes the open set of elements of $\Lambda^2\, \mathbf{R}_n = F_0^2 = \mathcal{F}_0^2$ (the vector space of jets of order 0 of 2-forms, which may or may be closed) with $n = 2p$ even ($n = 2p + 1$ odd, resp.) that have maximum rank $2p$ (the *corank* is then 0 or 1, respectively).

On the other hand, ζ_0 denotes the open set of F_1^1 (the space of jets of order 1 of Pfaff forms at the origin of $\mathbf{R}^n$) that is composed of the jets of class n (cf. II, 4.3.1).

4.1.1. THEOREM. – *Let* $\omega \in \mathcal{D}_k^2$ $(k \geq 1)$ *be a germ of a closed* 2-*form at the origin of* $\mathbf{R}^n$, *with* $n = 2p$ *such that* $j_0\omega \in \Sigma_0$. *Then* ω *is* C^k-*isomorphic to the germ that is defined by the expression:*

$$dx_1 \wedge dy_1 + \ldots + dx_p \wedge dy_p$$

(*in which* $x_1, \ldots, x_p, y_1, \ldots, y_p$, *denote the natural coordinates in* $\mathbf{R}^{2p}$).

Proof. – Consider the form ω_0 *with constant coefficients* that is equal to the expression for ω at the origin, and set:

$$\omega_t = \omega_0 + t \cdot (\omega - \omega_0) \quad \text{for any} \quad t \in [0, 1].$$

Therefore $\omega_1 = \omega$, and for any $t \in [0, 1]$, ω_t is of maximum rank at the origin.

The 2-form $\alpha = \omega - \omega_0$ is C^k and closed. There thus exists a Pfaff form β with C^k coefficients that one may take to be *null at the origin* and has the property that $\alpha = d\beta$.

Now, since the *ambient dimension is* $2p$ and the form ω_t has rank $2p$ in a neighborhood of the origin, the linear equation,

$$X_t \rfloor \omega_t = \beta,$$

determines the germ of a vector field X_t, which is null at the origin and of class C^k for any $t \in [0, 1]$.

For any t one then has the relation:

$$\theta(X_t)\,\omega_t = d(X_t \rfloor \omega_t) + X_t \rfloor d\omega_t = \alpha,$$

since ω_t is closed.

If one then considers the differential equation:

$$\frac{dx}{dt} = X_t(x) = X(t, x)\,,$$

and if one notates the solution that verifies $\varphi_0(x) = x$ by $\varphi_t(x)$, then one has $\varphi_t(x) = 0$ for any t, since $X(t, 0) = 0$. The function φ is therefore defined and C^k on $[0, 1] \times U$, in which U denotes a sufficiently small neighborhood of 0 in $\mathbf{R}^n$. Therefore, for any $t \in [0, 1]$, φ_t is a germ of a diffeomorphism that preserves the origin, and, from III.A, 1.3, one has, by construction:

$$\varphi_t^*(\omega_0) = \omega_t \quad \text{for any } t.$$

Therefore, φ_1 defines a C^k-isomorphism of ω onto ω_0. As a result, from I, 4.1, to arrive at the stated expression of the theorem, it suffices to make a *linear* change of variables in ω_0.

4.1.2. THEOREM. – *Let* $\omega \in \mathcal{D}_k^2$ $(k \geq 2)$ *be the germ of a closed* 2*-form at the origin of* $\mathbf{R}^n$ *with* $n = 2p + 1$ *such that* $j_0\,\omega \in S_1$. *Therefore,* ω *is* C^k*-isomorphic to the germ that is defined by the expression:*

$$dx_1 \wedge dy_1 + \ldots + dx_p \wedge dy_p$$

(*in which* $z, x_1, \ldots, x_p, y_1, \ldots, y_p$, *denote coordinates in* $\mathbf{R}^{2p+1}$).

Proof. – One has a direction field D that is defined at each point by the line associated to ω (which is of corank 1). This field is of class C^k.

Let S be a hypersurface element that is transverse to D at the origin (for example, a hyperplane). The restriction $\overline{\omega}$ of ω to S verifies the hypotheses of theorem 4.1.1. There thus exist functions $\overline{x}_1, \ldots, \overline{x}_p, \overline{y}_1, \ldots, \overline{y}_p$ that constitute a C^k system of local coordinates at the origin in S, such that:

$$\overline{\omega} = d\overline{x}_1 \wedge d\overline{y}_1 + \ldots + d\overline{x}_p \wedge d\overline{y}_p\,.$$

Now let $x_1, \ldots, x_p, y_1, \ldots, y_p$, be the first integrals of D that are obtained by prolonging $\bar{x}_1, \ldots, \bar{x}_p, \bar{y}_1, \ldots, \bar{y}_p$, which we complete into a system of coordinates:

$$C^k: (z, x_1, \ldots, x_p, y_1, \ldots, y_p).$$

In this system, one obviously has $\frac{\partial}{\partial z} \rfloor \omega = 0$. From III.A, 1.4.1, since the expression for ω is C^{k-1} with respect to its coordinates, one will have:

$$\omega = dx_1 \wedge dy_1 + \ldots + dx_p \wedge dy_p$$

provided that $k - 1 \geq 1$. Q.E.D.

4.1.3. THEOREM. – *Let* $\omega \in D_k^1$ $(k \geq 3)$ *be the germ of a Pfaff form at the origin of* $\mathbf{R}^n$ *with* $n = 2p + 1$ *such that* $j_1\omega \in \zeta_0$; *i.e.,* $\omega \wedge d\omega \neq 0$. ω *is then* C^k*-isomorphic to the germ that is defined by the expression,*

$$dz + dx_1 \wedge dy_1 + \ldots + dx_p \wedge dy_p .$$

Proof. – One has $d\omega \in \mathcal{D}_{k-1}^2$, and, from 4.1.2, there exist functions $x_1, \ldots, x_p, y_1, \ldots, y_p$ such that:

$$d\omega = \sum_{i=1}^{p} dx_i \wedge dy_i .$$

The differential form $\bar{\omega} = \sum_{i=1}^{p} x_i dy_i$ is therefore C^{k-2}, and such that $d(\omega - \bar{\omega}) = 0$. Therefore, $\omega - \bar{\omega} = dz$, in which z is a C^{k-1} function that is null at the origin. One has precisely:

$$\omega = dz + \sum_{i=1}^{p} x_i \, dy_i ,$$

because the functions $z, x_1, \ldots, x_p, y_1, \ldots, y_p$ are independent, since $\omega \wedge d\omega^p = p!\, dz \wedge dx_1 \wedge dy_1 \wedge \ldots \wedge dx_p \wedge dy_p \neq 0$.

4.1.4. THEOREM. – *Let* $\omega \in$ $(k \geq 3)$ *be the germ of a Pfaff form at the origin of* $\mathbf{R}^n$ *with* $n = 2p$ *such that* $j_1\omega \in \zeta_0$; *i.e.,* $\omega \neq 0$ *and* $d\omega_p \neq 0$. ω *is then* C^{k-2}*-isomorphic to the germ defined by the expression:*

$$(1 + x_1)\, dy_1 + x_2 dy_2 + \ldots + x_p dy_p .$$

Proof. – By hypothesis, the $(2p - 1)$-form $\omega \wedge d\omega^{p-1}$ is non-null and C^{k-1}. Let D be the direction field associated with this form. If X is a non-null vector field carried by D then one has $X \rfloor (\omega \wedge d\omega^{p-1}) = 0$, namely:

$$(X \rfloor \omega) \cdot d\omega^{p-1} - \omega \wedge (X \rfloor d\omega^{p-1}) = 0,$$

which implies $X \rfloor \omega = 0$ and $X \rfloor d\omega = f \cdot \omega$ since $d\omega$ is of rank $2p$.

Therefore, let S be an element of a hypersurface that is transverse to D at the origin. The restriction $\bar{\omega}$ of ω to S verifies the hypotheses of theorem 4.1.3. There thus exist C^{k-1} local coordinates $\bar{y}_1, \bar{x}_2, \bar{y}_2, \ldots, \bar{x}_p, \bar{y}_p$ at the origin in S such that:

$$\bar{\omega} = d\bar{y}_1 + \bar{x}_2\, d\bar{y}_2 + \ldots + \bar{x}_p\, d\bar{y}_p\,.$$

Let $y_1, x_2, \ldots, y_p$ be first integrals of D that are obtained by prolonging $\bar{y}_1, \bar{x}_2, \ldots, \bar{y}_p$. From the remark at the beginning of III.A, 1.4.2, one has:

$$\omega = h \cdot \omega',$$

in which $\omega' = dy_1 + x_2dx_2 + \ldots + x_pdy_p$ and h is a C^{k-2} function that is equal to 1 on S.

On the other hand, one has:

$$\begin{aligned} d\omega^p &= (dh \wedge \omega' + h\, d\omega')^p = \\ &= p(p-1)!\; h^{p-1}\, dh \wedge dy_1 \wedge \ldots \wedge dx_p \wedge dy_p \neq 0. \end{aligned}$$

Therefore, $h, y_1, x_2, \ldots, y_p$ are independent. The change of variables:

$$\begin{aligned} &X_1 = h - 1, && \\ &X_i = h \cdot x_i && \text{for} \quad 2 \leq i \leq p, \\ &X_j = y_j && \text{for} \quad 2 \leq j \leq p \end{aligned}$$

is C^{k-2} and puts ω into the required form.

4.2. We shall now examine the simplest singularities of the class of a Pfaff form and the rank of a closed 2-form.

4.2.1. As far as Pfaff forms are concerned, it is natural to first envision the case of germs ω that enjoy the following properties:

i) $j_1\omega \in \zeta_1$, i.e., the class of ω at 0 is $n - 1$.

ii) ω is generic from the point of view of II, 4.3.3, i.e., the map that is defined by $x \to j_1\omega(x)$ is transverse to ζ_1. The class of ω therefore remains equal to $(n - 1)$ at the points of a hypersurface $S = \zeta_1(\omega)$ that passes through the origin.

iii) The restriction of ω to S is of maximal class $(n - 1)$.

Indeed, these three conditions express properties of $j_2\omega$. More precisely, they define an *open set* of $\rho^{-1}(\zeta_1)$ (in which $\rho: F_2^1 \to F_1^1$ is the restriction of the homomorphism of the space of jets of order 2 into the jets of order 1), hence a submanifold of codimension 1 of F_2^1; i.e., a singularity of order 2. This singularity will be denoted by $\zeta_{1,0}$.

THEOREM. – ([12]) *Let* $\omega \in D_k^1$, *in which* $k \geq 7$ ($k \geq 6$, *resp.*), *be the germ of a* 1-*form at the origin of* $\mathbf{R}^n$, *in which* $n = 2p = 1$ ($n = 2p$, *resp.*), *such that* $j_2\omega \in \zeta_{1,0}$. ω *is therefore* C^{k-6}-*isomorphic* (C^{k-5}-*isomorphic, resp.*) *to one and only one of the germs defined by expressions*:

$$\pm z\, dz + (1 + x_1)\, dy_1 + x_2\, dy_2 + \dots + x_p\, dy_p\,,$$

(*to one and only one of the germs*:

$$\left(1 \pm \frac{x_1^2}{2}\right) dy_1 + x_2 dy_2 + \cdots + x_p dy_p\,,$$

resp.).

Proof.

a) $n = 2p + 1$. In this case, condition i) signifies that $\omega \wedge d\omega^p = 0$ and $d\omega^p \neq 0$ at the origin. Condition ii) signifies that the n-form $\omega \wedge d\omega^p$ is annulled *transversally* at the origin, and thus presents the singularity $\Sigma_{n,0}^n$ (in the sense of III, A, 2.2). *The locus of zeroes is a germ of the hypersurface S* of class C^{k-1}. Condition iii) signifies that the restriction $\bar{\omega}$ of ω to S verifies $\bar{\omega} \neq 0$ and $d\bar{\omega} \neq 0$ at the origin. In particular, *the direction field D that is associated with* $d\omega$ *is transverse to S*.

The form $\bar{\omega}$ is C^{k-1}. From theorem III.A, 4.1.4, there exist C^{k-4} local coordinates $\bar{x}_1, \dots, \bar{x}_p\ \bar{y}_1, \dots, \bar{y}_p$ at the origin of S such that:

$$\bar{\omega} = (1 + \bar{x}_1)\, d\bar{y}_1 + \bar{x}_2\, d\bar{y}_2 + \dots + \bar{x}_p\, d\bar{y}_p\,.$$

Let $x_1, \dots, x_p, y_1, \dots, y_p$ be the first integrals of D defined by the prolongations of $\bar{x}_1, \dots, \bar{x}_p\ \bar{y}_1, \dots, \bar{y}_p$. From Lemma III.A, 2.2, applied to $\omega \wedge d\omega^p$, there exists a C^{k-6} function such that:

$$\omega \wedge d\omega^p = \pm z\, dz \wedge dx_1 \wedge \dots \wedge dy_p\,. \tag{1}$$

One obviously has $d\omega = dx_1 \wedge dy_1 + \dots + dx_p \wedge dy_p$ (from III.A, 1.4.1). Therefore, if $\omega' = (1 + x_1)\, dy_1 + x_2\, dy_2 + \dots + x_p\, dy_p$ then one has $\omega - \omega' = df$, since f is a C^{k-5} function. df is null on S; one may thus take f to be *null on S*. Therefore:

$$\omega \wedge d\omega^p = df \wedge d\omega'^p = (p-1)!\,\frac{\partial f}{\partial z}\, dz \wedge dx_1 \wedge \dots \wedge dy_p$$

in the system of coordinates $(z, x_1, \ldots, x_p, y_1, \ldots, y_p)$. By comparison with (1), one immediately:

$$df = !\frac{z}{(p-1)!}\,dz.$$

The expression for ω in the C^{k-6} coordinate system:

$$(z, x_1, \ldots, x_p, y_1, \ldots, y_p)$$

is therefore:

$$\omega = \pm\frac{z}{(p-1)!}\,dz + (1 + x_1)dy_1 + x_2dy_2 + \ldots + x_pdy_p\,,$$

an expression that one easily transforms into the indicated form of the theorem.

b) $n = 2p$. This time, one has $d\omega_p = 0$ and $\omega \wedge d\omega^{p-1} \neq 0$ at the origin. $d\omega^p$ presents the singularity $\Sigma^n_{n,0}$. *Let S be the hypersurface of zeroes of* $d\omega^p$ (S is of class C^{k-1}). From iii), the restriction $\bar{\omega}$ of ω to S is such that $\bar{\omega} \wedge d\bar{\omega}^{\,p-1} \neq 0$. *The direction field D that is associated with the* $(n-1)$*-form* $\omega \wedge d\omega^{p-1}$ *is therefore transverse to S.*

If we apply theorem III.A, 4.1.3, to $\bar{\omega}$ then one chooses C^{k-3} local coordinates $\bar{y}_1, \bar{x}_2, \ldots, \bar{y}_p$ at the origin in S such that $\bar{\omega} = d\bar{y}_1 + \bar{x}_2\,d\bar{y}_2 + \ldots + \bar{x}_p\,d\bar{y}_p$. Let $y_1, x_2, \ldots, y_p$ be the first integrals of D that are defined by the prolongations of $\bar{y}_1, \ldots, \bar{y}_p$.

From lemma III.A, 2.2, applied to $d\omega^p$, there exists a C^{k-5} function x_1 such that:

$$d\omega^p = \pm\, x_1dx_1 \wedge dy_1 \wedge \ldots \wedge dx_p \wedge dy_p\,. \tag{1}$$

One easily shows that $\omega = f\cdot\omega'$, in which:

$$\omega' = dz + x_2\,dy_2 + \ldots + x_p\,dy_p\,,$$

and f is a C^{k-4} function that equals 1 on S. One then has:

$$d\omega^p = (df \wedge \omega' + fd\omega')^p = p(p-1)!\; f^{p-1}\frac{\partial f}{\partial z}\,x_1dx_1 \wedge dy_1 \wedge \ldots \wedge dx_p \wedge dy_p\,.$$

On account of (1), one deduces that $f^p = 1 \pm \dfrac{x_1^2}{2(p-1)!}$. The expression for ω in the C^{k-5} coordinate system $(x_1, y_1, \ldots, y_p)$ is therefore:

$$\omega = \left[1 \pm \frac{x_1^2}{2(p-1)!}\right]^{1/p}(dx_1 + x_2dy_2 + \ldots + x_pdy_p),$$

an expression that is easy to reduce to the indicated expression of the theorem.

Remarks.

1) In the case $n = 2p + 1$, the expressions:

$$\pm z\, dz + (1 + x_1)\, dy_1 + \ldots + x_p\, dy_p\,,$$

are easily distinguished geometrically by the behavior of the vector field X that is defined by the linear equation:

$$X \rfloor (\omega \wedge d\omega^p) = d\omega^p.$$

This field is defined only outside of the singular surface S ($z = 0$). *In one case, it points towards S, and in the other, it points away from S.*

One has analogous remarks for $n = 2p$.

2) The preceding theorem shows that the singularity $\zeta_{1,\,0}$ is the *disjoint union of two rigid singularities.*

4.2.2. In the case of *closed* 2-*forms*, one must consider the germs ω at the origin of $\mathbf{R}^n$, where $n = 2p$, which admits the following properties:

i) ω is of corank 2 at the origin; i.e., $j_0\omega \in \Sigma_2$, or furthermore, $\omega^p = 0$ and $\omega^{p-1} \neq 0$ at the origin.

ii) ω is generic from the standpoint of II, 2.4; i.e., ω is transverse to Σ_2. The set $\Sigma_2(\omega)$ of points where ω is of corank 2 is then a germ of a hypersurface.

iii) The restriction of ω to $\Sigma_2(\omega)$ is of maximum rank $2p - 1$.

These are properties of the jet of order 1 of ω at the origin, which defines a subset $\Sigma_{2,0} \subset \mathcal{F}_1^2$. This subset is clearly an open set of $\rho^{-1}(\Sigma_2)$, where $\rho: \mathcal{F}_1^2 \to \mathcal{F}_0^2$. It is therefore a submanifold of codimension 1.

THEOREM. – *Let* $\omega \in \mathcal{D}_k^2$ ($k \geq 6$) *be a germ of a closed* 2-*form at the origin of* $\mathbf{R}^n$ *with* $n = 2p$ *such that* $j_1\omega \in \Sigma_{2,0}$. ω *is then* C^{k-5}-*isomorphic to the germ*:

$$x_1\, dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_p \wedge dy_p\,.$$

Proof. – One easily finds a Pfaff form α such that $d\alpha = \omega$ and $j_2\omega \in \zeta_{1,0}$. One applies theorem III.A, 4.2.1, to α. One thus obtains:

$$\omega = d\alpha = \pm\, x_1\, dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \ldots + dx_p \wedge dy_p\,.$$

One makes the sign disappear by modifying y_1, perhaps.

4.3. We further mention an interesting to theorem III, A, 4.2.1, that relates to the *Pfaff equations in odd dimension,* $n = 2p + 1$.

Suppose we have a Pfaff equation σ that is generic from the standpoint of theorem II, 5.4. and $C_2(\sigma)$, viz., the set of points where σ is of co-class 2, is a hypersurface. Consider a point of $C_2(\sigma)$ such that the restriction of σ to $C_2(\sigma)$ is of maximal class $2p - 1$ at this point. One easily shows that in a neighborhood of such a point σ may be defined by a Pfaff form that presents the singularity $\zeta_{1,0}$ at this point. From this, it results that in a convenient local coordinate system σ is defined by the form:

$$z\,dz + (1 + x_1)\,dy_1 + x_2\,dy_2 + \ldots + x_p\,dy_p.$$

5.

I will now show several examples of singularities for differential forms, such that the classification of the germs that present these singularities involves parameters. They therefore amount to *non-rigid singularities* that admit an infinitude of models.

5. Examples of non-rigid singularities

5.1. For $n = 3$, consider the singularity of order 1 of the closed 2-forms $\Sigma_{3,0} \subset \mathcal{F}_1^2$ that is defined by the property: A germ $\omega \in \mathcal{D}^2$ presents the singularity $\Sigma_{3,0}$, (i.e., $j_1\,\omega \in \Sigma_{3,0}$) if and only if ω is annulled transverse to the origin.

Consider one such germ. Let Ω be a germ of a volume form, and let X be the vector field that is defined by $\omega = X \lrcorner\, \Omega$. Let Λ be the Jacobian matrix of X at the origin (X is annulled at this point). Λ *has null trace,* because ω is closed, so X is therefore unimodular. By the transversality condition, Λ *has rank* 3. The proper value equation for Λ is therefore of the form $\lambda^3 + p\lambda + q = 0$, where $q \neq 0$. *The number* $r = p \cdot q^{-2/3}$ *is then an invariant of* ω, *and therefore of* $j_1\,\omega \in \Sigma_{3,0}$. Indeed, if one makes another choice of volume for Ω then Λ is multiplied by a scalar and $r = p \cdot q^{-2/3}$ is unchanged.

We therefore set, for example:

$$\omega_p = (x_2 + p x_3)\,dx_2 \wedge dx_3 + x_3\,dx_3 \wedge dx_1 - x_1\,dx_1 \wedge dx_2\,,$$

where x_1, x_2, x_3 denote the natural coordinates in $\mathbf{R}^3$ and p is a scalar. For each p, ω_p presents the singularity $\Sigma_{3,0}$ at the origin and $r(\omega_p) = p$. ω_p and ω' might not be isomorphic if $p \neq p'$.

5.2. We now place ourselves in dimension 5. A germ $\omega \in \mathcal{D}^2$ will be said to present the singularity $\Sigma_{3,0} \subset \mathcal{F}_1^2$ if:

a) *ω is transverse to* $\Sigma_{3,0} \subset F_0^2$ (the set of forms of corank 3, hence, of rank 2). $\Sigma_3(\omega)$ is then (cf. II, 2.4.1, example 4) a germ of a *surface* (of codimension 3).
b) *The restriction of ω to $\Sigma_3(\omega)$ is non-null.*

Let Ω be a volume germ at the origin of $\mathbf{R}^5$, and define a vector field X by the equation $\omega^2 = X \rfloor \Omega$. X is annulled at the points of $\Sigma_3(\omega)$. For any $x \in \Sigma_3(\omega)$, the Jacobian Λ_x of X at x is of *rank* 3 and has *null trace*. Its proper value equation is of the form $\lambda^2(\lambda^3 + p(x)\lambda + q(x)) = 0$. One immediately verifies that the *function* $r(x) = p(x)q(x)^{-2/3}$ *is an invariant of the germ* ω (r is a germ of a function at the origin in $\Sigma_3(\omega)$).

A very large class of functions may be obtained in this manner: for example, consider the 2-forms:

$$\omega_p = (x_2 + px_3)\, dx_2 \wedge dx_3 + x_3\, dx_3 \wedge dx_1 - x_1\, dx_1 \wedge dx_2 \\ + \frac{\partial p}{\partial x_4} x_2 x_3\, dx_4 \wedge dx_3 + \frac{\partial p}{\partial x_5} x_2 x_3\, dx_5 \wedge dx_3 + dx_4 \wedge dx_5,$$

in which $x_1, \ldots, x_5$ denote the coordinates in $\mathbf{R}^5$, and:

$$p(x_4, x_5) = s(x_4) + t(x_5)$$

with s and t being arbitrary C^∞ functions.

B. STABILITY AND INFINITESIMAL STABILITY

In this part, all of the objects considered will be assumed to be C^∞.

1. Stability of the germ of a differential form.

1.1. Let M be a manifold $\omega \in D^p(M)$ ($\mathcal{D}^p(M)$, resp.), a p-form (closed p-form, resp.) on M, and x, a point of M. *ω will be called stable at x if the following condition is realized*: (compare with [9], pp. 44).

For any neighborhood U of x there exists a C^∞–neighborhood V of ω in $D^p(M)$ ($\mathcal{D}^p(M)$, resp.) such that for any $\omega' \in V$ there exists c U (?) *such that the germs of ω at x and ω' at x' are isomorphic.*

One easily recognizes that this property indeed depends on the germ of ω at x. One arrives at the definition of a *stable germ.*

1.2. THEOREM. – *All of the models indicated in the first part of this chapter define stable germs.*

Proof. – All of these cases realize the following situation: One has a singularity Σ of order k that is rigid, and a form ω in $\mathbf{R}^n$ such that:

a) $j_k\omega(0) \in \Sigma$,

b) $j_k\omega$ is transverse to S at 0.

A classical argument (which is analogous to the isotopy lemma) then shows that, for any neighborhood U of 0, there exists a C^{k+2}-neighorhood V of ω, such that for any $\omega' \in V$, there exist $x' \in U$, with $j_k\omega'(x') \in \Sigma$. Stability results immediately since Σ is rigid.

2. Infinitesimal stability.

2.1. In what follows, one denotes the space of germs of C^∞ vector fields at the origin of $\mathbf{R}^n$ by χ.

Let $\omega \in D^p$ ($\mathcal{D}^p$, resp.). One then lets T_ω: $\chi \to D^p$ (T_ω: $\chi \to \mathcal{D}^p$, resp.) denote the **R**-linear map defined by $T_\omega(X) = \theta(X)\omega$ for any $X \in \chi$ (not without having remarked that the Lie derivative of closed form with respect to a field is a closed form).

It is natural to make the:

DEFINITION. – Let $\omega \in D^p$ ($\mathcal{D}^p$, resp.). The germ ω is said to be *infinitesimally* stable if the map T_ω: $\chi \to D^p$ (T_ω: $\chi \to \mathcal{D}^p$, resp.) is surjective.

Remark. – One must take care to observe that the notions of infinitesimal stable closed and non-closed forms are distinct.

THEOREM. – *Let $\omega \in D^p$ be an infinitesimally stable germ. The germ $d\omega \in \mathcal{D}^{p+1}$ is therefore infinitesimally stable.*

Proof. – From the formula, it is obvious that:

$$d(\theta(X)\omega) = \theta(X)d\omega.$$

2.2. One may weaken the preceding definition by passing to jets of infinite order of forms and vector fields. One thus obtains the algebraic notion of *formal stability*.

One of the fundamental problems of the theory that is sketched out in this work is to establish the *equivalence* of the notions of stability, infinitesimal stability, and formal stability. I have not begun this problem.

By careful calculations that I will not impose on the reader, one may establish the infinitesimal stability of the germs defined by the models that were obtained in the first part of this chapter.

I prefer to conclude with the following proposition, which is interesting in itself, and which seems to me to give a good idea of the problems that are posed in this type of questions, as long as the situation envisioned in very simple.

PROPOSITION. – *The germs of closed* 2-*forms at the origin of* $\mathbf{R}^4$ *that are defined by the expressions*:

$$\omega = dx \wedge dy + z\, dy \wedge dz + d\left(xz + ty - \frac{z^3}{3} \right) \wedge dt,$$
$$\omega' = dx \wedge dy + z\, dy \wedge dz + d\left(xz - ty - \frac{z^3}{3} \right) \wedge dt$$

are formally stable. (*They amount to examples of the elliptic and hyperbolic points that were indicated in* II, 3.2.3).

Proof. – I will indicate this in the case of ω. The modifications that are necessary for the case of ω' are obvious.

a) The *infinitesimal* stability of ω is equivalent to the possibility of finding a germ of a vector field $X \in \chi$ such that:

$$\theta(X)\,\omega = d(X \rfloor \omega) = \tau = d\pi$$

for any germ of a closed 2-form $\tau = d\pi$ ($\pi \in D^1$).

Let M_ω *denote the module* (*over the ring* D^0 *of germs of* C^∞ *functions at the origin of* $\mathbf{R}^4$) *of germs Pfaff forms of the form* $X \rfloor \omega$. The infinitesimal stability of ω is then equivalent to existence of a function $f \in D^0$ such that $\pi - df \in M_\omega$ for any $\pi \in D^1$; i.e., the equality $D^1 = M_\omega + dD^0$.

b) LEMMA 1. – *A germ* $\pi \in D^1$ *belongs to* M_ω *if and only if* $\pi(m)$ *belongs to the support* $S_\omega(m)$ *of* ω *at m for any point m.*

The necessity is obvious since the interior products $X \rfloor \omega$ constitute the support $S_\omega(m)$ at each point *m*.

To show that the condition is sufficient, one may remark – without using the particular expression for ω – that the rank of the linear equation $X \rfloor \omega = \pi$, behaves *generically* (i.e., that ω is transverse to Σ_2), and use a recent theorem of J. Mather.

However, the direct proof is very simple:

One first verifies that one has $\omega = (dx - z\, dz - t\, dt) \wedge (dy + z\, dt)$ at the points of Σ_2 (i.e., such that $x = 0$). From this, one deduces that *a form* $\pi \in D^1$, $\pi = \alpha\, dx + \beta\, dy + \gamma\, dz + \delta\, dt$, *belongs to the support of* ω *at each point if and only if the functions* $z\alpha + \gamma$ *and* $t\alpha - z\beta + \delta$ *are annulled identically for* $x = 0$, i.e., are *divisible* (in the ring D^0) by the function *x*.

On the other hand, the equation $X \rfloor \omega = \pi$, in which:

$$X = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z} + d\frac{\partial}{\partial t},$$

is equivalent to:

$$a - zd = -\alpha, \tag{1}$$
$$a - zc - td = \beta, \tag{2}$$

$$zb - (x - z^2)d = \gamma \tag{3}$$
$$za + tb + (x - z^2)c = \delta \tag{4}$$

(1) and (3) give $xd = -(z\alpha + \gamma)$. From this, one deduces that d, $z\alpha + \gamma$ are divisible by x, and (1) gives b:

(1), (2) and (4) give $xc = t\alpha - z\beta + \delta$. One deduces c from this and (2) gives a.

Q.E.D.

c) Let $\overline{D}^0$ be the ring of germs of differentiable functions at the origin of the hyperplane $\Sigma_2(\omega)$ (defined by $x = 0$) and let $\overline{D}^1$ be the module of germs of Pfaff forms at the origin in $\Sigma_2(\omega)$. One lets $\overline{M}_\omega$ denote the $\overline{D}^0$ – viz., the module of *restrictions* of the elements of M_ω to $\Sigma_2(\omega)$

LEMMA 2. – *The infinitesimal stability of ω is equivalent to the condition*:

$$\overline{M}_\omega + d\overline{D}^0 = \overline{D}^1.$$

This necessity of this condition is obvious.

Sufficiency: We first remark that for any $\tau \in \overline{D}^1$ such that $\overline{\tau} = 0$ ($\overline{\tau}$ = restriction of τ to $\Sigma_2(\omega)$) there exists function h (which is null on $\Sigma_2(\omega)$) with $\tau - dh = 0$. Indeed, if $\tau = \alpha\, dx + \beta\, dy + \gamma\, dz + \delta\, dt$ then $\overline{\tau} = 0$ is equivalent to $\beta = \gamma = \delta = 0$ for $x = 0$. If $h = x \cdot \alpha$ then all of the coefficients of $\tau - dh$ are null on $\Sigma_2(\omega)$. Now let $\pi \in D^1$. By hypothesis, there exists a function f such that $\overline{\pi} - \overline{df} \in \overline{M}_\omega$, hence, $\overline{\pi - df} = \overline{\sigma}$, where $\sigma \in M_\omega$, namely, $\overline{\pi - df - \sigma} = 0$. From the preceding remark, there exists a function h such that $\pi - df - \sigma - dh \in M_\omega$, which gives $\pi - d(f + h) \in M_\omega$.

Q.E.D.

Remark. – One has not used the particular definition of M_ω here, but only the fact that it is defined by conditions at only the points of $\Sigma_2(\omega)$.

d) Let X be the vector field in the hyperplane $\Sigma_2(\omega)$ that is associated with the restriction ω' of ω; one has:

$$X = -z^2 \frac{\partial}{\partial y} - t\frac{\partial}{\partial z} + z\frac{\partial}{\partial t}$$

(cf. II. 3.2.3 a)).

LEMMA 3. – *A form $\pi \in \overline{D}^1$ belongs to $\overline{M}_\omega$ if and only if* $X \rfloor \pi = 0$.

The necessity is obvious.

Conversely, let $\pi = \beta\, dy + \gamma\, dz + \delta\, dt$ be such that $X \rfloor \pi = 0$, i.e., $-z^2\beta - t\gamma + z\delta = 0$ or $z(-z\beta + \delta) = t\gamma$. t is therefore divisible by z in D^0, and there exists a function $\alpha(y, z, t)$ such that $\gamma = -z\alpha$; but then $-z\beta + \delta = -t\alpha$. From a), it then results that the form $\alpha\, dx + \beta\, dy + \gamma\, dz + \delta\, dt$ belongs to M_ω.

Remark. – This is where one uses the fact that $j_1\omega$ is transverse to the singularity $\Sigma_{2,2}$.

e) From lemmas 2 and 3, the infinitesimal stability of ω is equivalent to the following condition:

For any $\pi \in \overline{D}^1$, there exists a function $f \in \overline{D}^0$ such that

$$X \rfloor df = X \rfloor \pi.$$

I.e.: the partial differential equation:

$$-z^2\frac{\partial f}{\partial y} - t\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial t} = -z^2\beta - t\gamma + z\delta\,, \tag{E}$$

has a C^∞ solution $f(y, z, t)$ for any functions β, γ, δ. The left-hand sides form *the ideal J of functions that are null on $\Sigma_{2,2}(\omega)$, precisely* $(z = t = 0)$.

By definition, one comes down to the study of the condition:

The partial differential equation:

$$-z^2\frac{\partial f}{\partial y} - t\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial t} = h \tag{E}$$

has a solution for any $h \in J$.

f) To my knowledge, there is no actual general method for the study of partial differential equations *with singularities*. In this particular case, I will simply show that it is possible to solve the preceding equation (E) in a *formal series*.

Let $[h] = \sum_{p,q,r\geq 0} a_{p,q,r}\, y^p z^q t^r$ be the Taylor development of the unknown function f.

By identifying the series developments of both sides of (E), one obtains the system of equations:

$$(E_{p,q,r}),\ p,\ q,\ r \geq 0: \quad -(p+1)b_{p+1,\,q-2,\,r} - (q+1)b_{p,\,q+2,\,r-1} + (r+1)\,b_{p,\,q-1,\,r+1} = a_{p,\,q,\,r}\,.$$

I shall call the *sum of the indices* of a coefficient its *height*.

I shall only give a summary of the proof, which is a little neater here.

It is clear that the system of equations that is being considered decomposes into two *disjoint* systems: $(E_{p,q,r})$, $q + r$ *odd*, and $(E_{p,q,r})$, $q + r$ *even*.

In the first case, one easily shows that for given p_0 and k_0 the system $(E_{p0,q,r})$, $q + r = k_0$ forms a system of equations that is independent of the height $p_0 + k_0$ in the unknowns. One then solves the first system by recurrence on p_0 and k_0.

In the second case, one envisions the subsystems:

$$S_{2k} = (E_{p,q,r};\ 2p + q + r = 2k).$$

They are disjoint systems. One shows that S_{2k} is an independent system of equations by recurrence on $q + r$.

Remarks.

1) In the case of the examples of "parabolic" points (cf. II, 3.2.4), one shows the equivalence of infinitesimal stability and stability in the same fashion, with the condition that the partial differential equation:

$$-z(z+2t)\frac{\partial f}{\partial y} \mp y\frac{\partial f}{\partial z} + z\frac{\partial f}{\partial t} = h$$

has a solution f for any $h \in J$, in which J is the ideal of functions that are null on $\Sigma_{2,2}(\omega)$ (i.e., h is null for $y - z = 0$).

I have acquired the conviction (without nevertheless writing down an explicit proof) that there is also formal stability in this case.

2) To me, these considerations seem to reasonably motivate the *conjecture*: *The generic singularities of a closed* 2-*form in dimension* 4 (cf., II, 3) *are all rigid.*

APPENDIX

TRANSVERSALITY THEOREMS FOR SECTIONS OF FIBER BUNDLES

1. Topologies on the space of sections.

1.1. All of the manifolds envisioned here will be finite-dimensional over **R**, C^∞ (unless stated to the contrary), and have a denumerable basis of neighborhoods.

Let M be a manifold and let $\pi: E \to M$ be a fiber bundle over M. One denotes the vector bundle of jets of order k of sections of E ($0 \leq k \leq \infty$) by $\pi^k: J^k \to M$.

Let D_k be the vector space of sections of E of class C^k. If $\omega \in D_k$ then one lets $j_k\omega$ denote the section of J^kE that is defined at each point $x \in M$ by the jet of order k of ω at x.

One will denote the vector space of C^∞ sections of E by D.

1.2. PROPOSITION – DEFINITION. – *The sets:*

$$V_U = \{\omega \in D_k : j_k\omega(M) \subset U\},$$

in which $U \subset J^kE$ is an open set, constitute the basis for a topology on D_k that is called the Whitney C^k*-topology.*

(If M is compact then this is the topology of "uniform convergence" on M of the sections of E and their partial derivatives up to order k.)

The Whitney C^∞–topology on D is defined as the projective limit of the C^k-topologies on the spaces D_k.

One recalls that D_k, when endowed with the C^k-topology ($0 \leq k \leq \infty$), is a *locally convex topological vector space that is, moreover, a Baire space.*

A subset of D_k will be called *residual* if it contains a denumerable intersection of dense open sets. A property that defines a residual set will be called *generic* on D_k.

1.3. PROPOSITION. – *For any $k' \geq k$, $D_{k'}$ is C^k-dense in D_k.*

This is a consequence of lemma 6 of [19].

1.4. Let $\pi: B \to M$ be an arbitrary C^∞ *locally trivial fiber bundle.* Let Γ_k be the set of C^k sections of this fiber bundle. One may define the C^k-topology on Γ_k in the same way as in 1.2 by using open sets of the fiber bundle J^kB.

One may also define a "manifold" structure on Γ_k ($0 \leq k \leq \infty$) that is modeled on the space of sections of the vector bundle in the following manner:

a) Let $V \to B$ be the vector bundle on B comprised of vectors that are *tangent to the fibers of* B. One gives, once and for all, a C^∞ "second order" differential equation on B such that for any $y \in B$ and any $v \in V$ with origin y the solution of this equation that is defined by the initial conditions (y, v) is traced out in the fiber $\pi^{-1}(\pi(y))$. One denotes the value of this solution at time $t = 1$ by $\exp(y, v)$ (for v sufficiently "small").

b) Let $\sigma \in \Gamma_k$ $(0 \leq k \leq \infty)$. One then considers the vector bundle $E_s \to M$, which is the inverse image of V by σ. It is clear that the exponential map defined in b) induces an *injection of a convenient open C^k-neighborhood of the null section of E_σ into Γ_k.*

One thus defines a chart on Γ_k that is modeled on $D_k(E_s)$. One easily verifies that these charts form an atlas on Γ_k whose subordinate topology is the C^k-topology precisely.

This viewpoint will be useful in what follows (cf. 4.3 and 6.2).

2. Distributions. Stratifications.

2.1. Let N be a manifold. One calls any function Δ that associates a subspace Δ_x of T_xN – viz., the tangent space to N at x – to each point $x \in N$ a *distribution* on N. *A distribution Δ will be called coherent if for every point x and every $v \in \Delta x$ there exists an open neighborhood U at x and a C^∞ vector field X on U such that $X(x) = v$ and $X(x') \in \Delta_{x'}$ for any $x' \in U$.*

PROPOSITION. – *Let V be a (C^∞) regular submanifold of N, and let F be a closed set of N that is included in V. The distribution Δ that is defined by $\Delta_x = T_xV$ for x in F and $\Delta_x = T_xN$ elsewhere is coherent.*

2.2. One defines a *stratification* of a manifold N to be any *partition* $S = (S_i)_{i \in I}$ of N such that:

a) Each S_i (or stratum) is a (C^∞) submanifold of N.
b) For any integer p, the union of the strata of dimension less than or equal to p is a closed subset of N.

The *codimension of a stratification* of N is the minimum of the codimensions of the non-open strata.

If S is a stratification of N, then for $x \in N$ one will let T_xS notate the subspace of T_xN that is tangent at x to the strata of S that contains x.

The distribution that is associated to a stratification S is defined by $\Delta_x = T_xS$.

A stratification will be called *denumerable* if the set of strata is denumerable, and if each stratum is a *regular* submanifold of N.

A stratification is called *coherent* if the associated distribution is, as well.

Examples.

1) A foliation of N defines a coherent stratification.

2) The orbits of a Lie group that acts on N constitute a coherent stratification; the images of the left-invariant vector fields on the group *globally* realize the required prolongations.

Remark. – One defines a stratification of class C^k on a manifold in an analogous fashion: the strata will be submanifolds of N of class C^k, and with the condition of coherence the vector fields will be of class C^{k-1} ($k \geq 1$).

2.3. PROPOSITION. – *Let S be a coherent stratification of a manifold N. S is then locally trivial.*

This signifies that for any point $x \in N$ there exists a chart φ: $U \to \mathbf{R}^n$ of N at x ($n = \dim N$) such that:

a) $\varphi(U) = B_{n-p} \times B_p$, in which B_{n-p}, B_p denote open cubes of $\mathbf{R}^{n-p}$ and $\mathbf{R}^p$, and p = dimension of the stratum that passes through x.
b) Each *plaque* $\varphi^{-1}(\{u\} \times B_p)$, $u \in B_{n-p}$ is contained in a stratum of S.

The proof is by recurrence on p (n arbitrary $\geq p$). The proposition is trivial for $p = 0$. Suppose that it has been established for dimension $(p - 1)$.

Now let $v \in T_xS$, $v \neq 0$. Since S is coherent there exists a local vector field X that is tangent to the strata such that $X(x) = v$. Let H be an element of the hypersurface in N that is transverse to X at x. Let S' be the stratification that is induced by S in H (indeed, H is transverse to the strata in a neighborhood of x). *S' is coherent.* If $v' \in \mathrm{T}_x\mathrm{S} \cap \mathrm{T}_x\mathrm{H}$, then one considers a local field Y (in N) that is tangent to S and prolongs v', and at each point $y \in H$ one projects $Y(y)$ into T_yH parallel to $X(y)$. One thus constructs a field in H that is tangent to S' and prolongs v'. In H, the stratum that passes through x is of dimension $p - 1$. From the recurrence hypothesis, let $\varphi' : U' \to \mathbf{R}^{n-1}$ be a chart of H at x that realizes conditions a) and b) for S'. It is clear that one may *prolong* to a chart φ:$U \to \mathbf{R}^{n-1} \times \mathbf{R}$ (perhaps by reducing U') such that $\varphi_*(X) = \dfrac{\partial}{\partial t}$, where t denotes the n^{th} coordinate. One therefore easily verifies that φ has properties a) and b) for S, because the integral curves of the field X are traced in the strata of S.

Remarks.

1) We have proved, in a precise way, that a *coherent stratification at a point x is locally trivial at x.*
2) The preceding proof also works for a coherent stratification of class C^k, $k \geq 2$ without modification; one will then obtain trivializations of class C_1. It is clear that a locally trivial stratification is coherent. The preceding proposition therefore shows that *coherence is equivalent to local triviality.*

3. Transversality.

3.1. Let M and N be manifolds, where N is endowed with a distribution Δ. A *differentiable* map f: $M \to N$ is called *transverse* to Δ at $x \in M$ if:

$$f'(T_xM) + \Delta_{f(x)} = T_{f(x)}N,$$

where f' denotes the linear map that is tangent to f. One may thus also speak of a 1-*jet transverse to* Δ. The map f is *transverse to* Δ *on a subset* $K \subset M$ if it is transverse to Δ at any point of K. If $K = M$ then one says that f is *transverse to* Δ.

Remark. – A *submersion* is transverse to any distribution.

3.2. Suppose that N is endowed with a stratification S. A map f: $M \to N$, which is assumed to be at least C^1, is called *transverse to the stratification* S if it is transverse to the distribution that is associated with S.

PROPOSITION. – *Let* f: $M \to N$ *be a map of class* C^k $(k \geq 1)$ *that is transverse to a stratification* $S = (S_i)_{i \in I}$ *of* N. *Let* $f^{-1}(S) = \Sigma = (\Sigma_i)_{i \in I}$ *be the partition of* M *that is defined by* $\Sigma_i = f^{-1}(S_i)$ *for any* $i \in I$. Σ *is a stratification of class* C^k *of* M, *and:*

$$\operatorname{codim}_M S_i = \operatorname{codim}_N S_i$$

if Σ_i *is non-vacuous. Moreover,* Σ *is denumerable* (*coherent, resp.*) *if* S *is denumberable* (*coherent, resp.*).

All of these assertions are obvious, except for the one that concerns the coherence of Σ. For this, upon using proposition 2.3, one is immediately reduced to the case where f is a *submersion,* and the conclusion is immediate in this situation since S is locally trivial, from the rank theorem.

Remark. – If f is transverse to the stratification S, and if $\operatorname{codim}_N S_i > \dim M$, then S_i is necessarily vacuous. In particular, if $\operatorname{codim}_N S > \dim M$ then Σ reduces to the trivial stratification (viz., just one stratum that equals M).

3.3. PROPOSITION. – *Let* Δ *be a coherent distribution on the manifold* N. *The set of 1-jets of* M *into* N *that are transverse to* Δ *is an open set of the manifold* $J^1(M, N)$.

COROLLARY. – *Let* $E \to M$ *be a vector bundle, and let* Δ *be a coherent distribution on the manifold* J^kE. *For* $r > k$, *the set of sections* ω *of* E *such that* ω_k *is transverse to* Δ *is a* C^k*-open set on* D_r.

These propositions are very easy to prove.

4. Isotopy theorem.

4.1. ISOTOPY LEMMA. – *Let* M *be a compact manifold, and let* N *be a manifold that is endowed with a coherent stratification* S. *Let:*

$$H: M \times [0, 1] \to N$$

be a C^k homotopy of M into N such that for any $t \in [0, 1]$ the map H_t: $M \to N$ that is defined by $H_t(x) = H(x, t)$ is transversal to S. One sets $S_t = H_t^{-1}(S)$. Therefore, if $k \geq 2$, there exists a C^{k-1} isotopy , of M such that $g_t(\Sigma_0) = \Sigma_t$ for any $t \in [0, 1]$.

One recalls that a C^k-isotopy of M is a C^k-homotopy g: $M \times [0, 1] \to M$ such that for any $t \in [0, 1]$, g_t is a (C^k) automorphism of M, and g_0 = identity of M.

On the other hand, $g_t(\Sigma_0) = \Sigma_t$ signifies that for any $x \in$ M, g_t goes from the stratum of Σ_0 that passes through x to the stratum of Σ_t that passes through $g_t(x)$.

Proof. – Let $\Sigma = H^{-1}(S)$ be the (C^k) stratification of $M' = M \times [0, 1]$ that is defined by the inverse image of S by H.

a) *There exists a vector field Y of class C^{k-1} on M' such that*:

1) *For any $y \in M'$, $Y(y)$ is tangent to the strata of S at y,*

2) *The "vertical" component (viz., the one tangent to the factor [0, 1]) of Y is $\dfrac{\partial}{\partial t}$ at each point.*

Indeed, there exists a tangent vector that has properties 1) and 2) at each point of M' because the strata of Σ are transverse to the "horizontals" $M \times \{t\}$. This remark, combined with the coherence of Σ (cf., proposition 3.2), permits us to locally construct a field that has properties 1) and 2). A partition of unity on then permits us to construct the stated field.

b) The integration of the field Y (which at least C^1 by hypothesis) obviously furnishes the desired isotopy (this is where the hypothesis of compactness intervenes).

4.2. ISOTOPY THEOREM. – *Let $E \to M$ be a vector bundle whose base M is compact, and let S be a coherent stratification of J^kE. Let $r > k + 2$. For $\omega \in T(S) \subset D$, viz., the set of sections ω such that $j_k\omega$ is transverse to S, one lets $S(\omega)$ notate the stratification of M that is the inverse image of S by $j_k\omega$. Then, for any $\omega \in T(S)$, there exists an open neighborhood V of ω in D_r such that*:

a) $V \subset T(S)$.

b) *For any $\omega' \in V$, there exists an isotopy φ_t of M such that $\varphi_1(S_\omega) = S_{\omega'}$.*

Indeed, from 3.3, $T(S)$ is open in D_r. Therefore, let V be a *convex* open neighborhood of ω that is included in $T(S)$. If $\omega' \in V$ then the homotopy H: $M \times [0, 1] \to J^kE$ that is defined by:

$$H(x, t) = (1 - t)j_k\omega(x) + tj_k\omega'(x)$$

verifies the hypotheses of lemma 4.1, and the theorem is established.

4.3. Extension to the case of an arbitrary fiber bundle. Let $\pi: B \to M$ be an arbitrary locally trivial fiber bundle. Let S be a coherent stratification of the fiber bundle J^kB. Theorem 4.2 remains valid in $T(S) \subset \Gamma_r$ ($r \geq k + 2$), viz., the set of sections σ of B such that $j_k\sigma$ is transverse to S. Indeed, one is immediately reduced to the preceding theorem by using a chart of Γ_r (cf., 1.4).

5. Sard's theorem and the fundamental lemma of the theory.

5.1. SARD'S THEOREM. – *Let M and N be manifolds – not necessarily C^k – of dimensions m and n, respectively. Let f: $M \to N$ be a differentiable map. If the differentiability class of f is greater than or equal to* Max$(1, m - n + 1)$ *then the set of regular values of f is residual in N* (*its complement has measure zero in N*).

(Recall that $y \in N$ is called a *regular value* of f if for any $x \in f^{-1}(y)$ the rank of f at x is equal to n.)

For a proof of this theorem, see R. Abraham, J. Robbin [1], pp. 37.

5.2. FUNDAMENTAL LEMMA. – *Let M, V, N be manifolds, and let H: $M \times V \to N$ be a C^k-morphism. If $v \in V$ then one lets H_v: $M \to N$ notate the morphism defined by $H_v(x) = H(x, v)$. Let S be a denumerable stratification of N.* If:

a) $k \geq$ Max$(1, m -$ codim $S + 1)$ *where* $m =$ dim M,

b) *H is transverse to S on $U \times V$, where U is an open set of M,*

then the set of $v \in V$ such that H_v is transverse to S on U is dense in V.

Indeed, let p: $M \times V \to V$ be the canonical projection, and let $\Sigma_i = H^{-1}(S_i) \cap (U \times V)$. One easily sees that for any $v \in V$, H_v is transverse to S_i on U if and only if v is a regular value of the restriction of p to the manifold S_i. Hypothesis a) permits us to apply Sard's theorem for each i, and the fundamental lemma is proved. (This is where one resorts to the hypothesis of the denumberability of S. In V, a denumerable intersection of residual sets, in the sense of 5.1, is dense.)

6. The transversality theorem.

6.1. THEOREM. – *Let $E \to M$ be a vector bundle, and let S be a denumerable stratification of J^kE. For $r \geq k + 1$, the set $T(S)$ of $\omega \in D_r$ such that $j_k\omega$: $M \to J^kE$ is transverse to S is a residual set of D_r.*

Proof (summary). – Since M has a denumerable basis, it suffices to show that for any $x \in M$ there exists a neighborhood U of x such that the set $T_U(S)$ of $\omega \in D_r$ that verify "$j_k\omega$ is transverse to S on U" is residual (since a denumerable intersection of residual sets is residual).

a) *Density* of $T_U(S)$. Let V be the J^kE at x. By a standard procedure – viz., using a trivialization of the bundle E – one constructs a *continuous* map P: $V \to D$, such that:

1) $j_kP(v) = v$ for any $v \in V$, i.e., that P is a prolongation.
2) $P(0) = 0$.
3) $P(v)$ is C^∞ for any v.
4) The map $\overline{H}$: $M\times V \to J^kE$, which defined by:

$$\overline{H}\,(y, v) = j_kP(v)(y),$$

is a *submersion* on $U\times V$, since U is a sufficiently small neighborhood of x in M.

Therefore, if $\omega \in D_{r'}$, in which:

$$r' \geq k + \mathrm{Max}(1, m - \mathrm{codim}\, S + 1) \qquad (m = \dim M),$$

then the map H: $M \times V \to J^kE$ that is defined by:

$$H(y, v) = j_k\omega(y) + j_kP(v)(y) = j_k(\omega + P(v))(y)$$

is, moreover, a submersion on $U\times V$. From the fundamental lemma 5.2, the set of v such that $j_k(\omega + P(v))$ is transverse to S on U is dense in V. Since P is continuous, one has proved the density of $T_U(S) \cap D_{r'}$ in $D_{r'}$. However, $D_{r'}$ is dense in D_r (Proposition 1.3), so $T_U(S)$ is dense in D_r.

b) One subsequently proves that $T_U(S)$ is a denumerable intersection of open sets (which, from a), are dense), and is therefore a residual set, in the following fashion: The strata of S have a denumerable basis of neighborhoods (as regular submanifolds of J^*E). One thus covers each stratum with a denumerable family of closed sets of J^kE. For each of these closed sets, the set of ω such that $j_k\omega$ is transverse on U to the corresponding distribution that is defined in proposition 2.1 is, from 2.1 and 3.3, an open set. It is clear that $T_U(S)$ is equal to the intersection of these open sets.

Remark. – In certain cases, one may add an interesting specialization of this.

1) *If* codim $S > m = \dim M$ *then* $T_U(S)$ *is a dense open set in* D_r *for* $r \geq k + 1$.

It suffices to justify "open." However, in this case, the phrase "$j_k\omega$ is transversal to S" signifies simply that for any $x \in M$, $j_k\omega(x)$ does not belong to any non-open stratum of J^kE. From property b) of the definition of a stratification, this defines an open set of J^kE.

One immediately deduces from this that $T(S)$ is an open dense set in D_r for $r \geq k$.

2) *If* S *is a coherent stratification of* J^kE *then* $T(S)$ *is a dense open set in* D_r *for* $r \geq k + 1$.

From the corollary 3.3., this is immediate.

6.2. Extension to the case of an arbitrary fiber bundle. If $\pi: B \to M$ is an arbitrary fiber bundle and S is a denumerable stratification of J^kB then one has a theorem that is identical to the preceding in Γ_r, $r \geq k + 1$:

If $\sigma \in \Gamma_r$ then one proves the theorem on a neighborhood of σ by reducing to the case of the preceding theorem with the aid of the chart defined in 1.4. (This amounts to a linearization technique.)

Remark. – This theorem is essentially the most general one that one may obtain (in finite dimensions). For the example, the classical transversality theorem, which relates to the maps of a manifold M into a manifold N that is endowed with a denumerable stratification S is a trivial consequence of it; one applies the preceding theorem to the sections of the trivial bundle $M \times N \to M$, when endowed with the stratification $S' = M \times S$.

7. Case of differential forms.

We always let M be a C^∞ manifold with a denumerable basis. The notations that relate to the cotangent bundle of M are those of II. 1. In this case, in addition to theorem 6.1, one must first state a transversality theorem that relates to *closed* differential forms.

The set of closed p-forms of class C^k, which we notate by $\mathcal{D}_k^p(M)$ ($k \geq 1$), is endowed with the C^k-topology that is induced from $D_k^p(M)$. It is also a locally convex topological vector space and a Baire space.

THEOREM. – *Let S be a denumerable stratification of $\Lambda^p T_k^* M$ (the bundle of k-jets of closed p-forms). The set $T(S)$ of $\omega \in S$ such that $j_k\omega$ is transverse to S is a residual set, provided that* $r \geq k + 1$.

The proof is analogous to that of theorem 6.1. It suffices to show that one may define a "family of perturbations" P for each point $x \in M$ that permits us to deform any closed form into a transverse closed form in a neighborhood of x. For this, let V be the fiber at x of $\Lambda^{p-1}T_{k+1}^* M$ (the bundle of jets of order $k + 1$ of not necessarily closed p-1-forms), and let $\tilde{P} : V \to D^{p-1}(M)$ be a "prolongation" that has the properties of 6.1a). Then the map $P: V \to \mathcal{D}^p(M)$ that is defined by $P(v) = d\tilde{P}(v)$ (in which d denotes the exterior derivative) has the required properties to achieve the proof as in 6.1.

Remarks.

1) As in 6.1, one may confirm, moreover, that $T(S)$ is open if the codimension of S is greater than the dimension of M, or if S is coherent.
2) The perturbations performed in the proof of the theorem involve only *exact* p-forms. One therefore says that $T(S)$ *is residual in each cohomology class of* $\mathcal{D}_k^p(M)$.

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