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On the canonical differential equations of mechanics

By ÉMILE MATHIEU

Translated by D. H. Delphenich

In this article, I propose to study certain properties of the canonical differential equations of mechanics. If one lets $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ denote the $2n$ variables of the problem and lets t denote time, and finally lets H denote a function of those variables then those equations will be included among the following two:

$$(1) \qquad \frac{dq_i}{dt} = \frac{dH}{dp_i}, \qquad \frac{dp_i}{dt} = -\frac{dH}{dq_i},$$

in which i can take the values 1, 2, …, n.

It is obvious that the integration of those equations can be more or less facilitated by the form of the function H in some particular cases. An interesting question is therefore to find a transformation of one canonical system to another that has no other effect than to change the form of the function H. I shall treat that question in this article.

Thus, imagine that the $2n$ variables q_i, p_i satisfy equations in terms of finite quantities that are $2r$ in number:

$$(2) \qquad f_1 = 0, \ f_2 = 0, \ \ldots, \ f_{2r} = 0,$$

in which $r < n$, and that one has, in addition, the equation:

$$(3) \qquad \frac{dq_1}{dt}\,\delta p_1 + \cdots + \frac{dq_n}{dt}\,\delta p_n - \frac{dp_1}{dt}\,\delta q_1 - \cdots - \frac{dp_n}{dt}\,\delta q_m = \delta H,$$

upon denoting the variations of the quantities q_i, p_i that are compatible with equations (2) by the symbol δ, and in which δH represents the variation of H that results from increases in those variables. In the case where r is zero, the variations δq_i, δp_i will be independent, and one will recover the system of equations (1). If one supposes that one has r conditional equations that include the n variables q_i, but do not include the variables p_i, in place of equations (2) then those equations, combined with (3) can, as I will show, relate to a problem in mechanics. In that case, it is easy to prove that the system of equations (2) and (3) can be reduced to a system of canonical equations whose number of variables reduces to 2 (n – r). However, it is interesting to example the more general case in which the given conditional equations include not only the variables q_i, but also the variables p_i . I shall now show that the system of equations (2) and (3) can be reduced to a system of 2 ($n – r$) canonical differential equations that depend upon the same function H with the aid of the Pfaff problem. I will then show certain cases in which that reduction might be effected more easily.

The Pfaff problem is a problem in integral calculus of great complexity, so there is good reason to study the system of equations (2) and (3) that presents it, and without reducing it to a canonical system. Now, I will show that the famous theorem of Poisson and Jacobi that relates to canonical equations and permits one to deduce a third integral from two of them, in general, is applicable to my system of equations. The statement of the theorem will just take a more complicated form.

In this article, I shall also give a theory of perturbations that includes several new considerations. Finally, I shall present some new properties of the functions that one represents by the symbol $[\alpha, \beta]$ in analysis.

Differential equations of mechanics. Very simple proof of the Hamiltonian equations.

1. – Consider a system of n material points. Let m_i denote the mass of each of those points, and let x_i, y_i, z_i denote its coordinates with respect to three rectangular axes, where i has the values 1, 2, …, n. Let:

$$F_1 = 0 , \qquad F_2 = 0 , \qquad \ldots, \qquad F_r = 0 \tag{1}$$

denote the equations that express the constraints to which the system is subject. Imagine a virtual displacement of all points of the system, and let δx_i, δy_i, δz_i be the variations of x_i, y_i, z_i, in general. Set:

$$\frac{dx_i}{dt} = x_i' , \qquad \frac{dy_i}{dt} = y_i' , \qquad \frac{dz_i}{dt} = z_i' ,$$

so the expression for the semi-*vis viva* will be:

$$T = \tfrac{1}{2} \sum m_i \, (x_i'^2 + y_i'^2 + z_i'^2) \; .$$

Suppose that there is a force function, which we shall represent by U, so the differential equations of the problem are contained in the formula:

$$\sum m_i \left(\frac{d^2 x_i}{dt^2} \delta x_i + \frac{d^2 y_i}{dt^2} \delta y_i + \frac{d^2 z_i}{dt^2} \delta z_i \right) = \delta U \; . \tag{2}$$

Upon differentiating T, one can write down the two formulas:

$$2\ \delta T = \delta \sum m_i \left(x_i' \frac{dx_i}{dt} + y_i' \frac{dy_i}{dt} + z_i' \frac{dz_i}{dt} \right),$$

$$\delta T = \sum m_i \left(x_i' \,\delta \frac{dx_i}{dt} + y_i' \,\delta \frac{dy_i}{dt} + z_i' \,\delta \frac{dz_i}{dt} \right) .$$

Upon changing the sign of equation (2) and then adding δT to both sides of it, we can put it into this form:

$$\delta \sum m_i \left(x_i' \frac{dx_i}{dt} + y_i' \frac{dy_i}{dt} + z_i' \frac{dz_i}{dt} \right) - \sum m_i \left(x_i' \,\delta \frac{dx_i}{dt} + y_i' \,\delta \frac{dy_i}{dt} + z_i' \,\delta \frac{dz_i}{dt} \right) - \sum m_i \left(\frac{dx_i'}{dt} \delta x_i + \frac{dy_i'}{dt} \delta y_i + \frac{dz_i'}{dt} \delta z_i \right)$$

$$= \delta T - \delta U .$$

Represent the function $T - U$ by H, so the preceding equation will become:

$$\delta \sum m_i \left(x_i' \frac{dx_i}{dt} + y_i' \frac{dy_i}{dt} + z_i' \frac{dz_i}{dt} \right) - \frac{d}{dt} \sum m_i \left(x_i' \,\delta x_i + y_i' \,\delta y_i + z_i' \,\delta z_i \right) = \delta H .$$

Represent the quantities x_i, y_i, z_i by the letter Q, affected with various indices, and represent the corresponding quantities $m_i\, x_i'$, $m_i\, y_i'$, $m_i\, z_i'$ by the letter P, affected with the same indices, in such a way that the preceding equation will become:

(3) $$\delta \left(P_1 \frac{dQ_1}{dt} + P_2 \frac{dQ_2}{dt} + \cdots \right) - \frac{d}{dt} \left(P_1 \,\delta Q_1 + P_2 \,\delta Q_2 + \cdots \right) = \delta H .$$

One can express the variables Q or x, y, z, which are $3n$ in number, in terms of only $3n - r$ variables by using the r conditional equations, and in such a manner that their expressions will satisfy equations (1) identically. Let q_i denote those $3n - r$ new variables, and choose the variables p_i to have the same number as the q_i and to satisfy the equation:

(4) $$p_1\ \delta q_1 + p_2\ \delta q_2 + \ldots + p_k\ \delta q_k = P_1\ \delta Q_1 + P_2\ \delta Q_2 + \ldots + P_k\ \delta Q_k$$

upon setting $3n - r = k$. Upon taking the virtual variations to be equal to the ones that the variables q_i, Q_i will effectively experience during the instant dt, one will deduce from the preceding equation that:

(5) $$p_1 \frac{dq_1}{dt} + \cdots + p_k \frac{dq_k}{dt} = P_1 \frac{dQ_1}{dt} + \cdots + P_{3n} \frac{dQ_{3n}}{dt} ,$$

and as a result, equation (3) will become:

$$\delta\left(p_1\frac{dq_1}{dt}+\cdots+p_k\frac{dq_k}{dt}\right)-\frac{d}{dt}(p_1\,\delta q_1+\cdots+p_k\,\delta q_k)=\delta H,$$

or rather:

(*A*) $$\frac{dq_1}{dt}\delta p_1+\cdots+\frac{dq_k}{dt}\delta p_k-\frac{dp_1}{dt}\delta q_1-\cdots-\frac{dp_k}{dt}\delta q_k=\delta H.$$

Now, there no longer exist conditional equations between the new variables, and since one has:

$$\delta H=\frac{dH}{dq_1}\delta q_1+\cdots+\frac{dH}{dq_k}\delta q_k+\frac{dH}{dp_1}\delta p_1+\cdots+\frac{dH}{dp_k}\delta p_k\ ,$$

one can conclude the *Hamiltonian* equations:

(*B*) $$\left\{\begin{array}{llll}\dfrac{dq_1}{dt}=\dfrac{dH}{dp_1}, & \ldots, & \dfrac{dq_k}{dt}=\dfrac{dH}{dp_k}, \\[2ex] \dfrac{dp_1}{dt}=-\dfrac{dH}{dq_1}, & \ldots, & \dfrac{dp_k}{dt}=-\dfrac{dH}{dq_k}.\end{array}\right.$$

2. – Let us examine the variables p. In equation (4), the variables δq are independent, and one concludes from this that:

(6) $$p_s=P_1\frac{dQ_1}{dq_s}+P_2\frac{dQ_2}{dq_s}+\cdots+P_{3n}\frac{dQ_{3n}}{dq_s}.$$

That is the formula that will generally permit one to pass from the variables in equation (3) to the ones in equation (*A*) and equations (*B*). However, upon recalling what the quantities Q, P represent, one can obtain another expression for the p_s. Indeed, one will then have:

$$p_s=\sum m_i\left(x_i'\frac{dx_i}{dq_s}+y_i'\frac{dy_i}{dq_s}+z_i'\frac{dz_i}{dq_s}\right).$$

Now, upon denoting the derivatives of q_1, q_2, … with respect to t by q_1', q_2', …, one will have:

(7) $$x_i'=\frac{dx_i}{dt}=\frac{dx_i}{dq_1}q_1'+\frac{dx_i}{dq_2}q_2'+\cdots.$$

One concludes the first of the following three equations from that:

$$\frac{dx_i'}{dq_s'} = \frac{dx_i}{dq_s}, \quad \frac{dy_i'}{dq_s'} = \frac{dy_i}{dq_s}, \quad \frac{dz_i'}{dq_s'} = \frac{dz_i}{dq_s},$$

and the other two are obtained similarly. One will then finally have:

$$p_s = \sum m_i \left(x_i' \frac{dx_i'}{dq_s'} + y_i' \frac{dy_i'}{dq_s'} + z_i' \frac{dz_i'}{dq_s'} \right) = \frac{dT}{dq_s'} . \tag{8}$$

From that, one will have the quantity p_s upon expressing T as a function of the variables q and their derivatives q', and taking the derivative of T with respect to q_s'. $H = T - U$ must be expressed in terms of the variables q_i, p_i in equations (*B*): One must then express T in terms of those variables. Now, one concludes from equation (5) that:

$$p_1 \, q_1' + \cdots + p_k \, q_k' = 2T .$$

It will then suffice to substitute the values of q_1', q_2', … that one infers from the k equations (8) in that equation.

What makes the preceding proof of Hamilton's equations simpler than the one that we are accustomed to seeing is that we have replaced equation (2) with equation (3), which is a more general form, but nonetheless includes the same properties that we wish to prove. The generalization then has the effect of allowing one to proceed to the objective by the most direct path. On the contrary, we should point out that the equation:

$$p_s = \frac{dT}{dq_s'}$$

is based upon the particular form of the quantities Q, P. Nevertheless, it is easy to show that this equation will be true whenever the given function H in equation (3) is composed of a function $-U$ that includes only the variables Q and a function T that is homogeneous of degree two with respect to the variables P but contains the variables Q arbitrarily.

3. – Equation (*A*) presents certain advantages over equations (*B*). Indeed, equation (*A*) is also applicable in the case where there is no force function in the problem of mechanics. It will suffice to agree that δT will not be a total differential in $\delta H = \delta T - \delta U$ unless one imagines that the variables are expressed as functions of t. A second advantage of equation (*A*) consists of the fact that it can also be used in the case where the variables q satisfy conditional equations.

Indeed, consider r' of the equations (1):

$$F_1 = 0 , \qquad F_2 = 0 , \qquad \ldots, \qquad F_{r'} = 0 , \tag{9}$$

in which $r' < r$ and can reduce to zero. It is possible to express the variables Q with the aid of $3n - r'$ variables that we denote by $q_1, q_2, \ldots, q_k$ by making $k = 3n - r'$, and in such a way that the expressions for the variables Q will satisfy equation (9) identically. Upon then substituting those expressions in the $r - r'$ equations (1), which have not yet been employed, one will have $r - r'$ conditional equations between the variables q :

$$L_1 = 0 , \qquad L_2 = 0 , \qquad \ldots$$

Upon assuming that equation (4) is true, one will then be once more led to equation (A), but in which the variations δq, δp will no longer be independent. The variables p_s will again be given by the formula:

$$p_s = \frac{dT}{dq'_s} .$$

Indeed, the variations δq will no longer be independent in equation (4), so equation (4) will not necessarily imply equation (6). However, it will be permissible to first assume the k equations that are included in equation (6), which will imply equation (4). Finally, equation (6) will lead to equation (8), as before.

4. – In the foregoing, we assumed that the constraints did not depend upon time t. It is easy to modify our analysis in order to make it applicable to the case in which those equations do contain time. In that case, we must note that the virtual variations cannot coincide with the effective variations, in such a way that equation (4) will no longer lead to equation (5).

The constraint equations are equivalent to equations that express the $3n$ variables Q_i in terms of the $3n - r$ variables q_i and presently include time t :

$$\begin{aligned} Q_1 &= \theta_1 (q_1, q_2, \ldots, q_k, t) , \\ Q_2 &= \theta_2 (q_1, q_2, \ldots, q_k, t) , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned}$$

and if one eliminates $q_1, q_2, \ldots$ from those $3n$ equations then one will have a system of r equations that is equivalent to the given constraint equations.

The variations δQ_1, δQ_2, … in equation (4) are obtained by varying $q_1, q_2, \ldots$, by while supposing that t is constant, as it would result from the principle of virtual velocities. One will then have:

$$\delta Q_i = \frac{d\theta_i}{dq_1}\delta q_1 + \frac{d\theta_i}{dq_2}\delta q_2 + \cdots + \frac{d\theta_i}{dq_k}\delta q_k .$$

If we let θ'_i denote the partial derivative of θ_i with respect to t then we will have:

$$\frac{dQ_i}{dt} = \theta_i' + \frac{d\theta_i}{dq_1} q_1' + \cdots + \frac{d\theta_i}{dq_k} q_k'$$

for the total derivatives. Since the variations δq are arbitrary, one can set:

$$\delta q_s = q_s' \, dt ,$$

in particular, and for all values of $s = 1, 2, \ldots, k$, and it will then result that:

$$\delta Q_i = \left(\frac{dQ_i}{dt} - \theta_i' \right) dt .$$

Equation (4) will no longer give equation (5) then, but the following one:

$$p_1 \frac{dq_1}{dt} + \cdots + p_k \frac{dq_k}{dt} = P_1 \frac{dQ_1}{dt} + \cdots + P_k \frac{dQ_k}{dt} - P_1 \theta_1' - \cdots - P_k \theta_k' .$$

From that, equation (3) will become:

$$\delta \left(p_1 \frac{dq_1}{dt} + \cdots + p_k \frac{dq_k}{dt} \right) - \frac{d}{dt} (p_1 \, \delta q_1 + \cdots + p_k \, \delta q_k) = \delta (H - P_1 \theta_1' - P_2 \theta_2' - \cdots) .$$

Equation (*A*) will still persist then, provided that one changes H into:

$$H - P_1 \theta_1' - P_2 \theta_2' - \cdots ,$$

and equations (*B*) will also persist with the same change. The variables p will continue to be given by equations (8), moreover.

Transformation of a canonical system of differential equations into a similar system.

5. – Consider equation:

$$\text{(1)} \qquad \frac{dq_1}{dt} \delta p_1 + \cdots + \frac{dq_n}{dt} \delta p_n - \frac{dp_1}{dt} \delta q_1 - \cdots - \frac{dp_n}{dt} \delta q_n = \delta H ,$$

in which H is a function of the $2n$ variables q, p that can even include t. We shall first suppose that the variables q, p are not coupled by any conditional equation, in such a way that the preceding equation will give rise to $2n$ equations that are included in the following two:

(2) $$\frac{dq_i}{dt} = \frac{dH}{dp_i}, \qquad \frac{dp_1}{dt} = -\frac{dH}{dq_i},$$

in which i can take the values 1, 2, …, n.

We propose to pass from the variables q_i, p_i to the variables Q_i, P_i, which are the same in number and which satisfy canonical equations that entirely similar to (2), in such a way that the function H will remain the same but take a different form.

In order to arrive at that, we return to equation (1), which we put into this form:

$$\delta\left(p_1\frac{dq_1}{dt}+\cdots+p_n\frac{dq_n}{dt}\right)-\frac{d}{dt}(p_1\,\delta q_1+\cdots+p_n\,\delta q_n) = \delta H .$$

It will then suffice to choose the variables Q, P to be functions of the former ones that satisfy the equation:

(3) $$P_1\,\delta Q_1 + \ldots + P_n\,\delta Q_n = p_1\,\delta q_1 + \ldots + p_n\,\delta q_n ,$$

because it will also result that:

$$P_1\frac{dQ_1}{dt}+\cdots+P_n\frac{dQ_n}{dt} = p_1\frac{dq_1}{dt}+\cdots+p_n\frac{dq_n}{dt} .$$

One will then have the equation:

$$\delta\left(P_1\frac{dQ_1}{dt}+\cdots+P_n\frac{dQ_n}{dt}\right)-\frac{d}{dt}(P_1\delta Q_1+\cdots+P_n\,\delta Q_n) = \delta H .$$

One concludes an equation that is similar to equation (1) from that, in which the symbols p, q are replaced by P, Q, resp., and as a result, one will have the equations:

$$\frac{dQ_i}{dt} = \frac{dH}{dP_i}, \quad \frac{dP_i}{dt} = -\frac{dH}{dQ_i} .$$

Let us now examine how we satisfy equation (3). It amounts to $2n$ equations that are included in the following two:

(4) $$P_i = p_1\frac{dq_1}{dQ_i}+p_2\frac{dq_2}{dQ_i}+\cdots+p_n\frac{dq_n}{dQ_i},$$

(5) $$0 = p_1\frac{dq_1}{dP_i}+p_2\frac{dq_2}{dP_i}+\cdots+p_n\frac{dq_n}{dP_i},$$

in which i can take the values 1, 2, …, n.

Upon eliminating $p_1, p_2, \ldots, p_n$ from the n equations (5), one will then conclude that the determinant:

$$\sum \pm \frac{dq_1}{dP_1}\frac{dq_2}{dP_2}\cdots\frac{dq_n}{dP_n}$$

is zero, and from a well-known theorem, it will then result that there exists a relation between the variables $q_1, q_2, \ldots, q_n$, which are functions of the variables Q_i, P_i, that does not include $P_1, P_2, \ldots, P_n$. That relation can be written:

$$\psi(q_1, q_2, \ldots, q_n, Q_1, Q_2, \ldots, Q_n) = 0\,, \tag{6}$$

in which ψ denotes an arbitrary function. Upon differentiating that equation with respect to Q_i and P_i in succession, one will have:

$$\left\{\begin{aligned} -\frac{d\psi}{dQ_i} &= \frac{d\psi}{dq_1}\frac{dq_1}{dQ_i} + \frac{d\psi}{dq_2}\frac{dq_2}{dQ_i} + \cdots + \frac{d\psi}{dq_n}\frac{dq_n}{dQ_i}\,, \\ 0 &= \frac{d\psi}{dq_1}\frac{dq_1}{dP_i} + \frac{d\psi}{dq_2}\frac{dq_2}{dP_i} + \cdots + \frac{d\psi}{dq_n}\frac{dq_n}{dP_i}\,. \end{aligned}\right. \tag{7}$$

Upon comparing the n equations that are included in the latter with the n equations (5), one will have:

$$\frac{d\psi}{dq_1} = \mu\, p_1\,, \qquad \frac{d\psi}{dq_2} = \mu\, p_2\,, \quad \ldots, \qquad \frac{d\psi}{dq_n} = \mu\, p_n\,, \tag{8}$$

in which μ is an indeterminate factor, and upon comparing (4) with (7), one will have:

$$P_i = -\frac{1}{\mu}\frac{d\psi}{dQ_i}\,,$$

or the n equations:

$$\frac{d\psi}{dQ_1} = -\mu\, P_1\,, \qquad \frac{d\psi}{dQ_2} = -\mu\, P_2\,, \qquad \ldots, \qquad \frac{d\psi}{dQ_n} = -\mu\, P_n\,. \tag{9}$$

One can infer $\mu, Q_1, Q_2, \ldots, Q_n$ from equations (6) and (8) and then infer $Q_1, Q_2, \ldots, Q_n$ from equations (9).

A particular case of the preceding solution is worthy of note, namely, the one in which one takes the variables q_i to be arbitrary functions of $Q_1, Q_2, \ldots, Q_n$, as in no. **1**. The n equations (5) will then be satisfied identically, and the P_i will be determined by the n equations (4). That particular case was given by Jacobi (**Jacobi**'s *Dynamik*, pp. 453).

As one knows, the solution to equation (1) can be replaced with that of a partial difference equation, in which H is a function of $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$:

$$H(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) .$$

Set:

$$p_1 = \frac{dV}{dq_1}, \qquad p_2 = \frac{dV}{dq_2}, \qquad \ldots$$

It will suffice to solve the partial difference equation:

$$H\left(\frac{dV}{dq_1}, \frac{dV}{dq_2}, \ldots, q_1, q_2, \ldots\right) = h , \tag{10}$$

in which h is an arbitrary constant, or rather, to find a complete solution. The transformation of equation (1) corresponds to a transformation of equation (10), and the formula:

$$P_1\,\delta Q_1 + P_2\,\delta Q_2 + \ldots = p_1\,\delta q_1 + p_2\,\delta q_2 + \ldots$$

will show that the function V is the same in equation (10) as it is in the one into which it transforms.

6. – Compare the rule that I just gave for passing from one canonical system to a similar one with the one that Jacobi gave for solving the same problem (*Dynamik*, pp. 447). Here is that rule:

Suppose that $2n$ *variables* q_i, p_i *are given by the Hamiltonian equations:*

$$\frac{dq_i}{dt} = \frac{dH}{dp_i}, \qquad \frac{dp_i}{dt} = -\frac{dH}{dq_i}, \tag{a}$$

in which i can take the values 1, 2, …, *n. Furthermore, let:*

$$\psi(q_1, q_2, \ldots, q_n, Q_1, Q_2, \ldots, Q_n)$$

be an arbitrary function of the n variables $q_1, q_2, \ldots, q_n$, *and n new variables* $Q_1, Q_2, \ldots, Q_n$. *Determine* $Q_1, Q_2, \ldots, Q_n$ *and some others* $P_1, P_2, \ldots, P_n$ *as functions of the former variables by means of the equations:*

$$\frac{d\psi}{dq_1} = p_1, \qquad \frac{d\psi}{dq_2} = p_2, \qquad \ldots, \qquad \frac{d\psi}{dq_n} = p_n , \tag{b}$$

(*c*) $$\frac{d\psi}{dQ_1} = -P_1, \quad \frac{d\psi}{dQ_2} = -P_2, \quad \ldots, \quad \frac{d\psi}{dQ_n} = -P_n .$$

The variables Q_i, P_i satisfy the equations:

(*d*) $$\frac{dQ_i}{dt} = \frac{dH}{dP_i}, \quad \frac{dP_i}{dt} = -\frac{dH}{dQ_i} .$$

One sees that the rule that I just gave above is closely analogous to Jacobi's rule. Equations (8) and (9) will become identical to equations (*b*) and (*c*) when one sets $\mu = 1$. Jacobi's solution includes one less unknown μ, and to compensate, it includes one less equation, which is:

$$\psi = 0 .$$

Jacobi's rule can be proved very simply as follows:

The system of equations (*a*) can be replaced with the following one:

(11) $$\delta\left(p_1 \frac{dq_1}{dt} + \cdots + p_n \frac{dq_n}{dt}\right) - \frac{d}{dt}\left(p_1\,\delta q_1 + \cdots + p_n\,\delta q_n\right) = \delta H .$$

Now, from equations (*b*):

$$\begin{aligned}
&\delta\left(p_1 \frac{dq_1}{dt} + \cdots + p_n \frac{dq_n}{dt}\right) - \frac{d}{dt}\left(p_1\,\delta q_1 + \cdots + p_n\,\delta q_n\right) \\
&\qquad = \delta\left(\frac{d\psi}{dq_1}\frac{dq_1}{dt} + \cdots + \frac{d\psi}{dq_n}\frac{dq_n}{dt}\right) - \frac{d}{dt}\left(\frac{d\psi}{dq_1}\delta q_1 + \cdots + \frac{d\psi}{dq_n}\delta q_n\right) \\
&\qquad = \delta\left(-\frac{d\psi}{dQ_1}\frac{dQ_1}{dt} - \cdots - \frac{d\psi}{dQ_n}\frac{dQ_n}{dt}\right) - \frac{d}{dt}\left(-\frac{d\psi}{dQ_1}\delta Q_1 - \cdots - \frac{d\psi}{dQ_n}\delta Q_n\right) ,
\end{aligned}$$

because the equality of the last two right-hand sides amounts to:

$$\delta\frac{d\psi}{dt} = \frac{d}{dt}\delta\psi .$$

Upon recalling equations (*c*), one will then see that the left-hand side of equation (11) is equal to the expression:

$$\delta\left(P_1 \frac{dQ_1}{dt} + \cdots + P_n \frac{dQ_n}{dt}\right) - \frac{d}{dt}\left(P_1\,\delta Q_1 + \cdots + P_n\,\delta Q_n\right),$$

which will then be equal to δH, and one will then conclude with equations (*d*).

7. – One can further imagine many other ways of passing from one canonical system to a similar one. We shall indicate some other ones.

The system of canonical equations is included in just one equation:

$$\frac{dq_1}{dt}\delta p_1+\cdots+\frac{dq_n}{dt}\delta p_n-\frac{dp_1}{dt}\delta q_1-\cdots-\frac{dp_n}{dt}\delta q_n = \delta H ,$$

and it can be put into these two forms:

$$\delta\left(p_1\frac{dq_1}{dt}+\cdots+p_n\frac{dq_n}{dt}\right)-\frac{d}{dt}\left(p_1\,\delta q_1+\cdots+p_n\,\delta q_n\right) = \delta H ,$$

$$-\ \delta\left(q_1\frac{dp_1}{dt}+\cdots+q_n\frac{dp_n}{dt}\right)-\frac{d}{dt}\left(q_1\,\delta p_1+\cdots+q_n\,\delta p_n\right) = \delta H .$$

Upon adding them, one will have:

$$\delta\left(p_1\frac{dq_1}{dt}-q_1\frac{dp_1}{dt}+p_2\frac{dq_2}{dt}-q_2\frac{dp_2}{dt}+\cdots\right)-\frac{d}{dt}\left(p_1\,\delta q_1-q_1\,\delta p_1+p_2\,\delta q_2-q_2\,\delta p_2+\cdots\right) = 2\ \delta H .$$

One easily concludes from this that the variables $Q_1, Q_2, \ldots, Q_n, P_1, P_2, \ldots, P_n$ will satisfy a similar canonical system if they satisfy the equation:

$$(A)\qquad P_1\,\delta Q_1-Q_1\,\delta P_1+\ldots+P_n\,\delta Q_n-Q_n\,\delta P_n = p_1\,\delta q_1-q_1\,\delta p_1+\ldots+p_n\,\delta q_n-q_n\,\delta p_n ,$$

and that will result from exchanging the big letters with the small ones.

One sees from the foregoing that there was an inadvertent slip in Jacobi's *Traité de Dynamique* on page 453, where he said that he had just given all possible transformations of one canonical system to another. That inadvertent slip was explained very easily, moreover, in a posthumous book.

Let us return to the transformation that is included in formula (A). In general, set:

$$P_i = R_i\cos\Theta_i , \qquad Q_i = R_i\sin\Theta_i ,$$
$$p_i = r_i\cos\theta_i , \qquad q_i = r_i\sin\theta_i ,$$

in which i can take the values 1, 2, …, n. Formula (A) will become:

$$(B)\qquad R_1^2\,d\Theta_1+\cdots+R_n^2\,d\Theta_n = r_1^2\,d\theta_1+\cdots+r_n^2\,d\theta_n .$$

More generally, one can argue with equation (B) as one did in no. **5** with the equation:

$$P_1\,\delta Q_1+\ldots+P_n\,\delta Q_n = p_1\,\delta q_1+\ldots+p_n\,\delta q_n .$$

Transforming a certain system of equations into a canonical system.

8. – Once more, suppose that the variables $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ satisfy the equation:

$$\text{(1)} \qquad \frac{dq_1}{dt}\delta p_1 + \cdots + \frac{dq_n}{dt}\delta p_n - \frac{dp_1}{dt}\delta q_1 - \cdots - \frac{dp_n}{dt}\delta q_n = \delta H .$$

However, suppose, in addition, that those variables are also coupled by $2r$ conditional equations:

$$\text{(2)} \qquad f_1 = 0 , \qquad f_2 = 0 , \qquad \ldots, \qquad f_{2r} = 0 .$$

I would like to prove that those $2n$ variables can be replaced with a system of $2n - 2r$ variables $Q_1, Q_2, \ldots, Q_{n-r}, P_1, P_2, \ldots, P_{n-r}$ that satisfy $2n - 2r$ canonical equations:

$$\text{(3)} \qquad \frac{dQ_i}{dt} = \frac{dH}{dP_i}, \qquad \frac{dP_i}{dt} = -\frac{dH}{dQ_i},$$

in which i can take the values $1, 2, \ldots, n - r$.

We might remark that this theorem was proved before in no. **1** of this article in the case where the conditional equations included only the variables $q_1, q_2, \ldots, q_n$.

Let us put equation (1) into that form, moreover:

$$\delta\left(p_1\frac{dq_1}{dt} + \cdots + p_n\frac{dq_n}{dt}\right) - \frac{d}{dt}(p_1\,\delta q_1 + \cdots + p_n\,\delta q_n) = \delta H .$$

If we choose new variables that satisfy the equation:

$$\text{(4)} \qquad p_1\,dq_1 + p_2\,dq_2 + \ldots + p_n\,dq_n = P_1\,dQ_1 + P_2\,dQ_2 + \ldots + P_n\,dQ_n$$

then we will have:

$$\delta\left(P_1\frac{dQ_1}{dt} + \cdots + P_n\frac{dQ_n}{dt}\right) - \frac{d}{dt}(P_1\,\delta Q_1 + \cdots + P_n\,\delta Q_n) = \delta H ,$$

and since the variations $\delta Q_1, \delta Q_2, \ldots, \delta P_1, \delta P_2, \ldots$ will be independent, one will conclude with equations (3).

Let us show how we can satisfy equation (4). Upon differentiating equations (2) according to the characteristic δ, we will have:

$$\frac{df_1}{dq_1}\delta q_1 + \cdots + \frac{df_1}{dq_{n-r}}\delta q_{n-r} + \frac{df_1}{dp_1}\delta p_1 + \cdots + \frac{df_1}{dp_{n-r}}\delta p_{n-r}$$

$$= -\frac{df_1}{dq_{n-r+1}}\delta q_{n-r+1} - \cdots - \frac{df_1}{dq_n}\delta q_n - \frac{df_1}{dp_{n-r+1}}\delta p_{n-r+1} - \cdots - \frac{df_1}{dp_n}\delta p_n ,$$

$$\frac{df_2}{dq_1}\delta q_1 + \cdots + \frac{df_2}{dq_{n-r}}\delta q_{n-r} + \frac{df_2}{dp_1}\delta p_1 + \cdots + \frac{df_2}{dp_{n-r}}\delta p_{n-r}$$
$$= -\frac{df_2}{dq_{n-r+1}}\delta q_{n-r+1} - \cdots - \frac{df_2}{dq_n}\delta q_n - \frac{df_2}{dp_{n-r+1}}\delta p_{n-r+1} - \cdots - \frac{df_2}{dp_n}\delta p_n ,$$

……………………………………………………………………………………

One can infer the $2r$ variations $\delta q_{n-r+1}, \ldots, \delta q_n, \delta p_{n-r+1}, \ldots, \delta p_n$ from those $2r$ equations, and upon substituting them in equation (4), one will have an equation of this form:

$$G_1\ \delta q_1 + G_2\ \delta q_2 + \ldots + G_{n-r}\ \delta q_{n-r} + L_1\ \delta p_1 + L_2\ \delta p_2 + \ldots + L_{n-r}\ \delta p_{n-r}$$
$$= P_1\ \delta Q_1 + P_2\ \delta Q_2 + \ldots + P_{n-r}\ \delta Q_{n-r} ,$$

From equations (2), $G_1, G_2, \ldots, L_1, L_2, \ldots$ can be reduced to functions that contain only the variables $q_1, q_2, \ldots, q_{n-r}, p_1, p_2, \ldots, p_{n-r}$.

The problem is thus reduced to the transformation of the differential expression on the left-hand side into another one that contains half as many differentials of the variables. That problem is known under the name of the *Pfaff problem*, and one knows that it is always possible to solve it.

It is useful to remark that the Pfaff problem can be solved by operations from the integral calculus only with great difficulty. Later on, we will show how one can avoid that problem in certain cases. Furthermore, the only possibility for reducing equations (1) and (2) to a canonical system will lead to some important consequences.

On the case where the transformation of the system of equations in no. 8 into a canonical system can be easily performed.

9. – We saw that the system of equations in the variables q_i, p_i :

(1) $$\frac{dq_1}{dt}\delta p_1 + \cdots + \frac{dq_n}{dt}\delta p_n - \frac{dp_1}{dt}\delta q_1 - \cdots - \frac{dp_n}{dt}\delta q_n = \delta H ,$$

(2) $$f_1 = 0 , \qquad f_2 = 0 , \quad \ldots, \quad f_r = 0 ,$$

(3) $$f_{r+1} = 0 , \qquad f_{r+2} = 0 , \ldots, \quad f_{2r} = 0$$

can always be transformed into a canonical system with the aid of the Pfaff problem. We shall examine some cases in which the transformation can be performed without any operation from integral calculus.

Suppose that the first r of the conditional equations (2) and (3) include only the variables q_i . From equations (2), one can then express those variables in terms of only the $n - r$ variables Q_1, Q_2, …, Q_{n-r}, in such a way that their expressions:

$$q_1 = \varphi_1 (Q_1, Q_2, \dots, Q_{n-r}) , \qquad \dots, \qquad q_n = \varphi_n (Q_1, Q_2, \dots, Q_{n-r}) \tag{4}$$

satisfy equations (2) identically, and upon eliminating Q_1, Q_2, …, Q_{n-r} from equations (4), one will find a system of r equations that is equivalent to equations (2). The variables q_i can be expressed in terms of the Q_i only by the formulas (4), while the Q_i are functions of the q_i in an infinitude of way.

Put equation (1) into this form:

$$\delta\left(p_1 \frac{dq_1}{dt} + \cdots + p_n \frac{dq_n}{dt} \right) - \frac{d}{dt} (p_1\, \delta q_1 + \cdots + p_n\, \delta q_n) = \delta H ,$$

and we see that it will suffice to determine the variables P_1, P_2, …, P_{n-r} that satisfy the equation:

$$P_1\, \delta Q_1 + \dots + P_{n-r}\, \delta Q_{n-r} = p_1\, \delta q_1 + p_2\, \delta q_2 + \dots + p_n\, \delta q_n . \tag{5}$$

By hypothesis, the variables q_i do not depend upon the variables P. That equation will then be equivalent to the following $n - r$ equations:

$$\left\{ \begin{array}{l} P_1 \;\; = p_1 \dfrac{dq_1}{dQ_1} + p_2 \dfrac{dq_2}{dQ_1} + \cdots + p_n \dfrac{dq_n}{dQ_1}, \\ \quad\;\; \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P_{n-r} = p_1 \dfrac{dq_1}{dQ_{n-r}} + p_2 \dfrac{dq_2}{dQ_{n-r}} + \cdots + p_n \dfrac{dq_n}{dQ_{n-r}}. \end{array} \right. \tag{6}$$

Multiply those equations by the unknown functions μ_1, μ_2, …, μ_r and subtract them from equation (5). Upon equating the coefficients of the variables δq to zero, we will have:

$$\left\{ \begin{array}{l} P_1 \dfrac{dQ_1}{dq_1} + P_2 \dfrac{dQ_2}{dq_1} + \cdots + P_{n-r} \dfrac{dQ_{n-r}}{dQ_1} = p_1 + \mu_1 \dfrac{df_1}{dq_1} + \cdots + \mu_r \dfrac{df_r}{dq_1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ P_1 \dfrac{dQ_1}{dq_n} + P_2 \dfrac{dQ_2}{dq_n} + \cdots + P_{n-r} \dfrac{dQ_{n-r}}{dQ_n} = p_n + \mu_1 \dfrac{df_1}{dq_n} + \cdots + \mu_r \dfrac{df_r}{dq_n}. \end{array} \right. \tag{6}$$

As we said, the Q_i are functions of the q_i in an infinitude of ways, and one will get any one of them by forming $n - r$ combinations of equations (4) and solving the resulting equations for Q_1,

$Q_2, \ldots, Q_{n-r}$. One will then have to substitute the expressions for the latter quantities in the left-hand sides of equations (7).

The n equations (7), combined with the r equations (3), will permit one infer $p_1, p_2, \ldots, p_n$ and $\mu_1, \mu_2, \ldots, \mu_r$ as functions of the variables P_i and q_i, and as a result, as functions of the variables P_i, Q_i .

It is obvious that the preceding method will be applicable whenever one can combine the given conditional equations in such a manner as to obtain r distinct equations that include only the system of variables q_i or that of the variables p_i, and they will present themselves whenever the number of variables p_i or that of the variables q_i that enter into the conditional equations (2) and (3) is not greater than r.

We further remark that each variable q_i is conjugate to a variable p_i, but the n variables q_i and the n variables p_i do not form two essentially-distinct systems.

Indeed, set:

$$p_s = q'_s , \qquad q_s = - p'_s ,$$

so equation (1) can be written:

$$\frac{dq_1}{dt}\delta p_1 + \cdots + \frac{dq_{s-1}}{dt}\delta p_{s-1} + \frac{dq'_s}{dt}\delta p'_s + \frac{dq_{s+1}}{dt}\delta p_{s+1} + \cdots$$
$$- \frac{dp_1}{dt}\delta q_1 - \cdots - \frac{dp_{s-1}}{dt}\delta q_{s-1} - \frac{dp'_s}{dt}\delta q'_s - \frac{dp_{s+1}}{dt}\delta q_{s+1} - \cdots = \delta H .$$

p_s then takes the place of q_s, and $- q_s$ takes that of p_s . It follows from this that the preceding method is also applicable whenever one can combine the given conditional equations in such a manner as to obtain a system of r distinct equations that does not include more than one variable from each of the n pairs:

$$q_1 , p_1 , \qquad q_2 , p_2 , \qquad q_3 , p_3 , \qquad \ldots$$

10. – Examine the case in which only r conditional equations are given:

$$f_1 = 0 , \qquad f_2 = 0 , \qquad \ldots, \qquad f_r = 0$$

that include only the variables q_i . Upon differentiating any one of them, one will have:

$$\frac{df_s}{dq_1}\frac{dq_1}{dt} + \frac{df_s}{dq_2}\frac{dq_2}{dt} + \cdots = 0 .$$

However, in the present case, equation (1) will give:

$$\frac{dq_1}{dt} = \frac{dH}{dp_1} , \qquad \frac{dq_2}{dt} = \frac{dH}{dp_2} , \qquad \ldots$$

One will then have:

$$\frac{df_s}{dq_1}\frac{dH}{dp_1}+\frac{df_s}{dq_2}\frac{dH}{dp_2}+\cdots=0\,,$$

which one can write more simply:

$$[f_s, H] = 0\,.$$

The preceding theory can then be applied, provided that one sets:

$$f_{r+1} = [f_1, H]\,,\quad f_{r+2} = [f_2, H]\,,\quad \ldots,\qquad f_{2r} = [f_r, H]\,.$$

Moreover, this case is found to be treated in no. **1**.

One can apply that particular case to the interesting paper by Bour "Sur les mouvements relatifs," J. Math. pures appl. **8** (1863). Upon applying formulas (6), one will immediately get the equations of relative motion for a system with arbitrary constraints in canonical form that Bour obtained by a comparatively-long calculation that extends from page 11 to page 19 in the cited volume.

Lowering the number of conditional equations.

11. – Consider the equation:

(1) $$\frac{dq_1}{dt}\delta p_1+\cdots+\frac{dq_n}{dt}\delta p_n-\frac{dp_1}{dt}\delta q_1-\cdots-\frac{dp_n}{dt}\delta q_n=\delta H$$

and the $2r$ conditional equations:

(2) $$f_1 = 0\,,\qquad f_2 = 0\,,\qquad \ldots,\qquad f_{2r} = 0\,.$$

We have seen that one can reduce that system of equations to a canonical system. One thus lowers the number r to zero and the number n to $n - r$. However, it is not without interest to examine the case in which one can lower the numbers n and r by a number of units that is less than r without performing any operation from the integral calculus, because one will then approach the canonical system.

Suppose that equations (2) include only k of the variables p_i that we denote by $p_1, p_2, \ldots, p_k$ and that k is less than $2r$ and greater than r. (If k is smaller than r then one will revert to the problem in no. **9**.)

When one eliminates $p_1, p_2, \ldots, p_k$ from equations (2) the result will be $2r - k$ equations:

(3) $$\varphi_1 = 0\,,\qquad \varphi_2 = 0\,,\qquad \ldots,\qquad \varphi_{2r-k} = 0\,,$$

which include only the variables $q_1, q_2, \ldots, q_k$. One then takes k of equations (2) that are distinct from equations (3) and one can, consequently, replace equations (2) with the latter, and divide them into two groups, which we arrange into two rows as follows:

(4) $$f_1 = 0\,, \quad f_2 = 0\,, \quad f_{2r-k} = 0\,,$$

(5) $$f_{2r-k+1} = 0\,, \ldots, f_k = 0\,.$$

We can express the n variables $q_1, q_2, \ldots, q_k$ in terms of only $n - 2r + k$ variables that we denote by $Q_1, Q_2, \ldots, Q_{n-2r+1}$ by means of equations (3). We then choose the variables $P_1, P_2, \ldots, P_{n-2r+k}$ in such a manner that we will have:

$$P_1\,\delta Q_1 + P_2\,\delta Q_2 + \ldots + P_{n-2r+k}\,\delta Q_{n-2r+1} = p_1\,\delta q_1 + \ldots + p_n\,\delta q_n\,.$$

That equation will amount to this one:

(6) $$\left\{ \begin{aligned} &P_1\left(\frac{dQ_1}{dq_1}\,\delta q_1 + \frac{dQ_1}{dq_2}\,\delta q_2 + \cdots + \frac{dQ_1}{dq_n}\,\delta q_n\right) + P_2\left(\frac{dQ_2}{dq_1}\,\delta q_1 + \cdots + \frac{dQ_2}{dq_n}\,\delta q_n\right) + \cdots \\ &\qquad\qquad = p_1\,\delta q_1 + \cdots + p_n\,\delta q_n\,. \end{aligned} \right.$$

The variables q_i are expressed uniquely in terms of the variables Q_i . On the contrary, the variables Q_i are expressed in terms of the q_i in an infinitude of ways. Imagine one of those ways, such that the derivatives of Q_i with respect to the q_i are then perfectly determined.

Differentiate equations (3) and then multiply them by the undetermined coefficients $\mu_1, \mu_2, \ldots, \mu_{2r-k}$. Finally, add them to equation (6). We will then deduce the following n equations:

(7) $$\left\{ \begin{aligned} &P_1\frac{dQ_1}{dq_1} + P_2\frac{dQ_2}{dq_1} + \cdots + P_{n-2r+k}\frac{dQ_{n-2k+k}}{dq_1} = p_1 + \mu_1\frac{d\varphi_1}{dq_1} + \cdots + \mu_{2r-k}\frac{d\varphi_{2r-k}}{dq_1}\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &P_1\frac{dQ_1}{dq_n} + P_2\frac{dQ_2}{dq_n} + \cdots + P_{n-2r+k}\frac{dQ_{n-2k+k}}{dq_n} = p_n + \mu_1\frac{d\varphi_1}{dq_n} + \cdots + \mu_{2r-k}\frac{d\varphi_{2r-k}}{dq_n}\,. \end{aligned} \right.$$

If we combine the n equations (7) with equations (4) then we will have $n + 2r - k$ equations for determining the $n + 2r - k$ variables $p_1, p_2, \ldots, p_n$ and $\mu_1, \mu_2, \ldots$ as functions of P_i and q_i . As a result, one can determine the variables p_i as functions of P_i and Q_i .

If we substitute the values of q_i and p_i as functions of the Q_i, P_i in the 2 $(k - r)$ equations (5) then we will have 2 $(k - r)$ conditional equations in the Q_i, p_i with the equation:

$$\frac{dQ_1}{dt}\,\delta P_1 + \cdots + \frac{dQ_{n-2k+k}}{dt}\,\delta P_{n-2r+k} - \frac{dP_1}{dt}\,\delta Q_1 - \cdots - \frac{dP_{n-2k+k}}{dt}\,\delta Q_{n-2r+k} = \delta H\,.$$

We will have then lowered the numbers n and r of equations (1) and (2), resp., by $2r - k$ units.

One must remark that if $2r$ is greater than n or equal to n then the preceding method will always be applicable when one sets $k = n$, because one can always eliminate the n variables p from equations (2) then. The number $n - 2r + k$ will become $2(n - r)$. Hence, the number n of equations (1) will be lowered to $2(n - r)$, and the number of conditional equations will reduce to the same number.

Introducing the principal derivatives of *H* into the system of equations that was just studied.

12. – Imagine any function φ of the variables $q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n$ and suppose that those variables satisfy $2r$ equations:

$$f_1 = 0\,, \qquad f_2 = 0\,, \qquad \ldots, \qquad f_{2r} = 0\,. \tag{1}$$

Upon differentiating the expression φ, we will have:

$$d\varphi = \frac{d\varphi}{dq_1}dq_1 + \frac{d\varphi}{dq_2}dq_2 + \cdots + \frac{d\varphi}{dq_n}dq_n + \frac{d\varphi}{dp_1}dp_1 + \cdots + \frac{d\varphi}{dp_n}dp_n\,,$$

and upon differentiating equations (1), we will have:

$$\begin{aligned}
0 &= \frac{df_1}{dq_1}dq_1 + \frac{df_1}{dq_2}dq_2 + \cdots + \frac{df_1}{dq_n}dq_n + \frac{df_1}{dp_1}dp_1 + \cdots + \frac{df_1}{dp_n}dp_n\,, \\
0 &= \frac{df_2}{dq_1}dq_1 + \frac{df_2}{dq_2}dq_2 + \cdots + \frac{df_2}{dq_n}dq_n + \frac{df_2}{dp_1}dp_1 + \cdots + \frac{df_2}{dp_n}dp_n\,, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

We multiply those equations by functions of the same variables that we denote by $\lambda_1, \lambda_2, \ldots$, respectively, and add them to the expression $d\varphi$. We will then obtain $d\varphi$ in a different form that depends upon the functions $\lambda_1, \lambda_2, \ldots$ The coefficients of $dq_1, dq_2, \ldots, dp_1, dp_2, \ldots$ form a group of *virtual* derivatives of the function φ. We represent them thus:

$$\left\{\begin{aligned}
\left(\frac{d\varphi}{dq_1}\right) &= \frac{d\varphi}{dq_1} + \lambda_1\frac{df_1}{dq_1} + \lambda_2\frac{df_2}{dq_1} + \cdots + \lambda_{2r}\frac{df_{2r}}{dq_1}, \\
\left(\frac{d\varphi}{dq_2}\right) &= \frac{d\varphi}{dq_2} + \lambda_1\frac{df_1}{dq_2} + \lambda_2\frac{df_2}{dq_2} + \cdots + \lambda_{2r}\frac{df_{2r}}{dq_2}, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right. \tag{2}$$

$$(3)\qquad \begin{cases} \left(\dfrac{d\varphi}{dp_1}\right) = \dfrac{d\varphi}{dp_1} + \lambda_1 \dfrac{df_1}{dp_1} + \lambda_2 \dfrac{df_2}{dp_1} + \cdots + \lambda_{2r} \dfrac{df_{2r}}{dp_1}, \\ \left(\dfrac{d\varphi}{dp_2}\right) = \dfrac{d\varphi}{dp_2} + \lambda_1 \dfrac{df_1}{dp_2} + \lambda_2 \dfrac{df_2}{dp_2} + \cdots + \lambda_{2r} \dfrac{df_{2r}}{dp_2}, \\ \dotfill \end{cases}$$

With the conventional notation, if u and v are arbitrary functions of the variables then set:

$$[u, v] = \frac{du}{dq_1}\frac{dv}{dp_1} + \frac{du}{dq_2}\frac{dv}{dp_2} + \cdots + \frac{du}{dq_n}\frac{dv}{dp_n} - \frac{du}{dp_1}\frac{dv}{dq_1} - \frac{du}{dp_2}\frac{dv}{dq_2} - \cdots - \frac{du}{dp_n}\frac{dv}{dq_n} .$$

If we then multiply equations (2) by $\frac{df_i}{dp_1}$, $\frac{df_i}{dp_2}$, …, respectively, and equations (3) by $\frac{df_i}{dq_1}$, $\frac{df_i}{dq_2}$, …, respectively, and subtract the sum of the second ones from the sum of the first ones then we will have:

$$\frac{du}{dq_1}\frac{dv}{dp_1} + \frac{du}{dq_2}\frac{dv}{dp_2} + \cdots + \frac{du}{dq_n}\frac{dv}{dp_n} - \frac{du}{dp_1}\frac{dv}{dq_1} - \frac{du}{dp_2}\frac{dv}{dq_2} - \cdots - \frac{du}{dp_n}\frac{dv}{dq_n}$$

$$= [\varphi, f_i] + [f_1, f_i]\,\lambda_1 + [f_2, f_i]\,\lambda_2 + \ldots + [f_{2r}, f_i]\,\lambda_{2r} ,$$

and since i can take values 1, 2, …, $2r$, that equation will include $2r$ distinct ones. Now choose the functions λ_1, λ_2, … to be the ones that annul the right-hand side of the preceding equation, and which will be, as a result, solutions to these $2r$ equations:

$$(4)\qquad \begin{cases} \qquad\qquad [f_2, f_1]\lambda_2 + [f_3, f_1]\lambda_3 + \cdots + [f_{2r}, f_1]\lambda_{2r} = [f_1, \varphi], \\ [f_1, f_2]\lambda_1 + \qquad\qquad + [f_3, f_2]\lambda_2 + \cdots + [f_{2r}, f_2]\lambda_{2r} = [f_2, \varphi], \\ \dotfill \\ [f_1, f_{2r}]\lambda_1 + [f_2, f_{2r}]\lambda_2 + [f_3, f_{2r}]\lambda_2 + \cdots + \qquad\qquad = [f_{2r}, \varphi]. \end{cases}$$

The virtual derivatives of φ will then satisfy $2r$ equations that are included in the following ones:

$$(5)\qquad \frac{df_i}{dp_1}\left(\frac{d\varphi}{dq_1}\right) + \frac{df_i}{dp_2}\left(\frac{d\varphi}{dq_2}\right) + \cdots + \frac{df_i}{dp_n}\left(\frac{d\varphi}{dq_n}\right) - \frac{df_i}{dq_1}\left(\frac{d\varphi}{dp_1}\right) - \frac{df_i}{dq_2}\left(\frac{d\varphi}{dp_2}\right) - \cdots - \frac{df_i}{dq_n}\left(\frac{d\varphi}{dp_n}\right) = 0 ,$$

in which i can take the values 1, 2, …, $2r$.

When the quantities λ have the values that are determined by equations (4), we give the name of *principal derivatives* to the virtual derivatives of φ and further refer to the expression:

$$\varphi' = \varphi + \lambda_i f_i + \lambda_i f_i + \ldots + \lambda_{2r} f_{2r}$$

by the name of *principal form* of the function φ. From equation (5), one will see that the principal form of φ satisfies the $2r$ equations:

$$[f_1, \varphi'] = 0, \quad [f_2, \varphi'] = 0, \quad \ldots, \quad [f_{2r}, \varphi'] = 0.$$

If the number of equations (1) is odd then the determinant of the equations (4) will be zero, and those equations will be incompatible. That is why we have supposed that the number is even.

13. – Let us return to the equation:

$$\frac{dq_1}{dt}\delta p_1 + \cdots + \frac{dq_n}{dt}\delta p_n - \frac{dp_1}{dt}\delta q_1 - \cdots - \frac{dp_n}{dt}\delta q_n = \delta H, \tag{1}$$

in which H is a function of the variables q_i, p_i, which we assume are coupled by $2r$ conditional equations:

$$f_1 = 0, \quad f_2 = 0, \quad \ldots, \quad f_{2r} = 0. \tag{2}$$

Upon introducing the undetermined multipliers $\lambda_1, \lambda_2, \ldots, \lambda_{2r}$, one will deduce from equations (1) and (2) that:

$$\left\{\begin{array}{l} \dfrac{dq_1}{dt} = \dfrac{dH}{dp_1} + \lambda_1 \dfrac{df_1}{dp_1} + \lambda_1 \dfrac{df_1}{dp_1} + \cdots + \lambda_{2r} \dfrac{df_{2r}}{dp_1}, \\ \dfrac{dq_2}{dt} = \dfrac{dH}{dp_2} + \lambda_1 \dfrac{df_1}{dp_2} + \lambda_1 \dfrac{df_1}{dp_2} + \cdots + \lambda_{2r} \dfrac{df_{2r}}{dp_2}, \\ \dotfill \end{array}\right. \tag{3}$$

$$\left\{\begin{array}{l} -\dfrac{dp_1}{dt} = \dfrac{dH}{dq_1} + \lambda_1 \dfrac{df_1}{dq_1} + \lambda_1 \dfrac{df_1}{dq_1} + \cdots + \lambda_{2r} \dfrac{df_{2r}}{dq_1}, \\ -\dfrac{dp_2}{dt} = \dfrac{dH}{dq_2} + \lambda_1 \dfrac{df_1}{dq_2} + \lambda_1 \dfrac{df_1}{dq_2} + \cdots + \lambda_{2r} \dfrac{df_{2r}}{dq_2}, \\ \dotfill \end{array}\right. \tag{4}$$

If one differentiates one of equations (2), say, $f_i = 0$, then one will have:

$$\frac{df_i}{dq_1}\frac{dq_1}{dt}+\frac{df_i}{dq_2}\frac{dq_2}{dt}+\cdots+\frac{df_i}{dp_1}\frac{dp_1}{dt}+\frac{df_i}{dp_1}\frac{dp_1}{dt}+\cdots = 0 ,$$

and upon replacing the derivatives of the q_i, p_i, from equations (3) and (4), one will have:

$$[f_i, H] + \lambda_1\,[f_i, f_1] + \lambda_2\,[f_i, f_1] + \ldots = 0 .$$

One concluses from this that the multipliers λ_1, λ_2, … satisfy the following $2r$ equations:

$$\begin{aligned}
&\qquad\qquad [f_2, f_1]\,\lambda_1 + [f_3, f_1]\,\lambda_3 + \ldots + [f_{2r}, f_1]\,\lambda_{2r} = [f_1, H] ,\\
&[f_1, f_2]\,\lambda_1 \qquad\qquad + [f_3, f_2]\,\lambda_3 + \ldots + [f_{2r}, f_2]\,\lambda_{2r} = [f_2, H] ,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

Now, it results from those equations that the right-hand sides of equations (3) and (4) will be the principal derivatives of the function H, and upon representing them in terms of the derivatives that are immediately placed within the parentheses, one will have:

$$\frac{dq_1}{dt} = \left(\frac{dH}{dp_1}\right), \qquad \frac{dq_2}{dt} = \left(\frac{dH}{dp_2}\right), \qquad \ldots, \qquad \frac{dq_n}{dt} = \left(\frac{dH}{dp_n}\right),$$
$$\frac{dp_1}{dt} = -\left(\frac{dH}{dq_1}\right), \qquad \frac{dp_2}{dt} = -\left(\frac{dH}{dq_2}\right), \qquad \ldots, \qquad \frac{dp_n}{dt} = -\left(\frac{dH}{dq_n}\right),$$

instead of equations (3) and (4).

Theorem on the variation of arbitrary constants.

14. – Consider the $2n$ equations:

(1)
$$\frac{dq_i}{dt} = \frac{dH}{dp_i}, \qquad \frac{dp_i}{dt} = -\frac{dH}{dq_i},$$

in which H is an arbitrary function of the q_i, p_i, and t, and i can take the values 1, 2, …, n. Those equations include the equations of dynamics, but they are more general, because in mechanics, H must be the sum of two functions, one of which includes only the variables q_i and time t, while the other one is homogeneous of degree two in the variables p_i .

Suppose that one has obtained the values of the q_i, p_i that depend upon $2n$ arbitrary constants and t. Adopt the symbol D in order to denote the increases in the quantities q_i, p_i, and the function H when those constants are subjected to certain variations.

Multiply equations (1) by $D\,p_i$ and $-D\,q_i$, add them, and sum over i. We will have the equation:

(2) $$\sum_i (dq_i\, Dp_i - dp_i\, Dq_i) = DH\, dt\,,$$

which is entirely analogous to equation (1) in the preceding section.

Let Δ denote a symbol that refers to the other variations of the arbitrary constants and differentiate the preceding equations using Δ. We will have:

$$\sum_i (d\Delta q_i\, Dp_i + dq_i\, \Delta Dp_i - d\Delta p_i\, Dq_i - dp_i\, \Delta Dq_i) = \Delta\, DH\, dt\,.$$

If we subtract those two equations from each other, while remarking that the two symbols $D\Delta$ and ΔD are equivalent, then we will get:

$$\sum_i d(\Delta q_i\, Dp_i - Dp_i\, \Delta q_i) = 0\,.$$

We then conclude that:

(3) $$\sum_i (\Delta q_i\, Dp_i - Dp_i\, \Delta q_i)$$

is independent of time. That important theorem was given by Lagrange (*Mécanique analytique*, section V, **§ 1**).

If the variations using Δ refer to the increases $\Delta\alpha$ in just one constant α, and the variations using D refer to the increases $D\beta$ in another constant β then Lagrange's theorem will express the idea that the formula:

$$\sum_i \left(\frac{dq_i}{d\alpha}\frac{dp_i}{d\beta} - \frac{dq_i}{d\beta}\frac{dp_i}{d\alpha}\right)$$

is independent of time.

One can generalize the preceding theorem by supposing that the q_i, p_i satisfy the equation:

(4) $$\frac{dq_1}{dt}\,\delta p_1 + \cdots + \frac{dq_n}{dt}\,\delta p_n - \frac{dp_1}{dt}\,\delta q_1 - \cdots - \frac{dp_n}{dt}\,\delta q_n = \delta H\,,$$

and that the variables q_i satisfy some conditional equations. That system of equations is also encountered in mechanics then. However, in order to generalize it even further, suppose that the conditional equations refer to not only the variables q_i, but also the values:

$$f_1 = 0\,, \qquad f_2 = 0\,, \qquad \ldots, \qquad f_{2r} = 0\,.$$

From what we saw in no. **15**, we will be led to the equations:

$$\frac{dq_i}{dt} = \left(\frac{dH}{dp_i}\right), \qquad \frac{dp_i}{dt} = -\left(\frac{dH}{dq_i}\right),$$

whose right-hand sides represent the principal derivatives of the function H. If one remarks that one has:

$$\left(\frac{dH}{dq_1}\right)Dq_1 + \cdots + \left(\frac{dH}{dq_n}\right)Dq_n + \left(\frac{dH}{dp_1}\right)Dp_1 + \cdots + \left(\frac{dH}{dp_n}\right)Dp_n = DH,$$

then one will again obtain (2), and upon following the method presented above exactly, one will also find that the expression (3) is independent of t.

On the theory of perturbations.

15. – Once more, suppose that the $2n$ variables q_i, p_i satisfy equation (4) and the $2r$ conditional equations:

(1) $$f_1 = 0, \; f_2 = 0, \; \ldots, \; f_{2r} = 0 .$$

One then concludes the $2n$ equations:

(2) $$\frac{dq_i}{dt} = \left(\frac{dH}{dp_i}\right), \qquad \frac{dp_i}{dt} = -\left(\frac{dH}{dq_i}\right).$$

The values of q_i, p_i as functions of t must contain only $2(n - r)$ arbitrary constants. Represent them by:

(3) $$\begin{cases} q_i = \varphi_i(t, \alpha_1, \alpha_2, \ldots, \alpha_{2(n-r)}), \\ p_i = \psi_i(t, \alpha_1, \alpha_2, \ldots, \alpha_{2(n-r)}). \end{cases}$$

Then imagine that H submits to an increase Ω and that one once more adopts the expressions (3) for the forms of the solutions, but while considering α_1, α_2, … to be functions of t. One will have:

$$\frac{dq_i}{dt} + \frac{dq_i}{d\alpha_1}\frac{d\alpha_1}{dt} + \frac{dq_i}{d\alpha_2}\frac{d\alpha_2}{dt} + \cdots = \left(\frac{dH}{dp_i}\right) + \left(\frac{d\Omega}{dp_i}\right),$$

$$\frac{dp_i}{dt} + \frac{dp_i}{d\alpha_1}\frac{d\alpha_1}{dt} + \frac{dp_i}{d\alpha_2}\frac{d\alpha_2}{dt} + \cdots = -\left(\frac{dH}{dq_i}\right) + \left(\frac{d\Omega}{dq_i}\right).$$

Subtract equations (2) from the preceding ones, and it will result that:

$$\frac{dq_i}{d\alpha_1}d\alpha_1 + \frac{dq_i}{d\alpha_2}d\alpha_2 + \cdots = \left(\frac{d\Omega}{dp_i}\right)dt\ ,$$
$$\frac{dp_i}{d\alpha_1}d\alpha_1 + \frac{dp_i}{d\alpha_2}d\alpha_2 + \cdots = -\left(\frac{d\Omega}{dq_i}\right)dt\ .$$

Multiply them by $-\delta p_i$, δq_i and sum over i. We will have:

$$\text{(4)} \qquad \delta\Omega = \sum_i \left(\frac{dq_i}{d\alpha_1}\delta p_i - \frac{dp_i}{d\alpha_1}\delta q_i\right)\frac{d\alpha_1}{dt} + \sum_i \left(\frac{dq_i}{d\alpha_2}\delta p_i - \frac{dp_i}{d\alpha_2}\delta q_i\right)\frac{d\alpha_2}{dt} + \cdots$$

Since the number of quantities α has been assumed to be only 2 $(n - r)$, they are not coupled by any conditional equations. Therefore, if one lets β denote any of the quantities α then one will have:

$$\frac{d\Omega}{d\beta} = \sum_i \left(\frac{dq_i}{d\alpha_1}\frac{dp_i}{d\beta} - \frac{dp_i}{d\alpha_1}\frac{dq_i}{d\beta}\right)\frac{d\alpha_1}{dt} + \sum_i \left(\frac{dq_i}{d\alpha_2}\frac{dp_i}{d\beta} - \frac{dp_i}{d\alpha_2}\frac{dq_i}{d\beta}\right)\frac{d\alpha_2}{dt} + \cdots$$

If one generally sets:

$$\sum_i \left(\frac{dq_i}{d\alpha}\frac{dp_i}{d\beta} - \frac{dp_i}{d\alpha}\frac{dq_i}{d\beta}\right) = (\alpha, \beta)$$

then that formula will become:

$$\frac{d\Omega}{d\beta} = (\alpha_1, \beta)\frac{d\alpha_1}{dt} + (\alpha_2, \beta)\frac{d\alpha_2}{dt} + \cdots,$$

or

$$\text{(5)} \qquad \frac{d\Omega}{d\beta} = \sum_{s=1}^{2(n-r)} (\alpha_s, \beta)\frac{d\alpha_s}{dt}\ .$$

From what we saw (no. **14**), the quantities (α_s, β) are independent of t. If we suppose that $r = 0$ then that equation will become the Lagrange perturbation formula. We will then see that the formula will persist without modification when we introduce the conditional equations (1).

If the functions α can be divided into two groups of $n - r$ quantities $\alpha_1, \alpha_2, \ldots, \alpha_{n-r}, \beta_1, \beta_2, \ldots, \beta_{n-r}$ that satisfy the equations:

$$\text{(6)} \qquad (\alpha_i, \alpha_k) = 0\ , \quad (\beta_i, \beta_k) = 0\ , \quad (\alpha_i, \beta_k) = 0\ , \quad (\alpha_i, \beta_i) = 1\ ,$$

in which i and k are two different numbers, then the 2 $(n - r)$ equations that are included in equation (5) will become:

$$\frac{d\alpha_i}{dt} = \frac{d\Omega}{d\beta_i}, \qquad \frac{d\beta_i}{dt} = -\frac{d\Omega}{d\alpha_i}\ .$$

16. – Therefore, suppose that one has integrated the $2n$ equations:

$$\frac{dq_i}{dt} = \left(\frac{dH}{dp_i}\right), \qquad \frac{dp_i}{dt} = -\left(\frac{dH}{dq_i}\right)$$

separately from the conditional equations. The q_i, p_i will no longer contain only $2(n - r)$ arbitrary constants then, as before, but they will be expressed in terms of t and $2n$ constants. Nonetheless, the number of arbitrary constants in the problem must reduce to $2(n - r)$. One will obtain the $2r$ relations that couple the constants that are found by substituting the expressions for q_i, p_i in equations (1). Hence, one concludes that t must be eliminated from it.

Therefore, in place of equations (3), we must write:

$$q_i = \varphi_i(t, \alpha_1, \alpha_2, \ldots, \alpha_{2n}),$$
$$p_i = \psi_i(t, \alpha_1, \alpha_2, \ldots, \alpha_{2n}),$$

in which $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$ are coupled by $2r$ conditional equations.

That problem will again lead to equation (4), which can put into this form:

$$\delta\Omega = \sum_\alpha (\alpha, \alpha_1)\frac{d\alpha}{dt}\,\delta\alpha_1 + \sum_\alpha (\alpha, \alpha_2)\frac{d\alpha}{dt}\,\delta\alpha_2 + \cdots + \sum_\alpha (\alpha, \alpha_{2n})\frac{d\alpha}{dt}\,\delta\alpha_{2n},$$

in which the symbol $\sum$ extends to all values of α, which are $\alpha_1, \alpha_2, \ldots, \alpha_{2n}$.

In the particular case where the functions α can be divided into two groups $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ that satisfy equations (6), one will have:

$$\delta\Omega = -\frac{d\beta_1}{dt}\,\delta\alpha_1 - \frac{d\beta_2}{dt}\,\delta\alpha_2 - \cdots - \frac{d\beta_n}{dt}\,\delta\alpha_n + \frac{d\alpha_1}{dt}\,\delta\beta_1 + \frac{d\alpha_2}{dt}\,\delta\beta_2 + \cdots + \frac{d\alpha_n}{dt}\,\delta\beta_n,$$

and as one knows, that equation is equivalent to the equations:

$$\frac{d\alpha_i}{dt} = \left(\frac{d\Omega}{d\beta_i}\right), \qquad \frac{d\beta_i}{dt} = -\left(\frac{d\Omega}{d\alpha_i}\right),$$

in which the principal derivatives of Ω with respect to the quantities α and β are taken from the $2r$ conditional equations that couple those quantities.

17. – Poisson gave some perturbation formulas that were solved for the derivatives of the perturbed elements in the case where the conditional equations were absent. Consequently, those formulas will provide the solution to equations (5) for the derivatives of the quantities α when r is zero. However, those formulas cease to be entirely applicable when one supposes that there are conditional equations between the variables of the problem. We shall look for the formulas that one must replace them with.

Once more, suppose that there are $2r$ finite equations:

$$f_1 = 0\,, \qquad f_2 = 0\,, \ldots, \qquad f_{2r} = 0 \tag{7}$$

between the $2n$ variables $q_1, q_2, \ldots, q_r, p_1, p_2, \ldots, p_r$, and the differential equations:

$$\frac{dq_i}{dt} = \left(\frac{dH}{dp_i}\right), \qquad \frac{dp_i}{dt} = -\left(\frac{dH}{dq_i}\right), \tag{8}$$

in which i can take the values 1, 2, …, n, and H can include t, in addition to the variables q_i, p_i .

Imagine that one has integrated those equations and that one has found the following $2n - 2r$ equations for their integrals:

$$\left\{\begin{array}{ll} \beta_1 & = \psi_1(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t), \\ \beta_2 & = \psi_2(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t), \\ \multicolumn{2}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ \beta_{2(n-r)} & = \psi_{2(n-r)}(q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n, t), \end{array}\right. \tag{9}$$

in which $\beta_1, \beta_2, \ldots, \beta_{2(n-r)}$ denote arbitrary constants that do not enter into the right-hand sides.

In mechanics, if the force function does not depend upon t then H will also be independent of it, and one can then suppose that t is present in only one of equations (9), and additively to the constant that enters into it.

Imagine then that one has solves the same problem, in which the function H is replaced with $H + \Omega$, in such a way that Ω is the perturbing function and Ω is generally a very small quantity. Equations (7) will again persist, but equations (8) will be replaced with:

$$\frac{dq_i}{dt} = \left(\frac{d\,(H+\Omega)}{dp_i}\right), \qquad \frac{dp_i}{dt} = -\left(\frac{d\,(H+\Omega)}{dq_i}\right). \tag{10}$$

Thus, the functions of the q_i, p_i that define the right-hand sides of equations (9) will no longer have constant values. However, their derivatives will generally be very small quantities that one must determine.

Those functions, which we denote by the symbol β, are defined by equations (9), and those equations, combined with the $2r$ equations (7), can serve to express the $2n$ variables q_i, p_i in terms of the quantities β and t, and in only one way.

Let $\alpha = \psi$ denote an integral of the *unperturbed* problem, i.e., of equations (9), in which α is the arbitrary constant. Upon differentiating that equation, one will have:

$$0 = \sum_{i=1}^{n} \left[\frac{d\psi}{dq_i} \left(\frac{dq_i}{dt} \right) + \frac{d\psi}{dp_i} \left(\frac{dp_i}{dt} \right) \right] + \frac{d\psi}{dt} \;,$$

and the last term will be zero if t is not explicitly present in ψ. Upon regarding equations (8), one will get:

$$0 = \sum_{i=1}^{n} \left[\frac{d\psi}{dq_i} \left(\frac{dH}{dp_i} \right) - \frac{d\psi}{dp_i} \left(\frac{dH}{dq_i} \right) \right] + \frac{d\psi}{dt} \,. \tag{11}$$

Let us now pass on to the *perturbed* problem. We must then regard α in the equation $\alpha = \psi$ as a function of t, and upon differentiating it, we will have:

$$\frac{d\alpha}{dt} = \sum_{i=1}^{n} \left(\frac{d\psi}{dq_i} \frac{dq_i}{dt} + \frac{d\psi}{dp_i} \frac{dp_i}{dt} \right) + \frac{d\psi}{dt} \,,$$

in which the derivatives of the variables q_i, p_i are provided by equations (10), and it will result that:

$$\frac{d\alpha}{dt} = \sum_{i=1}^{n} \left\{ \frac{d\psi}{dq_i} \left(\frac{d\,(H+\Omega)}{dp_i} \right) - \frac{d\psi}{dp_i} \left(\frac{d\,(H+\Omega)}{dq_i} \right) \right\} + \frac{d\psi}{dt} \;. \tag{12}$$

If we subtract the last two equations from each other and replace the symbol ψ with α in the result (which should not cause any confusion) then we will have:

$$\frac{d\alpha}{dt} = \sum_{i=1}^{n} \left[\frac{d\alpha}{dq_i} \left(\frac{d\Omega}{dp_i} \right) - \frac{d\alpha}{dp_i} \left(\frac{d\Omega}{dq_i} \right) \right] .$$

The function Ω can be expressed by means of the quantities β and t, and in only one way, and from a property that was proved in my article on principal derivatives (Bulletin de la Société mathématique, t. 1, pp. 164), the principal derivatives of Ω with respect to q_i, p_i are given by the formulas:

$$\left(\frac{d\,\Omega}{dq_i} \right) = \sum_{\beta} \frac{d\Omega}{d\beta} \left(\frac{d\beta}{dq_i} \right) , \qquad \left(\frac{d\,\Omega}{dp_i} \right) = \sum_{\beta} \frac{d\Omega}{d\beta} \left(\frac{d\beta}{dp_i} \right) .$$

One will then have:

$$\frac{d\alpha}{dt} = \sum_{\beta} \frac{d\Omega}{d\beta} \sum_{i=1}^{n} \left[\frac{d\alpha}{dq_i} \left(\frac{d\beta}{dp_i} \right) - \frac{d\alpha}{dp_i} \left(\frac{d\beta}{dq_i} \right) \right] .$$

Furthermore, one has:

$$\left(\frac{d\beta}{dq_i}\right) = \frac{d\beta}{dq_i} + \mu_1(\beta)\,\frac{df_1}{dq_i} + \mu_2(\beta)\,\frac{df_2}{dq_i} + \cdots + \mu_{2r}(\beta)\,\frac{df_{2r}}{dq_i} ,$$
$$\left(\frac{d\beta}{dp_i}\right) = \frac{d\beta}{dp_i} + \mu_1(\beta)\,\frac{df_1}{dp_i} + \mu_2(\beta)\,\frac{df_2}{dp_i} + \cdots + \mu_{2r}(\beta)\,\frac{df_{2r}}{dp_i} ,$$

when one takes $\mu_1(\beta)$, $\mu_2(\beta)$, … to be quantities that satisfy the $2r$ equations:

$$\begin{aligned} & + [f_2, f_1]\,\mu_2(\beta) + [f_3, f_1]\,\mu_3(\beta) + \ldots + [f_{2r}, f_1]\,\mu_{2r}(\beta) = [f_1, \beta] , \\ &[f_1, f_2]\,\mu_1(\beta) + + [f_3, f_2]\,\mu_3(\beta) + \ldots + [f_{2r}, f_2]\,\mu_{2r}(\beta) = [f_2, \beta] , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

Upon replacing the principal derivatives of the quantities β with the expressions above, one will finally have:

$$\frac{d\alpha}{dt} = \sum_\beta \frac{d\Omega}{d\beta} \sum_{i=1}^{n} \{[\alpha, \beta] + \mu_1(\beta)[\alpha, f_1] + \mu_2(\beta)[\alpha, f_2] + \cdots + \mu_{2r}(\beta)[\alpha, f_{2r}]\} ,$$

which is the desired formula.

The last formula can be considered to be something that provides the solutions to equations (5) for the derivatives of the perturbed elements. Therefore, since the quantities (α_1, β) are independent of time t, the same thing will be true of the expression:

$$[\alpha, \beta] + \mu_1(\beta)\,[\alpha, f_1] + \mu_2(\beta)\,[\alpha, f_2] + \ldots$$

The perturbation formula that was just given is obviously exact, no matter what limits within which Ω might vary. However, if Ω always remains very small, along with its derivatives, then the quantities $d\alpha / dt$ will themselves be very small, so the quantities α will vary only slightly, and one will get an approximate value for one of the quantities α by one quadrature upon regarding the coefficient of $d\Omega / d\beta$ as constant and regarding only the quantity t as variable in Ω, which is a function of the α and t.

Theorem concerning the integration of a system of equations of the type that was considered before.

18. – Suppose that the equation:

$$(a) \qquad \frac{dq_1}{dt}\delta p_1 + \cdots + \frac{dq_n}{dt}\delta p_n - \frac{dp_1}{dt}\delta q_1 - \cdots - \frac{dp_n}{dt}\delta q_n = \delta H ,$$

exists between the $2n$ variables q_i, p_i, along with the conditional equations:

$$(b) \qquad f_1 = 0\,, \qquad f_2 = 0\,, \ldots, \qquad f_{2r} = 0\,.$$

We proved (no. **8**) that one can replace the $2n$ variables q_i, p_i with $2n - 2r$ mutually-independent variables $Q_1, Q_2, \ldots, Q_{n-r}, P_1, P_2, \ldots, P_{n-r}$ that satisfy the $2n - 2r$ canonical differential equations that are included in the two equations:

$$(1) \qquad \frac{dQ_i}{dt} = \frac{dH}{dP_i}\,, \qquad \frac{dP_i}{dt} = -\frac{dH}{dQ_i}\,,$$

in which i can take the values $1, 2, \ldots, n - r$.

The quantity $[\alpha, \beta]$ is formed from the derivatives of α and β with respect to the variables q_i, p_i . Form a quantity that is composed in the same manner in terms of the derivatives of α and β with respect to the variables Q_i, P_i and set:

$$[\alpha, \beta]' = \frac{d\alpha}{dQ_1}\frac{d\beta}{dP_1} + \frac{d\alpha}{dQ_2}\frac{d\beta}{dP_2} + \cdots + \frac{d\alpha}{dQ_{n-r}}\frac{d\beta}{dP_{n-r}} - \frac{d\alpha}{dP_1}\frac{d\beta}{dQ_1} - \frac{d\alpha}{dP_2}\frac{d\beta}{dQ_2} - \cdots - \frac{d\alpha}{dP_{n-r}}\frac{d\beta}{dQ_{n-r}}\,.$$

Since there exist no conditional equations between the variables Q_i, P_i, the perturbation formula from the preceding section will change into the following one when one uses those variables:

$$(2) \qquad \frac{d\alpha}{dt} = \sum_{\beta} \frac{d\Omega}{d\beta}\,[\alpha, \beta]'\,.$$

The coefficients of the derivatives of Ω must be the same in the two formulas, and one concludes that:

$$(3) \qquad [\alpha, \beta]' = [\alpha, \beta] + \mu_1(\beta)\,[\alpha, f_1] + \mu_2(\beta)\,[\alpha, f_2] + \ldots + \mu_{2r}(\beta)\,[\alpha, f_{2r}]\,.$$

From Poisson's celebrated theorem, if:

$$\alpha = \text{const.}, \qquad \beta = \text{const.}$$

are two integrals of the canonical system of equations then the formula:

$$(4) \qquad [\alpha, \beta]' = \text{const.}$$

is likewise an integral of those equations, as long as the left-hand side is not identically constant.

Moreover, that theorem shows that the coefficients of the derivatives of Ω in formula (2) are independent of t, just as was remarked at the end of no. **17**.

If one returns to the variables q_i, p_i from the variables Q_i, P_i then one will get the following theorem:

Theorem:

Suppose that:

$$\alpha = \text{const.}, \qquad \beta = \text{const.}$$

are two integrals of the system of equations:

$$f_1 = 0\,, \quad f_2 = 0\,, \quad \ldots, \quad f_{2r} = 0\,,$$

$$\frac{dq_1}{dt} = \left(\frac{dH}{dp_1}\right), \qquad \frac{dq_2}{dt} = \left(\frac{dH}{dp_2}\right), \qquad \ldots, \qquad \frac{dq_n}{dt} = \left(\frac{dH}{dp_n}\right),$$
$$\frac{dp_1}{dt} = -\left(\frac{dH}{dp_1}\right), \qquad \frac{dp_2}{dt} = -\left(\frac{dH}{dp_2}\right), \qquad \ldots, \qquad \frac{dp_n}{dt} = -\left(\frac{dH}{dq_n}\right).$$

The equation:

$$[\alpha, \beta] + \mu_1(\beta)\,[\alpha, f_1] + \mu_2(\beta)\,[\alpha, f_2] + \ldots + \mu_{2r}(\beta)\,[\alpha, f_{2r}] = \text{const.}$$

will also be an integral of that system of equations then, as long as the left-hand side is not identically constant.

What adds to the importance of that theorem is that passing from the variables q_i, p_i to the variables Q_i, P_i can be accomplished only by some very difficult operations in integral calculus, in such a way that it would not be convenient to apply formula (4).

The preceding theorem is only a corollary to formula (3). Moreover, I have already pointed out that the formula (3) was given by Jacobi in the case where the conditional equations include only the variables q_i, and that formula then took a form that was very different and more complicated. Indeed, if one lets:

$$f_1 = 0\,, \quad f_2, = 0\,, \quad \ldots, \quad f_r = 0$$

denote the given equations of condition in that case, as in no. **10**, then one must set:

$$f_{r+1} = [f_1, H]\,, \quad f_{r+2} = [f_2, H]\,, \quad \ldots, \quad f_{2r} = [f_r, H]$$

in formula (3). It is also appropriate to observe that Jacobi's formula was proved on the basis of a particular transformation of the variables q_i, p_i into the variables Q_i, P_i, which obliged him to devote two entire pages to the statement of his problem (*Nova Methodus*, § **38**, *Werke*, t. III). On the contrary, the transformation from the q_i, p_i to the Q_i, P_i in my theorem is accomplished for an arbitrary canonical system, so its statement is very simple. One can further point out that the transformation of the variables q_i, p_i into the variables Q_i, P_i that Jacobi adopted is not applicable to the equation:

$$\frac{dq_1}{dt}\delta p_1 + \cdots + \frac{dq_n}{dt}\delta p_n - \frac{dp_1}{dt}\delta q_1 - \cdots - \frac{dp_n}{dt}\delta q_n = \delta H ,$$

when it is taken in full generality. It is based essentially upon the particular form that it takes in dynamics, where H is composed of the sum of a function $-U$ that includes only the variables q_i and a function T of the p_i, q_i that is homogeneous of degree two in the variables p_i . Nonetheless, one can recognize that from the considerations of § **49** in Jacobi's treatise, one can free oneself from the latter restriction.

From those ruminations, one sees how many different ways that formula (3) can be generalized, in such a way that I believe one can say that the theorem that is included in that formula was entirely new when I stated it for the first time [*Comptes rendus des séances de l'Académie des Sciences* **66** (1873), pp. 1193].

Properties of the expression $[\alpha, \beta]$.

19. – In no. **18**, we saw that the $2n$ variables q_i, p_i that satisfy equations (a) and (b) can be replaced with $2(n - r)$ variables Q_i, P_i as functions of the former that satisfy a system of $2(n - r)$ canonical differential equations, and upon setting:

$$[\alpha, \beta]' = \sum_{i=1}^{n-r}\left(\frac{d\alpha}{dQ_i}\frac{d\beta}{dP_i} - \frac{d\alpha}{dP_i}\frac{d\beta}{dQ_i}\right) ,$$

one will have the formula:

$$[\alpha, \beta]' = [\alpha, \beta] + \mu_1(\beta)[\alpha, f_1] + \ldots + \mu_{2r}(\beta)[\alpha, f_{2r}] . \tag{1}$$

It is important to prove that α and β can be considered to be two arbitrary functions of the variables Q_i, P_i or q_i, p_i in that formula, in such a way that the formula can serve to replace the expression $[\alpha, \beta]'$, in which α and β are two arbitrary functions of the variables Q_i, P_i with another expression that includes only the variables q_i, p_i .

Indeed, in the theory that was presented above:

$$\alpha = \text{const.}, \qquad \beta = \text{const.} \tag{2}$$

were two integrals of the canonical system that is included in the equations:

$$\frac{dQ_i}{dt} = \frac{dH}{dP_i} , \quad \frac{dP_i}{dt} = -\frac{dH}{dQ_i} ,$$

in which i can take the values $1, 2, \ldots, n - r$. Upon differentiating equations (2), one will then have:

(3) $$[\alpha, H]' + \frac{d\alpha}{dt} = 0 , \qquad [\beta, H]' + \frac{d\beta}{dt} = 0 .$$

Therefore, the preceding theory supposes that one can find a function H that satisfies those two equations, which limits the functions that one can take for α and β. However, imagine that one has obtained the formula that gives the expression $[\alpha, \beta]'$ in terms of the variables q_i, p_i when α and β are two arbitrary functions. The variables q_i, p_i are functions of the variables Q_i, P_i that do not depend upon the function H or t, as one can convince oneself from no. **8**. Similarly, the formula considered that gives $[\alpha, \beta]'$ is independent of H. Therefore, if one supposes that the α and β in that formula satisfy equations (3) then no reduction will ensue. Hence, conversely, the formula (1) that was found can be extended to two arbitrary functions under the hypothesis that the equations (3) are satisfied.

20. – Put the expression:

$$(\alpha) = \alpha + \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_{2r} f_{2r}$$

in formula (1) in place of the given function for α, while choosing λ_1, λ_2, … to be multipliers that give α its principal form. From what we said in no. **12**, we will have:

$$[(\alpha), f_1] = 0 , \quad [(\alpha), f_2] = 0 , \quad \ldots, \quad [(\alpha), f_{2r}] = 0 .$$

Therefore, all of the terms that follow the first one in the right-hand side of formula (1) will be annulled, and one will have:

(4) $$[\alpha, \beta]' = [\alpha + \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_{2r} f_{2r}, \beta] .$$

It is obvious that I did not put (α) in place of α in the left-hand side of that, because α is expressed in terms of the variables Q_i, P_i on the left-hand side, which can be true in only one way.

Similarly, if β has its principal form then one will have:

$$\mu_1 (\beta) = 0 , \quad \mu_2 (\beta) = 0 , \quad \ldots, \quad \mu_{2r} (\beta) = 0 ,$$

and one will again conclude from formula (1) that:

$$[\alpha, \beta]' = [\alpha, (\beta)]$$

or

(5) $$[\alpha, \beta]' = [\alpha, \beta + \mu_1 (\beta) f_1 + \mu_2 (\beta) f_2 + \ldots + \mu_{2r} (\beta) f_{2r}] .$$

The latter formula is verified quite easily, because upon developing it, one will get:

$$[\alpha, \beta]' = [\alpha, \beta] + \mu_1 (\beta) [\alpha, f_1] + \mu_2 (\beta) [\alpha, f_2] + \ldots + \mu_{2r} (\beta) [\alpha, f_{2r}] + \ldots,$$

and since $f_1, f_2, \ldots$ are zero, one will recover equation (1).

One can obviously combine formulas (4) and (5) with this other one:

$$[\alpha, \beta]' = [(\alpha), (\beta)] = [\alpha + \lambda_1 f_1 + \lambda_2 f_2 + \ldots, \beta + \lambda_1 f_1 + \lambda_2 f_2 + \ldots] ,$$

and the latter further amounts to the following one, which is formed by means of the principal derivatives of α and β:

$$[\alpha, \beta]' = \left(\frac{d\alpha}{dq_1}\right)\left(\frac{d\beta}{dp_1}\right) + \left(\frac{d\alpha}{dq_2}\right)\left(\frac{d\beta}{dp_2}\right) + \cdots - \left(\frac{d\alpha}{dp_1}\right)\left(\frac{d\beta}{dq_1}\right) - \left(\frac{d\alpha}{dp_2}\right)\left(\frac{d\beta}{dq_2}\right) - \cdots$$

Formulas (4) and (5) show that one can modify the forms of the functions α and β using the conditional equations in such a manner that one makes the expression $[\alpha, \beta]$ independent of the choice of variables q_i, p_i. From my paper on the principal derivatives, one can see that there exist other forms for the expressions for α and β that will likewise lead to that objective. Nonetheless, I consider the preceding formulas to be the simplest and most curious ones.

21. – In no. **15**, I found the perturbation formula:

$$\frac{d\Omega}{d\beta} = \sum_{s=1}^{2(n-r)} (\alpha_s, \beta) \frac{d\alpha_s}{dt}$$

by setting:

(6) $$(\alpha_s, \beta) = \sum_{i=1}^{n} \left(\frac{dq_i}{d\alpha}\frac{dp_i}{d\beta} - \frac{dp_i}{d\alpha}\frac{dq_i}{d\beta} \right) .$$

If one supposes that the variables q_i, p_i are transformed into the variables Q_i, P_i then the coefficient of $d\alpha_s / dt$ will generally be replaced in that formula with:

(7) $$(\alpha_s, \beta)' = \sum_{i=1}^{n-r} \left(\frac{dQ_i}{d\alpha}\frac{dP_i}{d\beta} - \frac{dP_i}{d\alpha}\frac{dQ_i}{d\beta} \right) .$$

One then concludes that the last two expressions are equal to each other, and that one has:

$$(\alpha_s, \beta)' = (\alpha_s, \beta) .$$

I shall conclude by proving that formula, into which only the variables q_i, p_i enter, which I suppose are subject to the conditional equations:

$$[\alpha_i,\ \alpha_1]'\ (\alpha_1,\ \alpha_k) + [\alpha_i,\ \alpha_2]'\ (\alpha_2,\ \alpha_k) + \ldots + [\alpha_i,\ \alpha_{2(n-r)}]'\ (\alpha_{2(n-r)},\ \alpha_k)$$

$$\begin{aligned} &= \ \ 0 \quad \text{if } i \text{ is different from } k\,, \\ &= -1 \quad \text{if } i = k\,, \end{aligned}$$

in which the first factors in the terms on the left-hand side are given by the formula:

$$(8) \qquad [\alpha_i,\ \alpha_s]' = \sum_{u=1}^{n} \left[\left(\frac{d\alpha_i}{dq_u}\right)\left(\frac{d\alpha_s}{dp_u}\right) - \left(\frac{d\alpha_i}{dp_u}\right)\left(\frac{d\alpha_s}{dq_u}\right) \right],$$

and the second factors are formed according to formula (6).

Indeed, from a known formula, upon expressing the 2 $(n - r)$ functions α in terms of the variables Q_i, P_i, one will have:

$$[\alpha_i,\ \alpha_1]'\ (\alpha_1,\ \alpha_k)' + [\alpha_i,\ \alpha_2]'\ (\alpha_2,\ \alpha_k)' + \ldots + [\alpha_i,\ \alpha_{2(n-r)}]'\ (\alpha_{2(n-r)},\ \alpha_k)'$$

$$\begin{aligned} &= \ \ 0 \quad \text{if } i \text{ is different from } k\,, \\ &= -1 \quad \text{if } i = k \end{aligned}$$

when one sets:

$$(9) \qquad [\alpha_i,\ \alpha_s]' = \sum_{u=1}^{n} \left[\frac{d\alpha_i}{dQ_u}\frac{d\alpha_s}{dP_u} - \frac{d\alpha_i}{dP_u}\frac{d\alpha_s}{dQ_u} \right],$$

and the $(\alpha_s,\ \alpha_k)'$ are formed according to formula (7).

Now, one has:

$$(\alpha_s,\ \alpha_k)' = (\alpha_s,\ \alpha_k)\,,$$

and the two expressions (8) and (9) are equivalent. One concludes the formula to be proved from that.