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The jittering electron and its dynamics

Drzacy electron i jego dynamika

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I. Zitterbewegung.
II. DE BROGLIE-waves and quantum laws in a kinematical light.

The new fundamental dynamical laws that are derived by the author include an additional term that depends upon the angular momentum in comparison to the classical equations. The new term has the effect that a particle of mass M and angular momentum Ω will possess a *proper frequency:* the center of mass of the particle oscillates with a frequency of $\nu = Mc^2 / 2\pi\Omega$ (Section I).

The phase of the zitterbewegung will depend upon the proper time of the particle and will not be a single-valued function of the coordinates, in general. The quantum laws of the older quantum theory have the consequence that the phase will be single-valued. If one introduces the PLANCK constant into the conditions for single-valuedness then that will necessarily lead to the value $h / 4\pi$ for the angular momentum of the electron. That will immediately yield a new conception of the DE BROGLIE-waves (Section II). In all of the considerations, we shall content ourselves with the non-relativistic approximation.

I. Zitterbewegung.

The foundations of the present work are defined by the equations of motion of a particle that the author has obtained from general relativity ["Neue Mechanik materialler System," Acta Phys. Pol. **6** (1937), 163; formulas (6.1) and (6.2)]:

$$M\dot{u}^{\alpha} + \Omega^{\alpha\nu}\dot{u}_{\nu} = P^{\alpha}, \tag{I.1}$$

$$\dot{\Omega}^{\beta\lambda} - \Omega^{\lambda\alpha}\dot{u}_{\alpha}u^{\beta} + \Omega^{\beta\alpha}\dot{u}_{\alpha}u^{\lambda} = 0. \tag{I.2}$$

The equations, when presented in this form, relate to MINKOWSKI space (velocity of light = 1) and allow a LORENTZ transformation. Along with the constant mass M, the four-velocity u^{α} of the center of mass (differentiation with respect to proper time will be denoted by a dot), the force P^{α}, and the components of angular momentum $\Omega^{\alpha\beta}$ will enter into our equation. Angular momentum will then be an antisymmetric tensor that satisfies the orthogonality relation:

$$\Omega^{\alpha\beta}u_{\alpha} = 0. \tag{I.3}$$

Its components in a LORENTZ coordinate system in which the center of mass is instantaneously at rest can be arranged into a vector $\mathfrak{w}$:

$$\Omega^{0i} = 0 \qquad (i = 1, 2, 3), \quad (\Omega^{23}, \Omega^{31}, \Omega^{12}) = (\mathfrak{w}_x, \mathfrak{w}_y, \mathfrak{w}_z). \tag{I.4}$$

The $\Omega^{\alpha\beta}$ term in (I.1), which will follow necessarily from general relativity, is of the greatest physical significance.

We replace equation (I.1) with its non-relativistic approximation. That will come about upon neglecting magnitudes of order v / c (v = velocity of the center of mass, $c = 3 \times 10^{10}$ cm sec^{-1}).

We would thus like to regard the angular momentum $\Omega^{\alpha\beta}$ as a constant; in fact, equation (I.2) will yield, in the non-relativistic approximation:

$$\frac{d\mathfrak{w}}{dt} = \frac{1}{c^2}[[\mathfrak{w}\dot{\mathfrak{v}}]\mathfrak{w}] \tag{I.5}$$

(square brackets the mean vector product, and *time* differentiation will be denoted by a dot), and the right-hand side will vanish when one passes to the limit of $v / c = 0$. $\mathfrak{w}$ and $\mathfrak{v}$ (the velocity of the center of mass) will be spatial vectors now.

We shall first consider an uncharged particle ($P^{\alpha} = 0$). Equation (I.1) will then yield:

$$M\dot{\mathfrak{v}} - \frac{1}{c^2}[\mathfrak{w}\ddot{\mathfrak{v}}] = 0 \tag{I.6}$$

in our non-relativistic approximation. If we now point the z-axis in the direction of constant $\mathfrak{w}$-vectors then equations (I.6) will read, in terms of their three components:

$$\begin{cases} \dot{v}_x = -k\ddot{v}_y, \\ \dot{v}_y = -k\ddot{v}_x, \\ \dot{v}_z = 0, \quad \dfrac{1}{k} = \dfrac{Mc^2}{\Omega}, \quad \Omega = |\mathfrak{w}|. \end{cases} \tag{I.7}$$

The first two equations yield:

$$\begin{cases} \dot{v}_x - k\ddot{v}_y = 0, \\ \dot{v}_y + k\ddot{v}_x = 0. \end{cases} \tag{I.8}$$

The third of equations (I.7) means that a motion with constant velocity takes place in the z-direction. If we disregard this inessential motion then the center of mass will remain in the xy-plane during its motion, where one introduces the integral of (I.8) (with two arbitrary constants) into (I.7), which will be a solution with two constants:

$$\dot{v}_x = a \sin\left(\frac{1}{k}+\alpha\right),$$

$$\dot{v}_y = a \cos\left(\frac{1}{k}+\alpha\right), \qquad z = 0.$$

The amplitude a and the initial α will be arbitrary constants. The center of mass regularly describes a circle with the period:

$$\Theta = 2\pi k.$$

In fact, we have:

(I.9)
$$\begin{cases} x = A\sin 2\pi\dfrac{1}{\Theta}, \\ y = A\cos 2\pi\dfrac{1}{\Theta}. \end{cases}$$

A is an arbitrary constant. We ignore the initial angle α and the regular motion in the xy-plane as inessential. The frequency will amount to:

(I.10)
$$\boxed{\nu = \frac{1}{\Theta} = \frac{Mc^2}{2\pi\Omega}.}$$

The center of mass of an isolated particle that possesses angular momentum performs a circular motion with a frequency ν (when one disregarded a uniform, rectilinear motion). The radius of the circle is arbitrary and its plane is orthogonal to the constant angular momentum (*).

For an electron, we will find a *zitterbewegung*; i.e., a high-frequency periodic motion. If:

(I.11)
$$\Omega = \frac{h}{4\pi}$$

(in the following section, we will see that the zitterbewegung itself will yield this value) then:

(I.12)
$$\boxed{\nu = \frac{2Mc^2}{h}, \quad \Theta = \frac{h}{2Mc^2}.}$$

The force P^α in (I.1) is the sum of two forces: The effect that all of the remaining electrons exert upon the electron and the radiation reaction of the electron. We consider (returning to its non-relativistic approximation) the radiation reaction force, in which we replace (I.7) by the following system:

(*) Ω is the angular momentum *about the center of mass*.

(I.13)
$$\begin{cases} \dot{v}_x = -q\ddot{v}_x - k\ddot{v}_y, \\ \dot{v}_y = k\ddot{v}_x - q\ddot{v}_y, \\ \dot{v}_y = -q\ddot{v}_z, \\ q = \dfrac{2}{3}\dfrac{e^2}{Mc^2} \qquad (e = \text{charge}). \end{cases}$$

We would again like to disregard any motion in the *z*-direction that converges to a uniform motion (or rest). The motion will then take place in the *xy*-plane, and the integral system of (I.3) will include an amplitude a and an initial angle α as arbitrary constants:

(I.14)
$$\begin{cases} \dot{v}_x = ae^{-\beta t}\sin\left(\dfrac{t}{k'} + \alpha\right), \\ \dot{v}_y = ae^{-\beta t}\cos\left(\dfrac{t}{k'} + \alpha\right). \end{cases}$$

If one introduces (I.14) into (I.13) then one will obtain the equations for k' and β:

$$\frac{k}{k'} + q\beta = 1, \quad -\frac{q}{k'} + q\beta = 0,$$

which will imply that:

(I.15)
$$k' = k\left(1 + \frac{q^2}{k^2}\right), \qquad \beta = \frac{q}{q^2 + k^2}.$$

The frequency will now be equal to:

(I.16)
$$\nu' = \frac{1}{2\pi k'} = \nu\left(1 - \frac{q^2}{k^2} + \cdots\right),$$

and will differ very slightly from the frequency (I.10) for the electron, for which, we will have:

$$\left(\frac{q}{k}\right)^2 = \frac{16}{9}\left(\frac{2\pi e^2}{hc}\right)^2.$$

If one integrates (I.14) twice then that will yield the orbit, which will correspond to a damped zitterbewegung of frequency ν' (up to a uniform, rectilinear motion). Since the initial amplitude and angle are arbitrary, the frequency and damping coefficient will be expressed in terms of constants of the electron. The radius vector, which points away from the point $x = y = 0$, performs a *uniform* rotation, just as it does in the case of undamped motion (I.9).

(I.17) The phase angle $\varphi - \varphi_0 = \dfrac{2\pi t}{\Theta} = 2\pi\nu\, t.$

The continuous β-spectrum. We have the conservation law:

$$\frac{c^2 M}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{[\mathfrak{v}\,\mathfrak{w}]\dot{\mathfrak{v}}}{c^2\sqrt{1-\frac{v^2}{c^2}}} = \text{constant} \tag{I.18}$$

for the energy of a free, non-radiating electron.

We would like to determine the order of magnitude of the second term on the right, namely, the energy of acceleration E_Ω . Let $v \ll c$ (slow electrons).

$$\pm E_\Omega = \frac{h}{4\pi}\frac{(2\pi\nu)^3 r^2}{c^2} = \frac{16\pi^2}{h^2} M^3 c^4 r^2 . \tag{I.19}$$

For zitterbewegung that corresponds to a velocity of $v = 2\pi\nu\, r = 1/3\, c$, we will have $r = h / 12\pi Mc$, so:

$$\pm E_\Omega = \frac{Mc^2}{9} \text{ erg.}$$

From the estimates (which need to be supplemented with a relativistic computation for large velocities), one will be led to the following consideration: If the β-electrons all have the same total energy (i.e., kinetic energy + energy of acceleration) due to their configuration about the nucleus then it cannot be said that they have the same kinetic energy if the energy of acceleration is different for different electrons. They will gradually lose their energy of acceleration by radiation, and will asymptotically approximate a motion with *different* constant velocities.

II. DE BROGLIE-waves and quantum laws in a kinematical light

Our treatment of zitterbewegung was non-relativistic; indeed, we assumed that v / c was small in a coordinate system in which the center of oscillation was at rest. Our fundamental equations, however, came from the relativistic school, so the consideration of our results in arbitrary LORENTZ coordinate systems must then follow from the laws of relativity. If x, y, z, t are the space-time coordinates in a LORENTZ coordinate system in which the center of oscillation moves with the constant velocity $\mathfrak{w}$ then one will have the formula:

$$\varphi - \varphi_0 = 2\pi\nu\, s = 2\pi\,\frac{s}{\Theta}, \tag{II.1}$$

instead of (I.17), in which s is the *proper time* (as measured from any initial point) of the center of oscillation. The angle φ must be measured in a *co-moving* coordinate system that always remains parallel to itself. We take $e^{i\varphi}$ to be the *phase* of the zitterbewegung. We have the following formula for the proper time element of the center of oscillation,

which is correct up to the order of magnitude $(v / c)^2$ (v means the velocity of the center of oscillation, now):

$$ds = dt \sqrt{1-\frac{v^2}{c^2}} = dt - \frac{1}{2}\frac{v^2}{c^2} dt.$$

Since:

$$dx = v_x\, dt, \qquad dy = v_y\, dt, \qquad dz = v_z\, dt,$$

$$v_x^2 + v_y^2 + v_z^2 = v^2,$$

we will then have:

(II.2) $$ds = dt - \frac{1}{2c^2}(v_x\, dx + v_y\, dy + v_z\, dz).$$

Since the components of the velocity remain constant, the difference between the phase angles at two world points will then equal:

$$\varphi - \varphi_1 = \frac{2\pi(s-s_1)}{\Theta} = 2\pi\frac{t-t_1}{\Theta} - \frac{\pi}{\Theta c^2}\,[v_x\,(x-x_1) + v_y\,(y-y_1) + v_z\,(z-z_1)].$$

Now, let the line along which the center of oscillation moves in space be covered with equidistant points; let the basic interval be λ. We ask: What is the smallest λ for which the phase will be a single-valued function of the position inside of an elementary interval? Thus, the space-dependent component of the phase angle in the last formula will be considered to be a time-dependent component. The answer to our question is clear: For two consecutive points (x, y, z) and (x_1, y_1, z_1) that correspond to each other inside of their respective intervals, and are thus separated from each other by λ, we will have:

$$\frac{v_x(x-x_1)+v_y(y-y_1)+v_z(z-z_1)}{2c^2\Theta} = \frac{v\lambda}{2c^2\Theta} = 1.$$

Due to (I.12), that will yield:

(II.3) $$\lambda = \frac{h}{Mv}.$$

Thus, we arrive at a simple explanation for the DE BROGLIE waves: *The difference between the system time and proper time increases by roughly one period* Θ *when the center of oscillation traverses the arc length* λ. That can be recognized from (II.2) immediately:

If a force field acts upon the electron then we will assume that the zitterbewegung, when considered from a coordinate system in which the center of oscillation is instantaneously at rest, can be replaced with the motion (I.9) with sufficient accuracy during a time interval that is short, but still includes many proper periods. We shall further assume that the plane of oscillation remains parallel to itself during the oscillation, so the position of the electrons relative to the given position of the center of oscillation will be determined by the phase angle. The force that will be necessary to stimulate

zitterbewegung with a frequency of $2Mc^2 / h$ is very large in comparison to the force that acts upon the electrons inside the atom, and it seems natural that we should disregard the influence of the atomic forces on the proper period. The zitterbewegung will then be similar to the workings of the clock of relativity theory. The frequency will then be constant when it is measured in the proper time of the center of oscillation. As for the phase angle, we will then have formula (II.1). From this Ansatz, which corresponds to an undamped zitterbewegung, the motion will be represented in a "time-integrated" form.

What we must now determine is the orbit of the center of oscillation. This related problem is dynamical in nature, and we would like to address it in a later treatise (in some other form). However, for the time being, we can still establish the following:

Theorem: *The quantum laws for the motion of the center of oscillation, namely:*

$$\text{(II.4)} \qquad \oint dS = n\,h, \qquad dS = M\,(v_x\,dx + v_y\,dy + v_z\,dz)$$

have the fact that the phase of the zitterbewegung is a single-valued function of the state coordinates x, y, z, v_x, v_x, v_x of the center of oscillation (up to a relativistic correction) as a consequence.

We assume that the center of oscillation returns to the same point of its orbit in state space infinitely often during its motion, or at least comes arbitrarily close to every point of its orbit (i.e., *periodic* vs. *quasi-periodic* motion, resp.).

Proof. We would like to take the broadest possible meaning for the quantum laws (II.4): The integral shall be an integral multiple of h for every closed path in state space. [One frees oneself from the assumption that there are separation variables for which the quantum constraints (II.4) are valid individually by means of this formulation. The path of integration does not need to be a path that actually goes through the center of oscillation in state space then in the case for which the equations of motion are presented in canonical form, which we do not need to assume.] However, the quantum law will then imply that:

$$\text{(II.5)} \qquad F(P) = e^{-2\pi i\,S/h} \quad \text{is single-valued} \qquad (\text{i.e., } S = \int_{P_0}^{P} dS\ );$$

i.e., F is single-valued in state space, when considered to be a function of the variable state P.

From formula (II.2), we will have:

$$\text{(II.6)} \qquad ds = dt - \frac{1}{2Mc^2}\,dS,$$

for the element of proper time ds of the center of oscillation, which is precise up to $(v / c)^4$, and in which dS is defined by (II.4). We will then get:

(II.7) $$\varphi - \varphi_0 = 2\pi \int_{P_0}^{P} \frac{ds}{\Theta} = \frac{4\pi Mc^2}{h}(t - t_0) - \frac{2\pi}{h}\int_{P_0}^{P} dS$$

for the difference between the phase angles at the points P_0 and P along the curve in state space when we integrate along that curve.

The phase will then be given by:

(II.8) $$e^{i(\varphi - \varphi_0)} = F(P)\; e^{\frac{4\pi i Mc^2}{h}(t - t_0)},$$

and our theorem will be proved by means of (II.5).

The theorem can be extended to several electrons with no further changes.

In the case of a lattice field, one must speak of passing to corresponding points in different cells, instead of returning to the same state. The coordinates that cover the respective cells will then enter into formula (II.7) (which will be equal for corresponding points).

The quantum law (II.4) connects with the purely kinematically-conceived single-valuedness demand, in addition to which, any dynamical equations must be assumed. If one takes NEWTONIAN dynamics, as well, then one will get the BOHR quantization process.

In order to come to (II.4), we see that it is necessary to introduce $2Mc^2 / h$ as the proper frequency of the electron. *However, from (I.10), that will yield the value* $h / 4\pi$ *for the angular momentum.*

The theory of relativity will appear for a second time as the fundamental prerequisite for quantum theory by means of formula (II.4). (The first fundamental quantum-mechanical consequence of the theory of relativity is zitterbewegung itself; its differential laws can be arrived at by using just the special theory of relativity and the theory of gravitation.) The relativistic distinction between proper time and system time comes into play in the quantum law (II.4). Using it, one can formulate the law that for each complete circuit the proper time will return to the system time after one, two, three, etc., proper periods. From (II.8), the phase that moves with the center of oscillation will take on values that belong to a *standing wave* in state space in the course of time. $F(P)$ will give the phase shift at the point P, when compared to an oscillation of all state space that happens with the frequency $2Mc^2 / h$.

Perhaps other quantum states will be realized physically for which the circuit of the center of oscillation is performed with the proper frequency in time, along with the one, two, …, n-quantum states.

Zitterbewegung experiences a strong damping. Namely, from (I.15), when one neglects q^2 in comparison to k^2, one will get the damping constant:

(II.9) $$\beta = \frac{q}{k^2} = \frac{8\pi}{3}\alpha\nu;$$

α and ν are the fine structure constant and the proper frequency, resp.:

$$\alpha = \frac{2\pi e^2}{hc} \equiv \frac{1}{137}, \qquad\qquad \nu = \frac{2Mc^2}{h}.$$

We assume that zitterbewegung will be continually stimulated by the surrounding radiation. The resonant frequency of the electron in a surrounding radiation field always exists: It is represented by an intensity that is, in general, insignificant, but sufficient for the stimulation of a small zitter-amplitude, which we can assume.

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