"Pseudogroupes de Lie transitif," Séminaire Bourbaki, May 1955.

Transitive Lie pseudogroups

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The object of this treatise is to present the fundamentals of E. CARTAN's theory of infinite transformation groups in terms of the ideas that were developed by EHRESMANN in his theory of jets.

1. Notations and terminology

The manifolds considered will be assumed to be differentiable of class C^∞, and verify the second countability axiom. For a Lie group G, this would signify that the number of connected components of G is denumerable.

Let $X = j_x^r f$ be an r-jet of a manifold V_n of dimension n in a manifold W_m of dimension m. The point x in V_n is the *source* of X and the point $f(x)$ in W_m is the *target* of X. The manifold $J^r(V_n, W_m)$ of all r-jets of V_n into W_m has three canonical projections: α, β, and γ, where $\alpha(X)$ is the source of X, $\beta(X)$ is the target of X, and $\gamma(X)$ is the pair $(\alpha(X), \beta(X))$. The r-jet $X = j_x^r f$ also determines a p-jet $\theta_x^p(X) = j_x^p f$, where $0 \leq p \leq r$. Let $X = j_x^r f$ be an r-jet of V_n into W_m, and let $Y = j_y^r g$ be an r-jet of V_n into U_p. If $\beta(X) = \alpha(Y)$ then one may define a jet $j_x^r g \cdot f$ of V_n into U_p that is independent of the choice of f and g such that $X = j_x^r f$ and $Y = j_y^r g$; one denotes that jet by $Y \cdot X$. An r-jet $X = j_x^r f$ of V_n into W_n with the same dimension is *invertible* if the map f is of maximum rank at the point x. The inverse of an invertible r-jet $X = j_x^r f$ is an r-jet $X^{-1} = j_y^r f^{-1}$, where $y = f(x)$. The set $\Pi^r(V_n)$ of all invertible r-jets of V_n into V_n is an open submanifold of $J^r(V_n, V_n)$. The set $L_n^r(V_n, x)$ of all jets $X \in \Pi^r(V_n)$ with source and target x is a connected Lie group. This group is isomorphic to $L_n^r(\mathbb{R}^n, 0)$ (which one denotes by L_n^r), where $\mathbb{R}^n$ is the numerical space of dimension n, and 0 is the origin of $\mathbb{R}^n$. The submanifold $\Pi^r(V_n)$ is a fiber bundle with base $V_n \times V_n$, projection γ, fiber $L_n^r \times L_n^r$, and group L_n^r. An invertible r-jet X of $\mathbb{R}^n$ into V_n with source 0 and target x is called an *r-frame* with origin at x in V_n. The set $H^r(V_n)$ of all r-frames in V_n is a principal fiber bundle of base V_n and structure group L_n^r. A local (differentiable) section of $H^r(V_n)$ that is defined over an open subset of V_n is called an *r-frame field.*

2. G-structures of order r

Let G be a Lie subgroup of L_n^r. One says that a manifold V_n is endowed with a *G-structure S of order r* if the following conditions are verified: There exists an open covering $\{U_\alpha\}$ of V_n and an r-frame field X^α that is defined on U_α such that $X^\beta(x) = X^\alpha(x) \cdot s^{\alpha\beta}(x)$ for $x \in U_\alpha \cap U_\beta$, where $s^{\alpha\beta}$ is a differentiable map of $U_\alpha \cap U_\beta$ into G. The set of pairs $\{(U_\alpha, X^\alpha)\}$ will be called an *atlas* for the structure S. Two atlases $\{(U_\alpha, X^\alpha)\}$ and $\{(V_\alpha, Y^\alpha)\}$ define the same structure if:

$$Y^\beta(x) = X^\alpha(x)\, t^{\alpha\beta}(x) \quad \text{for} \quad x = U_\alpha \cap U_\beta,$$

where $t^{\alpha\beta}$ is a differentiable map of $U_\alpha \cap U_\beta$ into G.

A subset Π of $\Pi^r(V_n)$ is a *sub-groupoid* of $\Pi^r(V_n)$ if it satisfies the following conditions:

1. If $X, Y \in \Pi$ and if the product $X \cdot Y$ is defined then $X \cdot Y \in \Pi$.
2. If $X \in \Pi$ then $X^{-1} \in \Pi$.

Let Ψ be a map of $\Pi \times \Pi$ into $V_n \times V_n$ such that:

$$\Psi(X, Y) = (\alpha(X), \beta(Y)),$$

and set $P(\Pi) = \Psi^{-1}(D)$, where D is the diagonal in $V_n \times V_n$. The product $X \cdot Y$ of two elements X and Y of P is defined if and only if $(X, Y) \in P(\Pi)$.

A *Lie sub-groupoid* of $\Pi^r(V_n)$ is a subset Π of $\Pi^r(V_n)$ that verifies the following conditions:

a. Π is a sub-groupoid of $\Pi^r(V_n)$.

b. Π is a submanifold of $\Pi^r(V_n)$, and the canonical projection γ of Π to $V_n \times V_n$ is onto and of maximal rank.

c. The map $(X, Y) \to X \cdot Y$ of $P(\Pi)$ into Π is differentiable. (By condition *b*, $P(\Pi)$ is a closed submanifold of $\Pi \times \Pi$).

d. The map $X \to X^{-1}$ of Π onto Π is differentiable.

We let $G(\Pi, x)$ denote the set of all elements X in a Lie sub-groupoid Π of $\Pi^r(V_n)$ that have source and target x. $G(\Pi, x)$ is a group that we shall call the *isotropy group* of Π at the point x. The isotropy groups of Π at different points are isomorphic.

PROPOSITION 2.1 – The group $G(\Pi, x)$ is a Lie sub-group of $L_n^r(V_n, x)$.

PROPOSITION 2.2 – A Lie sub-groupoid Π of $\Pi^r(V_n)$ is a fiber bundle with base $V_n \times V_n$, projection γ, fiber $G(\Pi, x_0)$, and structure group $G(\Pi, x_0) \times G(\Pi, x_0)$, where x_0 is a point of V_n, and the group $G(\Pi, x_0) \times G(\Pi, x_0)$ operates on $G(\Pi, x_0)$ in the following manner:

$$(\sigma, \tau) \cdot \xi = \tau \cdot \xi \cdot \sigma^{-1}.$$

Indeed, by condition *b*, there exists an open covering $\{U_\alpha\}$ of V_n and differentiable maps $\varphi_{\alpha,\beta}$ of $U_\alpha \times U_\beta$ into Π such that $\gamma(\varphi_{\alpha,\beta}(x, y)) = (x, y)$ for any $(x, y) \in U_\alpha \times U_\beta$. Suppose that $x_0 \in U_{x_0}$, and set $\varphi_\alpha(x) = \varphi_{\alpha_0,\alpha}(x_0, x)$ for any $x \in U_\alpha$. Let $\Psi_{\alpha,\beta}$ be a map of $U_\alpha \times U_\beta \times G(\Pi, x_0)$ into Π that is defined in the following fashion:

$$\Psi_{\alpha,\beta}(x, y, \sigma) = \varphi_\beta(x) \cdot \sigma \cdot \varphi_\alpha(x)^{-1}.$$

One easily verifies that the maps $\Psi_{\alpha,\beta}$ define the required fiber structure.

PROPOSITION 2.3 – With the same notations as in proposition 2.2, let Π_{x_0} be the set of all elements X of Π with source x_0. Π_{x_0} is a closed submanifold of Π and a principal fiber bundle with base V_n, structure group $G(\Pi, x_0)$, and projection β ($\beta(X)$ is the target of X).

Now let S be a G-structure of order r that is defined on a manifold V_n, and let (U_α, X^α) be an atlas of S. An r-frame X with origin x is called a *distinguished r-frame* of S if there exists an $\sigma \in G$ such that $X = X^\alpha(x) \cdot \sigma$, where $x \in U_\alpha$. This definition is independent of the choice of U_α such that $x \in U_\alpha$, and the choice of atlas. The set $H(S)$ of distinguished frames of S is a principal fiber bundle of base V_n and group G.

We let $\Pi(S)$ denote the set of all elements Y of $\Pi^r(V_n)$ that have the following property: If X is a distinguished frame of S with origin x (x being the source of Y) then $Y \cdot X$ is also a distinguished frame of S.

PROPOSITION 2.4. – $\Pi(S)$ is a Lie sub-groupoid of $\Pi^r(V_n)$.

Now, let Π be a Lie sub-groupoid of $\Pi^r(V_n)$, and let X_0 be a frame of origin x_0 in V_n. The map k of $L_n^r(V_n, x_0)$ onto L_n^r defined by $k(Y) = X_0^{-1} \cdot Y \cdot X_0$ is a differentiable isomorphism of $L_n^r(V_n, x_0)$ onto L_n^r. The Lie subgroup $G(\Pi, x_0)$ of $L_n^r(V_n, x_0)$ is mapped by k onto a Lie subgroup G of L_n^r.

PROPOSITION 2.5 – There exists a G-structure S such that $\Pi = \Pi(S)$ and X_0 is a distinguished frame of S.

With the same notations as in the proof of proposition 2.2., set $X^\alpha(x) = \varphi_\alpha(x) \cdot X_0$ for $x \in U_\alpha$. X^α is then an r-frame field on U_α, and the atlas $\{(U_\alpha, X^\alpha)\}$ defines a G-structure S such that $\Pi(S) = \Pi$. Moreover, one may assume that $\varphi_{\alpha_0}(x_0)$ = the neutral element of $G(\Pi, x_0)$. $X_0 = X^{\alpha_0}(x_0)$ is a distinguished frame in S.

3. Transitive Lie pseudogroups

Let V_n be a manifold, and let f be a bijective map of an open subset U of V_n onto an open subset V of V_n . f is a local automorphism of V_n with source U and target V if f and f^{-1} are differentiable.

A set Γ of local automorphisms of V_n is a *pseudogroup* if the following conditions are verified:

1. The identity map of V_n belongs to Γ.
2. If $f \in \Gamma$ then the inverse map f^{-1} belongs to Γ. If f and $g \in \Gamma$ then the composed map $f \cdot g$ belongs to Γ.
3. Let f be a local automorphism of V_n with source $U = \bigcup U_i$ such that the restrictions of f to U_i belong to Γ for each index i. One then has $f \in \Gamma$.

Γ is *transitive* on V_n if for any point (x, y) of $V_n \times V_n$ there exists an $f \in \Gamma$ such that $f(x) = y$.

We let $\Pi^r(\Gamma)$ denote the set of r-jets $j_x^r f$, where $f \in \Gamma$, and x ranges through the source of f. $\Pi^r(\Gamma)$ is a sub-groupoid of $\Pi^r(V_n)$. The set of elements of $\Pi^r(\Gamma)$ of source x will be denoted by $\Pi_x^r(\Gamma)$. The elements of $\Pi^r(\Gamma)$ with source and target x form a group $G^r(\Gamma, x)$ that we shall call the *isotropy group of order r* of Γ at the point x. A pseudogroup Γ is called *complete of order r* if the following condition is verified: If f is a local automorphism of V_n of source U such that $j_x^r f - \Pi^r(\Gamma)$ for any $x \in U$ then $f \in \Gamma$.

A *transitive Lie pseudogroup of order r* is a pseudogroup Γ that operates on V_n and verifies the following conditions:

1. Γ is complete of order r.
2. $\Pi^r(\Gamma)$ is a Lie sub-groupoid of $\Pi^r(V_n)$.

Since the canonical projection γ is a map of $\Pi^r(\Gamma)$ onto $V_n \times V_n$, Γ is transitive. The isotropy group of $\Pi^r(\Gamma)$ at the point x coincides with the isotropy group of r of Γ at the point x. $G^r(\Gamma, x)$ is a Lie subgroup of $L_n^r(V_n, x)$.

Now, let S be a G-structure of order r and let $\Gamma(S)$ be the set of all local automorphisms f of V_n that map the distinguished frames of S – i.e., for any point x of the source of f and for any distinguished frame X with origin x, $f^{(r)}(X) = j_x^r f \cdot X$ is a distinguished frame. $\Gamma(S)$ is a complete pseudogroup of order r that we shall call the *pseudogroup of local automorphisms of S*. The elements of $\Gamma(S)$ whose sources are in V_n form a group $G(S)$: the group of *automorphisms* of S. A local automorphism f of V_n belongs to $\Gamma(S)$ if and only if $j_x^r f \in \Pi(S)$ for any x in the source of f.

PROPOSITION 3.1 – Any transitive Lie pseudogroup of order r is the pseudogroup of local automorphisms of a G-structure of order r.

A G-structure S is called a *structure defined by a transitive Lie pseudogroup* Γ of order r if $\Pi(S) = \Pi^r(\Gamma)$. A G-structure S is *locally homogeneous* and *isotropic* if for any pair (X, Y) of distinguished frames of S there exists a $f \in \Gamma(S)$ such that $f^{(r)}(X) = Y$.

PROPOSITION 3.2 – S is locally homogeneous and isotropic if and only if it is defined by a transitive Lie pseudogroup. If S is locally homogeneous and isotropic then $\Gamma(S)$ is a transitive Lie pseudogroup.

EXAMPLE 1. – Let G be a Lie group, and let $V_n = G/H$ be a homogeneous space of G. Let Γ_G be the set of all local automorphisms f of V_n that enjoy the following property: The restriction to a sufficiently small neighborhood of any point of the source of f is a restriction of a transformation of G. Suppose that the following condition (p) is verified: If an element σ of G leaves invariant any point of a non-vacuous open subset of V_n then $\sigma = 1$. (If G is connected, and if G operates effectively on V_n then that condition is verified.) Γ_G is then a Lie pseudogroup of a certain order.

EXAMPLE 2. – Let V_{2n} be an analytic manifold that is endowed with an almost-complex structure S. S is locally homogeneous and isotropic if and only if it is a complex structure.

Two G-structures of order r are called *associated* ($S \sim T$) if $\Pi(S) = \Pi(T)$.

PROPOSITION 3.3 – If $S \sim T$ then $\Gamma(S) = \Gamma(T)$. If, conversely, $\Gamma(S) = \Gamma(T)$, and if S is locally homogeneous and isotropic then $S \sim T$. If S is locally homogeneous and isotropic and $S \sim T$ then T is also locally homogeneous and isotropic.

Now, let S be a G-structure of order r that is defined by an atlas $\{(U_\alpha, X^\alpha)\}$, and let σ be an element in the normalizer of $N(G)$ of G in L_n^r. The atlas $\{(U_\alpha, X^\alpha\sigma)\}$ then defines a G-structure $S\sigma$ (cf. [**1**]). $S\sigma = S\tau$ if and only if $\sigma\tau^{-1} \in G$.

PROPOSITION 3.4 – $S \sim T$ if and only there exists a $\sigma \in N(G)$ such that $T = S\sigma$.

A structure S_r of order r canonically defines a structure S_p of order p, $p < r$ such that $\Pi(S_p) = \theta_p^r(\Pi(S_r))$. In particular, if Γ is a Lie pseudogroup of order r then $\Pi^p(\Gamma) = \theta_p^r(\Pi^r(\Gamma))$ is a Lie sub-groupoid of $\Pi^p(V_n)$.

4. Prolongation of a structure of order 1 (cf. [3] and [5]).

A G-structure of order 1 will be defined by an atlas:

$$\{(U_\alpha;\ \theta_\alpha^1, \dots, \theta_\alpha^n)\},$$

where θ_α^i are linearly independent Pfaff forms on U_α such that:

$$\theta_\alpha^i(x) = a_j^i(S^{\alpha\beta}(x))\cdot\theta_\beta^j(x),$$

$\sigma \to (a_j^i(\sigma))$ being a matrix representation of L_n. Local charts $(U_\alpha, \varphi_\alpha)$ of $H(S)$ (the principal fiber bundle of S) are defined in the following fashion: φ_α is a map of $U_\alpha \times G$ onto $p^{-1}(U)$ (p being the projection of $H(S)$) such that $\varphi_\alpha(x, \sigma) = (X_1, \ldots, X_n)$, where $X_i = a_j^i(\sigma)\cdot X_j^\alpha(x)$, $(X_1^\alpha, \ldots, X_n^\alpha)$ being the frame field in U_α that is dual to (θ_α^i). Let η_α^i be Pfaff forms on $U_\alpha \times G$ such that $\eta_\alpha^i(x,\sigma) = a_j^i(\sigma^{-1})\theta_\alpha^j(x)$, and set $\omega_\alpha^i = \varphi_\alpha^{-1}(\eta_\alpha^1)$. One sees that $\omega_\alpha^i = \omega_\beta^i$ on $U_\alpha \cap U_\beta$, and one may thus define Pfaff forms ω^j on $H(S)$ that are defined globally by the conditions $\omega^j = \omega_\alpha^i(x)$ for $x \in U_\alpha$. ω^j are linearly independent.

Choose a basis $A_\rho = (a_{j\rho}^i)$ ($\rho = 1, \ldots, r$; $r = \dim G$) for the Lie algebra $L(G)$ of G, once and for all. On $p^{-1}(U_\alpha)$, one has:

$$\alpha\,\omega^j = a_{j\rho}^i\omega^j \wedge \pi_\alpha'^\rho + \tfrac{1}{2}c_{jk}'^i\omega^j \wedge \omega^k,$$

where $\pi_\alpha'^\rho$ and $c_{jk}'^i$ ($c_{jk}'^i = -\,c_{kj}'^i$) are Pfaff forms and functions on $p^{-1}(U_\alpha)$, respectively. Now, let ξ_i^ρ ($\rho = 1, \ldots, r$; $i = 1, \ldots, n$) be $r \cdot n$ independent variables, and let:

$$A_{jk}^i(\xi) = \sum_\rho (a_{j\rho}^i\xi_k^\rho - a_{k\rho}^i\xi_j^\rho)$$

be linear forms in ξ. Let A denote the indices $\begin{bmatrix} i \\ jk \end{bmatrix}$ ($i, j, k = 1, \ldots, n$; $j < k$). Let B be a subset of A that has the following property: The $A_{jk}^i(\xi)$ are such that $\begin{bmatrix} i \\ jk \end{bmatrix} \in B$ are linearly independent, and $A_{jk}^i(\xi)$, $\begin{bmatrix} i \\ jk \end{bmatrix} \in A - B$ is a linear combination of $A_{jk}^i(b(q)) = -\,c_{jk}'^i(q)$ for any $q \in p^{-1}(U_\alpha)$ and for any $\begin{bmatrix} i \\ jk \end{bmatrix} \in B$. Set $\pi = \pi_\alpha'^\rho + b_k^\rho\omega^k$. One then has:

$$d\omega^j = a_{j\rho}^i\omega^j \wedge \pi_\alpha^\rho + \tfrac{1}{2}c_{jk}^i\omega^j \wedge \omega^k,$$

where:

$$c_{jk}^i = c_{jk}'^i + A_{jk}^i(b).$$

Now, suppose that such a transformation has been associated with each α. One then verifies that the functions c_{jk}^i are functions that are defined globally on $H(S)$. CHERN [**5**] called these functions c_{jk}^i the *structure invariants of S.* The structure S is called *integrable* if one may choose a subset B of A in such a manner that the invariants c^i_{jk} are constants. The integrability condition depends only upon the structure S.

PROPOSITION 4.1 – A G-structure of order 1 that is locally homogeneous and isotropic is integrable.

Now, let $G^{(1)}$ denote the group of all matrices of the form:

$$\begin{pmatrix} I_n & 0 \\ B & I_n \end{pmatrix}, \qquad B = (b_i^\rho), \qquad (r = 1, \ldots, r;\ i = 1, \ldots, n),$$

where:

$$A^i_{jk}(b) = \sum_\rho (a^i_{j\rho} b^\rho_k - a^i_{k\rho} b^\rho_j) = 0.$$

The atlas $\{(p^{-1}(U_\alpha);\ \omega^j,\ \pi^\rho_\alpha)\}$ then defines a $G^{(1)}$-structure $\tilde{S}$ on $H(S)$. $\tilde{S}$ depends upon the choice of B. However, one may prove the following proposition:

PROPOSITION 4.2 – Suppose that S is integrable, and let T be a $G^{(1)}$-structure on $H(S)$ that is defined by an atlas:

$$(W_\beta, , \bar{\omega}^1, \ldots, \bar{\omega}^n, \bar{\pi}^1_\beta, \ldots, \bar{\pi}^r_\beta),$$

where $\bar{\omega}^i = s^i_j \omega^j$, s^i_j are constants.

Suppose that in W_β one has:

$$d\omega^j = a^i_{j\rho}\bar{\omega}^j \wedge \bar{\pi}^\rho_\beta + \tfrac{1}{2}\bar{c}^i_{jk}\bar{\omega}^j \wedge \bar{\omega}^k,$$

where $\bar{c}^i_{jk} = -\bar{c}^i_{kj}$ are constants.

The two structures T and $\tilde{S}$ are then associated.

One says that a $G^{(1)}$-structure T that verifies the conditions above is a *prolongation* of S with respect to a basis $A_\rho = (a^i_{j\rho})$ of $L(G)$, and that the constants $a^i_{j\rho}$ and $\bar{c}^i_{jk}$ that are associated with T form a system of *structure constants for the integrable structure* S. A prolongation of S is a prolongation of S with respect to a basis for $L(G)$. Two prolongations with respect to the same basis are associated.

Now, let S and $S' = S\tau$ be two associated integrable G-structures, where $\tau \in N(G)$. Let $(X'_1, \ldots, X'_n)$ be a distinguished frame of S', and let $X_i = a^j_i(\gamma) X'_j$. $(X_1, \ldots, X_n)$ is then a distinguished frame of S and the map $\alpha((X'_i)) = (X_i)$ is an isomorphism of the manifold $H(S')$ onto the manifold $H(S)$. Let T be a prolongation of S, and let $\alpha^*(T)$ be the reciprocal image of T. It then results that if S and S' are two associated integrable G-

structures then the families of systems of structure constants of S and S' that one may identify by the prolongations of S and S' are the same.

Now, let Γ be a transitive Lie pseudogroup of order 1 that operates on a manifold V_n, and let S be a G-structure of order 1 that is defined by Γ. S is locally homogeneous and isotropic, and therefore integrable. The family of systems of *structure constants of* Γ is, by definition, the family of systems of structure constants of the structure S. Since two G-structures that are defined by Γ are associated, the definition is independent of the choice of S.

Let $a^i_{\rho j}$, c^i_{jk} ($i, j, k = 1, \ldots, n$; $\rho = 1, \ldots, r$) be a system of structure constants of Γ. One sees that the following equations, which can be regarded as linear equations with respect to the unknowns $\gamma^\rho_{\sigma\tau}$, $u^\rho_{i\sigma}$, v^ρ_{ij} ($\gamma^\rho_{\sigma\tau} = -\gamma^\rho_{\tau\sigma}$, $v^\rho_{ij} = -v^\rho_{ji}$), are compatible:

(1) $$a^i_{j\tau}a^j_{k\sigma} - a^i_{j\sigma}a^j_{k\tau} = \gamma^\rho_{\tau\sigma}a^i_{k\rho},$$

(2) $$a^i_{kt}a^k_{s\rho} - c^i_{ks}a^k_{t\rho} + a^i_{k\rho}a^k_{ts} - a^i_{t\sigma}u^\sigma_{s\rho} + a^i_{s\sigma}u^\sigma_{t\rho} = 0,$$

(3) $$c^i_{ku}c^k_{st} + c^i_{ks}c^k_{tu} + c^i_{kt}c^k_{us} - a^i_{u\rho}v^\rho_{st} - a^i_{s\rho}v^\rho_{tu} - a^i_{t\rho}v^\rho_{us} = 0.$$

Equations (1) are equivalent to the equations:

$$[A_\tau, A_\sigma] = \gamma^\rho_{\tau\sigma}A_\rho .$$

It then results that $\gamma^\rho_{\tau\sigma}$ are the structure constants of the isotropy group of order 1 of Γ.

Now, let S^r be a G-structure of order r. Any $f \in \Gamma(S^r)$ may be prolonged to a local automorphism $f^{(r)}$ of $H(S^r)$ with source $p^{-1}(U)$ and target $p^{-1}(V)$, where U and V are the source and the target of f, respectively, and p is the projection of $H(S^r)$. Let Δ be a sub-pseudogroup of $\Gamma(S^r)$. We let $\Delta^{(r)}$ denote the set of all local automorphisms $\tilde{f}$ of $H(S^r)$ that enjoy the following property: The restriction to a sufficiently small neighborhood of any point of the source of $\tilde{f}$ is a restriction of an $f^{(r)}$, where $f \in \Delta$. $\Delta^{(r)}$ is a pseudogroup that we call the *prolongation* of Δ to $H(S^r)$. One may prove the following proposition:

PROPOSITION 4.3 – Let S be an integrable G-structure, and let T be a prolongation of S. One then has $\Gamma(T) = \Gamma(S)^{(1)}$. A local automorphism of $H(S)$ belongs to $\Gamma(S)^{(1)}$ if and only if it leaves invariant the Pfaff forms:

$$\omega^1, \ldots, \omega^n.$$

5. Reduction of a Lie pseudogroup of order $r > 1$ to a Lie pseudogroup of order 1.

Let Γ be a transitive Lie pseudogroup of order $r > 1$ that operates on a manifold V_n. One sees that Γ canonically determines a transitive Lie pseudogroup of order 1 that operates on the manifold $\Pi^{r-1}(\Gamma)$.

Let S^r be a G-structure of order r that is defined by Γ, let z be a point of V_n that is chosen once and for all, and let Z be a distinguished frame of S^r with its origin at z. The map $\sigma \to Z\,\sigma Z^{-1}$ of G into $L_n^r(V_n, z)$ is an isomorphism of G onto $G^r(\Gamma, z)$, the isotropy group of order r of Γ at the point z. Let $\{(U_\alpha, X^\alpha)\}$ be an atlas of S^r, and set $X'^\alpha(x) = X^\alpha(x) \cdot Z^{-1}$. One then has $X'^\alpha(x) \in \Pi_z^r(\Gamma)$ and $X'^\alpha(x) = X'^\beta(x) \cdot s'^{\alpha\beta}(x)$ for any $x \in U_\alpha \cap U_\beta$, where $s'^{\alpha\beta}$ is a differentiable map of $U_\alpha \cap U_\beta$ into $G^r(\Gamma, z)$. The atlas $\{(U_\alpha, X'^\alpha)\}$ thus defines a "$G^r(\Gamma, z)$-structure S'^r," and the principal fiber bundle of distinguished frames of S'^r is $\Pi_z^r(\Gamma)$. One sees, moreover, that $\Pi(S'^r) = \Pi(S^r)$ and that $\Gamma(S') = \Gamma(S) = \Gamma$.

We then, in turn, identify the structure S^r with S'^r.

The structure S^r determines a $G^r(\Gamma, z)$-structure S^p for any $p \leq r$ such that $\Pi(S^p) = \Pi^r(S)$ and that $H(S^p) = \Pi_z^p(\Gamma)$. Set $\Gamma^p = \Gamma(S^p)$. $\Pi^p(\Gamma_p) = \Pi^p(\Gamma)$ is then a Lie subgroupoid of $\Pi^p(V_n)$, and therefore Γ_p is a transitive Lie pseudogroup of order p that contains Γ. One sees that $\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_r = \Gamma$. One may thus define the prolongation $\Gamma_\tau^{(p)}$ of Γ_τ ($r \geq q \geq p$) to $H(S^p) = \Pi_z^p(\Gamma)$. $\Gamma_\tau^{(p)}$ is transitive. One says that $\Gamma^{(p)} = \Gamma_r^{(p)}$ is the *normal prolongation of order p of* Γ. We shall show that $\Gamma^{(r-1)}$ is a transitive Lie pseudogroup of order 1. Now, suppose that $p < r$, and set $Z_p = j_z^p I$, where I is the identity map of V_n. Let f and g be elements of Γ_{p+1} that are defined at the point z. If one utilizes the local coordinates of the manifold of jets then one verifies that $j_{Z_p}^1 f^{(p)} = j_{Z_p}^1 g^{(p)}$ if and only if $j_z^{p+1} f = j_z^{p+1} g$ (f and g being the prolongations of f and g to $H(S^p) = \Pi_z^p(\Gamma)$). One may thus define a map F_{p+1} of $\Pi_z^{p+1}(\Gamma_{p+1})$ ($= \Pi_z^{p+1}(\Gamma)$) in the manifold $\Pi_{Z_p}^1(\Pi_z^p(\Gamma))$ by $F_{p+1}(j_z^{p+1} f) = j_{Z_p}^1 f^{(p)}$. F_{p+1} is differentiable of maximal rank, and $F_{p+1}(\Pi_z^{p+1}(\Gamma)) = \Pi_{Z_p}^1(\Gamma^{(p)})$. Let $N^{p+1}(\Gamma, z)$ be the kernel of the canonical homomorphism of $G^{p+1}(\Gamma, z)$ onto $G^p(\Gamma, z)$. $\Pi_z^{p+1}(\Gamma)$ is then a principal fiber bundle with base $\Pi_z^p(\Gamma)$, projection θ_p^{p+1}, and group $N^{p+1}(\Gamma, z)$, and the following diagram is commutative:

$$
\begin{array}{ccc}
\Pi_z^{p+1}(\Gamma) & \xrightarrow{F_{p+1}} & \Pi_{Z_p}^1(\Gamma^{(p)}) \\
\Big\downarrow \theta_p^{p+1} & & \Big\downarrow \beta \\
\Pi_z^p(\Gamma) & \xrightarrow{\text{Identity}} & \Pi_z^p(\Gamma)
\end{array}
$$

Utilizing these facts, one may show that $\Pi^1(\Gamma^{(p)})$ is a Lie sub-groupoid of $\Pi^1(\Pi_z^p(\Gamma))$. The isotropy group of $\Pi^1(\Gamma^{(p)})$ at the point Z_p is $G^1(\Gamma^{(p)}, Z_p)$ and $\Pi^1(\Gamma^{(p)})$ thus defines a $G^1(\Gamma^{(p)}, Z_p)$-structure T_p of order 1 in the manifold $\Pi_z^p(\Gamma)$, which is locally homogeneous and isotropic, and $H(T_p) = \Pi_z^1(\Gamma^{(p)})$.

One may prove the following propositions:

1. $\Gamma(T_p) = \Gamma^p_{p+1}$ for $p < r$.
2. $F_{p+1} \cdot \Gamma(T_p)^{(1)} \cdot F^{(p+1)}_{p+1}$ for $p < r$,

where $\Gamma(T_p)^{(1)}$ is the prolongation of $\Gamma(T_p)$ to $H(T_p) = \Pi^1_{Z_p}(\Gamma^{(p)})$ and $F_{p+1} \cdot \Gamma(T_p)^{(1)} \cdot F^{(p+1)}_{p+1}$ is the pseudogroup of all local automorphisms h of $\Pi^{p+1}_z(\Gamma)$ of the form $h = F_{p+1} \cdot f \cdot F^{(p+1)}_{p+1}$, where $\tilde{f} \in \Gamma(T_p)^{(1)}$. For $p = r - 1$, one has $\Gamma_{p+1} = \Gamma$, and one thus obtains the following theorem:

THEOREM 5.1. – Let Γ be a transitive Lie peudogroup of order r that operates on a manifold V_n . The normal prolongation $\Gamma^{(r-1)}$ of order $r - 1$ of Γ is a transitive Lie pseudogroup of order 1 that operates on the manifold $\Pi^{r-1}_z(\Gamma)$, z being a point of V_n, which leaves invariant a G-structure T_{r-1} of order 1. The prolongation of $\Gamma^{(r-1)}$ to $H(T^{r-1})$ may be identified with $\Gamma^{(r)}$.

The family of systems of *structure constants of* Γ is, by definition, the corresponding family of the Lie pseudogroup $\Gamma^{(r-1)}$ of order 1.

The following theorem results from theorem 5.1 and proposition 4.3:

THEOREM 5.2 – The notations being those of theorem 5.1, the normal prolongation $\Gamma^{(r)}$ of order r of G is a pseudogroup of local automorphisms of a G-structure T_r of order 1 in the manifold $\Pi^r_z(\Gamma)$. T_r is defined by an atlas $\{(V_\alpha, \omega^1, \dots, \omega^m, \pi^1_\alpha, \dots, \pi^r_\alpha)\}$, where:

$$m = \dim \Pi^{r-1}_z(\Gamma),$$
$$r = \dim G^1(\Gamma^{(r-1)}, Z_{r-1}),$$

and:

$$d\omega^j = a^i_{j\rho}\omega^j \wedge \pi^\rho_\alpha + \tfrac{1}{2} c^i_{jk}\omega^j \wedge \omega^k .$$

$a^i_{j\rho}$, c^i_{jk} are the structure constants of Γ. A local automorphism of $\Pi^r_z(\Gamma)$ belongs to Γ if and only if it leaves the forms ω^j invariant.

REMARK. – One does not know, in general, if $\Gamma^{(r)}$ is a transitive Lie pseudogroup of order 1. This would be true if Γ is also a Lie pseudogroup of order $r + 1$ or if Γ has finite type or is involutive (see paragraphs 6 and 7).

6. Lie pseudogroups of finite type.

A pseudogroup Γ is a transitive Lie pseudogroup of *finite type* if there exists an integer $r \geq 1$ that enjoys the following properties:

1. Γ is a transitive Lie pseudogroup of order r.

2. There exists an integer $s \leq r$ such that dim $G^s(\Gamma, x)$ = dim $G^{s-1}(\Gamma, x)$. (We set $G^0(\Gamma, x)$ = dim (1).)

If Γ is of finite type then the canonical homomorphism of $G^s(\Gamma, x)$ into $G^{s-1}(\Gamma, x)$ is a local isomorphism. It results from this that the canonical map of $\Pi^s(\Gamma)$ onto $\Pi^{s-1}(\Gamma)$ is locally bijective. If one utilizes local coordinates on the manifold of jets then one can prove the following lemma:

LEMMA. – If $\Pi^s(\Gamma) \to \Pi^{s-1}(\Gamma)$ is locally bijective then $\Pi^s(\Gamma) \to \Pi^{s-1}(\Gamma)$ is bijective for $p > s$.

Now, let Γ be a pseudogroup that operates on V_n . One says that Γ is *simply transitive* if it is transitive and if an element *f* of Γ leaves invariant a point *x*, so the restriction of *f* to a neighborhood *U* of *x* is the identity map of *U*.

Let Γ be a Lie pseudogroup of finite type and order *r*. It results from the lemma that $\Pi_z^r(\Gamma)$ = dim $\Pi_z^{r-1}(\Gamma)$, and therefore that:

$$\dim \Pi^1(\Gamma^{(r-1)}, Z_{r-1}) \dim \Pi_z^r(\Gamma) - \dim \Pi_z^{r-1}(\Gamma) = 0.$$

The structure T_r in theorem 5.2 is thus defined by an atlas:

$$\{(U_\alpha, \omega^1, \ldots, \omega^n)\},$$

and $d\omega^j = \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k$. The following theorem then results:

THEOREM 6.1 – Let Γ be a Lie pseudogroup of finite type and order *r*. The normal prolongation $\Gamma^{(r)}$ of order *r* of Γ is then a simply transitive Lie pseudogroup of order 1. A local automorphism of the manifold $\Pi_z^r(\Gamma)$ belongs to $\Gamma^{(r)}$ if and only if it leaves invariant some linearly independent Pfaff forms $\omega^1, \ldots, \omega^n$, which are such that:

$$d\omega^j = \frac{1}{2} c^i_{jk} \omega^j \wedge \omega^k ,$$

where m = dim $\Pi_z^r(\Gamma)$, and c^i_{jk} are the structure constants of Γ.

EXAMPLE 3. – Let Γ_G be the pseudogroup in example 1. Γ_G is then a Lie pseudogroup of finite type and a certain order *r*. The number *r* is determined in the following manner: Let 0 be the point of $V_n = G/H$ that is represented by *H*. The map $\sigma \to j_0^p \sigma$, ($s \in H$), is a homomorphism of *H* onto $G^p(\Gamma_G, 0)$. By the condition (*p*) (see example 1), there exists an integer *p* such that dim *H* = dim $G^p(\Gamma_G, 0)$. The number $r - 1$ is then the smallest number *p* that enjoys that property. There exists an isomorphism θ of the manifold *G* onto the manifold $\Pi_0^r(\Gamma_G)$ such that $\theta^*(\omega^j)$ are left-invariant Pfaff forms on *G*, where ω^j are the forms in theorem 5.1.

7. Involutive Lie pseudogroups

For the definition of an involutive linear Lie algebra, see [**5**] and [**9**]. A linear Lie group is *involutive* if its Lie algebra is involutive and a G-structure of order 1 is *involutive* if G is involutive. A Lie pseudogroup of order 1 is *involutive* if its isotropy group of order 1 is involutive.

By using the theorem of the existence of solutions of systems in involution one proves:

PROPOSITION 7.1 – Let S be an involutive G-structure on an *analytic* manifold. If S is integrable then it is locally homogeneous and isotropic, and its prolongations are also integrable and involutive.

A transitive Lie pseudogroup of order r is involutive if its normal prolongation of order r- 1 is involutive. The following theorem results from theorem 5.2 and the proposition 7.1:

THEOREM 7.1 – If Γ is a transitive Lie pseudogroup of order r and involutive then the prolongation of order r of G is a transitive, involutive pseudogroup of order 1.

Now, let $\{a^i_{j\rho}, c^i_{jk}\}$ (i, j, k = 1, …, ; r = 1, …, r; $c^i_{jk} = -c^i_{kj}$) be a system of constants such that the linear equations (1), (2), and (3) of paragraph 4 are compatible. Suppose, moreover, that the Lie algebra L generated by the matrices $A\rho = (a^i_{j\rho})$ are involutive, and that G is the linear Lie group that is generated by L.

THEOREM 7.2 – Let $\{a^i_{j\rho}, c^i_{jk}\}$ be a system of constants that verify the conditions above. There then exists an integrable G-structure that is defined in a domain D of the numerical space $\mathbb{R}^n$ such that $\{a^i_{j\rho}, c^i_{jk}\}$ is a system of structure constants for that structure. In other words, there exists a transitive Lie pseudogroup of order 1 that operates on D such that $\{a^i_{j\rho}, c^i_{jk}\}$ is a system of structure constants of that pseudogroup (see [**2**]).

PROBLEMS:

1. One easily sees that a Lie pseudogroup Γ of order r is complete of order p for any $p > r$. Is Γ a Lie pseudogroup of order p for any $p > r$?

2. Is any transitive, analytic Lie pseudogroup involutive or of finite type?

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