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On some new oscillatory solutions of the equations of gravitation

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1. Introduction. – We know that in the gravitational field of relativity (i.e., the one that is defined by the equations $R_{\mu\nu} = 0$), the waves that are called "gravitational" propagate with the speed of light. We shall now see in the present paper that if one imposes an oscillation of very high frequency upon the potentials $g_{\mu\nu}$ in that same continuum then the result will be a wave of Maxwellian type. In other words, the elements of the curvature $R_{\alpha\beta\mu\nu}$ verify certain equations, namely, $R^{\alpha}{}_{\mu\beta\nu;\alpha} = 0$, which we shall deduce directly from $R_{\mu\nu} = 0$ and which reduce to Maxwell equations in an approximation that is based precisely upon the high value of the frequency. One will then find that the wave is Maxwellian in the first approximation, even if the curvature of the medium of propagation takes on a finite value, and for oscillations whose amplitude can be as small as one pleases. We emphasize that point because in the case of the classical result that is concerned with gravitational waves, the principal part of the curvature is due precisely to the wave, and it propagates in a medium that is almost Euclidian.

In view of obtaining the stated result, it would seem essential to separate space and time is such a way that we adopt the classical form for the element of arc length:

$$ds^2 = c^2\, dt^2 - d\sigma^2, \tag{1.1}$$

with

$$d\sigma^2 = a_{rs}\, dr^r\, dx^s \qquad (r, s = 1, 2, 3)$$

in which the functions a_{rs} and c depend upon four variables x^r and t (t stands for x^0). We will then see how it is possible to replace the equations $R_{\mu\nu} = 0$ with a system that is invariant with respect to the spatial variables x^r and in which time figures like a parameter that no longer has a tensorial character. That system will then provide some new equations in which the elements of curvature $R_{\alpha\beta\mu\nu}$ will be grouped into two tensors E_{rs} and H_{rs} that are tri-dimensional (r, s = 1, 2, 3) and have order two, in the first approximation, and from the standpoint of the structure of the equations, they play the same role that the vectors **E** and **H** do in Maxwell's equations.

2. Notation. – We shall encounter some equations that are invariant in space-time, as well as some other ones that are invariant only in spatial sections. In order to distinguish them, we agree that when Greek letters are employed as indices, they must take on the four values 0, 1, 2, 3, while Latin characters will be limited to only the values 1, 2, 3.

Furthermore, any tensorial symbol that is given an overbar, such as:

$$\bar{R}_{\mu\nu}\,,\ \bar{R}_{\alpha\beta\mu\nu}\,,\ \bar{\Gamma}_{\alpha\beta\nu}$$

for example, will represent a four-dimensional system in space-time, while the bar will be omitted for the systems that correspond to spatial sections. In that way, it will be possible to distinguish between a system in three-dimensional space such as $R_{u\upsilon rs}$, and another three-dimensional system, such as $\bar{R}_{u\upsilon rs}$, which will represent one subset of the components of the tensor $\bar{R}_{\alpha\beta\mu\nu}$.

Meanwhile, we shall employ two different letters to represent the metric tensor: namely, $g_{\mu\nu}$ for the space-time, and a_{rs} for the spatial sections. That means that one can always choose the coordinates to make $g_{0i} = 0$ in such a way that one will have the element of arc length (1.1), in which:

$$g_{00} = c^2, \qquad a_{rs} = -\,g_{rs}\,. \tag{2.1}$$

The system a_{rs} will then determine the metric of a whole family of spatial sections that are parameterized by time t, and the three-dimensional space thus-determined will often be represented by the symbol E (3).

An easy calculation gives:

$$\begin{gathered} g = -\,c^2\,a, \qquad a = \|\,a_{rs}\,\|, \\ g^{00} = c^{-2}, \qquad g^{00} = 0, \qquad g^{rs} = -\,a^{rs}. \end{gathered} \tag{2.1a}$$

We write:

$$A_{rs} = \frac{1}{2c}\frac{\partial a_{rs}}{\partial t}\,. \tag{2.2}$$

The system thus-defined is a tensor in E (3), since on the one hand, the elements $\partial x^i / \partial x'^k$ of a transformation do not depend upon time, and the operator $\partial\ /\ \partial t$ will permute with them, on the other. Meanwhile, it is important to note that the corresponding relation that is written in terms of the contravariant components will define a tensor of contrary sign. Essentially, if one derives the two sides of the identity:

$$a^{rs} = a_{mn}\,a^{mr}\,a^{ns}$$

then one will get:

$$-\frac{\partial a^{rs}}{\partial t} = a^{mr}\,a^{ns}\,\frac{\partial a_{mn}}{\partial t}\,;$$

i.e.:

$$A^{rs} = -\frac{1}{2c}\frac{\partial a^{rs}}{\partial t}\,. \tag{2.2a}$$

That tensor is obviously symmetric.

In $g_{00} = c^2$, the function $c\ (x^r,\ t)$ is an invariant of E (3), and its gradient can present itself as a vector in our invariant three-dimensional equations. We adopt the notation:

$$\frac{\partial c}{\partial x^n} = c_n\ , \qquad \frac{\partial c}{\partial t} = c_0\ .$$

As far as the Christoffel symbols are concerned, we adopt the notation:

$$\overline{\Gamma}_{\alpha\beta\nu} = \frac{1}{2}\left(\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right), \qquad \overline{\Gamma}_{\alpha\beta}{}^{\mu} = g^{\mu\nu}\,\overline{\Gamma}_{\alpha\beta\nu}\ .$$

With the aid of $g_{rs} = -\ a_{rs}$ in (2.1), we will now immediately have: $\overline{\Gamma}_{rst} = -\Gamma_{rst}$. Moreover:

$$\overline{\Gamma}_{000} = \frac{1}{2}\frac{\partial g_{00}}{\partial t} = c\ c_0\ ,$$

and in the same way, it is easy to verify the following results:

$$\overline{\Gamma}_{000} = \ c\ c_0\ , \qquad \overline{\Gamma}_{000} = \ \frac{1}{2}\frac{\partial a_{rs}}{\partial t}\ ,$$

$$\overline{\Gamma}_{00r} = -\ c\ c_r\ , \qquad \overline{\Gamma}_{r0s} = -\frac{1}{2}\frac{\partial a_{rs}}{\partial t}\ ,$$

$$\overline{\Gamma}_{r00} = \ c\ c_r\ , \qquad \overline{\Gamma}_{rst} = -\ \Gamma_{rst}\ ,$$

$$(2.3) \qquad \left\{ \begin{array}{ll} \overline{\Gamma}_{00}{}^{0} = \dfrac{c_0}{c}, & \overline{\Gamma}_{rs}{}^{0} = \dfrac{1}{c}A_{rs}, \\ \overline{\Gamma}_{00}{}^{r} = a^{rs} c\, c_s, & \overline{\Gamma}_{r0}{}^{t} = c\, A_r^t, \\ \overline{\Gamma}_{r0}{}^{0} = \dfrac{c_r}{c}, & \overline{\Gamma}_{rs}{}^{t} = \Gamma_{rs}{}^{t}. \end{array} \right.$$

3. The equations $\overline{R}_{\mu\nu} = 0$ in a form that is invariant in E (3). – Consider the system $\overline{R}_{urs\upsilon}$, which represents only one subset of the components of the tensor $\overline{R}_{\alpha\beta\mu\nu}$; we shall see that we will have in it a tensor in E (3) as long as nothing changes in the separation of space and time. That amounts to saying that we consider coordinate transformations of the type:

$$x'^0 = x^0 = t, \qquad \frac{\partial x^0}{\partial x'^0} = 1, \qquad \frac{\partial x^0}{\partial x'^r} = \frac{\partial x^r}{\partial x'^0} = 0.$$

The relation:

$$\bar{R}'_{urs\upsilon} = \bar{R}_{\alpha\beta\mu\nu}\frac{\partial x^{\alpha}}{\partial x'^{u}}\frac{\partial x^{\beta}}{\partial x'^{r}}\frac{\partial x^{\mu}}{\partial x'^{s}}\frac{\partial x^{\nu}}{\partial x'^{\upsilon}}$$

will then immediately give (since no significant term will be obtained when the dummy indices take the value 0):

$$\bar{R}'_{urs\upsilon} = \bar{R}_{abmn}\frac{\partial x^{a}}{\partial x'^{u}}\frac{\partial x^{b}}{\partial x'^{r}}\frac{\partial x^{m}}{\partial x'^{s}}\frac{\partial x^{n}}{\partial x'^{\upsilon}},$$

which proves our assertion.

The other components of $\bar{R}_{\alpha\beta\mu\nu}$ once more give two of those three-dimensional tensors whose order is equal to the number of indices that have not been replaced by the value 0. The verification of the tensorial character proceeds as before, and we will have, in the first place:

$$\bar{R}'_{r00s} = \bar{R}_{\alpha\beta\mu\nu}\frac{\partial x^{\alpha}}{\partial x'^{r}}\frac{\partial x^{\beta}}{\partial x'^{0}}\frac{\partial x^{\mu}}{\partial x'^{0}}\frac{\partial x^{\nu}}{\partial x'^{s}},$$

in which the sums give non-zero terms only for $\beta = \mu = 0$, in such a way that:

$$\bar{R}'_{r00s} = \bar{R}_{a00n}\frac{\partial x^{a}}{\partial x'^{r}}\frac{\partial x^{n}}{\partial x'^{s}}.$$

One finally shows that the system R_{r0mn} is a third-order tensor in E (3) in exactly the same way.

Note that one can always make use of the symmetry properties of $R_{\alpha\beta\mu\nu}$, which will give, for example:

$$\bar{R}_{r0mn} = -\bar{R}_{0rmn} = \bar{R}_{mnr0}.$$

As a consequence, all of the elements of the tensor $\bar{R}_{\alpha\beta\mu\nu}$ will then be found to be grouped into three three-dimensional tensors in space E (3). The passage to the contravariant components will be illustrated in the subsequent calculations.

We shall employ the same technique to separate the derivatives $\bar{R}_{\alpha\beta\mu\nu,\sigma}$ into a certain number of three-dimensional tensors, such as $\bar{R}_{urs\upsilon,\bar{t}}$, for example. Here, the last index is given an overbar in order to specify that it amounts to covariant differentiation with respect to space-time. The reader will have no trouble assuring himself that our argument in regard to the coordinate transformation in E (3) is valid again in order to establish the tensorial character of the five indices. Of course, that new tensor will be distinct from $\bar{R}_{urs\upsilon,t}$, which is obtained by derivation in E (3), and we shall see later on how one can develop the first one in such a fashion that everything will come down to derivation in the latter space.

We first address the separation of the equations $\bar{R}_{\mu\nu} = 0$ that we announced. One subset of the system can be written:

$$\overline{R}_{rs} = g^{\alpha\beta}\overline{R}_{\alpha rs\beta} = 0,$$

or, with the aid of (2.1a):

(3.1) $$-\ a^{uv}\overline{R}_{ursv} + c^{-2}\overline{R}_{0rs0} = 0.$$

As for the rest, one will have the equations:

$$\overline{R}_{0n} = g^{\alpha\beta}\overline{R}_{\alpha 0n\beta} = 0,$$

$$\overline{R}_{00} = g^{\alpha\beta}\overline{R}_{\alpha 00\beta} = 0,$$

which immediately gives:

(3.2) $$a^{uv}\overline{R}_{u0nv} = 0,$$

(3.3) $$a^{uv}\overline{R}_{u00v} = 0.$$

These equations represent a form for the system $\overline{R}_{\mu\nu} = 0$ that is invariant in E (3). One can write them as functions of the tensors a_{rs} and A_{rs} , but that would not be necessary for us to achieve the objective that we are pursuing.

4. Symmetry of the new tensors. – We utilize the tensor:

$$\varepsilon^{rst} = a^{-1/2}\, e^{rst}, \qquad \varepsilon_{rst} = a^{1/2}\, e_{rst}\,,$$

in which $e_{rst} = e^{rst}$ takes the following values:

0	if any two of the indices *rst* have the same value,
1	if *rst* represents an even permutation of the numbers 1, 2, 3,
– 1	if *rst* represents an odd permutation of the same numbers.

We would like to emphasize the fact that we have in this two types of coordinates for the same tensor, since the two systems coincide in normal coordinates (a = 1). We then intend that we can pass from one to the other indifferently when certain indices in our tensorial equations are raised or lowered.

Moreover, we have:

(4.1) $$\varepsilon_{uvr}\,\varepsilon^{uvs} = 2\,\delta_r^{\ s},$$

in which $\delta_r^{\ s}$ represents the identity matrix. The verification is easy, and it will result that the latter system is a mixed tensor.

Another mixed tensor that will be very useful is introduced as follows:

(4.2) $$\delta_{uv}^{\ \ rs} = \varepsilon_{uvr}\,\varepsilon^{uvs}\,.$$

One easily proves that the values of the components are:

$$\begin{array}{lllll} 1 & \text{if } u \neq \upsilon, & r = u, & s = \upsilon, \\ -1 & \text{if } u \neq \upsilon, & r = \upsilon, & s = u, \\ 0 & \text{in other cases.} \end{array}$$

We then set:

(4.3) $$4\,E^{rs} = \bar{R}_{u\upsilon mn}\varepsilon^{u\upsilon r}\varepsilon^{mns},$$

(4.4) $$2c\,H_r{}^s = \bar{R}_{r0u\upsilon}\varepsilon^{u\upsilon s}.$$

Since $\bar{R}_{u\upsilon mn} = \bar{R}_{mnw\upsilon}$, we have $E^{rs} = E^{sr}$, and of course that will imply the symmetry of the covariant components:

(4.5) $$E_{rs} = E_{sr}.$$

Equations (4.3) can be solved for the components $\bar{R}_{w\upsilon mn}$: Multiplying the two sides by $\varepsilon_{rab}\ \varepsilon_{xys}$ and making use of (4.2), one will have:

$$4\,E^{rs}\,\varepsilon_{rab}\,\varepsilon_{xys} = \bar{R}_{u\upsilon mn}\delta_{ab}^{u\upsilon}\delta_{xy}^{mn},$$

and since the only values that are used for the indices ($u\upsilon$) are (ab) and (ba), with the corresponding values $\delta_{ab}{}^{ab} = 1$ and $\delta_{ab}{}^{ba} = -1$, one will get:

$$\begin{aligned} 4E^{rs}\,\varepsilon_{abr}\,\varepsilon_{xys} &= (\bar{R}_{abmn} - \bar{R}_{bamn})\,\delta_{xy}^{mn} \\ &= \bar{R}_{abmn}\,\delta_{xy}^{mn}. \end{aligned}$$

Finally, one has:

(4.6) $$\bar{R}_{abmn} = E^{rs}\varepsilon_{abr}\,\varepsilon_{xys}.$$

Furthermore, (3.1) and (4.6) given, in turn:

$$\begin{aligned} c^{-2}\bar{R}_{0rs0} &= a^{u\upsilon}\bar{R}_{urs\upsilon} = a^{u\upsilon}E^{mn}\,\varepsilon_{urm}\,\varepsilon_{s\upsilon n} \\ &= E_m{}^n\,\varepsilon^{\upsilon im}\,a_{ir}\,\varepsilon_{s\upsilon n} = E_m{}^n\,\delta_{ns}^{im}\,a_{ir} \\ &= E_s{}^n\,a_{nr} - E_n{}^n\,a_{sr}. \end{aligned}$$

By contracting the last equality, and with the aid of (3.3), one will then have:

$$c^{-2}a^{rs}\,\bar{R}_{0rs0} = E_r{}^r - 3\,E_n{}^n = -2\,E_n{}^n = 0,$$

and thus, the following results:

(4.7) $$E_n{}^n = 0,$$

(4.8) $$E_{rs} = c^{-2}\bar{R}_{r00s} = a^{u\upsilon}\,\bar{R}_{urs\upsilon}.$$

One can note that the invariant $E_n^{\ n}$ is annulled as a result of the equations of relativity, while the tensor E_{rs} is identically symmetric. We shall recover some identical results in the case of H_{rs} , except that the order will be reversed.

One infers from the relation (4.4), in turn, that:

$$2c\,H^{rs} = a^{ri}\,\bar{R}_{i0u\upsilon}\varepsilon^{u\upsilon s},$$
$$2c\,H^{rs}\,\varepsilon_{rsn} = a^{ri}\,\bar{R}_{i0u\upsilon}\delta_{nr}^{u\upsilon} = 2a^{ri}\,\bar{R}_{i0nr},$$

which is annulled due to (3.2). However, $H^{rs}\,\varepsilon_{rsn}$ can be annulled only if:

(4.9) $$H^{rs} = H^{sr}.$$

On the other hand:

$$H_r^{\ r} = \bar{R}_{r0u\upsilon}\varepsilon^{u\upsilon r} = -\bar{R}_{0ru\upsilon}\varepsilon^{u\upsilon r},$$

which is annulled identically as a result of the classical identity:

$$\bar{R}_{\alpha\beta\mu\nu} + \bar{R}_{\alpha\mu\nu\beta} + \bar{R}_{\alpha\nu\beta\mu} = 0.$$

One will then have:

(4.10) $$H_r^{\ r} = 0.$$

Finally, another relation that will be useful in what follows is deduced from (4.4) by multiplying the sides by ε_{smn} :

$$2c\,H_r^{\ s}\,\varepsilon_{smn} = \bar{R}_{r0u\upsilon}\delta_{mn}^{u\upsilon} = 2\,\bar{R}_{r0mn},$$

so

(4.11) $$2\,\bar{R}_{r0s\upsilon} = c\,H_r^{\ n}\,\varepsilon_{ns\upsilon}.$$

The two tensors that we just introduced are at the basis of a new formalism that will permit us to shed light upon an aspect of the structure of the equations of gravitation that has been ignored up to now. Granted, we are less interested in those equations than we are in some of its differential consequences, which we shall look into at once.

5. A fundamental system of equations. – Start with the Bianchi identity:

$$\bar{R}_{\alpha\mu\nu\beta,\sigma} + \bar{R}_{\alpha\mu\beta\sigma,\nu} + \bar{R}_{\alpha\mu\sigma\nu,\beta} = 0.$$

After contracting with $g^{\alpha\beta}$, the first two terms will be annulled due to $\bar{R}_{\mu\nu} = 0$, and what will remain is:

(5.1) $$g^{\alpha\beta}\bar{R}_{\alpha\mu\sigma\nu,\beta} = 0.$$

We shall now address the separation of those equations according to the technique that was employed already. One subset of the system can be written $g^{\alpha\beta}\overline{R}_{\alpha rs\upsilon,\bar{0}} = 0$, or:

(5.2) $$-a^{un}\overline{R}_{urs\upsilon,\bar{n}} + c^{-2}\overline{R}_{0rs\upsilon,\bar{0}} = 0,$$

and we once more get:

(5.3) $$-a^{u\upsilon}\overline{R}_{uns0,\bar{\upsilon}} + c^{-2}\overline{R}_{0ns0,\bar{0}} = 0,$$

(5.4) $$a^{rs}\overline{R}_{r0u\upsilon,\bar{s}} = 0,$$

(5.5) $$a^{rs}\overline{R}_{r0n0,\bar{s}} = 0$$

for the complete set.

We shall now say that it is after having transcribed the system (5.2) to (5.5) into functions of the tensors E_{rs} and H_{rs} that one will recover the structure of Maxwell's equations. Of course, we are dealing with derivatives in space-time, and we would like to first develop the terms in such a way that everything comes down to derivatives in E (3). We then appeal to the classical development of the covariant derivative, and we will have:

$$\overline{R}_{urs\upsilon,\bar{n}} = \frac{\partial \overline{R}_{urs\upsilon}}{\partial x^n} - \overline{\Gamma}_{nu}^{\ \alpha}\overline{R}_{\alpha rs\upsilon} - \overline{\Gamma}_{nr}^{\ \alpha}\overline{R}_{u\alpha s\upsilon} - \overline{\Gamma}_{ns}^{\ \alpha}\overline{R}_{ur\alpha\upsilon} - \overline{\Gamma}_{n\upsilon}^{\ \alpha}\overline{R}_{urs\alpha},$$

for example. Compare this equality with:

$$\overline{R}_{urs\upsilon,\bar{n}} = \frac{\partial \overline{R}_{urs\upsilon}}{\partial x^n} - \overline{\Gamma}_{nu}^{\ i}\overline{R}_{irs\upsilon} - \overline{\Gamma}_{nr}^{\ i}\overline{R}_{uis\upsilon} - \overline{\Gamma}_{ns}^{\ i}\overline{R}_{uri\upsilon} - \overline{\Gamma}_{n\upsilon}^{\ i}\overline{R}_{ursi}$$

With the aid of $\overline{\Gamma}_{rs}^{\ t} = \Gamma_{rs}^{\ t}$, one will then infer that:

(5.6) $$\overline{R}_{urs\upsilon,\bar{n}} = \overline{R}_{urs\upsilon,n} - \overline{\Gamma}_{nu}^{\ 0}\overline{R}_{0rs\upsilon} - \overline{\Gamma}_{nr}^{\ 0}\overline{R}_{u0s\upsilon} - \overline{\Gamma}_{ns}^{\ 0}\overline{R}_{ur0\upsilon} - \overline{\Gamma}_{n\upsilon}^{\ 0}\overline{R}_{urs0}.$$

Moreover, one will get:

(5.7) $$\overline{R}_{0rs\upsilon,\bar{0}} = \frac{\partial \overline{R}_{0rs\upsilon}}{\partial t} - \overline{\Gamma}_{00}^{\ \alpha}\overline{R}_{\alpha rs\upsilon} - \overline{\Gamma}_{0r}^{\ \alpha}\overline{R}_{0\alpha s\upsilon} - \overline{\Gamma}_{0s}^{\ \alpha}\overline{R}_{0r\alpha\upsilon} - \overline{\Gamma}_{0\upsilon}^{\ \alpha}\overline{R}_{0rs\alpha}$$

directly.

We shall momentarily suspend our calculations, with a view towards introducing a method of approximation that will permit us to simplify things by neglecting a large number of terms. Those considerations will be the subject of the following section.

6. High-frequency waves. – We first consider the case of a wave whose frequency takes a very high value n, and it is upon that number n that our method will be based. Since the speed of propagation of our waves must likewise reveal the invariant c, we can profit from that fact in advance and choose the unit of time in such a way that the potentials c^2 and a_{rs} will all have the same order of magnitude. We will then adopt the centimeter as the unit of length, and we will choose the unit of time to be the interval that is necessary for light to traverse that distance, which will obviously imply that the value of the speed of light *in vacuo* will be equal to unity. Moreover, a frequency of 10^4 means 10^4 waves per centimeter, as well as an equal number of periods per unit time. In view of the considerations that will follow, it will then be important to note that our choice of units implies the following result: Not only will the potentials c^2 and a_{rs} have the same order of magnitude, but also their rates of variation with respect to time (as a result of the oscillation) have the same order of magnitude as the rates of variation with respect to distance. In other words, the first-order derivatives with respect to time and the ones with respect to the spatial variables take all of the values that have the same order of magnitude (except, of course, in the neighborhood of singularities; we shall consider the case of a domain in which the curvature remains finite). To fix ideas, we remark that a frequency of $n = 10^4$ corresponds to a wave in the infrared range that is close to the limits of the visible spectrum. On the other hand, and to avoid any confusion with the facts about light that are given experimentally, we once more recall that we would simply like to impose an oscillation upon the potentials $g_{\mu\nu}$ whose frequency is sufficiently high and then show that this will result in a wave that is characterized in the first approximation by equations that have the same structure as those of Maxwell's system.

To begin, it will be convenient for us to restrict ourselves to the case in which the intensity of oscillation of the potentials has order n^{-1}, and we take $n \gg 10^4$. Our potentials can then be represented by sinusoidal functions in the first approximation in a space-time domain that extends over a reasonable number of waves and periods of oscillation. That is, it will suffice to refer to the relation:

$$\frac{d}{dx}\left(\frac{1}{2\pi n}\sin 2\pi nx\right) = \cos 2\pi nx$$

for us to convince ourselves that if the intensity of oscillation of the potentials has the same order of magnitude as n^{-1} then the first-order derivatives of those same potentials will take on values with the same order as n^0 ; i.e., they are "finite" quantities from the standpoint of the scale that we defined in terms of powers of n. Moreover, the second-order derivatives will be quantities with the same order as n, and since the aforementioned derivatives present themselves linearly in the equations $R_{\mu\nu} = 0$, we will have terms of the same order as n for a first approximation.

Meanwhile, we are more interested in the system (5.2) to (5.5), in which the potentials appear with their derivatives up to order three, and the latter appear linearly. That amounts to saying that we have terms of the same order of magnitude as n^2. We shall shortly assure ourselves that those terms are the largest ones, and that we will get everything while preserving only the third-order derivatives. First of all, the coefficients a_{rs} and c^{-2} in (5.2) have an order of magnitude that is at most the same as n^0 or "finite." On the other hand, a simple examination of (5.6), (5.7) and the intermediate calculations will show that it still remains for us to decide what to do about terms such as:

$$\bar{\Gamma}_{nu}{}^{\alpha}\bar{R}_{\alpha rs\upsilon}\,, \qquad \bar{\Gamma}_{0r}{}^{\alpha}\bar{R}_{0\alpha s\upsilon}\,.$$

Now, these terms result from the product of a factor of “finite” order, such as $\Gamma_{nu}{}^{\alpha}$, with a factor of order one, such as:

$$\bar{R}^{n}{}_{rs\upsilon} = \frac{\partial\bar{\Gamma}_{r\upsilon}{}^{n}}{\partial x^{s}} - \frac{\partial\bar{\Gamma}_{rs}{}^{n}}{\partial x^{\upsilon}} + \bar{\Gamma}_{r\upsilon}{}^{\alpha}\,\bar{\Gamma}_{\alpha s}{}^{n} - \bar{\Gamma}_{rs}{}^{\alpha}\,\bar{\Gamma}_{\alpha\upsilon}{}^{n}\,. \tag{6.1}$$

We conclude that the covariant derivatives with respect to the two metrics – viz., the one on space-time and the one on E (3) – will differ only by terms that are negligible in the first approximation, since they have order at most one. Similarly, the right-hand side of (5.7) reduces to its first term. We shall appeal to these results in order to transcribe the system (5.2) to (5.5) into functions of the tensors E_{rs} and H_{rs} , and along the way, some other terms like $c_0\, R_{0rs\upsilon}$ and $c_{\upsilon}\, R_{0sun}$ will be neglected, and always as a result of the preceding considerations. We shall employ the symbol ~ to indicate an equality that is true only in the first approximation.

One first starts with equation (5.2), passes to derivatives in E (3), and then multiplies the sides of the equation by $\varepsilon^{ms\upsilon}$; it then results (while taking into account that $\bar{R}_{r0s\upsilon} = -\bar{R}_{0rs\upsilon}$) that:

$$a^{un}\bar{R}_{urs\upsilon,n}\varepsilon^{ms\upsilon} + c^{-2}\varepsilon^{ms\upsilon}\frac{\partial\bar{R}_{r0s\upsilon}}{\partial t} \sim 0.$$

The first term can be transformed with the aid of (4.6); (since $\varepsilon_{\alpha ur}$ plays the role of a constant with respect to the covariant derivative) one will get:

$$\begin{aligned} a^{un}\bar{R}_{urs\upsilon,n}\varepsilon^{ms\upsilon} &= a^{un}E^{ab}{}_{,n}\,\varepsilon_{aur}\,\varepsilon_{bs\upsilon}\varepsilon^{ms\upsilon} \\ &= 2\;\delta_b{}^{m}\,a^{un}\,E^{ab}{}_{,n}\,\varepsilon_{aur} \\ &= 2\;a^{un}\,E^{am}{}_{,n}\,\varepsilon_{aur}\,. \end{aligned}$$

Furthermore, upon utilizing (4.11) and neglecting terms such as $c_0\, H_r{}^{n}$ and:

$$H_r{}^{n}\frac{\partial\varepsilon_{ns\upsilon}}{\partial t},$$

one will get:

$$\varepsilon^{ms\upsilon}\frac{\partial\bar{R}_{r0s\upsilon}}{\partial t} \sim 2c\,\delta_n{}^{m}\,\frac{\partial H_r{}^{n}}{\partial t} \sim 2c\frac{\partial H_r{}^{m}}{\partial t}.$$

Our equation can then be written:

$$a^{un}\, E^{am}{}_{,\,n}\, \varepsilon_{aur} + c^{-1} \frac{\partial H_r{}^m}{\partial t} \sim 0,$$

or rather:

$$E^m{}_{a,\,n}\, \varepsilon^{anr} + c^{-1}\, a^{ir} \frac{\partial H_i{}^m}{\partial t} \sim 0.$$

Finally, as a result of relations such as:

$$a^{ir} \frac{\partial H_i{}^m}{\partial t} \sim \frac{\partial H^{rm}}{\partial t},$$

one will have:

$$E_{ma,\,n}\, \varepsilon^{anr} + c^{-1} \frac{\partial H_m{}^r}{\partial t} \sim 0.$$

Let us once more transcribe this result by appending equations (5.3) to (5.5), transformed in the same fashion, although much more easily, with the aid of (4.8) and (4.11). We will then get the complete system in the form:

$$(6.2) \qquad \begin{cases} E_{ru,\upsilon}\varepsilon^{u\upsilon s} + \dfrac{1}{c}\dfrac{\partial H_r{}^s}{\partial t} \sim 0, \\ H_{ru,\upsilon}\varepsilon^{u\upsilon s} - \dfrac{1}{c}\dfrac{\partial E_r{}^s}{\partial t} \sim 0, \\ E^{rs}{}_{,s} \sim 0, \\ H^{rs}{}_{,s} \sim 0. \end{cases}$$

It would be, so to speak, superfluous to emphasize here that one recovers the vectorial operations of divergence and rotation in the terms $E^{rs}{}_{,\,s}$ and $E_{ru,\,\upsilon}\, \varepsilon^{u\upsilon s}$ in our equations (6.2). As a consequence, one will easily recognize the structure of Maxwell's equations, but extended to the tensors E_{rs} and H_{rs} .

It is now important to note that in the second-order derivatives $E_{rs,\,u\upsilon}$, the indices $(u\upsilon)$ will permute in the first approximation, since (as we shall show in a moment) the difference between the covariant derivatives and ordinary derivatives will produce terms that drop out in the negligible part. We have essentially seen that the elements $E_{rs,\,u}$ have the same order of magnitude as n^2, while the $\Gamma_{u\upsilon}{}^r$ are finite like n^0 in such a way that the first term on the right-hand side of:

$$E^r{}_{s,\,u\upsilon} = \frac{\partial E^r{}_{s,u}}{\partial x^\upsilon} + \Gamma_{n\upsilon}{}^r E^n{}_{s,u} - \Gamma_{s\upsilon}{}^n E^r{}_{n,u} - \Gamma_{n\upsilon}{}^n E^r{}_{s,n}$$

will predominate with a value whose order is n^3. One will then have:

$$E^r{}_{s,uv} \sim \frac{\partial E^r_{s,u}}{\partial x^v} \sim \frac{\partial^2 E^r_s}{\partial x^v \partial x^u},$$

so

$$E^r{}_{s,vu} \sim E^r{}_{s,uv} .$$

As a consequence, one can refer to the classical calculation in order to solve the system (6.2) and to characterize the waves by means of the equation:

$$\Delta\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \sim 0. \tag{6.3}$$

Here, we intend that the function c that enters into the coefficients of our equations (6.2) and (6.3) can be considered to be a constant in the first approximation and in a neighborhood that can contain a certain number of waves. (On the one hand, the potentials oscillate with intensities that have order of n^{-1}, and on the other, the variations of the potential that are due to gravitation are also very small, except in the neighborhood of singularities.) That restriction to a neighborhood is obviously necessitated by the curvature of the medium of propagation, but it nonetheless allows us to recognize the essential characteristics of electromagnetic waves in this new aspect of the structure of the equations of gravitation.

7. Extension of the results. – Up to now, our results have been limited to the case in which the potentials oscillate with an amplitude of order n^{-1}. Our approximation procedure will no longer be valid for a larger amplitude (for example, one with the same order as the $g_{\mu\nu}$), because the last terms in (6.1) have the same order of magnitude as the second derivatives of the first terms. In reality, there is no reason to consider the extreme case for which the potentials periodically change sign.

By contrast, we shall show that the wave will remain Maxwellian for amplitudes as small as one desires and independently of the curvature of the medium of propagation. Consider the case then in which the amplitude of oscillation of the potential has order n^{-r} with $r > 1$, and set $g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$, in such a fashion that the $h_{\mu\nu}$ represent the oscillatory part of those potentials in the first approximation. Effectively, one can always arrange for the $h_{\mu\nu}$ to have the same order of magnitude as the amplitude of oscillation (namely, order n^{-r}), and for the $\gamma_{\mu\nu}$ to oscillate with amplitudes $\ll n^{-r-1}$, moreover.

We then note the following results:

7.1. The derivatives of order k of the $h_{\mu\nu}$ oscillate with amplitudes of order n^{-r+k}, and the order of magnitude of those quantities does not exceed that of their amplitude of oscillation.

7.2. The $\gamma_{\mu\nu}$ and their derivatives up to order two are at most “finite” quantities.

We now say (without worrying that we do not know what this will lead to, for the moment) that we shall separate equations (5.1) in such a fashion that we will keep only the terms that oscillate with the largest intensity. We shall see immediately that we will get all of those terms by keeping only the ones that contain the third-order derivatives of the $h_{\mu\nu}$, and that the intensity of the oscillation will then have order n^{-r+3}. A simple examination of (5.6), (5.7), and (6.1) will effectively show us that in the terms that do not contain third-order derivatives, only *one* of the factors can be a second derivative of the $h_{\mu\nu}$, and that all of the other factors have at most *finite* magnitudes, in such a way that the terms in question will oscillate with order at most $- r + 2$.

Furthermore, one can perform the development in (5.1) by first introducing the $E_{ru,\,\upsilon}$ and $H_{ru,\,\upsilon}$ as in the preceding section, and since the third-order derivatives enter linearly, one can split those tensors in such a fashion that one can represent them symbolically by:

$$E_{ru,\,\upsilon} = E_{ru,\,\upsilon}\,(\gamma) + E_{ru,\,\upsilon}\,(h)$$

in order to keep only the oscillating part with the largest intensity, namely, order $- r + 3$. The $E_{ru,\,\upsilon}$ and $H_{ru,\,\upsilon}$ are then tensors for which the metric is determined by the mean values $\gamma_{\mu\upsilon}$ in the first approximation.

Of course, the terms that we would now like to leave aside are not necessarily smaller than the other ones (indeed, the contrary is true for $r > 2$), but then if equations (5.2) to (5.5) must be verified then it is necessary that the terms of the oscillating part mutually cancel each other in the first approximation and that their sum (in each equation) reduces to a quantity that oscillates only with order $- r + 2$. Finally, the order of magnitude of the latter quantities (or sums) will not exceed $- r + 2$ if the $h_{\mu\nu}$ are chosen conveniently. Indeed, that leaves only the question of the part that is constant in the first approximation, which will go back to the $\gamma_{\mu\nu}$. Briefly, one is once more led to equations (6.2) for the oscillating part and the waves of small amplitude will always possess characteristics that involve the Maxwellian structure.

It is further possible to extend our results to the case in which the frequency is not necessarily very high, but this time we will be limited to an almost-Euclidian medium of propagation. Then set:

$$c^2 = 1 + h_{00}\,, \qquad\qquad g_{rs} = -\,\delta_r^{\ s} - h_{rs}\,,$$

in which the h_{rs} and h_{00} are elementary quantities of order one. We shall then appeal to a classical approximation procedure here that will permit us, in particular, to characterize the waves that are called gravitational by means of equation (6.3). Meanwhile, the introduction of our tensors E_{rs} and H_{rs} will permit us to endow that result with a new element.

With that method, one keeps only the linear terms in a first approximation. The reader will then have no trouble in assuring himself that equations (5.1) reduce to third-order derivatives of the $h_{\mu\nu}$ and that one will therefore once more recover the system (6.2), with the difference that the covariant derivatives will be replaced by ordinary derivatives. We deduce from this that in the radio-frequency domain, our waves will again be Maxwellian, provided that the curvature of the medium of propagation is very small.

It would not be senseless for us to make the distinction here that we exclude the gravitational waves, properly speaking, from our deductions. Indeed, in that case, the frequency will take on extremely small values (of order 10^{-18} in the case of the motion of the Earth around the Sun and with a unit of time as above), which will introduce a critical aspect into the problem that we would not like to discuss here.

8. Conclusion. – The structure of the Maxwell equations is then recovered in the first approximation and as a direct consequence of the equations $\bar{R}_{\mu\nu} = 0$ in order to characterize the waves that can exist the space-time continuum of relativity in almost all cases. In summary, if the frequency takes values that correspond to infrared or larger then our waves will be characterized by equations (6.2) and (6.3) independently of the curvature of the medium. By contrast, if the frequency is less elevated then our equations will remain valid only if the curvature is small. Note once more that in the final analysis, it is the elements of the curvature $\bar{R}_{\alpha\beta\mu\nu}$ that verify such equations and that this will be made possible simply because of the way (which is remarkable, at the very least) by which the aforementioned elements can be grouped into two tensors E_{rs} and H_{rs} that play the role of the vectors **E** and **H** in Maxwell's equations.

As a consequence, the invariance of the phenomenon cannot be in doubt, since the oscillations that one imposes arbitrarily upon the coordinate net cannot be characterized by invariant equations. We therefore have a wave, and we immediately compare it to the electromagnetic waves. Now, the latter can be represented by means of vectors (at least the aspects of them that we know of), while the new waves, the tensors E_{rs} and H_{rs} present a more complex structure and give us a glimpse of some great possibilities.

Can we now say that we have identified the electromagnetic waves in a new form in the continuum $\bar{R}_{\mu\nu} = 0$, but that we only know a few things about them? One must assert that for the moment any attempt to interpret them in that sense will pose some problems whose solution does not seem to be within our reach. First of all, do the waves that we just characterized even exist? More precisely, do there exist solutions of that type that are regular at infinity? That is a question that leaves us perplexed in view of the complexity of the equations $\bar{R}_{\mu\nu} = 0$. Nonetheless, it would be of greatest interest to know how to determine the possibilities from those equations in regard to the singularities that accompany a stationary oscillating wave or one that radiates energy, and it would undoubtedly imply a quantization phenomenon. Be that as it may, we conclude this article by presenting two other results that further extend the analogy with the equations of electromagnetism.

Note A. – In this section, we shall address two space-time invariants (always in the case of $g_{0r} = 0$, of course) and find a way of expressing them as functions of the tensors E_{rs} and H_{rs} .

First of all, one will effortlessly verify that:

$$
\begin{aligned}
\bar{R}_{\alpha\beta\mu\nu}\,\bar{R}^{\alpha\beta\mu\nu} &= \bar{R}_{rs\mu\nu}\,\bar{R}^{rs\mu\nu} + 2\bar{R}_{r0\mu\nu}\,\bar{R}^{r0\mu\nu} \\
&= \bar{R}_{rs\mu\nu}\,\bar{R}^{rs\mu\nu} + 2\bar{R}_{rs0\upsilon}\,\bar{R}^{rs0\upsilon} + 2\bar{R}_{r0u\upsilon}\,\bar{R}^{r0u\upsilon} + 4\bar{R}_{r00\upsilon}\,\bar{R}^{r00\upsilon} \\
&= \bar{R}_{rsu\upsilon}\,\bar{R}^{rsu\upsilon} + 4\bar{R}_{rs0\upsilon}\,\bar{R}^{rs0\upsilon} + 4\bar{R}_{r00\upsilon}\,\bar{R}^{r00\upsilon} .
\end{aligned}
\tag{A.1}
$$

On the other hand, with the aid of (2.1a) and (4.8), one will have:

$$\bar{R}^{r00\upsilon} = g^{r\alpha}\, g^{0\beta}\, g^{0\mu}\, g^{\upsilon\nu}\, \bar{R}_{s00u} = c^{-2}\, E^{r\upsilon},$$

so

(A.2) $$\bar{R}_{s00u}\bar{R}^{r00\upsilon} = E_{rs}\, E^{rs}.$$

If one is then given that $\bar{R}_{mnu\upsilon}$ is a tensor in E (3) then the relation (4.3)

$$4\, E^{rs} = \bar{R}_{mnu\upsilon}\ \varepsilon^{mnr}\ \varepsilon^{u\upsilon s}$$

will imply [taking (2.1a) into account], as before:

$$4\, E_{rs} = \bar{R}^{mnu\upsilon}\, \varepsilon_{mnr}\ \varepsilon_{u\upsilon s}$$

so

(A.3) $$16\, E_{rs}\, E^{rs} = \bar{R}^{mnu\upsilon}\, \bar{R}_{abxy}\, \delta_{mn}^{\ \ ab} \delta_{u\upsilon}^{\ \ xy} = 4\bar{R}_{mnu\upsilon}\, \bar{R}^{mnu\upsilon}.$$

Now recall (4.4):

$$2c\, H_r^s = \bar{R}_{r0u\upsilon}\, \varepsilon^{m\upsilon s}.$$

As in the preceding relations and taking (2.1a) into account, we also have:

$$2c\, H^r{}_s = -c^2 \bar{R}^{r0u\upsilon}\, \varepsilon_{u\upsilon s}.$$

Finally, by contraction, that will give:

$$4c^2\, H_r^s\, H^r{}_s = -c^2 \bar{R}_{r0u\upsilon} \bar{R}^{r0mn}\, \delta_{mn}^{\ \ u\upsilon},$$

or

(A.4) $$4\, H_{rs}\, H^{rs} = -\, 2\bar{R}_{r0u\upsilon} \bar{R}^{r0u\upsilon}.$$

The relations (A.2) and (A.4) will then permit one to write (A.1) in the form:

(A.5) $$\bar{R}_{\alpha\beta\mu\nu} \bar{R}^{\alpha\beta\mu\nu} = 8\, (E_{rs}\, E^{rs} - H_{rs}\, H^{rs}).$$

We similarly construct another invariant whose expression involves the universal tensor by way of:

$$\varepsilon_{\alpha\beta\gamma\delta} = g^{1/2}\, e_{\alpha\beta\gamma\delta},$$

in which $e_{\alpha\beta\gamma\delta} = 0$ or ± 1 according to the rules:

0	if any two of the indices $\alpha\beta\gamma\delta$ have the same value,
1	if $\alpha\beta\gamma\delta$ is an even permutation of the numbers 0, 1, 2, 3,
– 1	if $\alpha\beta\gamma\delta$ is an odd permutation of those same numbers.

The set of non-zero covariant components of that tensor can be easily represented by means of a tensor in E (3), as was explained in § **3**. The new tensor will have order three since one of the four indices $\alpha\beta\gamma\delta$ must take the value zero. One should first note that the permutations $0rst$ and rst have the same parity (i.e., the same sign) with respect to the numbers 0, 1, 2, 3, and 1, 2, 3, respectively. Furthermore, one will have $g^{1/2} = ic\, a^{1/2}$, with $i = (-1)^{1/2}$, in such a way that:

$$\varepsilon_{0rst} = i\, c\, \varepsilon_{rst}\,.$$

One will then have, in turn:

$$\begin{aligned}\bar{R}^{\alpha\beta\mu\nu}\,\bar{R}_{\alpha\beta}{}^{\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} &= \bar{R}^{mn\mu\nu}\,\bar{R}_{mn}{}^{\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} + 2\bar{R}^{n0\mu\nu}\,\bar{R}_{n0}{}^{\rho\sigma}\varepsilon_{\mu\nu\rho\sigma}\\ &= 2\bar{R}^{mnu\upsilon}\,\bar{R}_{mn}{}^{0s}\varepsilon_{\mu\nu 0s} + 2\bar{R}^{mn0\upsilon}\,\bar{R}_{mn}{}^{rs}\varepsilon_{0\upsilon rs} + 4\bar{R}^{n0u\upsilon}\,\bar{R}_{n0}{}^{0s}\varepsilon_{u\upsilon 0s} + 4\bar{R}^{n00\upsilon}\,\bar{R}_{n0}{}^{rs}\varepsilon_{0\upsilon rs}\\ &= 8\bar{R}^{n0u\upsilon}\,\bar{R}_{n0}{}^{0s}\varepsilon_{0su\upsilon} + 4\bar{R}^{mnu\upsilon}\,\bar{R}_{mn}{}^{0s}\varepsilon_{0su\upsilon}\\ &= 4ic\,(2\bar{R}^{n0u\upsilon}\,\bar{R}_{n0}{}^{0s}\varepsilon_{su\upsilon} + \bar{R}^{mnu\upsilon}\,\bar{R}_{mn}{}^{0s}\varepsilon_{su\upsilon})\,.\end{aligned}$$

The two terms in parentheses can be recovered, up to a factor, by contracting E_{rs} with H^{rs} in two different ways by starting from (4.3), (4.4), and (4.8). The calculations are analogous to the ones that gave (A.2) to (A.4) and finally imply that:

$$\bar{R}^{\alpha\beta\mu\nu}\,\bar{R}_{\alpha\beta}{}^{\rho\sigma}\varepsilon_{\mu\nu\rho\sigma} = 32i\,E_{rs}\,H^{rs}. \tag{A.6}$$

One must remark that the relations (A.5) and (A.6) will be possible as long as the form (1.1) of our element of arc length is preserved. Meanwhile, the invariants in the left-hand side permit us to conclude that the functions:

$$E_{rs}\,E^{rs} - H_{rs}\,H^{rs}, \qquad\qquad E_{rs}\,H^{rs}$$

will themselves be invariant under Lorentz transformations. We will then recover a property that the relativistic form of the electromagnetic equations confers upon the invariants:

$$\mathbf{E}^2 - \mathbf{H}^2, \qquad \mathbf{E}\cdot\mathbf{H}.$$

Note B. – Starting from (6.1), we set $r = 0$ and separate the sums that relate to the index a.

Then:

$$\bar{R}^n_{0s\upsilon} = \frac{\partial\bar{\Gamma}_{0\upsilon}{}^n}{\partial x^s} - \frac{\partial\bar{\Gamma}_{0s}{}^n}{\partial x^\upsilon} + \bar{\Gamma}_{0\upsilon}{}^i\,\bar{\Gamma}_{is}{}^n - \bar{\Gamma}_{0s}{}^i\,\bar{\Gamma}_{i\upsilon}{}^n + \bar{\Gamma}_{0\upsilon}{}^0\,\bar{\Gamma}_{0s}{}^n - \bar{\Gamma}_{0s}{}^0\,\bar{\Gamma}_{0\upsilon}{}^n\,.$$

With the aid of (2.3), we will have:

$$\bar{R}^n_{0s\upsilon} = \frac{\partial(c\,A^n_\upsilon)}{\partial x^s} - \frac{\partial(c\,A^n{}_s)}{\partial x^\upsilon} + c\,(\Gamma_{is}{}^n\,A^i_\upsilon - \Gamma_{i\upsilon}{}^n\,A^i{}_s) + c_\upsilon A^n_s - c_s A^n_\upsilon$$

$$= c\left(\frac{\partial A_{\upsilon}^{n}}{\partial x^{s}} - \frac{\partial A^{n}{}_{s}}{\partial x^{\upsilon}}\right) + c\,(\Gamma_{is}^{\ n}\, A_{\upsilon}^{i} - \Gamma_{i\upsilon}^{\ n}\, A^{i}{}_{s})\,.$$

If we add the term:

$$c\,(-\Gamma_{s\upsilon}^{\ n}\, A_{i}^{n} + \Gamma_{\upsilon s}^{\ i}\, A^{n}{}_{i})\,,$$

which is zero identically, to this last result then we will immediately recognize the development of the two covariant derivatives in such a way that:

(B.1) $$\overline{R}^{n}{}_{0s\upsilon} = c\,(A^{n}{}_{\upsilon,s} - A^{n}{}_{s,\upsilon})\,.$$

For $u = n$, we will recover the three equations $\overline{R}_{0s} = 0$ in the form:

(B.2) $$A^{n}{}_{n,s} - A^{n}{}_{s,n} = 0.$$

Furthermore, upon taking (2.1) into account, equation (B.1) can be written:

$$\overline{R}_{n0s\upsilon} = c\ (A_{ns,\ \upsilon} - A_{n\upsilon,\ s}),$$

so, after multiplying the two sides by $\varepsilon^{s\upsilon r}$ and substituting by means of (4.4):

$$2c\ H_{n}{}^{r} = c\ (A_{n\,s,\ \upsilon} - A_{n\upsilon,\ s})\ \varepsilon^{s\upsilon r} = 2c\ A_{n\,s,\ \upsilon}\ \varepsilon^{s\upsilon r}.$$

We will then finally have:

(B.3) $$H_{n}{}^{r} = A_{n\,s,\ \upsilon}\ \varepsilon^{s\upsilon r},$$

and one will recover the form of the classical relation **H** = rot **A**, but extended to the tensor A_{rs} .

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