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Variational principles and adiabatic transformations

By GIANDOMENICO MATTIOLI (in Padua)

Translated by D. H. Delphenich

Abstract. – The author establishes a variational principle that connects two arbitrary segments of adiabatically transformed trajectories, and he arrives at that conclusion by considering the adiabatic parameters to be supplementary Lagrangian coordinates. The substitutability of temporal means with spatial means is then proved by statistical considerations in the general quasi-ergodic case and in the case of STÄCKEL systems, from which the classical adiabatic invariants follow.

Consider a conservative dynamical system S with time-independent constraints, so from a geometric standpoint any of its trajectories lie on the surface H = constant that passes through the initial position. Suppose that the characteristic function is $H (p \mid q \mid a)$; i.e., it contains certain parameters that are indicated by a and are normally constants. One will then have a motion that has the aforementioned geometric character, but it can also be made to vary by means of suitable, but arbitrary, external interventions. That is equivalent to supposing that the a in H are equal to certain functions of time, so for the motions of this second type, H will no longer be constant, and the system S will transfer from one of the abovementioned surfaces H = const. to another in the time interval $t_1 - t_0$ during which the a vary. When the variation of the a is very slow, so they realize only infinitesimal increments δa in a finite time interval $t_1 - t_0$, the motion of S will experience an alteration with respect to the one that it possesses for the constant a's, and according to EHRENFEST ([1]), one calls that an *adiabatic transformation.* The main problem of that theory is the search for *adiabatic invariants* – viz., quantities that preserve the values that they had before the adiabatic transformation.

The adiabatic invariants that are known up to now are all attached to two particular base motions (among the ones for which the a are constants): periodic ones and ones that satisfy the condition of quasi-ergodicity, and the authors ([2]) that have studied them immediately adopted conditions that related to one or the other state of motion.

It would then be worthwhile to adopt a more general viewpoint, in the sense of studying the effect of an adiabatic transformation on a generic motion that is considered

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([1]) "Adiabatic invariants and the theory of quanta," Phil. Mag. **33** (1917), pp. 500.

([2]) BURGERS, Ann. Phys. (Leipzig) **52** (1917), pp. 195.
LEVI-CIVITA, "Drei Vorlesungen über adiabatische Invarianten," Abh. Math. Seminar, Hamburg **6** (1928), pp. 323; "Sugli invarianti adiabatici," Atti del Congresso int. dei Fisica (Como, 1927).
Also cf. the treatises:
BORN, *Vorlesungen über Atommechanik*,
JUVET, *Mécanique analytique et théorie des quanta.*

in an arbitrary interval $t_1 - t_0$ and not just one of the types above. The criterion seems advantageous to me, because on the one hand, it leads to formulas of general validity and on the other, it allows one to confirm the necessity of certain limitations that are imposed upon adiabatic transformations in order to verify the facts of ordinary statistics that are the only ones that lead to the actual construction of adiabatic invariants in the special cases that were first pointed out.

I found the method in an application of the variational principles of mechanics that is original in some of its details – for instance, in the initial definitions, where the adiabatic parameters (viz., the a quantities) are introduced as supplementary Lagrangian coordinates. From the formal standpoint, that allows one to treat both the base motions (a = const.), as well as those of adiabatic transformations (a variable), by a consistent procedure.

The variational method (which I believe to be new) with which I propose to treat the general problem of adiabatic transformations is developed in nos. **1-4**, and I will arrive at an identity – viz., (9) – that summarizes the effect of the slow and linear variation of the adiabatic parameter in a concise formal expression. An immediate and known application to the case of periodic systems is given in no. **5**. The formal modification of the identity in no. **7** will allow me to point out in no. **9** a feature that is assumed in the classical theory in regard to the HAMILTON-JACOBI method of integration in the case where an adiabatic parameter is present. Among other things, one will find a formula for the increment of energy that I believe to be worthy of mention, although I do not adopt it. In no. **8**, the ROUTH systems pass in review, and mainly for the purpose of concretely pointing out the importance that the duration of the adiabatic transformations has in the correct construction of adiabatic invariants. In no. **10**, the identity between the means of an arbitrary function along a dynamical trajectory that is dense on the surface H = const. is proved for arbitrary n, and on that surface one will consequently get the GIBBS invariant. In the succeeding numbers, the STAECKEL systems are examined and the adiabatic invariance of the SOMMERFELD integral is proved once more, which also makes the substitution of spatial and temporal means rigorous.

1. Observation about the asynchronous variations of the *vis viva* of a dynamical system. – Suppose that one has a dynamical system S with time-independent constraints. The corresponding *vis viva* is then a homogeneous quadratic function of the derivatives $\dot{q}_i$:

$$T = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}\,\dot{q}_i\,\dot{q}_j\,,$$

in which the a_{ij} are functions of only q. A particular motion is determined by the initial values of the q and $\dot{q}$. Consider two motions of S that correspond to initial values that are infinitely-close to those of q, $\dot{q}$. The two representative points in the space of coordinates q_i will remain infinitely close during the finite time intervals $[t_0, t_1]$ and $[t_0 + \delta t_0, t_1 + \delta t_1]$. Make the time t along the first trajectory correspond to an arbitrary time $t + \delta t$ along the second one, with the only restriction being that the previously-fixed initial and final instants must correspond. If one consequently associates the positions q_i and q_i

+ δq_i that are assumed by the system under the motions considered then one will realize an *asynchronous* variation (in general, when δt is not identically zero) of the base trajectory to the second one considered, in which the corresponding variation of the *vis viva* is given by:

$$\delta^* T = \delta T - 2T \frac{d\delta t}{dt}.$$

One agrees to let δ denote the increments that relate to the synchronous variations. In our case, we will have such a variation as long as the positions q_i , $q_i + \delta q_i$ are the ones that correspond to the instant t. However, the differential symbol δ^* relates to the asynchronous variation that was just defined. If one lets M_0 and M_1 denote two motions and adopt the same indices for quantities that relate to each of them then one will also have:

$$\delta^* T = T_1 (t + \delta t) - T_0 (t).$$

2. Interpretation of the parameters as Lagrangian coordinates. Consequent Lagrangian identity. – Therefore, one has a holonomic mechanical system in which some other parameters a_s that can be kept constant intervene along with the ones that are intrinsic to the system, which one calls q_i , as usual, and slowly varies them by means of suitable external influences (e.g., mass variations, constraints, forces, etc.). Assume that after any one of those variations, once it has returned to its constant value, the type of system has not changed, since it is always characterized by its own Lagrangian parameters. What will vary with the a_s will be the expressions for the *vis viva* and the forces that act upon the system, insofar as the a_s are contained in the analytical expressions for those quantities in a well-defined way.

The motion of the system is plainly determined by just the LAGRANGE equations that relate to the chosen coordinates, even when the a_s are varying, since one supposes that the a_s are specified as functions of time. However, nothing prevents one from also writing the corresponding Lagrangian equations for the parameters a_s in the form of *identities*, and in precisely the following way: The left-hand sides are deduced from the *vis viva* $T (q \mid \dot{q} \mid a \mid \dot{a})$ in the usual way, while the right-hand sides are set equal to what the first ones will become when the a_s , $\dot{a}_s$ in them are replaced with their known expressions in terms of t, $a_s (t)$, $\dot{a}_s(t)$. In addition, if one supposes that one has first integrated the LAGRANGE equations, properly-speaking, then they will represent the actual determination of the q_i , $\dot{q}_i$, $\ddot{q}_i$ as functions of t and the constants that were introduced by the integration. In conclusion, one therefore associates the equations of motion of the system with a number of identity relations that are equal to the ones for the parameters a_s and whose left-hand sides present themselves symmetrically to the analogous ones in the Lagrangian equations, while one can *think* that the right-hand sides are being well-defined functions of time, once one has specified the values of the arbitrary constants (i.e., one has chosen a particular motion). If one would wish that the actual knowledge of those functions should not be necessary then it would be enough to recognize that any individual motion that the system can exhibit exists in a uniquely-determined way.

Let us now develop the calculations. Let the mechanical system S depend upon n Lagrangian coordinates q_i $i = 1, 2, \ldots, n$, and let a_j , $j = 1, 2, \ldots, s$ be the adiabatic parameters, in addition. With no loss of generality, but solely for the sake of formal simplicity, assume that just one adiabatic parameter a is present; it is obvious how one would have to proceed otherwise. In addition, when a is kept constant, the constraints on the system will be fixed, in such a way that t will not enter into the *vis viva* T explicitly. One completes the assumed hypotheses, which correspond to the concrete cases (e.g., variable constraints) in which T also depends upon the derivative $\dot{a}$, one has a homogeneous quadratic function in the $\dot{q}_i$, $\dot{a}$ with coefficients that are functions of q_i , a. In addition, the forces are provided by a force function U that also depends upon only q_i , a.

Consistent with what was said above, write the complex of relations:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \frac{\partial U}{\partial q_i} \qquad i = 1, 2, \ldots, n, \tag{1}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{a}} - \frac{\partial T}{\partial a} = Q\,(t). \tag{1′}$$

The first n of them are the equations of motion, while the last one is the abovementioned identity that relates to the parameter a, and in which Q (t) denotes what the left-hand side will become for the particular motion that is considered; i.e., after replacing any quantity with its expression in time.

We repeat that (1) will suffice in any case to determine the motion [even when a varies, as long as a (t) is given]. As one will see in what follows, the consideration of (1′) will lead to a rapid evaluation of the contribution to the energy that is due to the variation of a, along with permitting a symmetric formulation of the problem, in the sense of treating the motions with a constant and the ones in which a varies (viz., *adiabatic transformations* of the system) in the same way.

From the formal standpoint, one can consider (1), (1′) to be the Lagrangian system of a dynamical problem with $n + 1$ degrees of freedom for the coordinates q_i, a. The definition of Q (t) for any individual motion is basically equivalent to asserting that the parameter a in the integral of (1), (1′) must prove to be the a (t) that was fixed to begin with.

3. Variational formulation of the problem. – The complete equivalence of a system of LAGRANGE equations with HAMILTON's variational principle is shown by a classical proof. To that end, one agrees to adopt the expression that relates to varied endpoints. One can then summarize the relations (1), (1′) (which present themselves formally as a Lagrangian system in $n + 1$ variables) in the variational formula:

$$\int_{t_0}^{t_1}\left(\delta T + \sum_{i=1}^{n}\frac{\partial U}{\partial q_i}\delta q_i + Q\,\delta a\right)dt = \left|\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\delta q_i + \frac{\partial T}{\partial \dot{a}}\delta a\right|_{t_0}^{t_1}, \tag{2}$$

which will persist for an arbitrary *synchronous* variation of the natural motion between varied extremes.

Now, introduce an *asynchronism* into the comparison of the varied (generally-virtual) motion that is based upon the natural motion (which can be either a motion with a = const. or one with a varying in some assigned way). If T is a homogeneous quadratic function of the $\dot{q}_i$, $\dot{a}$, and one lets δ^* denote the corresponding variations then one will have (cf., no. **1**):

$$\delta T = \delta^* T + 2T\,\frac{d\,\delta t}{dt},$$

and when one substitutes this in (2), one will get the variation principle:

$$\int_{t_0}^{t_1}\left(\delta^* T + 2T\,\frac{d\,\delta t}{dt} + \sum_{i=1}^{n}\frac{\partial U}{\partial q_i}\,\delta q_i + Q\,\delta a\right)dt = \left|\sum_{i=1}^{n}\frac{\partial T}{\partial \dot{q}_i}\,\delta q_i + \frac{\partial T}{\partial \dot{a}}\,\delta a\right|_{t_0}^{t_1}, \tag{3}$$

upon which our further consideration will be based and which is true for arbitrary variations of the natural trajectory, however asynchronous, and in which the variation δ^* of the *vis viva* must, of course, be calculated by treating T as a function of the a, $\dot{a}$ in a manner that is symmetric to the treatment of the q_i, $\dot{q}_i$.

4. Applying (3) to the calculation of adiabatic transformations. – Assign a constant value a_0 to a, and let M_0 denote a corresponding motion. In other words, determine certain initial conditions that will be specified, for ease of further reference, thus: For $t = t_0$, the q_i and $\dot{q}_i$ will become q_i^0, $\dot{q}_i^0$, respectively. Let P_0^0 comprehensively denote the initial state of motion thus-specified, and sometimes just the position of the system at that instant, as well.

Since the *vis viva* integral is valid, one will have:

$$T_0 = U_0 + E_0 \tag{4}$$

during M_0. It is clear from the chosen notation that the index 0 is intended to denote any quantity that relates to M_0.

If one starts from the state of motion at that instant t_0 and performs the *adiabatic transformation* that *very slowly* varies the parameter a in a specified way in time with an infinitesimal velocity, which is treated as a first-order quantity; i.e., one of the same order as the δq_i, δa that appear in (3). Let M denote the motion of the system that takes place (the *motion of the adiabatic transformation* or *intermediate motion*). Let t_1 be a value of time that follows t_0, and let P_0^1, P^1 be the states of motions of the system according to whether it has traversed the trajectory M_0 or M, resp. In addition, let δa be the total variation of the parameter over the time interval $t_1 - t_0$. The new constant value $a_1 = a_0 + \delta a$ of that parameter, together with the state of motion P^1, will uniquely determine a trajectory M_1 that can be traversed by the system S, which corresponds to a value of the parameter a that is infinitely close to the one a_0 that relates to M_0, and which will deviate from the trajectory M_0 by a quantity of first order for any of its finite lengths.

Choose a point P_1^0 on M_1 that is infinitely close to P_0^0, but otherwise arbitrary. Since time t does not enter into T or U explicitly, one can suppose that S traverses M_1 by starting from the state of motion P_1^0 at the instant $t_0 + \delta t_0$ (with δt_0 arbitrary, but infinitesimal, which one considers solely for the sake of taking into account all possible generalizations). Let $t_1 + \delta t_1$ be the instant at which S reaches the position $P_1^1 = P^1$ that was considered just now (final state of motion at the instant t_1 of the intermediate motion M). The *vis viva* integral is also valid along M_1. Let $E_1 = E_0 + \delta E$ be the value that the constant E assumes here. One will have:

(4) $$T_1 = U_1 + E_1 = U_1 + E_0 + \delta E$$

on M_1.

Apply equations (1), (1′) to the intermediate motion M. Multiply them by $\dot{q}_i$, $\dot{a}$, respectively, and sum them, while taking into account the form of T, one will get:

$$\frac{dT}{dt} = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} \dot{q}_i + Q\dot{a},$$

as everyone knows. As long as the potential U contains a, adding and subtracting $\frac{\partial U}{\partial a}\dot{a}$ will also give:

$$\frac{dT}{dt} = \frac{dU}{dt} - \left(\frac{\partial U}{\partial a} - Q\right)\dot{a},$$

and integrating along the trajectory M between t_0 and t_1 will give:

(5) $$\left| T \right|_{t_0}^{t_1} = \left| U \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\frac{\partial U}{\partial a} - Q\right)_M \dot{a}\, dt .$$

The index M indicates that the values of the integrand function are assumed to be functions of time at the points of M.

Let:

(6) $$T = \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}\, \dot{q}_i\, \dot{q}_j + \sum_{i=1}^{n} b_i\, \dot{q}_i\, \dot{a} + c\, \dot{a}^2$$

be the general expression for the *vis viva*; a_{ij}, b_i, c are functions of only q_i, a. We have assumed that the initial states of motion (i.e., for $t = t_0$) of the motions M_0, M correspond to the same values of q_i, $\dot{q}_i$, while $\dot{a} = 0$ for M_0, $\dot{a}$ = first-order quantity for M. If we let T^0, T_0^0 denote the initial *vis vivas* of M and M_0 then if we recall (6), we will have:

$$T^0 = T_0^0 + \left(\dot{a} \sum_{i=1}^{n} b_i \, \dot{q}_i \right)^0 + (c\,\dot{a}^2)^0 ,$$

in which the notation $(\ldots)^0$ is intended to mean that one must put the initial values q_i^0, $\dot{q}_i^0$, a_0, $\dot{a}_0$ for M in place of q_i, $\dot{q}_i$, a, $\dot{a}$ between the parentheses. Now compare T^1 – i.e., the *vis viva* of M at the end of that intermediate motion (which will be assumed at the instant t_1) – with T_1^1 = *vis viva* of M_1 for the state of motion = P^1 (and relative to the instant $t_1 + \delta t_1$). The states of motion coincide in regard to q_i, $\dot{q}_i$, while one will have $\dot{a} = \dot{a}^1 \neq 0$, in general, for M and $\dot{a} = 0$ for M_1. It will always follow from (6) that:

$$T^1 = T_0^1 + \left(\dot{a} \sum_{i=1}^{n} b_i \, \dot{q}_i \right)^1 + (c\,\dot{a}^2)^1 ,$$

in which the notation $(\ldots)^1$ has a meaning that is analogous to the one that was just explained.

The last terms in the preceding two relations are to be treated like second-order quantities, because $\dot{a}$ enters into them as a square. In addition, one can write:

$$\dot{a} \sum_{i=1}^{n} b_i \, \dot{q}_i = \dot{a} \frac{\partial T}{\partial \dot{a}} ,$$

while omitting the quantities of higher order in $\dot{a}$. By subtraction, one can then get:

$$\Big| T \Big|_{t_0}^{t_1} = T^1 - T^0 = T_1^1 - T_0^0 + \left| \dot{a} \frac{\partial T}{\partial \dot{a}} \right|_{t_0}^{t_1} + [2],$$

in which [2] indicates a quantity of order at least two in $\dot{a}$, and the notation $\Big| \cdots \Big|_{t_0}^{t_1}$ indicates that one must take the difference between the values that the corresponding quantity assumes at the end and the beginning of the motion M (i.e., at $P^1 = P_1^1$ and $P^0 = P_0^0$, resp.).

Obviously:

$$\Big| U \Big|_{t_0}^{t_1} = U_1^1 - U_0^0 ,$$

in which U_1^1, U_0^0 are the values of the force potentials at $P^1 = P_1^1$ and $P^0 = P_0^0$, respectively. Hence, by definition, (5) can be written:

$$(5') \qquad T_1^1 - T_0^0 + \left| \dot{a} \frac{\partial T}{\partial \dot{a}} \right|_{t_0}^{t_1} = U_1^1 - U_0^0 - \int_{t_0}^{t_1} \left(\frac{\partial U}{\partial a} - Q \right)_M \dot{a} \, dt + [2].$$

If one applies (4) and (4′) to the configurations P_0^0, P_1^1, respectively, then one will have:

$$T_0^0 = U_0^0 + E_0 ,$$
$$T_1^1 = U_1^1 + E_0 + \delta E.$$

If one substitutes this in (5′) then, by definition, one will get the increment in the *vis viva* constant when it passes from the motion M_0 to M_1 (with a constant in both cases) and one performs the adiabatic transformation M:

$$\delta E = -\left| \dot{a}\frac{\partial T}{\partial \dot{a}} \right|_{t_0}^{t_1} - \int_{t_0}^{t_1} \left(\frac{\partial U}{\partial a} - Q \right)_M \dot{a}\, dt + [2]. \tag{6′}$$

Now apply the variational formula (3) and assume precisely that M is the base motion in both cases and that M_0 and M_1, respectively, are the varied motions. Let δ_0, δ_1 be the corresponding variational symbols; one has:

$$\int_{t_0}^{t_1} \left(\delta_0^* T + 2T \frac{d\delta_0 t}{dt} + \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} + Q\,\delta_0 q_i + Q\,\delta_0 a \right)_M dt = \left| \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\delta_0 \dot{q}_i + \frac{\partial T}{\partial \dot{a}}\delta_0 a \right|_{t_0}^{t_1},$$

and an analogous one with δ_1 in place of δ_0 .

Subtract the first identity from the second one. Set:

$$\delta = \delta_1 - \delta_0 ,$$

in which δ is the variation symbol that relates to the (generally asynchronous) passage from the trajectory M_0 to M_1 . One will have:

$$\int_{t_0}^{t_1} \left(\delta^* T + 2T \frac{d\delta t}{dt} + \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} + Q\,\delta q_i + Q\,\delta a \right)_M dt = \left| \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\delta \dot{q}_i + \frac{\partial T}{\partial \dot{a}}\delta \dot{a} \right|_{t_0}^{t_1}, \tag{7}$$

in which the integral and the difference in the right-hand side is calculated for M.

Now recall the observation in no. **1**. If P_0 , P_1 are points that correspond to the instants t, $t + \delta t$, respectively, on M_0 and M_1 then one will have:

$$\delta^* T = T_1 (t + \delta t) - T_0 (t).$$

However, from (4), (4′), one will have:

$$T_0 (t) = U_0 (P_0) + E_0 ,$$
$$T_1 (t + \delta t) = U_0 (P_0) + E_0 + \delta E,$$

from which:

$$\delta^* T = U_1 (P_1) - U_0 (P_0) + \delta E = \delta U + \delta E.$$

Now:

$$\delta U = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i}\delta q_i + \frac{\partial U}{\partial a}\delta a\,,$$

and therefore:

$$\sum_{i=1}^{n} \frac{\partial U}{\partial q_i}\delta q_i = \delta^* T - \delta E - \frac{\partial U}{\partial a}\delta a\,,$$

so (7) will become:

$$(7') \qquad \int_{t_0}^{t_1}\left[\delta^* T + 2T\frac{d\delta t}{dt} - \delta E - \left(\frac{\partial U}{\partial a} - Q\right)\delta a\right]_M dt = \left|\sum_{i=1}^{n}\frac{\partial T}{\partial \dot q_i}\delta q_i + \frac{\partial T}{\partial \dot a}\delta a\right|_{t_0}^{t_1}.$$

If one sets δE equal to its value in (6′) and one takes into account the fact that δa is kept constant during the integration, in addition to δE, then one will have the identity ([1]):

$$(8) \qquad \begin{aligned}\delta^*\int_{t_0}^{t_1} 2T\,dt &= \int_{t_0}^{t_1}\left(2\delta^* T + 2T\frac{d\delta t}{dt}\right)dt \\ &= \delta a\int_{t_0}^{t_1}\left(\frac{\partial U}{\partial a} - Q\right)_M dt - (t_1 - t_0)\int_{t_0}^{t_1}\left(\frac{\partial U}{\partial a} - Q\right)_M \dot a\,dt - (t_1 - t_0)\left|\dot a\frac{\partial T}{\partial \dot a}\right|_{t_0}^{t_1} + \left|\sum_{i=1}^{n}\frac{\partial T}{\partial q_i}\delta q_i + \frac{\partial T}{\partial \dot a}\delta a\right|_{t_0}^{t_1},\end{aligned}$$

whose right-hand side provides the increment in the *action* that relates to the passage of the portion $P_0^0\,P_0^1$ of the trajectory M_0 to the corresponding one $P_1^0\,P_1^1$ of the trajectory M_1 (both of which traverse the system S with constant a and different δa), which are connected to each other by the adiabatic transformation M that is realized in the corresponding time interval $t_1 - t_0$.

The validity of (8) is completely general. In particular, in order to deduce it, one does not have to make any hypothesis in regard to the way that adiabatic parameter is varied, as long as one drops the oft-repeated demand that the derivative $\dot a$ is supposed to be a regular function of time between $t_1 - t_0$ and it must be treated as something that is as small as possible in the variational identities, like an infinitesimal quantity of the same order (viz., one) as the δq_i, etc. However, if (also within the scope of that restriction) the form that a (t) assumes in the course of the transformation is not further specified then (8), which also expresses an identity that follows from that transformation, will not permit any conclusion that is expressive, *in addition to being independent of the intermediate motions M*. Indeed, the essential intervention of M in the two integrals on the right-hand side is obvious.

The only way of making (8) *independent of M is to set* $\dot a = \varepsilon$ = *infinitesimal constant* (at least, in general). *In that way, one will specify the linear time evolution of the*

([1]) For the times when it is legitimate to transport the δ^* sign out of the integral, see LEVI-CIVITA and AMALDI, *Lezioni di Meccanica Razionale*, vol. $\mathrm{II}_{\mathrm{II}}$, pp. 507.

adiabatic parameter a. That is precisely what necessitates the definition that was mentioned in the introduction.

The verification is immediate, since in that case:

$$(t_1 - t_0)\ \dot{a} = (t_1 - t_0)\ \varepsilon = \delta a,$$

so one elides the two integrals in the right-hand side, but not the other two terms, and what will remain is:

$$\delta^* \int_{t_0}^{t_1} 2T\, dt = \left| \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} \delta q_i \right|_{t_0}^{t_1} \tag{9}$$

as the fundamental identity, which expresses the effect of an adiabatic transformation δa that is realized *linearly* and in the time interval $t_1 - t_0$.

5. Case of periodic systems. – As a simple and immediate application of (9) to the construction of *adiabatic invariants* (i.e., quantities whose values at the beginning of an adiabatic transformation remain unaltered), consider the case in which the two trajectories M_0 and M_1 are closed (and their motions are therefore periodic). If the variational formula (9) is applied asynchronously in such a way that the two complete orbits correspond then the right-hand side will be annulled (more rigorously, it will become infinitesimal of degree at least two), and one will have:

$$\delta^* \int_{t_0}^{t_1} 2T\, dt = 0.$$

One will recover a known result: viz., the adiabatic invariance of the *action* when it extends over one period (for periodic motions).

6. Observation about the mode of variation of adiabatic parameters. – It is apparent from the calculations that carried out in no. **4** that for the transformed trajectory M_1, assigning the increment δa of the adiabatic parameter and the time interval during which the transformation is performed is the same thing as determining the variation δE of the total energy of the system. In order to liberate the results of the intermediate motion M, one must, in addition, appeal to the hypothesis of linearity in time for the parameter a throughout M.

Now suppose, more generally, that other parameters besides a enter into the givens of the problem the *vis viva* and force functions, namely, $c_1, \ldots, c_\alpha, \ldots, c_m$, which are constant in the base motion and the transformed one, whereas for the motion of the transformation, they vary *in a well-defined way that is given by the base trajectory and the law of variation of a*. The c_α will then prove to be *not necessarily linear* during the interval $t_1 - t_0$ in which they are defined $c_\alpha(t)$. They are then supposed to be bounded and differentiable.

Even when the total variations of the c_α are infinitesimal, it is undoubtedly not therefore possible to treat them as further adiabatic parameters along with a, and if one wishes to do that then they must also be linear functions of t during the motion of transformation, except that it is possible to do that in one important case, and here is how:

Assign the variation δa of the parameter a and the interval $t_1 - t_0$ during which the motion of transformation occurs, but not the base motion and an origin on it. As we have seen, the transformed trajectory is determined uniquely and, in particular, it will correspond to certain increments δE, δc_α of the *vis viva* constant and the further constants c_α that are determined completely by that transformation. Conversely, suppose that the dynamical problem is such that *the values of the constants E, c_α specify the trajectory uniquely* (the temporal law is not important), so if a condition that one can recognize on a case-by-case basis is satisfied then it will be possible to consider the c_α to be adiabatic parameters whose time-dependency is – I repeat – not generally linear.

Indeed, let us try to treat the c_α as adiabatic parameters, like a, and as a result, replace the actual $c_\alpha(t)$ with the expressions:

$$c_\alpha(t) = c_\alpha^0 + \frac{\delta c_\alpha}{t_1 - t_0}(t_1 - t_0),$$

which are valid during the transformation.

It is clear that the increment that then results for c_α at the instant t_1 is equal to that of the actual transformation. Hence, *if the increment in the constant E for this virtual adiabatic transformation is equal to the actual one* (which might or might not be true) *then the hypothesis that was expressed above will be valid, and the transformed trajectory will coincide with the one that is reached by the actual transformation.*

Since the end points of the two trajectories in one mode of realizing the transformation or the other can be made to correspond, one concludes by affirming the possibility of treating the c_α as further (linearly-varying) adiabatic parameters (at the end of calculating the transformation).

In substance, when one proceeds in that way, one replaces the actual trajectory of the transformation with another one will be close to it (the expressions for c_α – namely, true and virtual – will differ in the concrete cases of infinitesimal quantities), but still have the same end points, and therefore one can make those two base trajectories correspond. Now, it is precisely that correspondence alone that we are interested in knowing about in order to evaluate the variations in the arbitrary mechanical quantities that are determined by the true adiabatic parameter. As long as it is conserved, no matter how one alters the intermediate motion, one can calculate those variations by referring to that virtual transformation.

An observation that is, in a sense, inverse to the preceding can be made in regard to the way that one realizes the variation δa of the parameter. In the preceding calculations, it was supposed to be linear. Now, if one assumes that a no longer linear, but otherwise well-defined, so the same δE, δC_α will follow from a given δa and that will once more determine the transformed trajectory uniquely, then it will be possible, if only for the sake of analytical convenience, to adopt that law of variation for $a(t)$. That observation can be useful when one agrees to replace time with a parameter that is not proportional to it.

7. Equivalent forms of the variational identity (9). – Introduce the Lagrangian function:

$$L = T + U.$$

The *vis viva* integral:

$$T - U = E$$

persists along the base trajectory M_0 and its variant M_1 with the values E_0, $E_0 + \delta E$, respectively, for the constants. Therefore, one will have:

$$2T = L + E.$$

Let T be a homogeneous quadratic function of only the $\dot{q}$ on M_0 and M_1 (recall that $\dot{a}$, which enters into the general expression for the *vis viva*, is zero on the aforementioned two trajectories). When one assumes that the canonical coordinates p_i , q_i relate to the reduced form (with $\dot{a} = 0$) of T, one will also have:

$$2T = \sum_{i=1}^{n} p_i\,\dot{q}_i\,,$$

so one can given one or the other of the following two equivalent forms to (9):

$$\text{(9}'\text{)}\qquad \delta^* \int_{t_0}^{t_1} \sum_{i=1}^{n} p_i\,\dot{q}_i\,dt = \left| \sum_{i=1}^{n} p_i\,\delta\dot{q}_i \right|_{t_0}^{t_1},$$

$$\text{(9}''\text{)}\qquad \delta^* \int_{t_0}^{t_1} (L+U)\,dt = \left| \sum_{i=1}^{n} p_i\,\delta\dot{q}_i \right|_{t_0}^{t_1}.$$

In certain cases, it can be convenient to formulate the dynamical problem from the Hamiltonian viewpoint. Therefore, consider the motions with constant a and then constant (possibly reduced) *vis viva T*.

Set $L + T + U$, so one will have, as is known:

$$H = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L = \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i}\dot{q}_i - L = \sum_{i=1}^{n} p_i\,\dot{q}_i - L.$$

The second group of canonical equations will yield:

$$\text{(10)}\qquad \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

One has $H = E$ for the motions considered, as well. If one knows the function H then one can then express $L + E$ in the following way:

(11) $$L + E = \sum_{i=1}^{n} p_i\,\dot{q}_i = \sum_{i=1}^{n} p_i\,\frac{\partial H}{\partial p_i},$$

which can be transformed, if desired, by replacing the p_i with the $\dot{q}_i$, after solving (10) for the p_i. It is possible to do that by supposing that the Hessian of the initial L with respect to the $\dot{q}_i$ is zero, and then, as one easily sees, that of H with respect to the p_i.

The observation to be gleaned from this is that when the dynamical problem is posed in Hamiltonian form, it is possible to get the corresponding function $L + E$ by algebraic calculations, and that is the function that is of interest in the fundamental variational principle of the adiabatic transformations (9″).

8. Routh systems. – Consider the elementary case in which some variables, which one calls $q_1, q_2, \ldots, q_m$, are ignorable; i.e., they do not enter into H (as always, for constant a). One will then have the corresponding first integrals:

$$p_j = c_j, \qquad j = 1, 2, \ldots, m.$$

It is also classical that the determination of the motion reduces to the integration of the canonical system that relates to the new function:

$$\mathcal{H} = H\,(q_{m+1}, \ldots, q_n\,;\,p_{m+1}, \ldots, p_n \mid c \mid a), \qquad j = 1, 2, \ldots, m$$

and the quadrature:

$$q_j = \int \frac{\partial H}{\partial c_j}\,dt, \qquad j = 1, 2, \ldots, m.$$

Two cases present themselves in regard to adiabatic transformations: Either $\dot{a}$ enters into the complete expression for the *vis viva* or it does not. In regard to the constants c_j, one sees that, in general (i.e., when the $q_1, \ldots, q_m$ are not ignorable, even in the terms in T that contain $\dot{a}$), at the end of the adiabatic transformation, in the first case, the c_j will be incremented by certain δc_j, while in the second case, they will remain constants.

Indeed, when the equations:

$$\dot{p}_j = -\frac{\partial H}{\partial q_j} = \frac{\partial L(q \mid \dot{q} \mid a, \dot{a})}{\partial q_j}, \qquad j = 1, 2, \ldots, m$$

are applied to the motion of the transformation, it will become apparent that if the $q_1, \ldots, q_m$ are ignorable in the *complete* expression for T (and therefore in the coefficients of $\dot{a}$, as well) then the right-hand side will be equal to zero for all of them, and therefore p_j = constant = c_j throughout the entire transformation. However, if $q_1, \ldots, q_m$ are ignorable only in the *reduced* form for T (viz., $\dot{a} = 0$) then the right-hand sides will no longer be zero, and a variation of the constants c_j will enter in.

In any event, (11) will become:

$$L + E = \sum_{r=m+1}^{n} p_r \dot{q}_r + \sum_{j=1}^{m} c_j \dot{q}_j ,$$

and therefore, (9″) will become:

$$\delta^* \left| \int_{t_0}^{t_1} \sum_{r=m+1}^{n} p_r \dot{q}_r \, dt + \sum_{j=1}^{m} c_j \int_{t_0}^{t_1} \dot{q}_j \, dt \right| = \left| \sum_{i=1}^{n} p_i \, \delta q_i \right|_{t_0}^{t_1} .$$

If one calculates the second integral and develops the variation then one will have:

$$\delta^* \int_{t_0}^{t_1} \sum_{r=m+1}^{n} p_r \dot{q}_r \, dt = \left| \sum_{j=1}^{m} c_j \, \delta q_j + \sum_{i=m+1}^{n} p_r \, \delta q_r \right|_{t_0}^{t_1} - \sum_{j=1}^{m} c_j \left| \delta q \right|_{t_0}^{t_1} - \sum_{j=1}^{m} \delta c_j \left| q \right|_{t_0}^{t_1} ,$$

and with some obvious simplifications:

$$\delta^* \int_{t_0}^{t_1} \sum_{r=m+1}^{n} p_r \dot{q}_r \, dt = \left| \sum_{r=m+1}^{n} p_r \, \delta q_r - \sum_{j=1}^{m} q_j \, \delta c_j \right|_{t_0}^{t_1} .$$

The importance of the duration of the adiabatic transformation in the correct construction of the adiabatic invariants was emphasized in the introduction. One can then give an example of that situation by specializing the case that was treated in that number and supposing that there are precisely $n - 1$ ignorable coordinates. Suppose that they are the first $n - 1$ of them, so the preceding relation will reduce to:

(12) $$\delta^* \int_{t_0}^{t_1} p_n \dot{q}_n \, dt = \left| p_n \, \delta q_n - \sum_{j=1}^{n-1} q_j \, \delta c_j \right|_{t_0}^{t_1} .$$

Suppose that $\dot{a}$ does not enter into T, and then suppose that $\delta c_j = 0$. In order to do that, set $\dot{q}_n \, dt = \delta q_n$ and replace the temporal integration in the integral with the corresponding integration in the plane p_n , q_n in phase space between the positions P_0 , P_1 that correspond to the instants t_0 , t_1 , resp., and one will get:

$$\delta^* \int_{P_0}^{P_1} p_n \, dq_n = \left| p_n \, \delta q_n \right|_{P_0}^{P_1} .$$

The motion will now be periodic with respect to the conjugate pair p_n , q_n for any value of the parameter a: If one extends the integration over the orbit γ that relates to that plane then the right-hand side will be zero, and what will remain will be:

$$\delta^* \int_{\gamma} p_n \, dq_n = 0,$$

with an obvious significance for the index on the integral sign, and one defines that integral to be an *adiabatic invariant.* It will then be apparent that this integral is invariant, as long as the adiabatic transformation δa that is assigned to the system *also lasts for just one period relative to* p_n , q_n .

Keep the hypothesis of periodicity in the plane p_n , q_n , so one can pass on to the other case: viz., T actually contains $\dot{a}$. Assume that our dynamical system possesses the so-called *Poisson stability*, which says that when the trajectory is not periodic, the system will certainly pass as close as one wants to any of its initial positions. Assume, as usual, that t_0 is the origin of the base motion M_0 , and in the identity (12), let t_1 be a value of time that corresponds to a position in the system that is close to the initial one, in such a way that the deviations Δp_i , Δq_i of the canonical variables p_i , q_i can be treated as first-order quantities. The projection of the trajectory onto that plane will be a closed curve γ of finite length. If one possibly restricts the agreed-upon limit to the deviation that was just mentioned then one can certainly do that in such a way that at the instant t_1 , the variables p_n , q_n are at a point on γ that is close to the one that they determine at t_0, so if τ is the period that related to p_n , q_n then $t_1 - t_0$ will differ by just a certain multiple $m\tau$ of τ. From the regularity of the motion at the instant:

$$t_1' = t_0 + m\tau,$$

which is close to t_1 , the system can be further evaluated at a deviated position by a quantity of first-order in t_0 . Now remove the prime that is affixed to t_1, and assume that t_1 is actually equal to $t_0 + m\tau$. The right-hand side of (12) will then be equal to zero because p_n , δq_n are equal at t_0 and t_1 , and the first-order difference:

$$q_j(t_1) - q_j(t_0) = \Delta q_j ,$$

which must be multiplied by δc_j , will give rise to a second-order quantity, which can then be neglected in the variational formulas. Therefore, one will have:

$$\delta^* \int_\gamma p_n \, dq_n = 0.$$

However, currently, in order for the indicated cyclic integral to be an *adiabatic invariant*, it will be necessary that the transformation δa must be realized at a time that can also be equal to a sufficiently large multiple of the period τ.

Further considerations can be carried out when T does not contain $\dot{a}$, since one can treat the integration constants c_j as further parameters in that case. Indeed, as was stated above, the proposed dynamical problem will give rise to the other one that relates to the Hamiltonian function:

$$(13) \quad \mathcal{H}(q_{m+1}, \ldots, q_n, p_{m+1}, \ldots, p_n \mid c \mid a) = H(q_{m+1}, \ldots, q_n, c_1, \ldots, c_m, p_{m+1}, \ldots, p_n \mid c \mid a),$$

and in addition, the c_j will remain constant under the transformation that relates to the parameter a, so they will be independent of it. Therefore, nothing in the dynamical problem that is defined by (13) directly forbids one from considering *a new adiabatic transformation* in which not only a is made to vary (linearly, by hypothesis), but also the quantities c_j . One will further have:

$$\delta^* \int_{t_0}^{t_1} \sum_{r=m+1}^{n} p_r \, \dot{q}_r \, dt = \left| \sum_{r=m+1}^{n} p_r \, \delta q_r \right|_{t_0}^{t_1}$$

between the base trajectory and the one that comes about as a result of the transformation [cf., (9′) and (11), which are to be evaluated with (13)], and in the case of $n - 1$ ignorable coordinates and a periodic motion in the remaining pair p_n, q_n , one will have:

$$\delta^* \int_{\gamma} p_n \, dq_n = 0$$

when the c_j are varied slowly (and linearly), as well as a, and a duration of one (or more) periods.

9. Relationship with the Hamilton-Jacobi method of integration. – The $(n + 1)^{\text{th}}$ Lagrangian equation (or more precisely, Lagrangian identity) will give ($\dot{a} = \varepsilon$ = constant from now on):

$$\varepsilon \int_{t_0}^{t_1} Q \, dt = \varepsilon \left| \frac{\partial T}{\partial \dot{a}} \right|_{t_0}^{t_1} - \varepsilon \int_{t_0}^{t_1} \frac{\partial T}{\partial a} dt \, .$$

However, one has [cf., (6′)]:

$$\delta E = \varepsilon \int_{t_0}^{t_1} \left(Q - \frac{\partial U}{\partial a} \right) dt - \varepsilon \left| \frac{\partial T}{\partial \dot{a}} \right|_{t_0}^{t_1} ,$$

from which, when one sets $L = T + U$, as usual, one will get:

$$0 = \delta E + \varepsilon \int_{t_0}^{t_1} \frac{\partial L}{\partial a} dt \, . \tag{14}$$

It will result from the sequence of calculations that the derivative of L with respect to a must be calculated by supposing that L is expressed in terms of q_i , $\dot{q}_i$, a. In addition, while $\partial L / \partial a$ must rigorously be taken along M, from the presence of the infinitesimal factor ε, one can calculate the preceding integral along the base trajectory M_0 with a negligible error. Add the preceding identity, multiplied by $t_1 - t_0$, to (9′), (9″) and get:

$$\delta^* \int_{t_0}^{t_1} 2T \, dt = \left| \sum_{i=1}^{n} p_i \, \delta q_i \right|_{t_0}^{t_1} + (t_1 - t_0) \, \delta E + \delta a \int_{t_0}^{t_1} \frac{\partial L}{\partial a} dt \, , \tag{15}$$

$$\delta^* \int_{t_0}^{t_1} (L + E)\, dt = \left| \sum_{i=1}^{n} p_i\, \delta q_i \right|_{t_0}^{t_1} + (t_1 - t_0)\, \delta E + \delta a \int_{t_0}^{t_1} \frac{\partial L}{\partial a}\, dt\,.$$

Now:

$$\delta^* \int_{t_0}^{t_1} E\, dt = (t_1 - t_0)\ \delta E + \left| E\, \delta t \right|_{t_0}^{t_1},$$

with which, the preceding identity will become (when one notes that $H = E$):

(15′) $$\delta^* \int_{t_0}^{t_1} L\, dt = \left| \sum_{i=1}^{n} p_i\, \delta q_i - H\, \delta t \right|_{t_0}^{t_1} + \delta a \int_{t_0}^{t_1} \frac{\partial L}{\partial a}\, dt\,.$$

The integrals on the right-hand sides of (15) and (15′) are the *action* and *Hamilton's principal function*, respectively. As usual, take:

$$A = \int_{t_0}^{t_1} 2T\, dt\,, \quad S = \int_{t_0}^{t_1} L\, dt\,,$$

so for the variations that correspond to the passage from the base trajectory to the transformed one that is mediated by the adiabatic transformation δa, when it is applied over a time interval $t_1 - t_0$ (which is equal to then one over which A and S are calculated), one will have:

(16) $$\delta^* A = \left| \sum_{i=1}^{n} p_i\, \delta q_i \right|_{t_0}^{t_1} + (t_1 - t_0)\, \delta E + \delta a \int_{t_0}^{t_1} \frac{\partial L}{\partial a}\, dt\,,$$

(16′) $$\delta^* S = \left| \sum_{i=1}^{n} p_i\, \delta q_i - H\, \delta t \right|_{t_0}^{t_1} + \delta a \int_{t_0}^{t_1} \frac{\partial L}{\partial a}\, dt\,.$$

Now take a to be constant and note that under certain qualitative conditions that are satisfied in dynamical problems, the action can be expressed as a function of the extreme coordinates of the trajectory and the energy E for conservative motions. That is, if one takes into account the fact that a parameter a will also enter into the present problem then one can write:

(17) $$A = A(q_i^0 \mid q_i^1 \mid E \mid a)\,,$$

while the principal function can be expressed in terms of the variables q^0, q^1, t_1, t_0. Note that for a = constant, H will not contain time explicitly, and S will depend upon t only by way of the difference $t_1 - t_0$. One will then have:

(18) $$S = S\,(q_i^0 \mid q_i^1 \mid E \mid a)\,.$$

In addition, it is known that when A, S are expressed in that way, one will have:

(19)
$$\frac{\partial A}{\partial q_i^1} = \frac{\partial S}{\partial q_i^1} = p_i^1,$$

$$\frac{\partial A}{\partial q_i^0} = \frac{\partial S}{\partial q_i^0} = -p_i^0,$$

$$\frac{\partial A}{\partial E} = t_1 - t_0,$$

$$\frac{\partial S}{\partial t_1} = H_1, \qquad \frac{\partial S}{\partial t_0} = -H_0.$$

If one substitutes these in accordance with the right-hand sides of (16), (16′) then one will get:

$$\delta^* A = \sum_{i=1}^{n} \frac{\partial A}{\partial q_i^1}\delta q_i^1 + \frac{\partial A}{\partial q_i^0}\delta q_i^0 + \frac{\partial A}{\partial E}\delta E + \delta a \int_{t_0}^{t_1} \frac{\partial L}{\partial a} dt,$$

$$\delta^* S = \sum_{i=1}^{n} \frac{\partial S}{\partial q_i^1}\delta q_i^1 + \frac{\partial S}{\partial q_i^0}\delta q_i^0 + \frac{\partial S}{\partial t_1}\delta t_1 + \frac{\partial S}{\partial t_1}\delta t_1 + \delta a \int_{t_0}^{t_1} \frac{\partial L}{\partial a} dt.$$

However, $\delta^* A$, $\delta^* S$ must be total differentials of the functions that are expressed as in (17), (18). One will then have:

(20)
$$\frac{\partial A}{\partial a} = \frac{\partial S}{\partial a} = \int_{t_0}^{t_1} \frac{\partial L}{\partial a} dt,$$

in which $\partial L / \partial a$ is calculated from $L\,(q \mid \dot{q} \mid a)$, and the integral extends along the base trajectory M_0.

As is known, the action satisfies the HAMILTON-JACOBI equation:

$$H\left(\frac{\partial A}{\partial q} \;\middle|\; q\right) = E,$$

and one proves that from the $n + 1$ constant parameters q_i^0, E that enter into A, it is always possible to choose n of them such that $A\,(q \mid q^0 \mid E \mid a)$ is a complete integral of the preceding equation.

In regard to that classical procedure for integrating the canonical equations, for the present problem in which one parameter a occurs, one will immediately get the increment of the energy E as a consequence of the adiabatic transformation δa that is applied during the time interval $t_1 - t_0$, where the system passed through q_i^0 and q_i^1 at the initial moment.

As a result of (14) and the first of (20), it follows that:

$$\frac{\partial A}{\partial a}\varepsilon = -\,\delta E,$$

so multiplying this by $t_1 - t_0$ and keeping the third of (19) in mind will give:

$$\frac{\partial A}{\partial a}\delta a = -\,\frac{\partial A}{\partial E}\delta E\,,$$

and therefore:

$$\delta E = -\,\frac{\dfrac{\partial A}{\partial a}}{\dfrac{\partial A}{\partial E}}\delta a\,.$$

Once one has solved the dynamical problem by the HAMILTON-JACOBI method, the calculation of δE will be immediate, and the quadrature that appears in (14) will not be necessary, either.

10. Gibbs adiabatic invariant. – The variational formula (9), or the equivalent one that was mentioned previously, contains everything that concerns the adiabatic transformation of a portion of an arbitrary trajectory, except that the more expressive results are realized in that theory for the closed (i.e., periodic) trajectories that were considered before in no. **5**, and the other ones that fill up the surface $H = E$ in phase space p_i , q_i densely or almost-densely. Now consider that second case. One can make two hypotheses: The surface $H = E_0$, which one calls Σ, upon which the base trajectory lies is closed, so the $2n$-dimensional volume V that is enclosed by the conjugate variables p_i , q_i will prove to be finite. In addition, the *quasi-ergodic* hypothesis is true; that is to say, as long as one considers M_0 in a sufficiently-large time interval, it will pass as close to any point of Σ, such that it will fill up Σ *densely*.

It is clear that one get a precise evaluation of the density δ by which the points of M_0 fill up Σ can be obtained only in the asymptotic case $t_1 - t_0 \to \infty$. However, one does not have to worry about the precise form that density will assume even when $t_1 - t_0$ is finite, as long as it is sufficiently large; i.e., such that the corresponding trace of M_0 will realize a covering of Σ that is dense, in practice.

The determination of the density δ was made by LEVI-CIVITA ([1]) with precise justifications in regard to uniqueness. Briefly, here is the calculation of that eminent author, which is based upon the theorem of LIOUVILLE that expresses the idea the volume in phase space that is transported by the dynamical trajectory is conserved.

([1]) "Drei Vorlesungen über Adiabatische Invarianten," *loc. cit.*

Let $H(p \mid q) = E$ be the equation of Σ, and then consider the analogous Σ' that relates to the value $E + \delta E$ of the constant (suppose that $\delta E > 0$, for convenience). Associate any point P of Σ with the corresponding P' of Σ' that is situated on the normal to Σ at P. Call the length of the relevant segment of the normal dn and let dp_i , dq_i be the corresponding increments in the variables. Set:

$$G = \left| \sqrt{\sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\right)^2 + \left(\frac{\partial H}{\partial q_i}\right)^2} \right| .$$

From the hypotheses that were made, $\delta E > 0$, so one will have:

$$\frac{dH}{dn} = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{dp_i}{dn} + \frac{\partial H}{\partial q_i}\frac{dq_i}{dn} = \frac{dE}{dn} > 0,$$

so one will have, in value and in sign:

$$dp_i = \frac{1}{G}\frac{\partial H}{\partial p_i}dn, \qquad dq_i = \frac{1}{G}\frac{\partial H}{\partial q_i}dn,$$

for the components of the segment of the normal along dn, and as a consequence:

$$dE = G\,dn.$$

If $d\sigma$ is the surface element on Σ around P then the corresponding volume that is found between Σ and Σ' will be:

$$dV = d\sigma \cdot dn,$$

and from the preceding:

$$dV = \frac{d\sigma}{G} dE .$$

Imagine that dV is transported along the dynamical trajectories so that dV is constant. The constancy of:

$$\frac{d\sigma}{G}$$

on Σ will then follow from the transport that operates in the trajectories that are situated on that Σ.

That says that the desired density δ by which those trajectories fill up Σ will be proportional to $1 / G$. That density is also true for the points of M_0 (always in the asymptotic case $t_1 - t_0 \to \infty$) by virtue of the quasi-ergodic hypothesis that was made in order for M_0 to cover all of Σ, so it alone can substitute for all of the trajectories that were considered just now.

Prof. LEVI-CIVITA also proved that $\delta = 1 / G$ will be the unique admissible density when one supposes that the canonical system does not admit any uniform integral other than $H = E$. One should refer to the cited paper for that proof.

Some further considerations lead us to see rigorously the identity of two particular means, which will constitute the keystone for the proof that we have in mind.

Cut out an elementary segment of (Euclidian) length $\Delta_0 s$ on an arbitrary trajectory of Σ. At the onset of an arbitrary time interval τ, $\Delta_0 s$ will be transported to another element of that trajectory of length $\Delta_1 s$ (in the sense that the two motions that simultaneously originate at the extremes of $\Delta_0 s$ will be represented by the extremes of $\Delta_1 s$ for $t = \tau$). One easily recognizes that:

$$\frac{\Delta s}{G}$$

is invariant in relation to that transport, and in fact:

$$G = \left| \sqrt{\sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i}\right)^2 + \left(\frac{\partial H}{\partial q_i}\right)^2} \right| = \left| \sqrt{\sum_{i=1}^{n} \dot{p}_i^{\,2} + \dot{q}_i^{\,2}} \right|$$

is the absolute value of the velocity with which the point P that represents the system moves in Euclidian phase space, so one can measure the arc length of the trajectory in the direction of motion by:

$$\Delta s = G\,\Delta t,$$

up to second-order quantities.

Since time does not enter explicitly into the equations of motion, it will be obvious that the aforementioned elements $\Delta_0 s$, $\Delta_1 s$ are traversed in the same infinitesimal interval Δt, and it will then follow from the preceding that:

$$\frac{\Delta_0 s}{G_0} = \frac{\Delta_1 s}{G_1}$$

(G_0 , G_1 are the values of G at the points of $\Delta_0 s$, $\Delta_1 s$, respectively), which express the invariance of $\Delta s / G$ that was claimed.

Now consider a number N of trajectory elements that are sufficiently large to fill an elementary region $\Delta_0\sigma$ of Σ densely; in addition, they all have equal length $\Delta_0 s$.

After time τ, that length will become:

$$\Delta_1 s = \frac{G_1}{G_0}\Delta_0 s,$$

and it will occupy a surface element of area:

$$\Delta_1\sigma = \frac{G_1}{G_0}\Delta_0\sigma,$$

by virtue of the invariance of the ratio $\Delta\sigma / G$ that was seen before. If one considers the fraction G_0 / G_1 on any trajectory element in the final position then one will get N elements that all have equal lengths $\Delta_0 s$ and occupy that fraction G_0 / G_1 of $\Delta_1\sigma$, i.e., an area that is equal to $\Delta_0\sigma$ and then equal to the one that covered by an equal number of segments of equal lengths in the initial position. One can then state that:

If one divides the trajectory that fills up Σ *densely into elementary segments of equal lengths* Δs*, and those elements can be thought of as the objects of a distribution on* Σ *then that density will be invariant along any trajectory.*

Make the hypothesis that the quantity G is never zero, or what amounts to the same thing, not all of the derivatives $\frac{\partial H}{\partial p_i}, \frac{\partial H}{\partial q_i}$ are annulled at the same time. The preceding conclusion will then be true with no restrictions for the length of the arc of the trajectory along which the transport of the density takes place. If that trajectory is then the M_0 that occupies all of Σ and one let s denote the arc length, measured in the direction of motion, then one can conclude that:

If M_0 *is divided into elements* Δs *of constant length then those linear elements will fill up* Σ *with a constant surface density.*

Clearly, that result can also be expressed in another way: If an aggregate A of *equidistant* points P is distributed on M_0, and if h is that distance then: *In the limit* $h \to 0$, $s_1 - s_0 \to \infty$ *that distribution, which is homogeneous on* M_0*, will also be (asymptotically) homogeneous on* Σ ([1]).

That amounts to saying that if one considers the aggregate A that relates to a certain h and $s_1 - s_0$, and if N is the total number of points in A, and r is the number of points that are contained in a portion of area $\Delta\sigma$ then, if Σ once more denotes the area of the surface Σ, one will have ([2]):

$$\lim_{\substack{N\to\infty \\ s_1 - s_0 \to\infty}} \frac{N\cdot\Delta\sigma}{\Sigma\cdot r} = 1$$

for any $\Delta\sigma$.

Now, let $F(p \mid q)$ be a function that is defined on Σ and is finite and continuous on it. It will also be determined at the point of M_0. We wonder how to calculate the mean $\overline{F}$ of F along the arc in M_0 of length $s_1 - s_0$. Obviously, from the definition of that definite integral, it will be:

([1]) E. Borel, *Méthodes et problèmes de la Théorie des fonctions*, 1922, pp. 30.
([2]) *Ibidem*, pp. 31.

$$\bar{F} = \frac{1}{s_1 - s_0}\int_{s_0}^{s_1} F(p \mid q)\, ds = \frac{1}{s_1 - s_0} \lim \Sigma_j\, F(p_j \mid q_j)\, \Delta s_j \,.$$

Assume that all of the elements of arc-length Δs_j are equal to h, and as with the points P_j, one must calculate $F(p_j \mid q_j)$ at the equidistant points of the (homogeneous) aggregate A at distances of h. If N is the number of points of A then one will have:

$$\Sigma_j\, F(p_j \mid q_j)\, \Delta s_j = \frac{\sum_{i=1}^{N} F(p_i \mid q_i)}{N},$$

which is, in fact, equal to the mean of the values of the function F at the points of A. Hence, since $N \to \infty$ for $h \to 0$, under the hypotheses of the existence of the two limits, one will have:

$$(\alpha) \qquad \bar{F} = \lim_{N\to\infty} \frac{\sum_{i=1}^{N} F(p_i \mid q_i)}{N}.$$

Now, calculate the surface mean of $F(p \mid q)$ over the closed surface Σ $(H = E)$, which is equal to:

$$\frac{1}{\Sigma}\int_{\Sigma} F(p \mid q)\, d\sigma = \frac{1}{\Sigma} \lim \Sigma_j\, F(p_j \mid q_j)\, \Delta\sigma_j \,.$$

Since the points of A end up being dense in Σ when $h \to 0$, $s_1 - s_0 \to \infty$, suppose that some of them lie in any $\Delta\sigma_j$. Specify that $\Delta\sigma_j$ must contain $r_j \geq 1$ and denote $P_j \equiv P_j^1$, P_j^2, …, $P_j^{r_j}$. Set:

$$F(p_j^s \mid q_j^s) = F(p_j \mid q_j) + \varepsilon_j^s, \qquad s = 2, 3, \ldots, r_j,$$

and assume that the p_j, q_j in the sum on the right-hand side of the penultimate relation are the coordinates of $P_j \equiv P_j^1$. If one sums the preceding and adds $F(p_j \mid q_j)$ to both sides then one will have (set $\varepsilon_j^1 = 0$):

$$\sum_{s=1}^{r_j} F(p_j^s \mid q_j^s) = r_j\, F(p_j \mid q_j) + \sum_{s=1}^{r_j} \varepsilon_j^s,$$

for which, one can also write:

$$\frac{1}{\Sigma} \Sigma_j\, F(p_j \mid q_j)\, \Delta\sigma_j = \frac{1}{\Sigma}\sum_j \frac{1}{r_j}\sum_{s=1}^{r_j} [F(p_j^s \mid q_j^s) - \varepsilon_j^s]\, \Delta\sigma_j \,,$$

or also, if one multiplies and divides by N in the right-hand side:

$$\frac{1}{\Sigma}\sum_j F(p_j \mid q_j)\,\Delta\sigma_j = \frac{1}{N}\sum_j\left\{\sum_{s=1}^{r_j}[F(p_j^s \mid q_j^s) - \varepsilon_j^s]\right\}\frac{N\Delta\sigma_j}{\Sigma\cdot r_j}.$$

Set:

$$\frac{N}{\Sigma}\frac{\Delta\sigma}{r_j} = 1 + \eta_j\,,$$

from which, one will know, in the meantime, that ([1]):

$$\lim \eta_j = 0 \qquad \text{when} \quad N \to \infty,\ s_1 - s_0 \to \infty,$$

and observe that:

$$\sum_j\sum_{s=1}^{r_j} F(p_j^s \mid q_j^s) = \sum_{i=1}^{N} F(p_i \mid q_i)$$

is nothing but the sum of the values of F over all points of A. One transforms the other sum of the same type analogously. One will then have:

$$(\beta) \qquad \frac{1}{\Sigma}\sum_j F(p_j \mid q_j)\,\Delta\sigma_j = \frac{1}{N}\sum_{i=1}^{N} F(p_i \mid q_i) + \frac{\sum_{i=1}^{N}\eta_i F(p_i \mid q_i)}{N} - \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i + \frac{\sum_{i=1}^{N}\eta_i\,\varepsilon_i}{N}\,,$$

in which one must suppose that the N points of A are ordered in the right-hand side in the same manner that they follow each other on M_0 .

Make $N \to \infty$, $s_1 - s_0 \to \infty$: The second and fourth terms on the right-hand side will obviously go to zero in relation to the asymptotic behavior of η_j . Hence, since the left-hand side is well-defined, under the hypothesis that the limit:

$$\lim_{\substack{N\to\infty \\ s_1-s_0\to\infty}} \frac{1}{N}\sum_{i=1}^{N} F(p_i \mid q_i)$$

exists, another limit that one calls $\mathcal{E}$ will exist, namely:

$$\lim_{\substack{N\to\infty \\ s_1-s_0\to\infty}} \frac{1}{N}\sum_{i=1}^{N}\varepsilon_i = \mathcal{E}.$$

One then makes $\Delta\sigma_j \to 0$, in order to realize the surface integral of $F(p \mid q)$ in the left-hand side of (β). If one supposes that F is continuous on Σ then the ε_j will tend to

([1]) From now on, the limit $N \to \infty$ will always be linked with $h \to 0$.

zero with the $\Delta\sigma_j$, so $\mathcal{E}$ can be made less than any number in modulus while one assumes that the $\Delta\sigma_j$ are sufficiently small. What will then remain is:

$$\frac{1}{\Sigma}\int_{\Sigma} F(p\,|\,q)\,d\sigma = \lim_{\substack{N\to\infty \\ s_1-s_0\to\infty}} \frac{1}{N}\sum_{i=1}^{N} F(p_i\,|\,q_i)\,,$$

and from (α), in which one passes to the asymptotic evaluation as $s_1 - s_0 \to \infty$:

$$(\gamma)\qquad \frac{1}{\Sigma}\int_{\Sigma} F(p\,|\,q)\,d\sigma = \lim_{s_1-s_0\to\infty} \frac{1}{s_1-s_0}\int_{s_0}^{s_1} F(p\,|\,q)\,ds\,,$$

which verifies the identity of the two means – viz., surface and asymptotic – along the trajectory under the single hypothesis of the existence of the limit in the right-hand side of (α) for $s_1 - s_0 \to \infty$, as well.

We now recall the variational identity (9′) and show, first of all, how its mode of application allows a noteworthy extension. The variation δ^* that was considered in it up to now corresponds to the passage of the base trajectory M_0 (which is situated in Σ) to the transformed one M_1 (which is Σ'). M_1 is a trajectory with a constant. Apply the *principle of varied action* to it ([1]). Under the passage of M_1 to any infinitely-close curve γ that is, like M_1, situated on Σ', we will have:

$$\delta_0^*\int_{t_0}^{t_1} 2T\,dt = \delta_0^*\int_{t_0}^{t_1}\sum_{i=1}^{n} p_i\,\dot q_i\,dt = \left|\sum_{i=1}^{n} p_i\,\delta_0 q_i\right|_{t_0}^{t_1},$$

in which, if one assumes that the p_i in the right-hand side relate to the extremes of M_0 then one will commit only a second-order error.

Take the sum of the preceding equation with (9′) and once more let δ^* denote the variation $\delta^* + \delta_0^*$, while recalling that (9′) (and therefore also the other equivalent identities) *continues to be valid under the passage from the trajectory M_0 to an arbitrary infinitely-close curve γ that is situated on the surface Σ'* ($H = E_0 + \delta E$) *that is the adiabatic transform of Σ.*

(9′), which was applied in the specified way just now, can take on the purely geometric aspect:

$$\delta\int_{s_0}^{s_1}\sum_{i=1}^{n} p_i\,dq_i = \left|\sum_{i=1}^{n} p_i\,\delta q_i\right|_{P_0}^{P_1},$$

in which the curvilinear integral is calculated along M_0, whose extremes are P_0, P_1, and the symbol of the asynchronous variation δ^* is replaced with δ, which denotes precisely the arbitrary variation, in the purely-geometric context, of M_0 to the arbitrary varied curve of Σ'.

([1]) LEVI-CIVITA and AMALDI, *Lezioni di Meccanica Razionale*, v. II, pp. 545.

If one develops the operation δ then one will get:

$$\int_{s_0}^{s_1} \sum_{i=1}^{n} (\delta p_i \, dq_i - p_i \, d\delta q_i) = \left| \sum_{i=1}^{n} p_i \, \delta q_i \right|_{P_0}^{P_1},$$

since the invertibility relation:

$$\delta dq_i = d\delta q_i$$

is obviously valid, and with one integration by parts that is applied to the second integrand:

$$\int_{s_0}^{s_1} \sum_{i=1}^{n} (\delta p_i \, dq_i - \delta q_i \, dp_i) = 0,$$

which can also be written:

$$\int_{s_0}^{s_1} \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \delta p_i + \frac{\partial H}{\partial q_i} \delta q_i \right) \frac{dt}{ds} ds \equiv \int_{s_0}^{s_1} \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \delta p_i + \frac{\partial H}{\partial q_i} \delta q_i \right) \frac{ds}{G} = 0,$$

by virtue of the canonical equations, and when one assumes that the arc-length s of the trajectory is the integration variable. The varied trajectory γ (to which M_0 is carried by δp_i, δq_i) will now be the projection of M_0 onto Σ that made along the normals to Σ.

Fix positive directions along those normals (towards the interior or the exterior of the volume that is enclosed by Σ) in such a way that when the value and sign of δn is given, which is the segment PP' that is found between Σ and Σ', one will have:

$$\delta p_i = \frac{1}{G} \frac{\partial H}{\partial p_i} \delta n, \qquad \delta q_i = \frac{1}{G} \frac{\partial H}{\partial q_i} \delta n.$$

With that, the preceding identity will become:

$$\int_{s_0}^{s_1} \delta n \, ds = 0,$$

which expresses a geometric situation that relates to any trajectory arc. That will lead to a conclusion that will be even more expressive when it is applied to an M_0 that satisfies the quasi-ergodic hypothesis.

Meanwhile, since the preceding identity is valid for any arc $s_1 - s_0$ of M_0 (i.e., it expresses a constant situation – viz., the vanishing of that integral – that relates to an adiabatic transformation δa that is realized along the arc $s_1 - s_0$ of the trajectory), one can obviously assume that it will also be valid in the limit as $s_1 - s_0 \to \infty$. That is, the integral in the left-hand side will also be zero for the (ideal) adiabatic transformation that is the limit of the transformations of infinitely-large duration. Obviously, one will also have:

$$\lim_{s_1-s_0\to\infty}\frac{1}{s_1-s_0}\int_{s_0}^{s_1}\delta n\,ds = 0,$$

from which, it will follow that:

$$\int_{s_0}^{s_1}\delta n\,ds = 0,$$

from the relation (γ).

The integral provides the variation of the volume V that is enclosed by the surface Σ. One can then see the invariance of that volume under adiabatic transformations that are infinitely-slow and linear and have durations such that corresponding M_0, with its points, will cover the surface Σ with a density that is sufficiently close to a constant. The volume V is the adiabatic invariant of GIBBS, who was essentially the first to recognize its invariant nature.

11. Dynamical systems that admit first integrals. – In order to adapt the method that was presented in nos. **1-4** to the treatment of the problems of adiabatic invariance that are of interest to theoretical physics, we shall first consider some aspects of the presence of first integrals for the equations with a constant.

We specialize the problem that we have treated in full generality up to now by supposing that a does not enter into the expression for the *vis viva,* for the sake of realizing a formal treatment that is more concise. Then, since the truly interesting questions in which one considers (or one can consider) adiabatic transformability, the parameters that determine it cannot be presented with their derivatives, as well.

Now adopt the Hamiltonian formulation of the problem. The equations:

(21) $$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i},$$

will obviously be valid for either the base motion (a = constant) or for the ones that realize the adiabatic transformation [$a = a_0 + \varepsilon\,(t - t_0)$].

Suppose that when a = constant, but arbitrary, equations (21) admit a first integral:

(22) $$F\,(p \mid q \mid a) = c,$$

along with the *vis viva* integral $H = E$. Assign the interval $t_1 - t_0$ during which the adiabatic transformation is performed that is characterized by the increment δa of the parameter, and that integral will have a value that the end of it that gives a new value of the constant c.

Let us evaluate the principal value of the increment δc.

As before, let M_0 denote a base motion that relates to a constant value a_0 of the parameter a, and let M denote the motion of the system during the associated adiabatic transformation that is also performed over the time interval $t_1 - t_0$. During M, one will have:

(23) $$a = a_0 + \varepsilon\,(t_1 - t_0),$$

with

$$\varepsilon = \frac{\delta a}{t_1 - t_0}.$$

ε must then be treated like a first-order constant quantity, along with δa.

Follow the variation of F along M. One will have:

$$\frac{dF}{dt} = \sum_{i=1}^{n} \frac{\partial F}{\partial p_i}\dot{p}_i + \frac{\partial F}{\partial p_i}\dot{q}_i + \frac{\partial F}{\partial \varepsilon}\varepsilon,$$

and upon taking the canonical equations (21) into account:

$$\frac{dF}{dt} = (F, H)_M + \left(\frac{\partial F}{\partial a}\right)_M \varepsilon,$$

in which the index M indicates that the corresponding quantity is calculated at the points of M.

Now, (F, H) is *identically zero* with respect to all of the variables that it depends upon, namely, p_i , q_i , a, since F is a first integral for any value of a. In particular, that bracket, which depends upon t by way of a, will be annulled at the points of M, so what will remain is:

$$\frac{dF}{dt} = \varepsilon\left(\frac{\partial F}{\partial a}\right)_M,$$

and up to second-order quantities, one can replace the values of $\partial F / \partial a$ that are calculated on M with the ones that are calculated at the points of M_0 , so, within the limits of that approximation:

$$\frac{dF}{dt} = \varepsilon\left(\frac{\partial F}{\partial a}\right)_{M_0}.$$

If one drops the index M_0, for simplicity, and integrates from t_0 to t_1 along the base trajectory M_0 then one will get:

(24)
$$\delta F = \delta c = \varepsilon \int_{t_0}^{t_1} \frac{\partial F(p \mid q \mid a)}{\partial a} dt,$$

which will provide the requisite increment of the integration constant c.

12. Systems that admit separation of variables. Stäckel case. – We now move on to consider systems of STÄCKEL type, which are, as is known, characterized by the fact that they admit n quadratic first integrals (including that of energy), and for which one adopts the following expressions:

(25) $$F_\alpha = \sum_{\beta=1}^{n} \Phi^{\alpha\beta} \left(\tfrac{1}{2} p_\beta^2 - U_\beta\right) = c_\alpha , \qquad \alpha = 1, 2, \ldots, n,$$

in which the $\Phi^{\alpha\beta}$ are the reciprocal in the determinant $\| \Phi^{\alpha\beta} \| \neq 0$ of the elements $\Phi_{\alpha\beta}$, and are, at the same time, functions of only the coordinate q_β that corresponds to the second index.

Agree, in addition, that $\alpha = n$ corresponds to the integral of the *vis viva*, so:

$$H = F_n \qquad \text{and} \qquad E = c_n .$$

As a consequence of the corresponding equation (25), one will have:

$$\dot{q}_h = \frac{\partial H}{\partial p_h} = \Phi^{nh} p_h ,$$

and when one expresses the p in terms of the $\dot{q}$, it will result that:

$$H = F_n = \sum_{\beta=1}^{n} \left(\frac{\dot{q}_\beta^2}{2\Phi^{n\beta}} - \Phi^{n\beta} U_\beta \right),$$

and it will appear that the *vis viva* possesses the expression:

$$T = \tfrac{1}{2} \sum_{\beta=1}^{n} \frac{\dot{q}_\beta^2}{\Phi^{n\beta}} .$$

Since T must be a positive-definite form in the $\dot{q}_\beta$ in dynamical problems, it will be clear that the condition:

(26) $$\Phi^{n\beta} > 0$$

must be imposed upon the functions $\Phi^{n\beta}$ in the entire domain of existence of any solution.

If one solves (25) with respect to the p_h then one will get:

(25′) $$p_h = \sqrt{2\left(U_h + \sum_{\alpha=1}^{n} \Phi_{\alpha h} \, c_\alpha \right)},$$

such that, in particular, the p_h will be functions of the corresponding q_h .

Assuming that the equations:

$$U_h + \sum_{\alpha=1}^{n} \Phi_{\alpha h} \, c_\alpha = 0$$

have two simple roots $q_h = a_h$, $q_h = b_h$, between which, the initial value q_h^0 is contained. One then notes that any variable q_h will perform successive librations between the corresponding extremes a_h , b_h in the course of the motion of the system, and in addition, since the sign of the root in (25′) must change for any semi-oscillation, the representative curve of the two variables in the plane of p_h , q_h (which is then the projection of the system trajectory onto that plane) will be a closed curve that is symmetric with respect to the axis q_h . Denote it briefly by γ_h . Its equation can also be written in a form that is equivalent to (25′):

$$\tag{25″} \tfrac{1}{2} p_h^2 - U_h - \sum_{\alpha=1}^{n} \Phi_{\alpha n}\, c_\alpha = 0,$$

and one will see that γ_h is *clearly specified by the values that are attributed to the constants* c_α (and to the adiabatic parameter a that enters into $\Phi_{\alpha\beta}$ and U_β).

One will now assume a base motion M_0 that corresponds to the values c_α^0 of the constants c_α, a_0 of the parameter a and the values q_h^0 of the variables q_h at time t_0 , and follows it in the interval $t_1 - t_0$. From the preceding, the corresponding trajectory projects onto the planes of the conjugate variables p_h , q_h along the closed curves γ_h^0 that relate to the values of the constants c_α^0 , a_0 , and each of which can also be traversed many times when t is contained in (t_0, t_1). Next, introduce the adiabatic parameter a, to which the increment δa is attributed and realized in the usual way in the interval (t_0, t_1); i.e., according to the linear law:

$$a = a_0 + \frac{\delta a}{t_1 - t_0}(t_1 - t_0) = a_0 + \varepsilon\,(t_1 - t_0) .$$

The relevant transformed motion M, which begins with the corresponding values q_h^0 , c_α^0 , a_0 , will take the system to a configuration that is situated on a well-defined trajectory M_1 (with a and c_α constants), along which the constants a, c_α assume the values $a_0 + \delta a$, $c_\alpha^0 + \delta c_\alpha$.

The δc_α will soon be determined. For the moment, note that M_1 also projects onto the planes p_h , q_h along certain curves that are *specified uniquely* by the new values $a_0 + \delta a$, $c_\alpha^0 + \delta c_\alpha$. In that regard, it is not essential to know the precise point of M_1 to which the motion M transports the system at the end of the transformation. It is only the increment in the c_α that is of interest.

Now, evaluate δc_α, for which it is enough to apply (24), while taking into account the expressions (25) for the c_α , and get:

$$\tag{27} \delta c_\alpha = \varepsilon \int_{t_0}^{t_1} \sum_{\beta=1}^{n} \left[\frac{\partial \Phi^{\alpha\beta}}{\partial a} \left(\tfrac{1}{2} p_\beta^2 - U_\beta \right) - \Phi^{\alpha\beta} \frac{\partial U_\beta}{\partial a} \right] dt ,$$

and from (25″):

$$\delta c_\alpha = \varepsilon\int_{t_0}^{t_1}\sum_{\beta=1}^{n}\left[\sum_{\gamma=1}^{n}\frac{\partial\Phi^{\alpha\beta}}{\partial a}\Phi_{\gamma\beta}\,c_\gamma - \Phi^{\alpha\beta}\frac{\partial U_\beta}{\partial a}\right]dt\,.$$

Therefore, it will follow from:

$$\sum_{\beta=1}^{n}\Phi^{\alpha\beta}\,\Phi_{\gamma\beta} = \varepsilon_\gamma^\alpha = \begin{cases} 1 & \text{for} \quad \alpha = \gamma, \\ 0 & \text{for} \quad \alpha \neq \gamma \end{cases}$$

that

$$\sum_{\beta=1}^{n}\frac{\partial\Phi^{\alpha\beta}}{\partial a}\Phi_{\gamma\beta} = -\sum_{\beta=1}^{n}\Phi^{\alpha\beta}\frac{\partial\Phi_{\gamma\beta}}{\partial a},$$

and one will have, by definitive:

$$(27') \qquad c_\alpha(t_1) - c_\alpha(t_0) = \delta c_\alpha = -\,\varepsilon\int_{t_0}^{t_1}\sum_{\beta,\gamma=1}^{n}\Phi^{\alpha\beta}\left(\frac{\partial\Phi_{\gamma\beta}}{\partial a}c_\gamma + \frac{\partial U_\beta}{\partial a}\right)dt\,,$$

in which the integral must be calculated by taking the values for q_h and c_γ that they take along the trajectory M. If one desires the increment in c_α at the generic instant t that is found between t and t_1 then it will be enough to replace the upper limit t_1 with variable t, and then differentiate:

$$(28) \qquad \frac{dc_\alpha}{dt} = -\varepsilon\sum_{\beta,\gamma=1}^{n}\Phi^{\alpha\beta}\left(\frac{\partial\Phi_{\gamma\beta}}{\partial a}c_\gamma + \frac{\partial U_\beta}{\partial a}\right),$$

in which the c_γ are defined by means of the left-hand sides of (25), and the latter equation is satisfied identically along the transformed motion M.

Observe that, in regard to (27), (27′), if one treats ε like a first-order constant then the integrals in the right-hand sides can also be calculated along the base trajectory M_0. Hence, by definition, in the first-order limit, the base motion M_0, along with the variation δa that is imposed upon the parameter a, will determine the increments in the constants c_α, independently of whether one knows the intermediate motion, and therefore, from what was just said, the curve γ_h^1 into which the initial curve γ_h^0 is transformed, as well.

13. Adiabatic transformation of the Sommerfeld integrals. – As is known, the integrals $\int_{\gamma_h} p_h\,dq_h$ that extends along the curves γ_h that were considered just now play an important role in the criteria for the quantization of dynamical systems, which is, in part, justified by their adiabatic invariance.

Fix any variable q_h and then the corresponding γ_h and consider the dynamical problem with one degree of freedom whose characteristic function is:

(29) $$\mathcal{H} = \tfrac{1}{2}\,p_h^2 - U_h - \sum_{\alpha=1}^{n} \Phi_{\alpha h}\,c_\alpha \;.$$

For reasons that will become clear shortly, let τ denote the temporal variable that relates to that auxiliary dynamical problem. Treat the c_α and a as constants whose values, along with the initial values of q_h , p_h at the time t_0, are those $(c_\alpha^0, a_0, p_h^0, q_h^0)$ of the base motion of the STÄCKEL system that was considered before. One will then have [cf., (25″)]:

(30) $$\mathcal{H} = 0$$

initially, and since $\mathcal{H}$ does not include time, (30) will represent the determination of the *vis viva*:

$$\mathcal{H} = \mathcal{E}$$

for the motion that was defined just now, and that one calls m_0 . (In other words, the *vis viva* constant $\mathcal{E}$ is zero for it.)

Since the problem has one degree of freedom, (30) will define the trajectory of m_0 , which will then be the curve γ_h^0 that was considered above, due to (25″), which is identified with (30). That trajectory will then be traversed by an orbital law that is different in the two cases.

Indeed, from the last of (25), one will get:

(30) $$\frac{dq_h}{dt} = \frac{\partial H}{\partial p_h} = \Phi^{nh}\,p_h$$

for the STÄCKEL system, while (29) will imply that:

(31′) $$\frac{dq_h}{dt} = \frac{\partial \mathcal{H}}{\partial p_h} = p_h \;.$$

With the substitution:

(32) $$\Phi^{nh}\,dt = d\tau,$$

by virtue of (26), as well as noting that the motion M_0 gives rise to a one-to-one relationship between the two variables t and τ:

(33) $$\tau = f(t),$$

and the system (31) transforms into (31′):

Now, from (30), (31′) can be written:

(34) $$\frac{dq_h}{d\tau} = \sqrt{2\left(U_h + \sum_{\alpha=1}^{n} \Phi_{\alpha n}\, c_\alpha\right)},$$

and if one supposes that the radical admits two roots a_h , b_h , between which q_h^0 is located, the motion m_0 of the auxiliary system will be periodic in the variable τ.

Now, vary the a and c_α, *which are considered to be precisely those functions of t that will result by replacing t with* τ *by means of* (33) *in their respective expressions in terms of t* (that are valid during the transformed motion M). In general, it is obvious that a (τ) will not be linear in τ. Let m denote the motion that results from m_0 by starting from those initial data (for $\tau = \tau_0$). The system of trajectories will be transported to another one that is *determined completely by the increments that the* c_α *and the energy constant* $\mathcal{E}$ *of the present problem experience.* Let m_1 denote the motion along γ_h^1 .

Now, one will have:

$$\mathcal{H} = 0$$

along m_0 , while one will have:

$$\mathcal{H} = \mathcal{E}$$

along m_1 .

Now, evaluate the increment $d\mathcal{E} = \mathcal{E}_1$ of the *vis viva* constant. One has ([1]):

$$\frac{d\mathcal{H}}{dt} = \frac{\partial\mathcal{H}}{\partial t} = \frac{\partial\mathcal{H}}{\partial a}a' + \sum_{\alpha=1}^{n} \frac{\partial\mathcal{H}}{\partial c_\alpha}c'_\alpha .$$

When that is integrated along m or m_0, as usual, given that a' and c'_α are once more infinitesimal, one will get:

$$\delta\mathcal{E} = \int_{\tau_0}^{\tau_1}\left(\frac{\partial\mathcal{H}}{\partial a}a' + \sum_{\alpha=1}^{n} \frac{\partial\mathcal{H}}{\partial c_\alpha}c'_\alpha\right)d\tau .$$

From (33):

$$\frac{d}{d\tau} = \frac{1}{\Phi^{nh}}\frac{d}{dt};$$

hence, if one calls the expression (28) c_α and recalls that $a = \varepsilon$, one will have:

$$\frac{d\mathcal{H}}{dt} = -\left(\frac{\partial U_h}{\partial a} + \sum_{\alpha=1}^{n}\frac{\partial \Phi_{\alpha\beta}}{\partial a}c_\alpha\right)\frac{\varepsilon}{\Phi^{nh}} + \frac{\varepsilon}{\Phi^{nh}}\sum_{\alpha,\beta,\gamma=1}^{n}\Phi_{\alpha h}\Phi^{\alpha\beta}\left(\frac{\partial U_\beta}{\partial a} + \frac{\partial \Phi_{\gamma\beta}}{\partial a}c_\gamma\right).$$

However, since $\Phi_{\alpha\beta}$, $\Phi^{\alpha\beta}$ are reciprocal, when the sum over α in last term is expanded, it will reduce to:

([1]) The derivative with respect to τ is denoted by a *prime*.

$$\frac{\varepsilon}{\Phi^{nh}}\left(\frac{\partial U_h}{\partial a}+\sum_{\gamma=1}^{n}\frac{\partial \Phi_{\gamma\beta}}{\partial a}c_\gamma\right),$$

which will vanish, from the preceding. What will remain along m is then:

$$\frac{d\mathcal{H}}{dt}=\frac{d\mathcal{E}}{dt}=0\,.$$

One concludes from this that the energy will remain constant for the intermediate motion m, due to the assumed law of variation of the a and c_α, and if it is initially zero then that will also be true for the final m; viz., for the transformed motion m_1.

Under the passage from the trajectory γ_h^0 to γ_h^1, one will then have:

$$\delta\mathcal{E} = 0,$$

and therefore, the present γ_h^1 will be just the same as the curve that relates to the motion M_1 of the STÄCKEL system.

At this point, we apply the variational principle (7′), *which does not depend upon the way that the adiabatic parameters are varied*, to our auxiliary system. Presently, the base motion m_0 and the transformed one m_1 are both periodic: The interval $t_1 - t_0$ will then be precisely equal to the period of m_0, and in addition, the asynchronism in the correspondence between the two motions will chosen in such a way that the respective periods will correspondence. The adiabatic parameters are: a, c_α.

When (7′) is completed with terms that relate to the c_α, and all of the analogous things that are written for a, and when one notes that presently:

$$Q = -\frac{\partial \mathcal{T}}{\partial a} \qquad \text{and} \qquad \delta\mathcal{E} = 0,$$

(7′) will become:

$$\delta^*\int_{\tau_0}^{\tau_1} 2\mathcal{T}\,d\tau - \delta a\int_{\tau_0}^{\tau_1}\frac{\partial \mathcal{L}}{\partial a}d\tau - \sum_{\alpha=1}^{n}\delta c_\alpha\int_{\tau_0}^{\tau_1}\frac{\partial \mathcal{L}}{\partial c_\alpha}d\tau = 0.$$

One will have invariance of the action, when extended over the period $\tau_1 - \tau_0$ – viz., the integral $\int_{\gamma_h} p_h\,dq_h$ – if and only if:

$$\delta a\int_{\tau_0}^{\tau_1}\frac{\partial \mathcal{L}}{\partial a}d\tau - \sum_{\alpha=1}^{n}\delta c_\alpha\int_{\tau_0}^{\tau_1}\frac{\partial \mathcal{L}}{\partial c_\alpha}d\tau = 0. \tag{35}$$

It would be useful to make the left-hand side of (35), which we will call A, more explicit. From (29) and (31′), we have:

$$\mathcal{L} = p_h\, q_h' - \mathcal{H} = \frac{q'^2}{2} + U_h + \sum_{\alpha=1}^{n} \Phi_{\alpha h}\, c_\alpha ,$$

from which, if we express the integral on the right-hand side of (27) that provides δc_α in terms of τ:

$$A = \delta a \int_{\tau_0}^{\tau_1} \left(\frac{\partial U_h}{\partial a} + \sum_{\beta=1}^{n} \frac{\partial \Phi_{\beta h}}{\partial a} c_\beta \right) - \varepsilon \sum_{\alpha,\beta=1}^{n} \int_{\tau_0}^{\tau_1} \Phi^{\alpha\beta} \left(\frac{\partial U_\beta}{\partial a} + \sum_{\gamma=1}^{n} \frac{\partial \Phi_{\gamma\beta}}{\partial a} c_\gamma \right) \frac{d\tau}{\Phi^{nh}} \int_{\tau_0}^{\tau_1} \Phi_{\alpha h}\, d\tau .$$

The question of recognizing whether A is or is not zero is reduced completely to the correct evaluation of the three preceding integrals. We see that this calculation (which is asymptotically rigorous) is possible only by supposing that the base trajectory M_0 along which we calculate the values of the functions in the integrands will fill up an n-dimensional region in the space of q_i *densely*. [Therefore, in the finite time interval (t_0, t_1), it will be dense enough for us to be able to consider the density that we will discuss to have been realized in practice.] That is, with the terminology that was introduced in quantum theory, under the hypothesis that the base motion M_0 presents no *degeneracy*. In particular, the substitutability of the spatial mean for the temporal one will still be proved, although the proof of that in the case of $n > 1$ will still be missing.

14. Determination of the density of the points of M_0 . – It is known that for the motion of a STÄCKEL system, any variable q_h will perform its excursions (in a finite time) between certain extremes a_h , b_h . Hence, in the space of the q_h , which is assumed to be endowed with the Euclidian metric, the trajectory is contained in the (n-dimensional) parallelepipeds:

(36) $$a_h \le q_h \le b_h , \qquad h = 1, 2, \ldots, n.$$

CHARLIER ([1]) has shown that when some relations with integer coefficients between certain moduli of periodicity are not verified, the trajectory will fill up the volume that is represented by (36), and which we call V, densely. One then asks: What is the density with which the points of the trajectory M_0 will fill up the volume V as time increases indefinitely? We shall be guided by a hydrodynamical picture. Let P be any point of V. The velocity with which a point of the trajectory passes through it will be:

(37) $$\frac{dq_i}{dt} = \Phi^{ni} \sqrt{2\left(U_1 + \sum_{\alpha=1}^{n} \Phi_{\alpha i}\, c_\alpha \right)} = \Phi^{ni} \sqrt{(i)} ,$$

in which, we have set:

$$2\left(U_1 + \sum_{\alpha=1}^{n} \Phi_{\alpha i}\, c_\alpha \right) = (i),$$

([1]) *Mechanik des Himmels*, Bd. 1, pp. 97, *et seq.*

to abbreviate the writing, and that will prove to be a function of only q_i . The sign of $\sqrt{(i)}$ will invert in the course of the motion any time that q_i reaches the extremes a_i, b_i . Hence, a finite number ν of trajectories will pass through one of its points P, which will correspond to all of the possible choices for the sign of the radical.

Consider out base trajectory M_0 and associate all of its segments along which the n roots $\sqrt{(i)}$ have the same sign. One will then get ν systems of trajectory elements, any one of which can be assumed to cover the volume V densely.

Indeed, STÄCKEL has shown that the motion of a dynamical system in which any variable q_i performs librations will be *quasi-periodic* ([1]), and more precisely, that the system will pass as close as one desires to one of its initial positions after *an integer number* of librations for any coordinate. Now, since the sign of $\sqrt{(i)}$ inverts whenever q_i touches the corresponding extreme a_i , b_i of the libration, it will just so happen that after an integer number of complete librations, that radical will recover its original sign. Hence, as long as the time interval $t_1 - t_0$ in which M_0 is defined is sufficiently large, the portions of the trajectories that belong to any of the aforementioned ν systems will be sufficiently close to each other. That statement that was just made that any of the ν will realize a dense covering of V is then valid.

We shall now determine the density with which the points of M_0 cover V.

Let ρ_r , $r = 1, 2, \ldots, n$ be the density that relates to the r^{th} of the aforementioned ν systems of segments, and let $\Delta_0 V$ be any volume inside of V. The points of $\Delta_0 V$ that belong to the segments of M_0 in question will be transported to a volume $\Delta_1 V$ after a time T that is small enough for the signs of the radicals to be preserved. The conservation of the number of points demands that one must write:

$$\int_{\Delta_0 V} \rho_r \, dV = \int_{\Delta_1 V} \rho_r \, dV ,$$

so the functions ρ_r (P) behaves like the mass density in hydrodynamics. Since the transport of $\Delta_0 V$ to $\Delta_1 V$ comes about according to the velocity law (37) (in which one assumes the r^{th} system of signs for the radical above), it will happen that ρ_r satisfies the corresponding continuity equation in a form that would be true for permanent motions, in which time does not enter explicitly in (37). One will then have the equation for ρ_r :

$$(38) \qquad \sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\rho_r \frac{dq_i}{dt}\right) = \sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\rho_r \Phi^{ni} \sqrt{(i)}\right) = 0 \qquad r = 1, 2, \ldots, N.$$

In relation to the ν values of r, one can write down just as many equations (38): *All n of them admits the same uniform integral inside of V*; represent it by:

([1]) See CHARLIER, *loc. cit.*

(39)
$$\rho = \frac{|D|}{\left|\prod_{i=1}^{n}\sqrt{(i)}\right|},$$

in which D is the non-zero determinant:

$$D = \| \Phi_{hk} \|,$$

and the denominator is the absolute value of the product of n radicals $\sqrt{(i)}$. The function ρ is then positive and finite inside of V, except that on its boundary, ρ will become infinite of order 1/2, since the quantity (i) will have simple roots there. It is therefore suitable to represent a density.

Let us verify the preceding assertion. One has:

$$\sum_{j=1}^{n}\Phi^{nj}\sqrt{(j)}\frac{\partial\rho}{\partial q_j} = \rho\sum_{j=1}^{n}\left[\sum_{e=1}^{n}\Phi^{ej}\frac{d\Phi_{ej}}{dq_j} - \frac{1}{2(j)}\frac{d(j)}{dq_j}\right]\Phi^{ej}\sqrt{(j)},$$

$$\rho\sum_{j=1}^{n}\frac{\partial}{\partial q_j}(\Phi^{nj}\sqrt{(j)}) = \rho\sum_{j=1}^{n}\left[\sqrt{(j)}\frac{d\Phi^{nj}}{dq_j} + \frac{\Phi^{nj}}{2\sqrt{(j)}}\frac{d(j)}{dq_j}\right].$$

If one sums then the last two terms will vanish, and if one lets B denote the left-hand side of (38) then what will remain is:

$$B = \rho\sum_{j=1}^{n}\sqrt{(j)}\left(\frac{\partial\Phi^{nj}}{\partial q_j} + \Phi^{nj}\sum_{e=1}^{n}\Phi^{ej}\frac{d\Phi_{ej}}{dq_j}\right).$$

Now, since the Φ^{hk} are the reciprocal elements of Φ_{hk} in the determinant D, one will have:

$$\Phi^{nj} = \frac{\partial D}{\partial\Phi_{nj}}\frac{1}{D},$$

and then, if one recalls the fact that Φ_{ej} is a function of only the q_j, so $\frac{\partial D}{\partial \Phi_{nj}}$ will be independent of q_j :

$$\frac{\partial\Phi^{nj}}{\partial q_j} = -\frac{\partial D}{\partial\Phi_{nj}}\sum_{e=1}^{n}\Phi^{ej}\frac{d\Phi_{ej}}{dq_j} = -\Phi^{nj}\sum_{e=1}^{n}\Phi^{ej}\frac{d\Phi_{ej}}{dq_j}.$$

If one substitutes this in the expression for B then one will see that the result will be:

$$B = 0;$$

i.e., the function ρ that was defined by (39) is an integral of equation (38) *when written for any of the* ν *determinations of the right-hand sides of* (37).

One then concludes that, in particular, when one assumes that $\rho_r = \rho$, the density of the r^{th} system of elements of the trajectory M_0 will be transported invariantly inside of V, along with any other of the remaining $\nu - 1$ systems. Hence, the total density with which M_0 covers V, which is the sum of the ν partial ones, will also be proportional to ρ. Obviously, one can set the proportionality factor equal to unity.

15. Uniqueness of the density. – ρ is obviously a uniform integral of (38). Is it suitable then to represent the desired density of points in M_0? Yes, because we easily see that two (or more) uniform integrals of (38) can exist. Indeed, a ρ that satisfies (38) will be a *multiplier* of the differential system (37), and it is known that if one knows two (uniform) multipliers then their ratio, which will obviously be uniform, will be an integral of the system (37). Now, we have made the hypothesis that the trajectory M_0 covers the volume V densely (viz., the *quasi-ergodic* hypothesis), and that is why we exclude the possibility that the system (37) will admit a uniform integral for the constants c_α. ρ will therefore be determined uniquely, and its expression will have the form that was given in (39).

16. Rigorous calculation of the temporal means. – One is given a function $F(q_1, \ldots, q_n)$, and the q_i are such that:

$$q_i = q_i(t),$$

which are the equations of the trajectory M_0. One would like to calculate the mean $\bar{F}$ of F over an interval of time (t_0, t_1):

$$\bar{F} = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} F[q(t)]\,dt. \tag{40}$$

Recall the convention (32) with which we defined a variable τ that always increases with t, and then substitute it for t in any regard. In particular, if τ_0, τ_1 correspond to the extremes t_0, t_1 of t then we will have:

$$t_1 - t_0 = \int_{\tau_0}^{\tau_1} \frac{d\tau}{\Phi^{nh}[q(\tau)]}, \tag{41}$$

with which:

$$\bar{F} = \frac{\displaystyle\int_{\tau_0}^{\tau_1} F[q(\tau)]\frac{d\tau}{\Phi^{nh}}}{\displaystyle\int_{\tau_0}^{\tau_1} \frac{d\tau}{\Phi^{nh}}}, \tag{40′}$$

in which all of the q (t) must be expressed in terms of the new variable τ.

The calculation of any $\overline{F}$ will then lead to the calculation of a definite integral of the type:

$$\overline{f} = \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} f[q(\tau)]\, d\tau \,,$$

in which the:

$$q_i = q_i\,(\tau)$$

are once more the equations of the trajectory M_0 .

We now move one to calculate those integrals when relying upon the fact that the points of the trajectory M_0 occupy the volume V with a density that will get closer to ρ as the time interval $t_1 - t_0$ (or the corresponding one $\tau_1 - \tau_0$) in which the motion is defined gets larger. Consequently, the procedure that was adopted will appear to be rigorously exact when one passes to the asymptotic evaluation for $\tau_1 - \tau_0 \to \infty$. In practice, the values are realized for the mean in question are approximate, but they will approach the asymptotic ones as $\tau_1 - \tau_0$ gets larger, and under the hypothesis that $\tau_1 - \tau_0$ is like that, they can be replaced with the aforementioned asymptotic means.

Consider a value of q_h that is well-defined, but arbitrary, and found between the limits of the respective libration:

(42) $$a_h \le q_h \le b_h \,,$$

and the corresponding hyperplane π_h :

$$q_h = \text{const.}$$

The quantity (h) [cf., (37)] will be non-zero on π_h since it vanishes for only $q_h = a_h$, $q_h = b_h$, and therefore the trajectory M_0 will never be tangent to π_h .

Fix $\tau_1 - \tau_0$, and let $2N$ be the number of times that crosses π_h . Since q_h varies in just one sense between a_h and b_h , and *vice versa*, $2N$ will be equal (up to unity) to the number of semi-librations of q_h that are contained in $\tau_1 - \tau_0$, and therefore the varying of q_h between the limits (42) will also be constant (always up to unity). (34) will then give the duration T of such a semi-libration in the reduced time τ as:

$$T = \int_{a_h}^{b_h} \frac{dq_h}{\sqrt{(h)}} \,,$$

so:

(43) $$\frac{\tau_1 - \tau_0}{2N} = \int_{a_h}^{b_h} \frac{dq_h}{\sqrt{(h)}} \,.$$

That equivalence is valid asymptotically as $\tau_1 - \tau_0 \to \infty$; i.e., it is enough that we remember it further. For finite $\tau_1 - \tau_0$, it is sufficiently exact when N is very large.

Consider a hyperplane π_h , along with an analogous one π'_h that is infinitely close; i.e., it relates to the value $q_h + \delta q_h$ of the coordinates q_h . π_h and π'_h cut out precisely $2N$

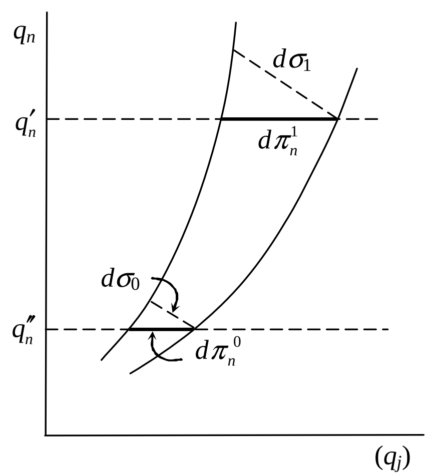
infinitesimal segments along M_0 , and on the basis of (34), *dq_h will be constant for any of them, so $d\tau$ will also be constant*, that is to say, independently of the values of the other $n - 1$ variables q_i . The contribution of all such elements of M_0 to the integral:

$$\int_{\tau_0}^{\tau_1} f[q]\,d\tau$$

is therefore:

$$d\tau \sum_{r=1}^{2N} f[q_i^{\,r}] = \sum_{r=1}^{2N} f[q_i^{\,r}] \frac{dq_h}{\sqrt{(h)}}, \tag{44}$$

also from (34), in which the $q_i^{\,r}$ for $r = 1, 2, \ldots, 2N$ are the values of the coordinates of the points P_r where M_0 crosses π_h .



17. Density of the points P_r on π_h . – Let q_h^0 and q_h^1 be two values of q_h . A tube Σ of trajectories (which belong to one of the ν systems that were mentioned many times) of infinitesimal section intersects the planes $q_h = q_h^0$, $q_h = q_h^1$ in the elements $d\pi_h^0$, $d\pi_h^1$, respectively. In addition, if $d\sigma_0$, $d\sigma_1$ are the corresponding normal sections of Σ, while υ_0, υ_1 are the velocities, and ρ_0, ρ_1 are the mass densities of the points in the two positions considered ([1]). Since the transport of points that operates on the trajectory is permanent, it will give rise to the equivalence:

$$\upsilon_0\,\rho_0\,d\sigma_0 = \upsilon_1\,\rho_1\,d\sigma_1\,. \tag{45}$$

The normal to $d\sigma$ is the direction of the relative velocity υ of $q_i = \Phi^{ni}\sqrt{(i)}$, whose direction cosine with q_h is:

([1]) In the figure, the indices h are replaced with n.

$$\frac{\dot{q}_h}{\sqrt{\sum_i \dot{q}_i^{\,2}}} = \frac{\Phi^{nh}\sqrt{(h)}}{\upsilon}.$$

One will then have:

$$d\pi_h^0 = \frac{\upsilon_0}{(\Phi^{nh}\sqrt{(h)})_0}\, d\sigma_0\,, \qquad d\pi_h^1 = \frac{\upsilon_1}{(\Phi^{nh}\sqrt{(h)})_1}\, d\sigma_1\,,$$

from which, (45) will imply that:

$$\frac{d\pi_h^0}{d\pi_h^1} = \frac{\rho_1(\Phi^{nh}\sqrt{(h)})_1}{\rho_0(\Phi^{nh}\sqrt{(h)})_0}.$$

If δ is the density of the points P_r of intersections of the hyperplane q_h = constant with the trajectory in question then one will have:

$$\delta_0\, d\pi_h^0 = \delta_1\, d\pi_h^1,$$

so, from the preceding relation:

$$\frac{\delta_1}{\delta_0} = \frac{\rho_1(\Phi^{nh}\sqrt{(h)})_1}{\rho_0(\Phi^{nh}\sqrt{(h)})_0},$$

which leads one to take the desired density to be:

$$\delta = \rho\,\Phi^{nh}\sqrt{(h)}\,. \tag{46}$$

A constant factor is obviously inessential.

One will arrive at that result for any of the ν systems of trajectories, so, if one agrees to assume that the root is positive, which will also make δ essentially positive, then (46) will represent the density of the points P_r at which the trajectory M_0 crosses the generic hyperplane q_h = const.

18. Return to the problem in no. 16. – We now move on to the evaluation of the sum that enters into formula (44). We are supported by the statistical criterion that as long as N is sufficiently large, the points P_r (as long as they are finite in number) can be replaced with a continuous distribution on p_h that has a density equal to δ. One can then assume that an approximate value of that summation is its asymptotic value:

$$\frac{1}{2N}\sum_{r=1}^{2N} f[q_i^{\,r}] = \frac{\int_{\pi_h} \delta\, f[q][dq]^h}{\int_{\pi_h} \delta[dq]^h},$$

in which the convention:

$$[dq]^h = dq_1\, dq_2 \,\ldots\, dq_{h-1}\, dq_{h+1} \,\ldots,\, dq_n\,,$$

has been adopted for brevity of notation, and the integrals extend over the section of the volume V with plane q_h = const; i.e., the n – 1-dimensional region:

$$a_i \leq q_i \leq b_i \qquad \text{for} \qquad i = 1, 2, ..., h-1, h+1, \ldots, n.$$

By definition, upon referring to (44) and what was said before, one will have:

(47)
$$\int_{\tau_0}^{\tau_1} f[q]\,d\tau = 2N \int_{a_h}^{b_h} \frac{dq_h}{\sqrt{(h)}} \frac{\int_{\pi_h} \delta f[q][dq]^h}{\int_{\pi_h} \delta [dq]^h}.$$

We show that the integral:

$$\int_{\pi_h} \delta [dq]^h = \int_{\pi_h} \frac{|D|}{\left|\Pi_i^h \sqrt{(i)}\right|} \Phi^{nh} [dq]^h$$

is independent of q_h . [The notation $\Pi_i^h \sqrt{(i)}$ is intended to mean the product of the roots $\sqrt{(i)}$, *excluding* $\sqrt{(h)}$]

Indeed, one has:

$$D\, \Phi^{nh} = \frac{\partial D}{\partial \Phi_{nh}},$$

and since the right-hand side does not contain the h^{th} column, it will be independent of q_h. The sign of the fraction in (47) can also be moved under the first integral then. If one recalls (40), (40′) then one can write:

$$\bar{F} = \frac{\int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} F[q][dq]}{\int_{\pi_h} \frac{|D|}{\left|\Pi_i^h \sqrt{(i)}\right|} \Phi^{nh} [dq]^h} : \frac{\int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} [dq]}{\int_{\pi_h} \frac{|D|}{\left|\Pi_i^h \sqrt{(i)}\right|} \Phi^{nh} [dq]^h},$$

and therefore, by definition, one will have the formula for the asymptotic calculation of the temporal mean:

(48)
$$\bar{F} = \frac{\int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} F[q][dq]}{\int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} [dq]}.$$

The preceding shows, with full rigor, that the temporal mean (*asymptotic*, so it will then exist) can be replaced with the spatial mean that one calculates by assuming that the density is $\rho = \dfrac{|D|}{\left|\Pi_i \sqrt{(i)}\right|}$.

BURGERS already assumed that formula, but justified it with the presumption that certain variables were developed into series. As shown in the text, effectively knowing the motion is not really necessary when its resulting statistical situation is based upon the only motions that are possible under the supposed conditions.

19. Proof of the adiabatic invariance of the Sommerfeld integrals. – Finally, we return to the evaluation of the quantity A in no. **13**, which is rewritten with the abbreviated notation [cf., (37)]:

$$2A = \delta a \int_{\tau_0}^{\tau_1} \frac{\partial(h)}{\partial a} d\tau - \varepsilon \sum_{\alpha,\beta=1}^{n} \int_{\tau_0}^{\tau_1} \Phi^{\alpha\beta} \frac{\partial(\beta)}{\partial a} \frac{d\tau}{\Phi^{nh}} \cdot \int_{\tau_0}^{\tau_1} \Phi_{\alpha h}\, d\tau .$$

By virtue of (47), (48), we will have [$\partial(h) / \partial a$ and $\Phi_{\alpha h}$ are functions of only q_h]:

$$\int_{\tau_0}^{\tau_1} \frac{\partial(h)}{\partial a} d\tau = 2N \int_{a_h}^{b_h} \frac{\partial(h)}{\partial a} \frac{dq_h}{\sqrt{h}} ,$$

$$\frac{1}{t_1 - t_0} \int_{\tau_0}^{\tau_1} \Phi^{\alpha\beta} \frac{\partial(\beta)}{\partial a} \frac{d\tau}{\Phi^{nh}} = \frac{\displaystyle\int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} \Phi^{\alpha\beta} \frac{\partial(\beta)}{\partial a} [dq]}{\displaystyle\int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} [dq]} ,$$

$$\int_{\tau_0}^{\tau_1} \Phi_{\alpha h}\, d\tau = 2N \int_{a_h}^{b_h} \frac{\Phi_{\alpha h}}{\sqrt{(h)}} dq_h = 2N\, \Omega_{\alpha h} .$$

Let $\Omega_{\alpha h}$ denote the generic element of the determinant:

$$\Omega = \| \Omega_{\alpha h} \| = \left\| \int_{a_i}^{b_i} \frac{\Phi_{\alpha i}}{\sqrt{(i)}} dq_i \right\| ,$$

which is undoubtedly non-zero, because since it is the volume inside of the parallelepiped:

$$a_i \le q_i \le b_i$$

one will have:

(49) $$\Omega = \int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} dq_1\, dq_2 \,\ldots\, dq_n\,,$$

and in order to have $D \neq 0$ in V, it must have the same sign everywhere. Note that:

$$|D|\,\Phi^{\alpha\beta} = \frac{\partial |D|}{\partial \Phi_{\alpha\beta}}$$

is independent of q_β, so one can write:

$$\int_V \frac{|D|}{\left|\Pi_i \sqrt{(i)}\right|} \Phi^{\alpha\beta} \frac{\partial(\beta)}{\partial a} [dq] = \int_{a_\beta}^{b_\beta} \frac{\partial(\beta)}{\partial a} \frac{dq_\beta}{\sqrt{(\beta)}} \int_{\pi_\beta} \frac{\partial |D|}{\partial \Phi_{\alpha\beta}} \frac{1}{\left|\Pi_i^\beta \sqrt{(i)}\right|} [dq]^\beta\,.$$

Now the integral that extends over π_β (i.e., the region $a_i \leq q_i \leq b_i$ for $i = 1, 2, \ldots, \beta - 1, \beta + 1, \ldots, n$) is equal to:

$$\frac{\partial |\Omega|}{\partial \Omega_{\alpha\beta}} = |\,\Omega\,|\,\Omega^{\alpha\beta},$$

in which the usual notation for reciprocal elements has been adopted. Consequently, also from (49), one will have:

$$\frac{1}{t_1 - t_0} \int_{\tau_0}^{\tau_1} \Phi^{\alpha\beta} \frac{\partial(\beta)}{\partial a} \frac{d\tau}{\Phi^{nh}} = \Omega^{\alpha\beta} \int_{a_\beta}^{b_\beta} \frac{\partial(\beta)}{\partial a} \frac{dq_\beta}{\sqrt{(\beta)}}\,.$$

Recalling that:

$$\varepsilon = \frac{\delta a}{t_1 - t_0}\,,$$

and then substituting that in the expression for A, one will have:

$$2A = 2N\,\delta a \left[\int_{a_h}^{b_h} \frac{\partial(h)}{\partial a} \frac{dq_h}{\sqrt{(h)}} - \sum_{\alpha,\beta=1}^{n} \int_{a_\beta}^{b_\beta} \frac{\partial(\beta)}{\partial a} \frac{dq_\beta}{\sqrt{(h)}} \cdot \Omega^{\alpha\beta}\, \Omega_{\alpha h} \right].$$

If one develops the sum over α then one will see that β cannot take on the value h, such that:

$$A = 0,$$

which expresses the adiabatic invariance of the SOMMERFELD integrals, when one refers to no. **13** for the significance of A.
