HISTORY

OF

THE PRINCIPLE OF LEAST ACTION

ACADEMIC INAUGURAL LECTURE

BY

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Distinguished attendees!

For all of the inspiration and interest that we can glean from the biographies of the great men of science, it is still even more instructive to pursue the manner by which an important theory or a significant theorem came about and how it developed and was redefined in course of time. It is just such a path of development that I would like to present to you, and what led me choose precisely the principle of least action as the topic of my talk is not in the least the obscurity and complexity that has woven a mysterious veil around that principle, along with its relationship to history and science, and which already come to light characteristically in the German way of referring to the law as the *Princip der kleinsten Wirkung*, insofar as that term basically says precisely the opposite of what was true for a long time of the essential content of the principle and that great fundamental law of nature.

We get a clear picture of the way that the principle of least action came about from the letters of **Daniel Bernoulli** to **Leonhard Euler** that **Euler**'s great-grandson, **Paul Heinrich von Fuss**, published in the second volume of his *Correspondance mathématique et physique de quelques cèlébres geomètres du XVIII*ème *siècle*, while **Euler**'s responses have unfortunately been lost.

Ever since **Johann Bernoulli** posed the celebrated problem of the brachistochrone, or the curve of fastest descent, towards the end of the 17th century, the mathematicians preferred to devote their attention to the isoperimetric problem, i.e., those problems for which a curve is to be determined by the constraint that a given integral should be greater or smaller for it than for all other neighboring curves, and among the great number of men of the first rank of that great era, there were none of them that had not pursued the definition and solution of such problems many times. Even in the thirtieth year of the 18th century, we also find **Daniel Bernoulli** concerned with a certain difficult problem of statics – namely, the problem of the elastic string and rod – that led him back to the isoperimetric problem. It is always greatly to one's advantage when such a reformulation gives one two completely different methods for treating those problems, and **Daniel Bernoulli** found that his static problem could be treated more easily by the isoperimetric method than with the help of known mechanical principles. Now, should it not be possible that what worked for equilibrium might also work for motion? At the time, most of the interest in dynamics was directed towards the problem of motion under the influence of central forces. It was therefore entirely natural that **Daniel Bernoulli** would pose the question of whether one could also address the *orbitas circa centra virium* with the *methodo isoperimetrica* (*), and it was likewise natural that he presented that question to the man who had also already far outstripped all of his predecessors in the theory of the isoperimetric problem at the time.

The answer to the question did not seem easy to **Euler**, since after the course of almost two years (**), **Daniel Bernoulli** came back to it and with more urgency. However, **Euler** found the solution no later than March of 1743, since **Daniel Bernoulli** had congratulated him on his beautiful discovery in a letter on 23 April of that year.

Now, that discovery, which **Euler** had, however, first published in the Fall of 1744 in *de motu projectorum* in the second Appendix to his celebrated work on the isoperimetric problem, was the simplest case of the correct and precise form of the principle of least action. We will see that he abandoned that correct form in the course of time and that it soon lost as much in power as it seemed to gain in generality.

(*) Letter, v. 28, Jan. 1741.
(**) Letter, v. 12, Dec. 1742.

However, rather than needing to say the same thing twice, might I be permitted (with no hindsight of the sequence of events) to now present the true principle of least action in its full generality, as it was presented almost a hundred years later by **Jacobi**'s direct generalization of **Euler**'s theorem (*).

When a system of material points is subject to only mutual attractions and attractions to fixed centers, the magnitude and direction of the resultant of all forces that act upon any point of the system can be expressed very simply by means of a single function – viz., the *force function* – of the masses and coordinates of the points, and the motion of the system in a space with no resistance will then obey the celebrated principle of the conservation of *vis viva*; i.e., as long as no collisions occur in the system, its *vis viva* will always be the same, or one-half the sum of the products of the mass of each point with the square of its velocity will differ from the force function only by a constant quantity. The value of that constant is determined completely whenever the magnitude of the *vis viva* is given in any position of the system, which I will assume in what follows, and the law will remain unchanged regardless of whether the system is free or subject to any sort of constraints, assuming only that the equations that express those constraints do not include time.

Along with the principle of the conservation of *vis viva*, the principle of least action is always likewise true, and it says:

If one multiplies the sum of the products of each mass with the square of its arc-element with the value that the principle of vis viva implies for twice the vis viva then when the integral of the square root of that product is taken between two given positions of the system, it will have the property that it will be smaller for the actual path of the system than it is for all other paths that the system can traverse between the two given positions without violating the constraints that are prescribed for it (**).

I can easily ignore the fact that a certain restriction must actually be observed in that way insofar as that integral does not necessarily need to always be a true minimum, since that is entirely irrelevant for the sake of dynamics. The true meaning of the law is of a purely formal nature: It consists solely of the fact that as long as the principle of *vis viva* is valid, it shows one how to reduce the determination of the path, and ultimately (since time is obtained from the latter principle by merely a quadrature when the path has been found) the motion of the system, as well, to a problem in the calculus of variations, and the complicated form of the integrals that appear in it already suggests that it might probably be impossible to discover any sort of metaphysical origin for the principle.

The law also appeared in each of the examples that **Euler** examined in succession in precisely the same form, except that an isolated point entered in place of the system, and he arrived at the theorem that he proved from it in such a way that he proved how one would obtain the same curves (differential equations, resp.) that the rules of dynamics imply for the path of the point from the demand that the integral should be a minimum. However, **Euler** expressed the general result of his investigation differently.

(*) Among **Jacobi**'s handwritten accounts, one finds several attempts (which were, sadly, always truncated) at a thorough history of the principle of least action that I have found quite stimulating and instructive. Namely, they also contain some very attractive considerations that seek to make the principle itself more intuitive in such a way that it reduces the action to time and the dynamical problem to a brachistochrone.

(**) **Jacobi**, *Vorlesungen über Dynamik*, pp. 45.

Namely, when one substitutes the value of the force function that follows from the *vis viva* equation in the integral that was given above, and in that way once more introduces the *vis viva* itself into the integral, it will be represented as the integral over time of twice the *vis viva* or also as the integral of the sum of the product of the mass of each point with its velocity and its path element, and one then also express the theorem by saying that under all of the same conditions as before, the latter integral will be a minimum, assuming that one has expressed everything under the integral that relates to only the path of the system in terms of such quantities by means of the *vis viva* principle (*).

Euler generally expressed his theorem in that form for the motion of an isolated point, and he then quite expressly emphasized that, first of all, it could be valid only when the *vis viva* principle was [so, e.g., it could be impossible for motion in a resisting medium (**)], and secondly, that one must express the velocity in the integral *ex viribus sollitantibus per quantitates ad curvam pertinentes* (***).

But how did the theorem come to be called "the principle of least action?" In order to discover the answer to that question, once we have come to understand, e.g., the beginning and the end of the principle, let us rather examine what fate had in store for it since then.

Whereas **Euler**'s work first appeared in Fall 1744, as was remarked before, in 15 April of the same year, **Maupertuis** of the Paris Academy published a treatise in which he replaced the current opposing presentation of **Fermat**'s principle of shortest time with another arbitrary hypothesis (†). Here, **Maupertuis** was still always speaking only of the motion of light. However, two years later, once he had accepted the presidency of the Berlin Academy (which delegated quite extraordinary powers to him, among other things), he extended that hypothesis to any motion at all and announced it in the Berliner Memoiren of 1746 as a universal principle of equilibrium and motion. For motion, he expressed it as:

When any change takes place in nature, the amount of action that is required for that change will be the least possible,

and as the fundamental theorem of equilibrium, he assumed that at rest, the body:

that maintains equilibrium must assume positions such that when one imparts a small motion to it, the amount of action that it creates would be smallest.

In that, following the lead of **Leibniz**, he understood *action* [which is a word that is, however, used here not in the sense of effect (*Wirkung*), but rather of *activity* (*Tätigkeit*)] to mean the product of the mass of the body with its velocity and with the space that it traverses.

(*) **Jacobi**, *Vorlesungen über Dynamik*, pp. 44.

(**) *De motu projectorum*, 13, 14, 16.

(***) *Ibid.*, 2. [DHD: "in terms of the forces that act upon quantities along the curve."]

(†) It follows from **Fermat**'s principle that under the transition from one medium to another, light will be refracted in such a way that the sines of the angles of incidence and refraction are in the same proportion as the velocities of light in the first and second medium. By contrast, according to the theory of emission, one finds precisely the opposite behavior. If one would like to arrive at that then one would need only (and this is the entire gist of **Maupertuis** treatise) to set the sum of the product of the arc length and velocity equal to a minimum instead of the time that the ray requires in order to go from a given point of the one medium to a given point of the other; i.e., in place of the sum of the arc lengths that the ray traverses in one and the other medium, each arc length is divided by the constant velocity with which it traversed it.

In order to prove his universal principle, **Maupertuis** derived it from the known laws of collision for completely hard and completely elastic bodies. And that, as well as the previous derivation of the laws of refraction and reflection, is everything that **Maupertuis** put forth as the foundation for his *principle of the least amount of action* as a great principle of economy in nature, and whose wondrous wisdom should prove the existence of God more clearly and worthily than the structure of plants and animals or the motion of planets.

By contrast, in order to understand the weakness in the foundations of that principle, which had been announced with such great fanfare, one needs only to leaf through the two articles that the celebrated **Chevalier d'Arcy** had written in the Paris Memoirs in 1749 and 1752 in order to present the principle of the conservation of surface area, and in them he showed how arbitrarily **Maupertuis** had avoided defining what he called the amount of action necessary for change, and how (when one wished to connect the word with any sort of well-defined sense at all) one must mostly understand it to mean something entirely different from what **Maupertuis** had set equal to a minimum, due to the fact that he had otherwise not arrived at the known formulas. Thus, for light, it is the sum of the actions before and after the refraction or reflection that must be a minimum, while for collisions that does not happen for the sum of the actions before and after the collision, but rather in order to get the law of collisions, one must make the *vis viva* of the motion that is lost by the collision a minimum, while one can regard the amount of action that the change produces during the collision (so consistently only the action that is used by the collision) as what is to be set to a minimum, but that would imply an entirely absurd result ([*]). In that way, **d'Arcy** convincingly contradicted the great discovery that it was action that nature was so thrifty with by a single figure, and in it he showed that under the reflection of light, **Maupertuis**'s principle could just as well give a maximum as a minimum, such that whether the great economist of nature proceeded with the greatest thrift or the most outrageous waste depended upon solely the lesser or greater concavity of the reflecting mirror ([**]). Even the law of equilibrium (or rather, the way that **Maupertuis** applied it to the lever) was subject to a compelling criticism. In fact, one sees in that application that **Maupertuis** had no idea of the fact that one could give a more correct sense to the latter law, namely, as **Euler** showed later, when one reduced the law of *vis viva* to the fact that the force function would be a maximum or a minimum in the equilibrium position, which is a property of equilibrium that **Maupertuis** himself first pointed out in in the simplest cases and communicated to the Paris Academy in 1740 as the *law of rest*. However, the fact that the law was first made applicable to another known law by that reduction likewise explained why the name of "the principle of least action" could not be retained as the law of statics, but rather should be restricted exclusively to describing a dynamical principle as a result.

Maupertuis's principle had only a purely superficial similarity in form with **Euler**'s theorem, insofar as for a motion of the same form, **Euler**'s integral of the product of mass, velocity, and arc-length would be equal to the action (so that integral itself would also be called the *action* later on), but that similarity would vanish entirely when one recalled the true meaning of **Euler**'s integral. Moreover, **Euler** assumed the *vis viva* principle as a necessary condition, and **Maupertuis** looked for the universality of his principle in precisely the fact that it was more general than the former. Finally, the conception that **Maupertuis** had given to his principle of motion makes sense only for sudden changes of velocity, and he also always got only finite

([*]) The article "Cosmologie" in the *Encyclopédie* also includes similar objections.
([**]) Mémoires de l'Academie de Paris (1752), pp. 511.

equations from it, while **Euler**'s theorem referred exclusively to a continuous motion and delivered differential equations for the path (*).

Therefore, if **Maupertuis** made mention of **Euler**'s discovery as a beautiful application of his principle in the preliminaries to his treatise, from which he wished, however, to infer truths of a higher and more important sort, then one should expect that **Euler** would have had to regard that decidedly as an entirely unjustified pretense that he indeed could probably not properly reject in light of his powerful position as an academic president, but which he would at least remain completely silent about. Only **Euler** remained silent. However, in the letters to the Berlin Academy in 1751 (which one should probably observe first appeared in 1753), he could not find enough words to express his unbounded admiration for **Maupertuis**'s principle of least action and its universal meaning, while, at the same time, he allowed his own theorem to be true for only an entirely special case of the larger principle.

In order to explain **Euler**'s quite peculiar behavior, I must speak of another reply to **Maupertuis**'s treatise that appeared in 1751 in the March issue of the *Leipziger Acta Eruditorum*, and which gave rise to a bitter dispute from which the victor did not really emerge heaped in glory.

The author of those two replies, **Samuel Koenig**, had led a rather restless life up to that point. A student of **Maupertuis**, as well as **Johann Bernoulli** (**), he was later banned from his father city of Bern due to his political activities, and was instructed in mathematics for several years by **Voltaire**'s famous friend the **Marquise du Chatelet**, but was once more rejected from that endeavor, and he finally obtained a permanent position as a professor at the university of **Franeker**. He also a foreign member of the Berlin Academy on the basis of **Daniel Bernoulli**'s warm recommendation (***). I do not need to go into more details here on his, for the most part very well-founded, factual objection that he directed superbly against the concept of equilibrium as the only position for which the action would be a minimum and that it should be, by contrast, more general than the *vis viva* principle. That is because it was not by way of the latter, but in a fragment of an apocryphal letter from **Leibniz** to **Hermann**, that **König** enclosed his dissertation, that the memorable argument would be ignited whose evolution would be described more less thoroughly in either the biography of **Voltaire** or **Fredrick the Great.**

Leibniz had already remarked in regard to that letter that the action would ordinarily be a maximum or a minimum in the modification of motions, and that one could derive the path of a body that was attracted more or less to a center from that fact, along with other important consequences.

Maupertuis, as well as **Euler**, made the priority of their laws controversial with that. Now, it was, however, quite conceivable from the outset that **Leibniz** made some meaningful discoveries that should have never been published, however. The notion of a forgery must then suggest itself, and in fact, that letter fragment was undoubtedly fabricated. However, a proof of its falsehood could not be produced, since **König** confessed that he possessed only a copy of the letter, and that the person who made that copy was beheaded in Bern due to his attempts at fomenting rebellion. Inquiries into the papers that were seized from the man who had been executed, as well as into **Hermann**'s handwritten accounts, likewise did not produce an actual material proof of when the letter had been sent. Despite that lack of an actual proof of **Maupertuis**'s claim, on 13 April 1752,

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(*) On this latter point, see: **Carnot**, *Principes fondamentaux de d'équilibre et du mouvement*, Paris, 1803, Préface, pp. 6.

(**) **Fuss**, *Correspondance*, t. II, pp. 426. *Histoire de l'académie de Berlin*, 1759, pps. 473 and 479.

(***) Letter to **Euler** in 13 June 1744.

the Berlin Academy declared the letter in question to be false and its contents null and void (*), which raised a general storm of displeasure against that process and above all against its instigators.

Already embittered in regard to **Maupertuis** through prior events, but also out of sympathy for the oppressed, **Voltaire** took the side of the latter, and on the other hand, **König**, who had indeed read **Voltaire**'s biting satires with great satisfaction, but did not wish to see the president of the academy that he had installed become a general laughingstock, went to the barricades for **Maupertuis**, and after publicly burning **Voltaire**'s second masterful lampoon, namely, his *Diatribe du docteur Akakia*, everything ended with the rift between **Friedrick** and **Voltaire**, and in that way the commotion about **Maupertuis** and the judgment of the Berlin Academy had been communicated to ever-broader circles.

I must present a brief account of this whole unproductive argument in order to make it more understandable how **Euler** could be led to champion a common cause with **Maupertuis**, and despite the completely intrinsic differences between the two theorems, to completely subordinate his significant discovery to **Maupertuis**'s vague principle. The bitterness had reached such a high level that finding a middle ground between the two parties had become impossible, and one could not overlook the fact that **Euler**, in addition to his position and the common question of priority, and also due to what was put forth in **König**'s treatise against the minimum of action in the case of equilibrium, was necessarily brought over to **Maupertuis**'s camp. Only one thing could perhaps make us wonder why, in fact, **Euler** completely ignored **d'Arcy**'s important presentation in the essays that were written against **König** and in defense of **Maupertuis**. However, that is explained most naturally by the remark that the letters to the Paris Academy from 1749 showed a publication date of 1753 on the title page.

However, **Euler** decidedly had a second ulterior motive. In his essay on the principle of least action, once he had set down how his theorem was only a special case of **Maupertuis**'s general principle and only after it had been published, added (**):

"Moreover, I did notice that beautiful property *a priori*, but *a posteriori*."

I believe that his respect for the *a priori*, his known penchant for metaphysical speculations, is a factor that should not be underestimated as something that attracted him increasingly to **Maupertuis**'s principle. That already came to light very wonderfully in the concluding words of his essay *De motu projectorum* (***), and **Daniel Bernoulli** fought vainly against it, who once wrote to him, e.g. (†):

"Herr **Rampseck** has written to my father that you have entered into various *controversiis metaphysicis publicis*. You should not bother with such matters, because one expects nothing but sublime things from you, and it is not possible for you to excel at it."

We then also come across **Euler**'s actual proof in the essays that were provoked by the **König-Maupertuis** dispute only for the case of equilibrium, while motion was addressed with the bold conclusion:

(*) *Histoire de l'académie de Berlin*, 1750, pp. 64.

(**) Mémoires de l'académie de Berlin (1751), pp. 214.

(***) Cf., **Jacobi**, *Vorlesungen über Dynamik*, pp. 43. **Dühring**, *Kritische Geschichte der allgemeinen Principien der Mechanik*, Berlin, 1873, pp. 298.

(†) Letter on 29 April 1747.

"If it is nature's intention to skimp on the exertion of force in equilibrium then that intent must also extend to motion (*)."

Not satisfied with the fact that he had based the law of rest only in a strictly mechanical way, he even sought to also give a purely metaphysical demonstration of it (**).

However, it is was self-explanatory that in those treatises of **Euler**, the principle of least action appeared in a much more reasonable form that it did in **Maupertuis**, anyway. Action was defined to be the integral of the product of mass, velocity, and the arc-length element, and nothing was said about how the principle could be true without the *vis viva* principle. However, nothing further was said about the essential constraint that everything under the integral must be reduced to merely spatial quantities by means of the latter principle, and nothing probably could be said, since citing it would have immediately disturbed the entire appearance of similarity between **Euler**'s discovery and **Maupertuis**'s principle. However, the fact that omitting that constraint would make our theorem completely absurd is illuminated directly by the fact that the forces that act upon the system do not occur at all in the action integral. It might then be impossible for the problem of making the action a minimum to lead to a solution for the path of the system when it depends upon the forces in an essential way.

Whereas **Euler** himself thus sought to make the principle of least action emerge from metaphysical necessity, and increasingly at the cost of rigor, it is interesting to see how **Daniel Bernoulli** viewed the matter in a completely different way and how correctly he recognized the true nature of the law. Hence, he wrote to **Euler**, e.g., on 25 Dec. 1743:

"I doubt whether one can ever show *a priori* that the *elastica* must generate a *maximum solidum*. I consider that to be a property that the calculus exhibits and that no man could have foreseen *ex principiis novis*, much less for the *identitatem isochronae et brachystochronae*. That sort of *proprietates* are *ratione nostri* accidental, as it were, and I also consider the *observatuam proprietatem orbitarum*, etc., to rest upon that same basis."

Indeed, it does not sound as if **Daniel Bernoulli** had guessed the complete perfection of analytical form that mechanics was to accrue from the calculus of variations (***) when he said on another occasion (12 Dec. 1742), with special reference to the problem of central motion:

"One cannot investigate the *principia maximorum et minimorum* enough… In my opinion, it is *argumentum inter omnia pure analytica utilissimum* and that is a true example of how *vel sola proposition problematis*, when one does not have the solution either, *saepe maxima laude Digna sey*."

However, the task of following through on what **Daniel Bernoulli** had in mind was left to someone else. Just as the 23-year-old **Lagrange** freed the calculus of variations from the shackles of geometry and gave it its present analytical form, he must also have necessarily directed his attention to **Euler**'s theorem, whose great formal meaning his sharp eyes, which were not clouded by any sort of metaphysical inclination, could not avoid. In the Turin Miscellenea of 1760-61, **Lagrange** not only extended the principle of least action, as he himself called it, to all problems

(*) Mémoires de l'académie de Berlin (1751), pp. 175.
(**) *Ibid.*, pp. 246.
(***) Remark by **Jacobi**.

in mechanics for which only mutual attractions and attractions to fixed centers were in effect, but also inferred almost all of the great results that he had derived later in *Mécanique analytique* in a different way from it, such that, as **Jacobi** (*) justifiably remarked, **Lagrange**'s principle of least action was the mother of all of our analytical mechanics.

However, just as the curse of ambiguity rested upon **Maupertuis**'s meddling in the principle of least action, here, we see the miraculous phenomenon that we are not at all able to specify which law **Lagrange** had actually called the general principle of least action (at least, not with absolute certainty). Indeed, in my opinion, the principle that **Lagrange** had to thank for such significant results, and that he later once more completely pushed aside so undeservedly, just as if he had sensed that ambiguity, was a theorem that was already completely different from the **Euler-Jacobi** principle of least action in the conditions for its validity, and it is indeed that same theorem that has served as the starting point for the latest and greatest unfolding of the analytical aspects of mechanics under entirely different names.

Thus, in fact, the way that **Lagrange** had expressed the principle was meaningless, because he completely omitted the necessary condition that one must eliminate time from the action integral by means of the *vis viva* equation. Thus, **Lagrange** might have also intended that the theorem should be impossible. Therefore, the editor of the 3rd edition of *Mécanique analytique*, **Joseph Bertrand**, had already added the remark that **Lagrange**'s conception of the principle of least action was imprecise and that from the proof that **Lagrange** gave, the theorem should rather read:

When taken from a given starting point to a given endpoint, the integral of twice the vis viva should be smaller for the actual motion than it is for all other motions, after posing the new constraint that the system can go from the given initial position to the given final position without altering the equation of vis viva (**).

In fact, from a purely mathematical standpoint, one can regard that theorem as an equivalent form of the **Euler-Jacobi** principle. However, when **Lagrange** prescribed the *vis viva* equation for the motions to which the actually-occurring motion was to be compared in the given way as the constraint equation, as I have proved on another occasion (***), his proof of the principle would be false, and indeed false in such a way that one would need to continue the argument by only one more step in order to show that no motion at all besides rest would result because of that principle. Thus, **Lagrange** could not have concluded that. In my opinion, only one way of avoiding that remained, namely, that **Lagrange**, far from considering the *vis viva* equation to be the prescribed constraint equation, employed it only to reduce the integral that represented the action to another integral, and indeed to the integral over time of the *vis viva* plus the force function (†). However, if that conception by which **Lagrange**'s proof becomes clear and rigorous were correct then the principle that we have to thank for **Lagrange**'s great mechanical discoveries would be nothing but that celebrated law that **Hamilton** (††) first actually expressed in 1835, which reduced all problems in mechanics in which all forces that are in effect are representable in terms of a force function to the calculus of variations, independently of whether the *vis viva* principle is or is not true, namely,

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(*) *Vorlesungen über Dynamik*, pp. 2.

(**) *Mécanique analytique*, 3rd ed., t. I, pp. 277. Cf., also *Handbuch der theoretischen Physik* by **Thomson** and **Tait**, v. I, Th. 1, pp. 259.

(***) Jahrbuch über die Fortschritte der Mathematik, Jahrgang 1871, pp. 174.

(†) It was not really the integrals itself that **Lagrange** has reduced to each other, but their variations.

(††) Philosophical Transactions, 1835, I, pp. 99.

it would be nothing but **Hamilton**'s principle (*), which was simultaneously discovered and concealed for 75 years, to use **Jacobi**'s phraseology.

We then see that the principle of least action played a very vague and ambivalent role in history, as in science. In its youth, it was complicated by a dispute that was essentially due to only the initial ambiguity in the concept that **Maupertuis** had introduced, but that made it possible for its final consequences to enter into history itself, and in a more mature state of development, it made its greatest contributions to science, while being confused with another and greater principle. When we now ask ourselves what it means to us today, we can only answer that question with:

The principle of least action, as **Euler** and **Jacobi** conceived it, is a much more beautiful, and insofar as it leads directly to the differential equations of the path while saving one two integrations, a much more useful theorem, but to which **Hamilton**'s principle and the older principles of mechanics could not probably raise any actual objection to calling it by the lofty name of a "principle." By contrast, as far as **Bertrand**'s form of the principle is concerned (which is the first time that a proof was given, moreover), it should certainly not be overlooked for any purpose or use, since it can still accomplish everything that **Hamilton**'s principle achieves much more simply and naturally in all cases, but only to a lesser extent, despite a disproportionate cost, so to speak.

Now that we have clarified the mechanical meaning of the principle of least action, in conclusion, let us say a few words about its basic metaphysical ideas! Is that ancient representation of nature as an economical worker actually just a beautiful illusion that dissipates upon closer inspection? No, of course not! Rather, the great worker of nature is always thinking of how to achieve its purpose with the least-possible means, but it does not economize on action, nor *vis viva*, nor energy, but rather, as we have learned from **Gauss**'s wonderfully intuitive representation of the fundamental principle of mechanics, it economizes on something that we humans would most dearly like to conserve, namely, *constraint*. However, going further in that direction would not belong to the history of the principle of least action!

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(*) **Jacobi**, *Vorlesungen über Dynamik*, pp. 58.