"Die Kriterien des Maximums und Minimums der einfacher Integrale in den isoperimetrischen Problemen," Math. Ann. **13** (1878), 55-68.

The criteria for the maxima and minima of simple integrals in isoperimetric problems (*)

By A. MAYER in Leipzig

Translated by D. H. Delphenich

According to the theory that is assumed in the textbooks, the isoperimetric problem of finding the relative greatest or smallest value of the given integral:

$$V = \int_{x_0}^{x_1} f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx$$

when the only functions $y_1, \ldots, y_n$ that should be taken under consideration are the ones for which a series of other given integrals:

$$V_\kappa = \int_{x_0}^{x_1} f_\kappa(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx \qquad \kappa = 1, 2, \ldots, m$$

maintain prescribed values is completely identical to the problem of making the integral:

$$\int_{x_0}^{x_1} (f + \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m)\, dx$$

an absolute maximum or minimum, in which $\lambda_1, \lambda_2, \ldots, \lambda_m$ mean undetermined constants that will be determined later in such a way that the integrals V_κ will assume given values. Now, that is generally quite correct as long as one merely deals with the problem, viz., the determination of the unknown functions y. By contrast, if one wishes to also answer the question of whether, and within what limits, the functions y that are found will produce a true maximum or minimum in the same way in both problems then that would imply, for example, that the center of mass of a homogeneous string that hangs from both ends is not at all the lowest-possible position that it assumes for each position of its endpoints, which is obviously absurd. It is therefore clear that it is

(*) From the Ber. der Kgl. Sächs. Ges. d. Wiss. July 1877.

impossible for the criteria for the maximum and minimum to be the same in both problems. For the case of a single unknown function y, but for which differential quotients of arbitrarily-higher orders could appear in the integrals, that was emphasized in the treatise by the Swedish mathematician **Lundström** in 1869 (which was also, unfortunately, the year that he died): "Distinction des maxima et des minima dans un problème isopérimétrique," Nova acta reg. soc. Sc. Upsaliensis, series 3, vol. VII, in which the correct criteria for a maximum or minimum for the isoperimetric problem were likewise exhibited.

If one overlooks the single fact that as a result of his imprecise expression of his conclusions, and formulas that were, in part, not entirely correct, his conclusions were just hard to understand, and it would probably be impossible to prove that the criteria that were known to be necessary would be, at the same time, sufficient in **Lundström**'s way. Namely, in my opinion, that last, most difficult, point can be resolved only by the procedure of **Jacobi** and **Clebsch** that puts the second variation of the integral in question into its simplest form, and it was just that transformation, which depends upon the integration of differential equations, that **Lundström** intentionally avoided as too complicated.

Now, in my Habilitationsschrift (*), I used the **Clebsch** reduction to develop criteria for the maximum or minimum for the general problem that includes all problems in the calculus of variations for which only a single independent variable appears, as **Lagrange** (**) and **Clebsch** (***) had shown. Therefore, the special criteria for the isoperimetric problem must be included in those general criteria for a maximum or minimum. It was the wish to establish the latter criteria by deriving them from the stronger ones, and in that way to simultaneously show the applicability of my general criteria to the various special classes of problems in the calculus of variation in the most important example, that gave rise to the present note. I shall refer to the paper: "Ueber die Kriterien des Maximums und Minimums der einfachen Integrale," Borchardt's J., **69**, in which the investigations of my Habilitationsschrift were reproduced, in a partially-altered representation, and summarize that derivation in § **1.**

Moreover, a very remarkable reciprocity theorem comes to light in the isoperimetric problems, according to which, any isoperimetric problem with m isoperimetric conditions is equivalent to m other isoperimetric problems in such a way that not only the solution, but also the limits within which the solution will produce an actual maximum or minimum are common to all $m + 1$ problems. That reciprocity theorem is merely a consequence of **Euler**'s rule for solving the isoperimetric problem and the form that the second variation presents before each reduction in those problems. The derivation of the reciprocity theorem from the criteria in § **1** will then be considered to be a welcome confirmation of those criteria.

Finally, in the last section, the application of the criteria and the reciprocity theorem shall explain just that example of the equilibrium figure of a massive homogeneous string, or what amounts to the same thing, the problem of the curve of given length and lowest center of mass.

In what follows, I shall always consider only the simplest case, in which the limits x_0 and x_1, as well as the values that the unknown functions y assume at those two limits, are fixed, because all other cases can be reduced to that case. In regard to that reduction, I shall refer to my article

(*) "Beiträge zur Theorie der Maxima und Minima der einfachen Integrale," Leipzig 1866.
(**) *Leçons sur le calcul des fonctions*, 1806 edition, pps. 466 and 469.
(***) Borchardt's Journal, **55**, pp. 336.

that was cited above, which will also easily shed light upon the question of how one must proceed when the unknown functions y are subject to given differential equations, in addition to the isoperimetric conditions, or when higher-order differential quotients of the y appear in the integrals, which is only a special case of that.

§ 1.

Criteria for a maximum or minimum.

In the cited article, whose page numbers shall be referenced in brackets in what follows, I treated the problem:

I. *Determine the functions $y_1, \ldots, y_m$, between which m condition equations are prescribed:*

$$\varphi_\kappa (x, y_1, \ldots, y_n, y_1', \ldots, y_n') = 0 \qquad \kappa = 1, 2, \ldots, m,$$

such that the integral:

$$V = \int_{x_0}^{x_1} f(x, y_1, \ldots, y_n, y_1', \ldots, y_n')\, dx,$$

with given limits and limiting values, will be a relative minimum or maximum,

and obtained [pp. 260] the following criteria for the maximum or minimum:

Problem I. will be solved by the $n + m$ ordinary differential equations in the independent variable x and the $n + m$ dependent variables $y_1, \ldots, y_n, \lambda_1, \ldots, \lambda_m$:

(1) $$\frac{\partial \Omega}{\partial y_h} = \frac{d}{dx}\frac{\partial \Omega}{\partial y_h'}, \qquad \varphi_\kappa = 0,$$

in which:

(2) $$\Omega = f + \lambda_1 \varphi_1 + \lambda_2 \varphi_2 + \ldots + \lambda_m \varphi_m ,$$

and in order for one to be able to satisfy the $2n$ limit conditions, the functions $y_1, \ldots, y_n$ that are obtained by the complete integration of those equations must include $2n$ arbitrary constants $a_1, \ldots, a_{2n}$. If one has expressed those constants in terms of the given limiting values then (except for the special cases that always occur [pps. 241, 260]) in order for the functions thus-obtained to produce an actual relative maximum or minimum, it is sufficient and (also, at least in general) necessary that the upper limit x_1 (which I always assume to be > x_0) should remain between x_0 and the next root of the limit equation:

(3) $$\sum \pm \frac{\partial y_1}{\partial a_1} \cdots \frac{\partial y_n}{\partial a_n} \frac{\partial y_{10}}{\partial a_{n+1}} \cdots \frac{\partial y_{n0}}{\partial a_{2n}} = 0$$

after x_0, and that the homogeneous function of degree two:

(4) $$2W = \sum_{h=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 \Omega}{\partial y'_h \, \partial y'_i} U_h U_i \, ,$$

whose n arbitrary arguments $U_1, \ldots, U_n$ are subject to the m condition equations:

(5) $$\sum_{h=1}^{n} \frac{\partial \varphi_\kappa}{\partial y'_h} U_h = 0 \, ,$$

cannot change sign between x_0 and x_1.

In formulas (3), (4), (5), one understands $y_1, \ldots, y_n, \lambda_1, \ldots, \lambda_m$ to mean those functions of x, $a_1, \ldots, a_{2n}$ that are obtained by the complete integration of equations (1). One must ascribe the fixed values to the integration constants $a_1, \ldots, a_{2n}$ themselves that one would get from the $2n$ limit conditions, and finally y_0 denotes the value of the function y_h for $x = x_0$.

I based the following upon that result and considered the isoperimetric problem, moreover:

II. *Determine functions* $y_1, \ldots, y_n$ *of* x *that are subject to the* m *isoperimetric conditions:*

$$\int_{x_0}^{x_1} f_\kappa (x, y_1, \ldots, y_n, y'_1, \ldots, y'_n) \, dx = \lambda_\kappa \, , \qquad \kappa = 1, 2, \ldots, m$$

and keep the same values at the two given limits x_0 *and* x_1 *such that the integral:*

$$V = \int_{x_0}^{x_1} f \, (x, y_1, \ldots, y_n, y'_1, \ldots, y'_n) \, dx$$

will be a relative maximum or minimum (in which, obviously, m is no longer subject to the restriction that $m < n$, as in Problem I).

If one follows **Lagrange**'s procedure and introduces m new variables $u_1, \ldots, u_m$ by the substitutions:

$$u_\kappa = \int f_\kappa \, dx$$

then one can replace the isoperimetric conditions with the m condition equations:

$$f_\kappa - u'_\kappa = 0 \, ,$$

coupled with the $2m$ limit conditions:

$$[u_\kappa]_{x=x_0} = \alpha_\kappa\,, \qquad [u_\kappa]_{x=x_1} = \alpha_\kappa + l_\kappa\,,$$

and consider the initial values α_κ in the latter to be given quantities, with which Problem II assumes the form:

III. *Determine the* $n + m$ *functions* $y_1, \ldots, y_n, u_1, \ldots, u_m$, *which are coupled by* m *given condition equations:*

(6) $$\varphi_\kappa = f_\kappa - u'_\kappa = 0\,,$$

in such a way that the integral V *will be a relative maximum or minimum for given limiting values of* $x, y_1, \ldots, y_n, u_1, \ldots, u_m$.

However, Problem III is only a special case of Problem I, and one can then apply the rule that was given above to it.

According to it, the differential equations in Problem III will be:

$$\frac{\partial \Omega}{\partial y_h} = \frac{d}{dx}\frac{\partial \Omega}{\partial y'_h}\,, \qquad \frac{\partial \Omega}{\partial u_\kappa} = \frac{d}{dx}\frac{\partial \Omega}{\partial u'_\kappa}\,, \qquad \varphi_\kappa = 0\,.$$

However, when one sets:

$$F = f + \lambda_1\,\varphi_1 + \lambda_2\,\varphi_2 + \ldots + \lambda_m\,\varphi_m\,,$$

they will automatically reduce to the equations:

$$\frac{\partial F}{\partial y_h} = \frac{d}{dx}\frac{\partial F}{\partial y'_h}\,, \qquad 0 = -\frac{d\lambda_\kappa}{dx}\,, \qquad f_\kappa - u'_\kappa = 0\,.$$

Problem III will be solved as follows by the n differential equations:

(7) $$\frac{\partial F}{\partial y_h} = \frac{d}{dx}\frac{\partial F}{\partial y'_h}\,,$$

in which $\lambda_1, \lambda_2, \ldots, \lambda_m$ are regarded as undetermined constants, and after integrating those equations, that will yield the u_κ in terms of quadratures:

$$u_\kappa = c_\kappa + \int f_\kappa\, dx\,.$$

In order for one to obtain the required number of arbitrary constants for one to be able to satisfy the 2 ($n + m$) prescribed limit conditions, it is necessary and sufficient that the n equations (7) are soluble for the n second differential quotients y_1'', …, y_n''.

Since one further has:

$$\frac{\partial^2\Omega}{\partial u_k' \, \partial y_h'} = \frac{\partial^2\Omega}{\partial u_k' \, \partial u_i'} = 0 ,$$

from (2) and (3), the function $2W$ for the Problem III will then reduce to:

$$2W = \sum_{h=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 F}{\partial y_h' \, \partial y_i'} U_h U_i ,$$

and the m condition equations (5) will become:

$$\sum_{h=1}^{n} \frac{\partial f_\kappa}{\partial y_h'} U_h = V_\kappa .$$

However, those m equations determine only the quantities V_1, …, V_m, which do not enter into the function $2W$ at all, as functions of the arguments U_1, …, U_n . They therefore do not restrict the arbitrariness of those arguments in any way and can thus be omitted completely.

Finally, when one understands the a_1, …, a_{2n} to now mean $2n$ arbitrary constants that come with the complete integration of equations (7), the limit equation in Problem III will become:

$$\sum \pm \frac{\partial y_1}{\partial a_1} \cdots \frac{\partial y_n}{\partial a_n} \frac{\partial y_{10}}{\partial a_{n+1}} \cdots \frac{\partial y_{n0}}{\partial a_{2n}} \frac{\partial u_1}{\partial \lambda_1} \cdots \frac{\partial u_1}{\partial \lambda_1} \frac{\partial u_{10}}{\partial c_1} \cdots \frac{\partial u_{m0}}{\partial c_m} = 0 .$$

However, since:

$$\frac{\partial y_h}{\partial c_k} = \frac{\partial y_{h0}}{\partial c_k} = \frac{\partial u_h}{\partial c_k} = \frac{\partial u_{h0}}{\partial c_k} = 0$$

and

$$\frac{\partial u_k}{\partial c_k} = \frac{\partial u_{k0}}{\partial c_k} = 1 ,$$

that equation will reduce to:

$$\sum \pm \frac{\partial y_1}{\partial a_1} \cdots \frac{\partial y_n}{\partial a_n} \frac{\partial y_{10}}{\partial a_{n+1}} \cdots \frac{\partial y_{n0}}{\partial a_{2n}} \frac{\partial v_1}{\partial \lambda_1} \cdots \frac{\partial v_m}{\partial \lambda_m} = 0 ,$$

in which:

(8) $$v_k = u_k - u_{k0} = \int_{x_0}^{x_1} f_k \, dx ,$$

and one will then obtain the following criteria for a maximum or minimum for the isoperimetric problem II from the rule that was cited for Problem I:

IV. *Problem II is solved by the n differential equations:*

$$\frac{\partial F}{\partial y_h} = \frac{d}{dx}\frac{\partial F}{\partial y'_h},$$

in which:

$$F = f + \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_m f_m ,$$

and the λ mean undetermined constants. The complete integration of those equations, assuming that the second differential quotients $y''_1, \ldots, y''_n$ cannot be eliminated from them, will yield $y_1, \ldots, y_n$ as functions of x, the m isoperimetric constants $\lambda_1, \ldots, \lambda_m$, and 2n integration constants $a_1, \ldots, a_{2n}$. If one has determined those $2n + m$ constants from the m isoperimetric ones and the 2n limit conditions then (except for exceptions that can occur only in special cases and by their nature do not obey the general rules) *the functions $y_1, \ldots, y_n$ thus-obtained will yield a true relative maximum or minimum of the integral V when the homogeneous function of degree two in the n independent variables $y_1, \ldots, y_n$:*

$$2W = \sum_{h=1}^{n}\sum_{i=1}^{n} \frac{\partial^2 F}{\partial y'_h \,\partial y'_i} U_h U_i ,$$

always has the same sign within the limits of integration, and as long as the upper limit x_1 remains between x_0 and the next root of the limit equation:

$$\Delta(x_0, x_1) = \sum \pm \frac{\partial y_1}{\partial a_1}\cdots\frac{\partial y_n}{\partial a_n}\frac{\partial y_{10}}{\partial a_{n+1}}\cdots\frac{\partial y_{n0}}{\partial a_{2n}}\frac{\partial v_1}{\partial \lambda_1}\cdots\frac{\partial v_m}{\partial \lambda_m} = 0 ,$$

in which the functions v_k are calculated by the quadratures:

$$v_k = \int_{x_0}^{x_1} f_k \, dx .$$

By contrast, if the first condition is not fulfilled then there will be neither a maximum nor a minimum, and in general the same thing will also be true when x_1 attains or exceeds the given limit.

§ 2.

The reciprocity theorem for the isoperimetric problem.

Problem II, to which Theorem IV refers, can be reproduced briefly in symbols thus:

$$(\alpha)\qquad \begin{cases} V = \int_{x_0}^{x_1} f\, dx = \text{max., min.}, \\ \int_{x_0}^{x_1} f_1\, dx = l_1, \quad \int_{x_0}^{x_1} f_2\, dx = l_2, \quad \cdots \quad \int_{x_0}^{x_1} f_m\, dx = l_m . \end{cases}$$

We now compare that with another isoperimetric problem, which is included in the formulas:

$$(\beta)\qquad \begin{cases} V_1 = \int_{x_0}^{x_1} f_1\, dx = \text{max., min.}, \\ \int_{x_0}^{x_1} f\, dx = l, \quad \int_{x_0}^{x_1} f_2\, dx = l_2, \quad \cdots \quad \int_{x_0}^{x_1} f_m\, dx = l_m . \end{cases}$$

I assume that the same fixed values for the limits and limiting values have been prescribed in both problems.

If we introduce homogeneous isoperimetric constants for the sake of ease of comparison, i.e., we set:

$$\lambda_k = \frac{\mu_k}{\mu},$$
$$\mu F = \mu f + \mu_1 f_1 + \ldots + \mu_m f_m = \Phi ,$$

and imagine that the determinant $\Delta(x, x_0)$ will change by only a constant factor when we introduce any $2n + m$ independent functions of the constants $a_1, \ldots, a_{2n}, \lambda_1, \ldots, \lambda_m$ in place of those constants as new constants then we can also express Theorem IV as:

Problem (α) will be solved by the n differential equations:

$$\frac{\partial \Phi}{\partial y_h} = \frac{d}{dx}\frac{\partial \Phi}{\partial y'_h}, \tag{9}$$

whose complete integration will determine the $2n$ integration constants $a_1, \ldots, a_{2n}$ and the ratios of the $m + 1$ isoperimetric constants $\mu_1, \mu_2, \ldots, \mu_m$ from the $2n$ limit conditions and the m isoperimetric conditions:

$$\int_{x_0}^{x_1} f_1\, dx = l_1, \quad \int_{x_0}^{x_1} f_2\, dx = l_2, \quad \ldots, \quad \int_{x_0}^{x_1} f_m\, dx = l_m, \tag{10}$$

and the functions $y_1, \ldots, y_n$ thus-obtained will produce a true maximum or minimum for the problem (α) as long as the upper limit x_1 remains between x_0 and the next root of the equation:

$$\Delta_\alpha (x_0 , x_1) = \sum \pm \frac{\partial y_1}{\partial a_1} \cdots \frac{\partial y_n}{\partial a_n} \frac{\partial y_{10}}{\partial a_{n+1}} \cdots \frac{\partial y_{n0}}{\partial a_{2n}} \frac{\partial v_1}{\partial \mu_1} \frac{\partial v_2}{\partial \mu_2} \cdots \frac{\partial v_m}{\partial \mu_m} = 0 ,$$

assuming, moreover, that the homogeneous function of degree two:

$$2 W_\alpha = \frac{1}{\mu} \sum_{h=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial y'_h \, \partial y'_i} U_h U_i$$

always keeps the same sign between x_0 and x_1 .

If we now move on to problem (β) then equations (9) and the limit conditions will remain completely unchanged for it, and the isoperimetric conditions (10) will change only to the extent that now, in place of the condition:

$$\int_{x_0}^{x_1} f_1 \, dx = l_1 ,$$

we will find:

$$\int_{x_0}^{x_1} f \, dx = l .$$

In general, the solution to the problem (β) will be different from problem (α). If we assume that the solution of problem (α) for a maximum or minimum of the integral V gives:

$$V = \kappa$$

then under the assumption that:

$$l = \kappa , \tag{11}$$

the functions $y_1, \ldots, y_n$ that were obtained from problem (α) will also be, at the same time, solutions to problem (β) and will yield the value l_1 for the integral V_1 here. That is because, by assumption, those functions and the values of the constant ratios $\mu : \mu_1 : \ldots : \mu_m$ that are obtained when those functions are found will simultaneously fulfill equations (9) and the $2n$ limit conditions that are common to both problems, and they will satisfy the $m + 1$ equations:

$$\int_{x_0}^{x_1} f \, dx = \kappa , \quad \int_{x_0}^{x_1} f_1 \, dx = l_1 , \quad \ldots, \quad \int_{x_0}^{x_1} f_1 \, dx = l_1 ,$$

moreover, which include the isoperimetric conditions for the first, as well as the second problem, from the assumption (11).

That immediately implies that one can always solve problem (β) with just algebraic operations as often as one has solved problem (α) for the undetermined values of the constants $l_1, \ldots, l_m$.

Furthermore, under the assumption (11), with the common solution to both problems, the limit equation in problem (β) will be:

$$\Delta_\beta(x, x_0) = \sum \pm \frac{\partial y_1}{\partial a_1} \cdots \frac{\partial y_n}{\partial a_n} \frac{\partial y_{10}}{\partial a_{n+1}} \cdots \frac{\partial y_{n0}}{\partial a_{2n}} \frac{\partial v_1}{\partial \mu_1} \frac{\partial v_2}{\partial \mu_2} \cdots \frac{\partial v_m}{\partial \mu_m} = 0,$$

in which the y_h and v_k, as well as the a_i and μ_k, have the same values that they had in the determinant $\Delta_\alpha(x, x_0)$, but the function:

(12) $$v = \int_{x_0}^{x_1} f\, dx$$

enters in place of v_1. However, since:

$$\Phi = \mu f + \mu_1 f_1 + \ldots + \mu_m f_m,$$

it will follow from (12) and (8) that:

$$\mu v + \mu_1 v_1 + \ldots + \mu_m v_m = \int_{x_0}^{x_1} \Phi\, dx,$$

and differentiating that with respect to a_i and μ_k will give:

$$\mu \frac{\partial v}{\partial a_i} + \mu_1 \frac{\partial v_1}{\partial a_i} + \cdots + \mu_m \frac{\partial v_m}{\partial a_i} = \int_{x_0}^{x_1} dx \sum_{h=1}^{n} \left(\frac{\partial \Phi}{\partial y_h} \frac{\partial y_h}{\partial a_i} + \frac{\partial \Phi}{\partial y'_h} \frac{\partial y'_h}{\partial a_i} \right),$$

$$v_\kappa + \mu \frac{\partial v}{\partial \mu_\kappa} + \mu_1 \frac{\partial v_1}{\partial \mu_\kappa} + \cdots + \mu_m \frac{\partial v_m}{\partial \mu_\kappa} = \int_{x_0}^{x_1} dx \left\{ f_\kappa + \sum_{h=1}^{n} \left(\frac{\partial \Phi}{\partial y_h} \frac{\partial y_h}{\partial \mu_\kappa} + \frac{\partial \Phi}{\partial y'_h} \frac{\partial y'_h}{\partial \mu_\kappa} \right) \right\}.$$

When one understands c to mean any of the constants $a_1, \ldots, a_{2n}, \mu, \mu_1, \ldots, \mu_m$, one will then have:

$$\mu \frac{\partial v}{\partial c} + \mu_1 \frac{\partial v_1}{\partial c} + \cdots + \mu_m \frac{\partial v_m}{\partial c} = \int_{x_0}^{x_1} dx \sum_{h=1}^{n} \left(\frac{\partial \Phi}{\partial y_h} \frac{\partial y_h}{\partial c} + \frac{\partial \Phi}{\partial y'_h} \frac{\partial y'_h}{\partial c} \right).$$

Now, one has:

$$\frac{\partial \Phi}{\partial y_h} \frac{\partial y_h}{\partial c} + \frac{\partial \Phi}{\partial y'_h} \frac{\partial y'_h}{\partial c} = \left(\frac{\partial \Phi}{\partial y_h} - \frac{d}{dx} \frac{\partial \Phi}{\partial y'_h} \right) \frac{\partial y_h}{\partial c} + \frac{d}{dx} \left(\frac{\partial \Phi}{\partial y'_h} \frac{\partial y_h}{\partial c} \right),$$

or from (9):

$$= \frac{d}{dx} \left(\frac{\partial \Phi}{\partial y'_h} \frac{\partial y_h}{\partial c} \right),$$

so one will get:

$$\frac{\partial v}{\partial c} = -\frac{\mu_1}{\mu}\frac{\partial v_1}{\partial c} - \frac{1}{\mu}\left\{\sum_{k=2}^{m}\mu_k\frac{\partial v_k}{\partial c} - \sum_{h=1}^{n}\left(\frac{\partial \Phi}{\partial y_h'}\frac{\partial y_h}{\partial c} - \left[\frac{\partial \Phi}{\partial y_h'}\right]_0 \frac{\partial y_{h0}}{\partial c}\right)\right\},$$

in which the index 0 suggests the substitution $x = x_0$, as before.

However, those formulas express only the element $\partial v / \partial c$ in the determinant $\Delta_\beta(x, x_0)$ in terms of the quantity $-\frac{\mu_1}{\mu}\frac{\partial v_1}{\partial c}$, plus the corresponding elements of the other rows in that determinant, multiplied by the same factor. One then has:

$$\Delta_\beta(x, x_0) = -\frac{\mu_1}{\mu}\,\Delta_\alpha(x, x_0)$$

identically. Their limit equations for the common solution to the two problems are also the same under the assumption (11).

Finally, the function:

$$2W_\beta = \frac{1}{\mu_1}\sum_{h=1}^{n}\sum_{i=1}^{n}\frac{\partial^2\Phi}{\partial y_h'\,\partial y_i'}\,U_h U_i$$

enters into problem (β) in place of the homogeneous function $2W_\alpha$ of problem (α). For the common solution in question, one then has:

$$2W_\beta = \frac{\mu}{\mu_1}\,2W_\alpha = \frac{1}{\lambda_1}\,2W_\alpha\,.$$

If we combine those results together then we can state the following theorem:

V. *If one has solved the isoperimetric problem:*

$$(\alpha)\qquad \begin{cases} V = \displaystyle\int_{x_0}^{x_1} f\,dx = \text{max., min.,} \\ \displaystyle\int_{x_0}^{x_1} f_1\,dx = l_1, \quad \int_{x_0}^{x_1} f_2\,dx = l_2, \quad \cdots \quad \int_{x_0}^{x_1} f_m\,dx = l_m \end{cases}$$

by completely integrating the n differential equations:

$$(7)\qquad \frac{\partial F}{\partial y_h} = \frac{d}{dx}\frac{\partial F}{\partial y_h'}\,,$$

in which one has:

$$F = f + \lambda_1 f_1 + \lambda_2 f_2 + \ldots + \lambda_m f_m\,,$$

and the λ *are undetermined constants whose values are then determined from the m isoperimetric conditions of the problem, and one has then found the maximum or minimum value of the integral V to be:*

$$V = l$$

then the solution of that problem will be, at the same time, also the solution to the reciprocal problem:

$$(\beta) \qquad \begin{cases} V_1 = \int_{x_0}^{x_1} f_1\,dx = \text{max., min.}, \\ \int_{x_0}^{x_1} f\,dx = l, \quad \int_{x_0}^{x_1} f_2\,dx = l_2, \quad \cdots \quad \int_{x_0}^{x_1} f_m\,dx = l_m, \end{cases}$$

assuming that one has prescribed the same fixed values for their limits and limiting values in both problems.

Moreover, if:

$$V = l$$

is a true maximum or minimum of the integral V in problem (α) *then:*

$$V_1 = l_1$$

will be, at the same time, a true maximum or minimum of the integral V_1 *in problem* (β) *when one has:*

$$\lambda_1 > 0\ ,$$

while it will be a minimum or maximum of V_1 *when:*

$$\lambda_1 < 0\ .$$

Finally, when the value l that is found for V in problem (α) *is neither a maximum nor a minimum, the same thing will be true for the value* l_1 *of* V_1 *in problem* (β).

One sees from this *reciprocity theorem for the isoperimetric problem* that (for fixed, but undetermined values of the constants l) one has found the solution to only any given isoperimetric problem and needs to decide whether, and within what limits, that solution will imply a true maximum or minimum in order to also be able to answer the same questions for any reciprocal problem with no further analysis. Furthermore, it is self-explanatory that in order to be able to apply the reciprocity theorem, it is not necessary to have solved the given problem (α) in precisely the way that was given. Rather, if one has ascertained the solution in any other way (e.g., by geometric considerations) then one needs only to inversely determine the signs of the isoperimetric constants $\lambda_1, \ldots, \lambda_m$. For the isoperimetric problems of the form:

$$\int_{x_0}^{x_1} f(x, y)\,dx = \text{max., min.,} \qquad \text{e.g.,} \qquad \int_{x_0}^{x_1} dx\sqrt{1+y'^2} = l_1\,,$$

one has:

$$2W = \frac{\lambda_1}{\left(\sqrt{1+y'^2}\right)^3}\,U_1^2\,.$$

A solution that produces a maximum will then belong to a negative value of λ_1 here, and a solution that produces a minimum will belong to a positive one (*).

§ 3.

Example.

Find the curve of given length and given endpoints whose center of mass lies deepest.

I take the z-axis to be vertical and have the opposite sense to gravity, and for the sake of convenience, lay the xy-plane through the given starting point of the curve. The problem can then be expressed analytically by the formulas:

$$(1) \qquad \left\{ \begin{aligned} &\int_{x_0}^{x_1} z\sqrt{1+y'^2+z'^2}\,dx = \text{min.},\\ &\int_{x_0}^{x_1} \sqrt{1+y'^2+z'^2}\,dx = l_1. \end{aligned} \right.$$

One will then have:

$$(2) \qquad F = (z+\lambda_1)\sqrt{1+y'^2+z'^2}$$

here, and obtain the catenary as the solution to the problem:

(*) The considerations above prove the reciprocity theorem only for the case of fixed given limits and limiting values. However, one can show that Theorem V is also true with no changes for arbitrary limit conditions, but naturally only if one assumes that those limiting conditions are the same in both problems. That is connected with the reciprocity relationship that the maxima and minima of inverse functions exhibit. Moreover, the reciprocity theorem is not the only property that the isoperimetric problems have beyond the other problems in the calculus of variations. Rather, there is another extremely important theorem that is true for them that one might call the *law of the invariability of the isoperimetric constants*. Namely, if one forces them to split up into branches by imposing conditions on the limits of the curves y then the integration constants a will vary from one branch to another, but the isoperimetric constants λ will keep the same values everywhere. In the special case where one seeks the closed curve of greatest area for a given circumference, or smallest circumference for a given area, and further demands that the curve should lie inside of a given polygon, the latter law will coincide with the known theorem of **Steiner** that all free parts of such curves must be arcs of equal circles.

$$(3) \qquad \begin{cases} y = \dfrac{a_1}{a_2} x + a_3 , \\[2ex] z + \lambda_1 = \tfrac{1}{2}\sqrt{a_1^2 + a_2^2}\left\{\exp\left(\dfrac{x - a_4}{a_2}\right) + \exp\left(-\dfrac{x - a_4}{a_2}\right)\right\}, \end{cases}$$

from which, the integral:

$$v_1 = \int_{x_0}^{x} \sqrt{1 + y'^2 + z'^2}\, dx$$

will take the value:

$$(4) \qquad v_1 = \tfrac{1}{2}\sqrt{a_1^2 + a_2^2}\left\{\exp\left(\frac{x - a_4}{a_2}\right) - \exp\left(-\frac{x - a_4}{a_2}\right) - \exp\left(\frac{x_0 - a_4}{a_2}\right) - \exp\left(-\frac{x_0 - a_4}{a_2}\right)\right\} .$$

The five constants a_1, a_2, a_3, a_4, λ_1 are determined from the given positions of the two endpoints and the given arc-length l_1, and in fact one will get two different systems of values for them that belong to two equal and opposite values of the constant a_2 .

Since, according to (4):

$$\frac{dv_1}{dx} = \frac{1}{2}\frac{\sqrt{a_1^2 + a_2^2}}{a_2}\left\{\exp\left(\frac{x - a_4}{a_2}\right) + \exp\left(-\frac{x - a_4}{a_2}\right)\right\}$$

will always have the same sign as $\sqrt{a_1^2 + a_2^2} / a_2$, one must give $\sqrt{a_1^2 + a_2^2}$ the same sign as a_2 in formulas (3) and (4) in order for the arc-length to be positive.

Moreover, from (1), the function:

$$2W = \frac{\partial^2 F}{\partial y' \partial y'} U_1^2 + 2\frac{\partial^2 F}{\partial y' \partial z'} U_1 U_2 + \frac{\partial^2 F}{\partial z' \partial z'} U_2^2$$

will become:

$$2W = \frac{z + \lambda_1}{\left(\sqrt{1 + y'^2 + z'^2}\right)^3} \left\{U_1^2 + U_2^2 + (z' U_1 - y' U_2)^2\right\} ,$$

and from (3), it will continually have the same sign as $\sqrt{a_1^2 + a_2^2}$ then. Therefore, in order to get a minimum, we must take that root, and a_2 with it, to be positive, i.e., of the two catenaries of given length through the two given endpoints that can be drawn in the vertical plane from one point to the other, we must take the one whose convex side points downward. As a result of the assumption $z_0 = 0$, from (3), we will then have $\lambda_1 > 0$.

Finally, the limit equation:

(5)
$$\sum \pm \frac{\partial y}{\partial a_1}\frac{\partial y_0}{\partial a_2}\frac{\partial z}{\partial a_3}\frac{\partial z_0}{\partial a_4}\frac{\partial v_1}{\partial \lambda_1} = 0$$

due to the formulas that follow from (3) and (4):

$$\frac{\partial y}{\partial a_3} = 1\,, \qquad \frac{\partial y}{\partial a_4} = 0\,, \qquad \frac{\partial y}{\partial \lambda_1} = 0\,,$$
$$\frac{\partial z}{\partial a_3} = 0\,, \qquad \frac{\partial z}{\partial \lambda_1} = -1, \qquad \frac{\partial v_1}{\partial a_3} = 0\,, \qquad \frac{\partial v_1}{\partial \lambda_1} = 0\,,$$

will next reduce to the equation:

(6)
$$\begin{vmatrix} \frac{\partial (y-y_0)}{\partial a_1} & \frac{\partial (y-y_0)}{\partial a_2} & 0 \\ \frac{\partial (z-z_0)}{\partial a_1} & \frac{\partial (z-z_0)}{\partial a_2} & \frac{\partial (z-z_0)}{\partial a_4} \\ \frac{\partial v_1}{\partial a_1} & \frac{\partial v_1}{\partial a_2} & \frac{\partial v_1}{\partial a_4} \end{vmatrix} = 0\,.$$

If one further sets, for the moment:

(7)
$$\frac{x-a_4}{a_2} = \xi\,, \qquad \tfrac{1}{2}(e^{\xi}+e^{-\xi}) = p\,, \qquad \tfrac{1}{2}(e^{\xi}-e^{-\xi}) = q\,,$$

then from (3) and (4), that will give:

$$y - y_0 = a_1\,(\xi - \xi_0)\,,$$
$$z - z_0 = \sqrt{a_1^2+a_2^2}\,(p-p_0)\,,$$
$$v_1 = \sqrt{a_1^2+a_2^2}\,(q-q_0)\,.$$

If one differentiates the latter formulas partially with respect to a_1, a_2, a_4 and substitutes the values of the differential quotients in equation (6) then, after some simple reductions and dropping the constant factor $(a_1^2+a_2^2)/a_2^2$, one will get the equation:

$$(\xi - \xi_0)\,[(p\,q_0 - p_0\,q)\,(\xi - \xi_0) - (p - p_0)^2 + (q - q_0)^2] = 0\,,$$

such that when one further sets:

$$\xi - \xi_0 = \frac{x - x_0}{a_2} = \Theta ,$$

the limit equation of the problem will ultimately become:

$$\Theta\, \Psi\, (\Theta) = 0 ,$$

in which:

$$\Psi\, (\Theta) = e^{\Theta} + e^{-\Theta} - \frac{\Theta}{2}(e^{\Theta} - e^{-\Theta}) - 2 .$$

However, a consideration of the functions $\Psi'\,\Theta$ and $\Psi''\,\Theta$ will immediately show that this equation admits only the one real root $\Theta = 0$, i.e., $x = x_0$. The equation then implies no sort of restriction for the upper limit x_1 . Rather, it shows that an unrestricted minimum will exist for the catenary that is convex-downward, and we have arrived at the theorem:

Among all curves of given length and given endpoints, the catenary always has the lowest center of mass.

Since $\lambda_1 > 0$, the following theorem will emerge directly from that theorem when one applies the reciprocity theorem:

Among all curves of given endpoints whose centers of mass lie on one and the same horizontal plane, the catenary always has the shortest length.

By contrast, in the absolute problem:

$$\int_{x_0}^{x_1} (z + \lambda_1) \sqrt{1 + y'^2 + z'^2}\, dx = \text{min.} , \tag{8}$$

which can be solved by the same catenary, assuming that one is gives the same value to the constant λ_1 that it had in the isoperimetric problem that was just treated, the equation:

$$\sum \pm \frac{\partial y}{\partial a_1} \frac{\partial y_0}{\partial a_2} \frac{\partial z}{\partial a_3} \frac{\partial z_0}{\partial a_4} = 0$$

will enter in place of equation (5), which will reduce to:

$$(q\, \xi - p)\, q_0 - q\, (q_0\, \xi_0 - p_0) = 0$$

after dropping the factor $\frac{a_1^2 + a_2^2}{a_2^2}(\xi - \xi_0)$. Now, from (3) and (7), one will have:

$$p = \frac{2(z+\lambda_1)}{\sqrt{a_1^2+a_2^2}}, \qquad q = \frac{2a_2\, z'}{\sqrt{a_1^2+a_2^2}}, \qquad \xi - \xi_0 = \frac{x-x_0}{a_2}\ .$$

In this absolute problem, we will then get the limit equation:

$$x - \frac{z+\lambda_1}{z'} = x_0 - \frac{z_0+\lambda_1}{z_0'},$$

i.e., when we advance from the given starting point along the catenary that solves the problem, in order for a true minimum to exist, we cannot extend the integral up to the point whose tangent once more intersects the line $z = - \lambda_1$ in the vertical plane that connects the two given endpoints at the same point as the tangent to the starting point. Thus, where we obtain an unbounded minimum in the isoperimetric problem (1), we will get only a bounded minimum in the unconstrained problem (8), which clearly illustrates the difference between the two problems that possess the same solution.