"Ueber die Jacobi-Hamilton'sche Integrationsmethode der partiellen Differentialgleichungen," Math. Ann. **3** (1871), 436-452.

On the Jacobi-Hamilton method for integrating first-order partial differential equations.

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Once **Pfaff** had shown that one could reduce the solution of any first-order partial differential equation to the integration of several systems of ordinary differential equations, that theory advanced to the next essential step by the work of **Cauchy** (*) and **Jacobi** (**), who both arrived at the same simplification of **Pfaff**'s result along very different paths and showed that the complete integration of the first **Pfaff** system by itself would suffice for one to obtain a complete solution of the partial differential equation.

Jacobi, who found that theorem by generalizing **Hamilton**'s discoveries in mechanics (which is why he himself called the process the **Hamiltonian** method) had reproduced it in a somewhat-different form and with a somewhat-different method of proof in his *Vorlesungen über Dynamik* (***), and there especially the type of derivation was much simpler and more concise than when one goes down the trail that **Cauchy** had blazed. Moreover, the **Jacobi** method has the advantage over **Cauchy**'s that it not only leads to a complete solution of the partial differential equation, but at the same time, it also shows how one can obtain all integral equations of the corresponding system of ordinary differential equations from that solution by mere partial differentiation. With **Cauchy**'s method, it is only when it is used in the correct way for discovering a complete solution to all first-order partial differential equations that one can pose without exception that one will very soon discover classes of equations that can be inferred from completely integrating the equation by that method from a closer scrutiny of **Jacobi**'s rule, at least in its later formulation.

Indeed, **Jacobi** had later overshadowed all of the previous integration methods for first-order partial differential equations quite conclusively with his brilliant "new method" (**Crelle**'s Journal, v. 60). Nonetheless, **Jacobi**'s older method still remained most interesting, on the one hand, because it had the shortest derivation, but then mainly due to the fact that it defined the most-natural starting point for all of **Jacobi-Hamilton** theory.

Based upon that, it seems desirable to me for investigate whether one might not be able to modify the aforementioned method in such a way that it might subsume all first-order partial differential equations, and that modification is found by a direct application of the rule that **Cauchy**

(*) *Execises d'Analyse et de Physique mathématique*, t. II.

(**) **Crelle**'s Journal, 17.

(***) Lecture 19, as well as pp. 364, especially.

gave for deriving the *general* solution to the simpler problem of finding a *complete* solution to the partial differential equation:

$$\frac{\partial V}{\partial x} + H\left(x, x_1, \ldots, x_n, \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) = 0\ .$$

It is only necessary to consider equations of that form since one can reduce any first-order partial differential equation that does not contain the unknown function itself to one in which the unknown function no longer appears explicitly.

Namely, if:

$$f\left(x_1, \ldots, x_n, x, \frac{\partial x}{\partial x_1}, \ldots, \frac{\partial x}{\partial x_n}\right) = 0$$

is the given partial differential equation then, to that end, one needs only to initially inquire indirectly about whether an equation exists from which the desired solution x can be determined algebraically instead of starting from the direct determination of x.

In fact, when one imagines that that this solution is defined by an equation of the form:

$$V(x_1, \ldots, x_n, x) = \text{const.}$$

then the function V must satisfy the equation:

$$F\left(x_1, \ldots, x_n, x, -\frac{\frac{\partial V}{\partial x_1}}{\frac{\partial V}{\partial x}}, \ldots, -\frac{\frac{\partial V}{\partial x_n}}{\frac{\partial V}{\partial x}}\right) = 0$$

identically, and conversely when one has found any solution to the latter equation, the value of x that one gets from the substitution:

$$V = \text{const.}$$

will be a solution of the given equation because one will have (*):

$$\frac{\partial x}{\partial x_i} = -\frac{\frac{\partial V}{\partial x_i}}{\frac{\partial V}{\partial x}}$$

for it.

(*) **Jacobi** had later abandoned the reduction method that he himself had proposed (**Crelle**'s J., v. 23, pp. 18) in a remarkable way and replaced it in his great treatise (**Crelle**'s J., v. 60, pp. 1) with another one that can indeed be quite useful for certain transformations but is illusory as a direct method of reduction. On that topic, cf., **Imschenetsky**, (**Grunert**'s Archiv, t. 50, pp. 315).

Now, when **Jacobi** applied that to first-order differential equations that do not include the unknown function itself in his *Vorlesungen über Dynamik*, pp. 364, he gave the following formulation to the **Hamilton**ian method:

When the problem that was posed is that of finding a complete solution to the given first-order partial differential equation:

$$\frac{\partial V}{\partial x} + H\left(x, x_1, \ldots, x_n, \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) = 0 , \tag{1}$$

and one writes:

$$p_1, \ldots, p_n \qquad \text{for} \qquad \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n},$$

resp., one can next pose the following system of $2n$ *first-order differential equations:*

$$\frac{dx_k}{dx} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dx} = -\frac{\partial H}{\partial x_k}, \tag{2}$$

and integrate it completely. In order to do that, one introduces the values:

$$a_1, \ldots, a_n, b_1, \ldots, b_n$$

of the quantities:

$$x_1, \ldots, x_n, p_1, \ldots, p_n ,$$

resp., for the arbitrarily-chosen initial values a of x in place of the arbitrary constants in the integral equations of that system, and one finally expresses the integral:

$$V = \int_a^x dx \left[\sum_{k=1}^{m} p_k \frac{\partial H}{\partial p_k} - H\right] \tag{3}$$

in terms of the quantities $x, x_1, \ldots, x_n, a_1, \ldots, a_n$ *by means of those equations. One will then have the* $2n$ *equations:*

$$\frac{\partial V}{\partial x_k} = p_k , \qquad \frac{\partial V}{\partial a_k} = -b_k , \tag{4}$$

which one can consider to be the complete integral equations of the system (2), *and at the same time, the given expression for V will be a complete solution of the partial differential equation* (1) *when one adds an arbitrary solution of the partial differential equation* (1).

One sees immediately that this theorem is subject to some exceptions. For example, as long as H is a homogeneous function of degree one in $p_1, \ldots, p_n$ (and that case will always occur when equation (1) is obtained from a partial differential equation in the way that given above, in which the unknown function itself occurred), it will yield only the entirely-useless value $V = 0$ in place of the desired solution, and it would be impossible for it to lead to the complete integral equations of the system (2).

I would next like to seek to reveal the true basis for those exceptions, and to that end, **Jacobi**'s method of proof will be subjected to a closer scrutiny in **§ 1.** The discussion of the **Jacobi-Hamilton** integration method that is not restricted by any sort of exceptions will then follow in **§ 2**. I shall expressly point out that it is only one special case (and indeed the simplest one) of the more general **Cauchy** method. **Cauchy** himself has also emphasized a particular complete solution from all of the ones that can be obtained by his procedure as the simplest one, although it was a different one: *viz.*, **Hamilton**'s solution. That is not precise. The derivation of the general **Cauchy** method from **Jacobi**'s principles in **§ 3** will show that the consistent application of that method can never lead to **Hamilton**'s solution. Finally, some theorems about the determination of the solution to a first-order partial differential equation by a boundary condition will be suggested, and how one can also arrive at Cauchy's more theorems by the direct method in **§ 2**.

§ 1.

Jacobi's proof of the cited theorem is based essentially upon the following fact:

When one varies the initial values $a_1, \ldots, a_n, b_1, \ldots, b_n$ and denotes the variations that arise in that way by the symbol δ, one will arrive directly at the formula:

$$\delta V = \sum_{k=1}^{n} (p_k\,\delta x_k - b_k\,\delta a_k)$$

from the defining equation (3) by an easy calculation that I will omit here, since an entirely-similar one will be found in the following **§**. However, when one initially considers V to be a function of the quantities $x_1, \ldots, x_n, a_1, \ldots, a_n$, one will:

$$\delta V = \sum_{k=1}^{n} \left(\frac{\partial V}{\partial x_k}\delta x_k + \frac{\partial V}{\partial a_k}\delta a_k \right) .$$

As a result, for all values of the variations:

$$\delta a_1, \ldots, \delta a_n, \delta b_1, \ldots, \delta b_n,$$

one will have the identity:

$$\sum_{\lambda=1}^{n} \left[\left(\frac{\partial V}{\partial x_\lambda} - p_k \right)\delta x_k + \left(\frac{\partial V}{\partial a_\lambda} + b_k \right)\delta a_k \right] = 0 .$$

Now, **Jacobi** deduced equations (4) from that formula with no further analysis. However, that deduction is correct only when the variations:

$$\delta x_1, \ldots, \delta x_n, \delta a_1, \ldots, \delta a_n$$

are independent of each other, just as the original ones:

$$\delta a_1, \ldots, \delta a_n, \delta b_1, \ldots, \delta b_n$$

were, and since one will have:

$$\delta x_k = \frac{\partial x_k}{\partial b_1}\delta b_1 + \cdots + \frac{\partial x_k}{\partial b_n}\delta b_n$$

when one sets all $\delta a = 0$, that independence will exist only when the determinant:

$$\sum \pm \frac{\partial x_1}{\partial b_1}\frac{\partial x_2}{\partial b_2}\cdots\frac{\partial x_n}{\partial b_n}$$

is non-zero, i.e., when the n constants $b_1, \ldots, b_n$ can all be determined from the n solutions $x_1, \ldots, x_n$, or in other words, when the $2n$ arbitrary constants of the complete integral equations of the system (2) can be expressed in terms of the initial and final values of the variables $x, x_1, \ldots, x_n$ alone.

Now, that is indeed always the case when the function H possesses the property that the determinant:

$$\sum \pm \frac{\partial^2 H}{\partial p_1\,\partial p_1}\frac{\partial^2 H}{\partial p_2\,\partial p_2}\cdots\frac{\partial^2 H}{\partial p_n\,\partial p_n}$$

is non-zero because the n equations:

$$\frac{\partial H}{\partial p_k} = \frac{dx_k}{dx}$$

will then determine all n quantities $p_1, \ldots, p_n$ as functions of:

$$x, x_1, \ldots, x_n, \frac{dx_1}{dx}, \ldots, \frac{dx_n}{dx},$$

and therefore, in that case, the system of $2n$ first-order differential equation (2) can always be reduced to a system of n second-order differential equations in terms of $x_1, \ldots, x_n$, and x alone and whose complete integration must necessarily involve $2n$ arbitrary constants.

By contrast, when that determinant is zero, the quantities $p_1, \ldots, p_n$ can be eliminated completely from the n equations above, and a relation that is free from arbitrary constants will then exist between the variables $x_1, \ldots, x_n$ and their first differential quotients, so in general, the complete solutions $x_1, \ldots, x_n$ will no longer include $2n$ arbitrary constants. For example, when H is linear in the $p_1, \ldots, p_n$, only n arbitrary constants will enter into those solutions.

However, as soon as one no longer has the right to pose the equations:

$$\frac{\partial V}{\partial x_k} = p_k ,$$

the justification for the conclusion that:

$$\frac{\partial V}{\partial x} + H = 0$$

will also break down. That is because that equation is obtained from the double manner of expressing the differential quotients of the function V:

$$\frac{dV}{dx} = \sum_{k=1}^{n} p_k \frac{\partial H}{\partial p_k} - H$$

and

$$\frac{dV}{dx} = \sum_{k=1}^{n} \frac{\partial V}{\partial x_k}\frac{dx_k}{dx} + \frac{\partial V}{\partial x}$$

as merely a consequence of those equations.

One then sees that **Jacobi**'s method of proof tacitly includes an assumption that is not fulfilled for an arbitrary function H, and it is therefore clear from the outset that **Jacobi**'s rule cannot imply a generally-valid method of integrating first-order partial differential equations.

§ 2.

Now that the inadequacy of the **Jacobi-Hamilton** method and the basis for the flaw in it has been explained, I would now like to show how one can arrive at an entirely-general integration method by a very slight alteration of **Jacobi**'s argument. For the sake of clarity, I will then go to work somewhat more thoroughly than I did before.

Let:

$$H = H(x, x_1, \ldots, x_n, p_1, \ldots, p_n)$$

be a given arbitrary function of the $2n + 1$ variables $x, x_1, \ldots, x_n, p_1, \ldots, p_n$, and let the following $2n$ first-order ordinary differential equations be given:

$$(1) \qquad \frac{dx_k}{dx} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dx} = -\frac{\partial H}{\partial x_k} .$$

One integrates those equations completely, and once one has expressed the $2n$ integration constants in terms of the values:

$$a_1, \ldots, a_n, b_1, \ldots, b_n$$

that the variables:

$$x_1, \ldots, x_n, p_1, \ldots, p_n$$

assume for the arbitrarily-chosen initial value a of x, one will find:

(2) $$x_k = [x_k] , \qquad p_k = [p_k] ,$$

in which the $[x_k]$ and $[p_k]$ are then certain functions of x, a, a_1, …, a_n, b_1, …, b_n that reduce to a_k and b_k, resp., for $x = a$.

For the sake of better understanding, the substitution of the solutions (2) will be suggested by enclosing a variable in square brackets.

The determinant:

$$\sum \pm \frac{\partial [x_1]}{\partial a_1} \frac{\partial [x_2]}{\partial a_2} \cdots \frac{\partial [x_n]}{\partial a_n}$$

can never be zero because it takes the value 1 for $x = a$. It will then follow from this that the n equations:

(3) $$[x_k] = x_k$$

must always be soluble for the n quantities a_1, …, a_n .

Having done that, one calculates the expression:

(4) $$V = \sum_{k=1}^{n} a_k b_k + \int_a^x \left[\sum_{k=1}^{m} p_k \frac{\partial H}{\partial p_k} - H \right] dx$$

as a function of x, a, a_1, …, a_n, b_1, …, b_n by a simple quadrature.

Upon differentiating that expression with respect to c, which one understands to means any one of the constants a_k or b_k, one will next get:

$$\frac{\partial V}{\partial c} = \frac{\partial}{\partial c} \sum_{k=1}^{n} a_k b_k + \int_a^x dx \sum_{k=1}^{m} \left\{ [p_k] \frac{\partial}{\partial c} \left[\frac{\partial H}{\partial p_k} \right] - \left[\frac{\partial H}{\partial x_k} \right] \frac{\partial [x_k]}{\partial c} \right\} .$$

However, as a consequence of equations (1), one has:

$$\left[\frac{\partial H}{\partial x_k} \right] = - \frac{d[p_k]}{dx}$$

and

$$\frac{\partial}{\partial c} \left[\frac{\partial H}{\partial p_k} \right] = \frac{\partial}{\partial c} \frac{d[x_k]}{dx} = \frac{d}{dx} \frac{\partial [x_k]}{\partial c}$$

identically. The expression under the integral sign will then become:

$$= \sum_{k=1}^{m} \left\{ [p_k] \frac{d}{dx} \frac{\partial [x_k]}{\partial c} + \frac{d[p_k]}{dx} \frac{\partial [x_k]}{\partial c} \right\}$$
$$= \frac{d}{dx} \sum_{k=1}^{m} [p_k] \frac{\partial [x_k]}{\partial c},$$

and one will find by performing the integration that:

$$\frac{\partial V}{\partial c} = \frac{\partial}{\partial c} \sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{m} \left\{ [p_k] \frac{\partial [x_k]}{\partial c} - b_k \frac{\partial a_k}{\partial c} \right\} .$$

When one first sets $c = a$ and $c = b$, that will give:

$$\frac{\partial V}{\partial a_i} = \sum_{k=1}^{m} [p_k] \frac{\partial [x_k]}{\partial a_i}$$
$$\frac{\partial V}{\partial b_i} = a_i + \sum_{k=1}^{m} [p_k] \frac{\partial [x_k]}{\partial b_i} .$$

When one compares that with the previous values, it will then follow that one must have:

(5) $$\sum_{k=1}^{m} \left[\frac{\partial (V)}{\partial x_k} - p_k \right] \frac{\partial [x_k]}{\partial a_i} = 0 ,$$

(6) $$\sum_{k=1}^{m} \left[\frac{\partial (V)}{\partial x_k} - p_k \right] \frac{\partial [x_k]}{\partial a_i} + \left[\frac{\partial (V)}{\partial b_k} \right] = 0$$

identically.

For $i = 1, 2, \ldots, n$, equation (5) represents a system of n equations that are linear and homogeneous in the n quantities:

$$\left[\frac{\partial (V)}{\partial x_k} - p_k \right] .$$

From the previous remarks, the determinant of this system is non-zero. The system therefore consist of nothing but the single equation:

$$\left[\frac{\partial (V)}{\partial x_k} - p_k \right] = 0 ,$$

from which it will follow from (6) that one must likewise have:

$$\left[\frac{\partial (V)}{\partial b_i}\right] - a_i = 0 .$$

One will then see that the complete solutions (2) of the system (1) of $2n$ equations must satisfy:

(7) $$\frac{\partial (V)}{\partial x_k} = p_k ,$$

(8) $$\frac{\partial (V)}{\partial b_k} = a_k$$

identically. Those equations are not identical, *per se*, because the quantities p_k and a_k do not enter into the left-hand sides at all. As a result, they are integral equations of the system (1). Moreover, they are $2n$ in number, and they include $2n$ arbitrary constants, namely, the $2n$ initial values $a_1, \ldots, a_n, b_1, \ldots, b_n$. Finally, none of the equations (7) and (8) can be a consequence of the remaining ones because a quantity p_k or a_k will enter into each of them that is missing from all remaining ones.

Equations (7) and (8) will then define a system of complete integral equations for the differential equations (1).

In the same double way by which we formed the partial differential quotients of V with respect to a_i and b_i, we can also form the differential quotients of this function with respect to x now. The definition (4) implies directly:

$$\frac{dV}{dx} = \left[\sum_{k=1}^{n} p_k \frac{\partial H}{\partial p_k} - H\right].$$

On the other hand, one will obtain it indirectly from the function (V) when one imagines substituting the values (3) for $x_1, \ldots, x_n$:

$$\frac{dV}{dx} = \left[\frac{\partial (V)}{\partial x}\right] + \sum_{k=1}^{n}\left[\frac{\partial (V)}{\partial x_k}\right]\frac{d[x_k]}{dx},$$

or from (1) and (7):

$$\frac{dV}{dx} = \left[\frac{\partial (V)}{\partial x} + \sum_{k=1}^{n} p_k \frac{\partial H}{\partial x_k}\right].$$

Subtracting those two formulas will show that the complete solutions to the system (1) must also fulfill the equation:

$$0 = \frac{\partial (V)}{\partial x} + H$$

identically. From (7), one can write this as:

(9) $$0 = \frac{\partial (V)}{\partial x} + H\left(x, x_1, \dots, x_n, \frac{\partial (V)}{\partial x_1}, \dots, \frac{\partial (V)}{\partial x_n}\right).$$

Initially, all that we know about that equation, in which the variables p no longer occur at all, is that it must be fulfilled identically under the substitution of the values (3). However, when we assign the values to the n quantities $a_1, \dots, a_n$ that follow from equations (3) in the identity that arises in that way, which is true for all arbitrary values of the a_k and b_k, that substitution will, in turn, cancel out, which explains the fact that equation (9) must be an identity in its own right.

The function (V) will then satisfy the partial differential equation (9), and indeed when one adds an arbitrary constant to it, one will have a complete solution to that equation because it includes n arbitrary constants $b_1, \dots, b_n$, and they cannot be eliminated from the partial differential quotients:

$$\frac{\partial (V)}{\partial x_1}, \frac{\partial (V)}{\partial x_2}, \dots, \frac{\partial (V)}{\partial x_n}$$

since otherwise one of the n equations (7) would be a consequence of the remaining ones.

If one summarizes the results that were obtained then that will give the following, generally-valid, theorem:

I. *When one deals with the problem of solving the given first-order partial differential equation:*

(α) $$0 = \frac{\partial (V)}{\partial x} + H\left(x, x_1, \dots, x_n, \frac{\partial (V)}{\partial x_1}, \dots, \frac{\partial (V)}{\partial x_n}\right)$$

then one can generally write:

$$p_k, \text{ instead of } \frac{\partial V}{\partial x_k}$$

in the function H and use that function of $x, x_1, \dots, x_n, p_1, \dots, p_n$ *to define the system of* $2n$ *ordinary differential equations:*

(β) $$\frac{dx_k}{dx} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dx} = -\frac{\partial H}{\partial x_k}.$$

If one has integrated that system completely and expressed the $2n$ *integration constants in its solutions in terms of the values:*

$$a_1, \dots, a_n, b_1, \dots, b_n$$

of the quantities:

$$x_1, \dots, x_n, p_1, \dots, p_n$$

for the arbitrarily-chosen values a of x then one substitutes those solutions in the expression:

$$\sum_{k=1}^{n} p_k \frac{\partial H}{\partial p_k} - H$$

and calculates:

$$V = \sum_{k=1}^{n} a_k\, b_k + \int_a^x dx \left[\sum_{k=1}^{n} p_k \frac{\partial H}{\partial p_k} - H \right]$$

as a function of x, a_1, …, a_n, b_1, …, b_n *by a simple quadrature. Finally, if one eliminates the quantities* a_1, …, a_n *from that function by means of the values of* x_1, …, x_n *that were found and denotes the resulting function of* x, x_1, …, x_n, b_1, …, b_n *by* (V) *then the formula:*

$$V = (V) + \text{const.}$$

will yield a complete solution to the partial differential equations (α), *and at the same time, one will have the complete integral equations for the ordinary differential equations* (β) *in the form of* $2n$ *equations:*

$$\frac{\partial\,(V)}{\partial x_k} = p_k\,, \qquad \frac{\partial\,(V)}{\partial b_k} = a_k\,.$$

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If one applies that theorem to a partial differential equation of the form:

$$\frac{\partial z}{\partial x} + F\left(z, x, x_1, \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n} \right) = 0\,,$$

once one has previously reduced it to an equation in which the unknown function no longer occurs in the given way, then one will be led to the following general theorem that will replace Theorem V on pp. 366 of **Jacobi**'s *Vorlesungen*, which is true to only a limited extent, and which one can more easily prove directly in the way that is given there:

II. *Let:*

$$F\,(z, x, x_1, \ldots, x_n, p_1, \ldots, p_n)$$

be an arbitrarily-given function $2n + 2$ *quantities* z, x, x_1, …, x_n, p_1, …, p_n, *and let the following system of* $2n + 1$ *ordinary differential equations between them be given:*

$$\frac{dx_k}{dx} = \frac{\partial F}{\partial p_k}\,, \qquad \frac{dp_k}{dx} = -\,\frac{\partial F}{\partial x_k} - p_k \frac{\partial F}{\partial z}\,, \qquad \frac{dz}{dx} = \sum_{k=1}^{n} p_k \frac{\partial F}{\partial p_k} - F\,.$$

One has to integrate that system completely and introduce the values:

$$a_1, \ldots, a_n, b_1, \ldots, b_n \quad \text{and} \quad c$$

into the $2n + 1$ *integral equations in place of the arbitrary constants, and they will contain the quantities:*

$$x_1, \ldots, x_n, p_1, \ldots, p_n, \quad z - (x_1\, p_1 + \ldots + x_n\, p_n),$$

resp., for the arbitrarily-chosen initial values a of x. If one then expresses the quantity z in terms of only $x, x_1, \ldots, x_n, a, b_1, \ldots, b_n$ *by means of the integral equations then the expression that one obtains:*

$$z = Z$$

will be a complete solution of the first-order partial differential equation:

$$\frac{\partial z}{\partial x} + F\left(z, x, x_1, \ldots, x_n, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right) = 0\,,$$

and at the same time, the $2n + 1$ *equations:*

$$\frac{\partial Z}{\partial x_k} = p_k\,, \qquad \frac{\partial Z}{\partial b_k} = \frac{\partial Z}{\partial c}\, a_k\,,$$
$$x = Z$$

will be the complete integral equations of the ordinary differential equations above.

Finally, when one applies that theorem to a partial differential equation in the unsolved form:

$$\varphi\left(x_1, \ldots, x_n, z, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right) = 0\,,$$

one will arrive at the following theorem, which substitutes for the first formulation of the **Jacobi-Hamilton** method (**Crelle**'s J., v. 17):

III. *Let:*

$$\varphi\,(x_1, \ldots, x_n, x, p_1, \ldots, p_n)$$

be an arbitrarily-given function of the $2n + 1$ *variables* $x_1, \ldots, x_n, x, p_1, \ldots, p_n$, *and let the following* $2n$ *first-order ordinary differential equations in those variables be given:*

$$\frac{dx_1}{\dfrac{\partial\varphi}{\partial p_1}} = \frac{dx_2}{\dfrac{\partial\varphi}{\partial p_2}} = \ldots = \frac{dx_n}{\dfrac{\partial\varphi}{\partial p_n}} = -\frac{dp_1}{\dfrac{\partial\varphi}{\partial x_1} + p_1\dfrac{\partial\varphi}{\partial z}} = \ldots = -\frac{dp_n}{\dfrac{\partial\varphi}{\partial x_n} + p_n\dfrac{\partial\varphi}{\partial z}} = \frac{dz}{p_1\dfrac{\partial\varphi}{\partial p_1} + \cdots + p_n\dfrac{\partial\varphi}{\partial p_n}}\,,$$

one integral of which is φ = const. *If one has found the* $2n - 1$ *remaining integrals of that systems, and after eliminating* p_1 *by means of the equation* $\varphi = 0$, *one expresses its* $2n - 1$ *arbitrary constants in terms of the values:*

$$a_2, \ldots, a_n, c, b_2, \ldots, b_n$$

of the quantities:

$$x_2, \ldots, x_n, z, p_2, \ldots, p_n$$

for the arbitrarily-chosen value a_1 *of* x_1 *then one sets:*

$$c = b_1 + a_2\, b_2 + \ldots + a_n\, b_n$$

and eliminates the $2n - 2$ *quantities:*

$$a_2, \ldots, a_n, p_2, \ldots, p_n$$

from the $2n - 1$ *integrals. In that way, one will get an equation between* $x_1, \ldots, x_n, z, b_1, b_2, \ldots, b_n$ *that will yield a complete solution with the n arbitrary constants* $b_1, b_2, \ldots, b_n$ *of the first-order partial differential equation:*

$$\varphi\left(x_1, \ldots, x_n, z, \frac{\partial z}{\partial x_1}, \ldots, \frac{\partial z}{\partial x_n}\right) = 0$$

when it is solved for z.

§ 3.

Above all, the foregoing considerations had the goal of replacing the **Jacobi-Hamilton** method with another one that is as simple as possible and shares the same brevity and clarity of derivation as the latter method but is no longer subject to any exceptions. However, one will easily see, at the same time, that those considerations also include the means for proving **Cauchy**'s more-general rule, i.e., solving the problem of finding the solution to the partial differential equation:

$$\frac{\partial V}{\partial x} + H\left(x_1, \ldots, x_n, \frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right) = 0$$

that reduces to a given function of $x_1, x_2, \ldots, x_n$ for $x = a$.

To that end, one does in fact need only to replace the initial values $b_1, \ldots, b_n$ in the previous § with any other n arbitrary constants $\alpha_1, \ldots, \alpha_n$ by means of n equations of the form:

$$b_1 = \frac{\partial f}{\partial \alpha_1}, \ldots, b_n = \frac{\partial f}{\partial \alpha_n},$$

in which f is an arbitrarily-given function of the initial values $a_1, \ldots, a_n$, and the new constants $\alpha_1, \ldots, \alpha_n$, and in place of formula (4) as the definition of V as a function of $x, a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n$, one poses the formula:

$$V = f + \int_a^x \left[\sum_{k=1}^n p_k \frac{\partial H}{\partial p_k} - H \right] dx .$$

If one lets:

$$x_k = [x_k] , \qquad p_k = [p_k]$$

denote the complete solutions, as expressed in terms of x and the $2n$ constants $a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n$, to the $2n$ first-order ordinary differential equations that are equivalent to the partial differential equation above then the determinant of the functions $[x_1], \ldots, [x_n]$ relative to the initial values $a_1, \ldots, a_n$ will still be non-zero now, and indeed to the same degree. When one now, in turn, takes the differential quotients of the function V with respect to a_i, α_i, and x in the double way, one will be led to the following theorem by almost the same argument as before *verbatim*, which includes **Cauchy**'s solution to the stated problem as a special case, as well as being a special case of Theorem I in the previous §:

IV. *Let:*

$$f(a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n)$$

be an arbitrarily-given function of the $2n$ *quantities* $a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n$ *whose partial differential quotients:*

$$\frac{\partial f}{\partial a_1}, \ldots, \frac{\partial f}{\partial a_n}$$

are independent of each other relative to $\alpha_1, \ldots, \alpha_n$. *Moreover, let:*

$$H(x, x_1, \ldots, x_n, p_1, \ldots, p_n)$$

be an arbitrary given function of $x, x_1, \ldots, x_n, p_1, \ldots, p_n$, *and let the system of* $2n$ *first-order ordinary differential equations in those variables be given:*

$$(1) \qquad \frac{dx_k}{dx} = \frac{\partial H}{\partial p_k}, \qquad \frac{dp_k}{dx} = -\frac{\partial H}{\partial x_k} .$$

One integrates that system completely and expresses the $2n$ *integration constants in terms of the values:*

$$a_1, \ldots, a_n, b_1, \ldots, b_n$$

of the quantities:

$$x_1, \ldots, x_n, p_1, \ldots, p_n$$

resp., for $x = a$. *When one sets:*

(2) $$b_1 = \frac{\partial f}{\partial a_1}, \dots, b_n = \frac{\partial f}{\partial a_n},$$

one will get the complete solutions to the system (1):

(3) $$x_k = [x_k],$$ (4) $$p_k = [p_k],$$

as functions of x, $a_1, \dots, a_n$, $\alpha_1, \dots, \alpha_n$, *and one will get:*

(5) [*sic*] $$V = f + \int_a^x \left[\sum_{k=1}^n p_k \frac{\partial H}{\partial p_k} - H \right] dx$$

by a simple quadrature. If one eliminates the quantities $a_1, \dots, a_n$ *from* V *by means of the* n *equations* (3) *and defines the* $2n$ *equations:*

$$\frac{\partial V}{\partial x_k} = p_k, \qquad \frac{\partial V}{\partial \alpha_k} = \frac{\partial f}{\partial \alpha_k}.$$

with the function V *of* x, $x_1, \dots, x_n$, $\alpha_1, \dots, \alpha_n$ *thus-obtained then they will be the complete integral equations of the system* (1), *and at the same time, that function* V *will itself be the complete solution of the first-order partial differential equation:*

(6) $$\frac{\partial V}{\partial x} + H\left(x_1, \dots, x_n, \frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right) = 0$$

that assumes the given value:

$$V = f(x_1, \dots, x_n, \alpha_1, \dots, \alpha_n)$$

for $x = a$.

The latter will follow immediately from formula (5) when one adds that equations (3), from which one determines the initial values $a_1, \dots, a_n$, will reduce to:

$$x_n = a_n$$

for $x = a$.

I have expressed that theorem for the special of the complete solution to equation (6) in order to simultaneously show how to represent the complete integral equations of the system (1) by means of it. However, it is clear, with no further analysis, that the part of the theorem that refers to the solution of the partial differential equation (6) will also not be true in general when one assigns certain constants to the quantities $\alpha_1, \dots, \alpha_n$, i.e., one takes f to be an entirely-arbitrary function of $a_1, \dots, a_n$, since the only change that assumption will produce in the derivation of the theorem is that the differentiation of V with respect to the α will drop out. However, one must not overlook the fact that certain special exceptions can occur in the latter case, namely, when complete solutions of the system (1) take on infinite or indeterminate values under the substitutions (2) for the function f that was just chosen.

Moreover, one can derive that theorem from the following theorem, which shows how to find the solution of equation (6) that is equal to an arbitrarily-given function of $x_1, \ldots, x_n$ for $x = a$ from any given complete solution to it:

V. *Let an arbitrary first-order partial differential equation be given between V and the independent variables $x, x_1, \ldots, x_n$ and into which V itself does not enter, and let:*

$$V = \varphi\,(x, x_1, \ldots, x_n, c_1, \ldots, c_n) + \text{const.}$$

be a complete solution of it. If one then sets:

$$V = \varphi - \varphi_a + f_a\,, \tag{7}$$

in which φ_a arises from φ when one switches $x, x_1, \ldots, x_n$ with $a, a_1, \ldots, a_n$, resp., and:

$$f_a = f\,(a_1, \ldots, a_n)$$

is an arbitrarily-given function of $a_1, \ldots, a_n$, and one eliminates the quantities:

$$a_1, \ldots, a_n, \qquad c_1, \ldots, c_n$$

from it with the help of the $2n$ equations:

$$\text{(8)} \quad \frac{\partial \varphi}{\partial c_k} = \frac{\partial \varphi_a}{\partial c_k}\,, \qquad \text{(9)} \quad \frac{\partial f_a}{\partial a_k} = \frac{\partial \varphi_a}{\partial a_k}\,,$$

then the resulting function V of $x, x_1, \ldots, x_n$ will have the two properties: It satisfies the given partial differential equation, and it takes the value:

$$V = f\,(x_1, \ldots, x_n)$$

for $x = a$.

Namely, if one considers the quantities $a_1, \ldots, a_n, c_1, \ldots, c_n$ in (7) to be functions of the independent variables then differentiating that formula with respect to x_i will give:

$$\frac{\partial V}{\partial x_i} = \frac{\partial \varphi}{\partial x_i} + \sum_{k=1}^{n} \left(\frac{\partial \varphi}{\partial c_k} - \frac{\partial \varphi_a}{\partial c_k} \right) \frac{\partial c_k}{\partial x_i} + \sum_{k=1}^{n} \left(\frac{\partial f_a}{\partial a_k} - \frac{\partial \varphi_a}{\partial a_k} \right) \frac{\partial a_k}{\partial x_i}\,.$$

For those functions $a_1, \ldots, a_n, c_1, \ldots, c_n$ that satisfy equations (8) and (9), one will then have:

$$\frac{\partial V}{\partial x_i} = \frac{\partial \varphi}{\partial x_i},$$

just as one had for the original complete solution. The given value of V will then be a solution to the given partial differential equation, and it will reduce to $f(x_1, \ldots, x_n)$ when one sets $x = a$ because equations (8) will be fulfilled by:

$$a_1 = x_1, \quad \ldots, \quad a_n = x_n$$

for $x = a$.

One can ultimately derive many other solutions of the same first-order partial differential equation from a known complete solution with that help of that theorem. However, not every arbitrary solution can be obtained in that way. In particular, it should be emphasized that neither the theorem above nor **Cauchy**'s method can ever lead to **Hamilton**'s solution whose arbitrary constants are the initial values $a_1, \ldots, a_n$, and which reduce to zero or a constant for $x = a$. Rather, one will get **Hamilton**'s solution (if it even exists) by eliminating the constants $c_1, c_2, \ldots, c_n$ from $V = \varphi - \varphi_a$ by means of the n equations:

$$\frac{\partial \varphi}{\partial c_k} = \frac{\partial \varphi_a}{\partial c_k}.$$

If one demands that the desired solution V should take the given value:

$$V = f(x, x_1, \ldots, x_n),$$

not for $x = a$, but for an arbitrarily-given equation:

$$\psi(x, x_1, \ldots, x_n) = 0,$$

then one can reduce that problem to the previous one by imagining that one has introduced a new variable u into the partial differential equations in place of x by means of the equation $y = u$, along with its complete solution and the given function f. In that way, one will arrive at the following theorem:

VI. *In order to use the complete solution:*

$$V = \varphi(x, x_1, \ldots, x_n, c_1, \ldots, c_n) + \text{const.}$$

to a first-order partial differential equation between x, $x_1, \ldots, x_n$, and V in order to derive the solution to that equation that will take the given value:

$$V = f(x, x_1, \ldots, x_n)$$

as soon as one poses the given condition equation:

$$\psi(x, x_1, \ldots, x_n) = 0$$

between the independent variables, one exhibits the $2n + 3$ *equations:*

$$V = \varphi - \varphi_a + f_a, \qquad \psi_a = 0,$$

$$\frac{\partial \varphi}{\partial c_k} = \frac{\partial \varphi_a}{\partial c_k}, \quad \frac{\partial f_a}{\partial a_k} - \frac{\partial \varphi_a}{\partial a_k} = \lambda \frac{\partial \psi_a}{\partial a_k},$$

in which $k = 1, 2, \ldots, n$, $a_k = a, a_1, \ldots, a_n$, *and eliminates the* $2n + 2$ *quantities:*

$$a, a_1, \ldots, a_n, c_1, \ldots, c_n, \lambda$$

from it. The resulting value of V *will be the desired solution.*

Here, as in what follows, in general, F_a will always be understood to mean the function that emerges from a given function F of $x, x_1, \ldots, x_n$ by the substitution:

$$x = a, \quad x_1 = a_1, \ldots, \quad x_n = a_n.$$

Since one can ultimately reduce any first-order partial differential equation that includes the unknown function itself to another partial differential equation in which the new unknown function no longer enters explicitly, and any solution of which will produce a solution to the original equation when it is set equal to a constant, whereas conversely any solution to the given equation that includes an arbitrary constant will lead to a solution to the transformed equation that one can complete by multiplication by an arbitrary constant by solving the given equation for the arbitrary constant, Theorem VI will show one how to solve the following problem:

Determine the solution to a first-order partial differential equation between $x, x_1, \ldots, x_n$ such that for an arbitrarily-given equation:

$$\psi(x, x_1, \ldots, x_n) = 0,$$

another, likewise-given, equation:

$$f(x, x_1, \ldots, x_n) = 0$$

will exist.

That most-general problem will be solved by the $2n + 5$ formulas:

$$(10) \qquad \begin{cases} F = 0, \quad F_a = 0, \quad f_a = 0, \quad \psi_a = 0, \\ \dfrac{\partial F}{\partial c_k} + \lambda \dfrac{\partial F_a}{\partial c_k} = 0, \\ \dfrac{\partial F_a}{\partial a_k} + \mu \dfrac{\partial f_a}{\partial c_k} + \nu \dfrac{\partial \psi_a}{\partial a_k} = 0, \end{cases}$$

from which the $2n + 4$ quantities:

$$a, a_1, \ldots, a_n, c_1, \ldots, c_n, \lambda, \mu, \nu$$

are eliminated. In so doing, one assumes that a complete solution to the given partial differential equation is defined by the equation $F = 0$ between $x, x_1, \ldots, x_n$, and the n arbitrary constants $c_1, \ldots, c_n$.

Here, I shall highlight the following two more-specialized theorems that include the previous ones V and VI as special cases, and can be easily verified directly in a similar way:

VII. *If:*

$$x = X(x_1, \ldots, x_n, c_1, \ldots, c_n)$$

is the complete solution to a first-order partial differential equation between $x, x_1, \ldots, x_n$ *then one will obtain the solution to that equation that will become equal to the given function* $f(x_1, \ldots, x_{n-1})$ *for* $x_n = a$ *when one eliminates the* $2n$ *quantities* $a_1, \ldots, a_{n-1}, c_1, \ldots, c_n, \lambda$, *from the* $2n + 1$ *equations:*

$$x = X, \qquad X_a = f_a,$$

$$\frac{\partial X}{\partial c_k} = \lambda \frac{\partial X_a}{\partial c_k}, \qquad \frac{\partial X_a}{\partial a_k} = \frac{\partial f_a}{\partial a_k},$$

in which $k = 1, 2, \ldots, n$, $h = 1, 2, \ldots, n - 1$.

VIII. *From the complete solution:*

$$x = X(x_1, \ldots, x_n, c_1, \ldots, c_n)$$

to a first-order partial differential equation between $x, x_1, \ldots, x_n$, *one will obtain the solution* x *to that equation that assumes the given value:*

$$x = f(x_1, \ldots, x_n)$$

as soon as the given condition equation:

$$(x, x_1, \ldots, x_n) = 0$$

exists when one eliminates the $2n + 2$ *quantities:*

$$a_1, \ldots, a_n, c_1, \ldots, c_n, \lambda, \mu$$

from the $2n + 3$ *equations:*

$$x = X, \quad X_a = f_a, \quad \psi_a = 0,$$

$$\frac{\partial X}{\partial c_k} = \lambda \frac{\partial X_a}{\partial c_k}, \qquad \frac{\partial X_a}{\partial a_k} - \frac{\partial f_a}{\partial a_k} = \mu \frac{\partial \psi_a}{\partial a_k}.$$

Theorem VII is interesting because it mediates the transition from the method for integrating first-order partial differential equations that was given in Theorem III to the more general **Cauchy** method.

One can give Theorems V-VIII a more concise formulation with the help of the remark that, as the formulas in question show, the solution that is produced by those theorems will always correspond to a certain maximum or minimum of the given one.

They simplify appreciably for the special case of linear partial differential equations and lead to a symmetric expression for the known **Jacobi** rules for determining the solution of such an equation by a boundary condition (**Crelle**'s J., v. 23, pp. 26).

Leipzig, October 1870.