# The existence conditions for a kinetic potential 

By A. Mayer, o. M.<br>Translated by D. H. Delphenich

In the article "Ueber die physikalische Bedeutung des Princips der kleinsten Wirkungen," HELMOLTZ had emphasized ${ }^{( }{ }^{1}$ ) that a kinetic potential $H$ for the motion of a system of material points with the independent determining data $p_{1}, \ldots, p_{n}$ will always exist as long as the LAGRANGIAN expressions for the driving forces $P_{1}, \ldots, P_{n}$ of the system are functions of $p, p^{\prime}$, $p^{\prime \prime}$ that are linear in the $p^{\prime \prime}$, i.e., the variables $p_{1}, \ldots, p_{n}$ and their first and second differential quotients with respect to time $t$, and between which the relations exist identically:

$$
\frac{\partial P_{i}}{\partial p_{k}^{\prime \prime}}=\frac{\partial P_{k}}{\partial p_{i}^{\prime \prime}}, \quad \frac{\partial P_{i}}{\partial p_{k}^{\prime}}+\frac{\partial P_{k}}{\partial p_{i}^{\prime}}=2 \frac{d}{d t} \frac{\partial P_{k}}{\partial p_{i}^{\prime \prime}}, \quad \frac{\partial P_{i}}{\partial p_{k}}+\frac{\partial P_{k}}{\partial p_{i}}=\frac{1}{2} \frac{d}{d t}\left(\frac{\partial P_{i}}{\partial p_{k}^{\prime}}-\frac{\partial P_{k}}{\partial p_{i}^{\prime}}\right)
$$

or they are such that under those assumptions, there is always a function $H$ of the $p$ and $p^{\prime}$ that fulfills the $n$ equations identically:

$$
-\frac{\partial H}{\partial p_{\lambda}}+\frac{d}{d t} \frac{\partial H}{\partial p_{\lambda}^{\prime}}=P_{\lambda}
$$

HELMHOLTZ likewise suggested that this assertion can be proved by extending certain theorems from the theory of potential functions to the space of $n$ dimensions but added: Since the fact of a proof would be interesting in its own right, it did not seem appropriate to him to do that only incidentally, and for that reason, he preferred to give the proof on another occasion.

In the immensely stimulating book Ueber die Principien der Mechanik, KOENIGSBERGER gave a different path $\left({ }^{2}\right)$ along which one could prove HELMHOLTZ's theorem and the investigation can even be extended to the question of necessary and sufficient conditions for $n$ functions $P_{1}, \ldots, P_{n}$ of the variables $p_{1}, \ldots, p_{n}$, and their differential quotients with respect to $t$ up to order $2 v$ to be expressible by a single function $H$ of the $p$ and their differential quotients up to order $v$ in the form:

[^0]$$
P_{\lambda}=-\left\{\frac{\partial H}{\partial p_{\lambda}}-\frac{d}{d t} \frac{\partial H}{\partial p_{\lambda}^{\prime}}+\cdots+(-1)^{v} \frac{d^{v}}{d t^{v}} \frac{\partial H}{\partial p_{\lambda}^{(\nu)}}\right\} .
$$

However, the proof was actually performed for only the case of $n=2, v=1\left({ }^{1}\right)$.
In what follows, I shall communicate another proof of HELMHOLTZ's remark that is shorter and entirely direct, and is completely independent of whether time does or does not appear in the $P_{\lambda}$, and that proof is arranged in such a way that from the very beginning one will see, with no further discussion, how one can also resolve the more general question that KOENIGSBERGER raised with a minimum of calculation. For the sake of that latter purpose, I shall take my starting point to be the principle by which JACOBI showed how to calculate the partial derivatives of complete differential quotients by mere variation $\left({ }^{2}\right)$, although in the present simple case, direct calculation will lead to the desired formulas almost as rapidly. However, when one develops this fertile principle a little bit further, it will allow one to derive all of the interesting formulas in KOENIGSBERGER's treatise with the greatest of ease.

From the foregoing, one addresses the complete answer to the question:
What conditions must $n$ given functions $P_{1}, \ldots, P_{n}$ of the variables $p_{1}, \ldots, p_{n}$, their first and second differential quotients with respect to $t$, and possibly also the independent variable $t$ itself fulfill in order for a function $H$ of $t, p_{1}, \ldots, p_{n}, p_{1}^{\prime}, \ldots, p_{n}^{\prime}$ to exist that satisfies the $n$ equations:

$$
\begin{equation*}
-\frac{\partial H}{\partial p_{\lambda}}+\frac{d}{d t} \frac{\partial H}{\partial p_{\lambda}^{\prime}}=P_{i} \tag{1}
\end{equation*}
$$

identically?
If one generally lets $\delta V$ denote the variation that any function of $t$, the $p$, and their differential quotients experiences when one assigns the arbitrary variations $\delta p$ to the variables $p$ then one will always have:

$$
\delta \frac{d V}{d t} \equiv \frac{d \delta V}{d t}
$$

( ${ }^{1}$ ) A small mistake crept into it. Namely, on pp. 932, the two conditions:

$$
\begin{aligned}
& \frac{\partial f_{01}}{\partial p_{1}^{\prime}}=\frac{\partial f_{11}}{\partial p_{1}} p_{1}^{\prime}+\frac{\partial f_{11}}{\partial p_{2}} p_{2}^{\prime}, \\
& \frac{\partial f_{02}}{\partial p_{2}^{\prime}}=\frac{\partial f_{22}}{\partial p_{1}} p_{1}^{\prime}+\frac{\partial f_{22}}{\partial p_{2}} p_{2}^{\prime}
\end{aligned}
$$

were left out. However, the first of them makes it possible to determine function $\omega$ from equations (121) on pp. 934, since they say that $R_{1}$ and $R_{2}$ must be free of $p_{1}^{\prime}$ and $p_{2}^{\prime}$.
$\left(^{2}\right)$ JACOBI, Werke, Bd. IV, pp. 496.
so in particular, when $V$ includes no higher differential quotients of the $p$ than the first:

$$
\begin{gathered}
\sum_{\lambda=1}^{n}\left\{\frac{\partial \frac{d V}{d t}}{\partial p_{\lambda}} \delta p_{\lambda}+\frac{\partial \frac{d V}{d t}}{\partial p_{\lambda}^{\prime}} \frac{d \delta p_{\lambda}}{d t}+\frac{\partial \frac{d V}{d t}}{\partial p_{\lambda}^{\prime \prime}} \frac{d^{2} \delta p_{\lambda}}{d t^{2}}\right\} \\
\equiv \frac{d}{d t} \sum_{\lambda=1}^{n}\left(\frac{\partial V}{\partial p_{\lambda}} \delta p_{\lambda}+\frac{\partial V}{\partial p_{\lambda}^{\prime}} \frac{d \delta p_{\lambda}}{d t}\right) \\
\equiv \sum_{\lambda=1}^{n}\left\{\frac{\partial \frac{d V}{d t}}{\partial p_{\lambda}} \delta p_{\lambda}+\left(\frac{\partial V}{\partial p_{\lambda}}+\frac{d \frac{\partial V}{\partial p_{\lambda}^{\prime}}}{d t}\right) \frac{d \delta p_{\lambda}}{d t}+\frac{\partial V}{\partial p_{\lambda}^{\prime}} \frac{d^{2} \delta p_{\lambda}}{d t^{2}}\right\},
\end{gathered}
$$

and upon comparing the coefficients of $\delta p_{\lambda}$ and its differential quotients on both sides, that will lead immediately to the relations:
(A)

$$
\frac{\partial}{\partial p_{\lambda}} \frac{d V}{d t} \equiv \frac{d}{d t} \frac{\partial V}{\partial p_{\lambda}}
$$

(B)

$$
\frac{\partial}{\partial p_{\lambda}^{\prime}} \frac{d V}{d t} \equiv \frac{\partial V}{\partial p_{\lambda}}+\frac{d}{d t} \frac{\partial V}{\partial p_{\lambda}^{\prime}},
$$

(C)

$$
\frac{\partial}{\partial p_{\lambda}^{\prime \prime}} \frac{d V}{d t} \equiv \frac{\partial V}{\partial p_{\lambda}^{\prime}} .
$$

Having established that, with the $n$ substitutions:

$$
\frac{\partial H}{\partial p_{i}^{\prime}}=\psi_{i}
$$

I can convert the $n$ equations (1) into the $2 n$ following ones:

$$
\begin{equation*}
\frac{\partial H}{\partial p_{i}^{\prime}}=\psi_{i}, \quad \frac{\partial H}{\partial p_{i}}=\frac{d \psi_{i}}{d t}-P_{i} . \tag{2}
\end{equation*}
$$

Since $H$ is free of the $p^{\prime \prime}$, from (C), those equations next demand that:

$$
\begin{equation*}
\frac{\partial \psi_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime \prime}}=0 \tag{3}
\end{equation*}
$$

and therefore, as equations (1) also show already, that likewise demands that the $P_{i}$ must be linear in the second differential quotients of the $p$.

If one further defines the integrability conditions:

$$
\frac{\partial \frac{\partial H}{\partial p_{i}^{\prime}}}{\partial p_{\kappa}^{\prime}}=\frac{\partial \frac{\partial H}{\partial p_{\kappa}^{\prime}}}{\partial p_{i}^{\prime}}, \quad \frac{\partial \frac{\partial H}{\partial p_{\kappa}^{\prime}}}{\partial p_{i}}=\frac{\partial \frac{\partial H}{\partial p_{i}}}{\partial p_{\kappa}^{\prime}}, \quad \frac{\partial \frac{\partial H}{\partial p_{i}}}{\partial p_{\kappa}}=\frac{\partial \frac{\partial H}{\partial p_{\kappa}}}{\partial p_{i}}
$$

which are based upon formulas (A) and (B) using the values (2), then one will see that: It is, in addition, also necessary, but not likewise sufficient, for the existence of a function $H$ that satisfies all $2 n$ equations (2) that the:

$$
\frac{1}{2} n(n-1)+n^{2}+\frac{1}{2} n(n-1)
$$

identities must exist:

$$
\begin{align*}
& \frac{\partial \psi_{i}}{\partial p_{\kappa}^{\prime}}=\frac{\partial \psi_{\kappa}}{\partial p_{i}^{\prime}}  \tag{4}\\
& \frac{\partial \psi_{i}}{\partial p_{\kappa}}=\frac{\partial \psi_{\kappa}}{\partial p_{i}}+\frac{d}{d t} \frac{\partial \psi_{\kappa}}{\partial p_{i}^{\prime}}-\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}} \\
& \frac{d}{d t} \frac{\partial \psi_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}=\frac{d}{d t} \frac{\partial \psi_{\kappa}}{\partial p_{i}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}
\end{align*}
$$

However, it follows from (3) and (4) that:

$$
\begin{equation*}
\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime \prime}}=\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime \prime}}, \tag{7}
\end{equation*}
$$

and one gets from (5) the system of two equations that are collectively equivalent to the $n^{2}$ equations (5):

$$
\begin{align*}
& \frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}+\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}=\frac{d}{d t}\left(\frac{\partial \psi_{i}}{\partial p_{\kappa}^{\prime}}+\frac{\partial \psi_{\kappa}}{\partial p_{i}^{\prime}}\right)  \tag{5.a}\\
& \frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}=\frac{d}{d t}\left(\frac{\partial \psi_{i}}{\partial p_{\kappa}}-\frac{\partial \psi_{\kappa}}{\partial p_{i}}\right) \tag{5.b}
\end{align*}
$$

and from (3), the first of them, which is also fulfilled for $\kappa=i$, can then be written ${ }^{1}$ ):

$$
\begin{equation*}
\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}+\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}=\frac{d}{d t}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime \prime}}+\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime \prime}}\right) \tag{8}
\end{equation*}
$$

Finally, (6) gives:
( ${ }^{1}$ ) Due to (7), that is just HELMHOLTZ's second condition.

$$
\frac{\partial P_{i}}{\partial p_{\kappa}}-\frac{\partial P_{\kappa}}{\partial p_{i}}=\frac{d}{d t}\left(\frac{\partial \psi_{i}}{\partial p_{\kappa}}-\frac{\partial \psi_{\kappa}}{\partial p_{i}}\right)
$$

and therefore, from (5.b):

$$
\begin{equation*}
\frac{\partial P_{i}}{\partial p_{\kappa}}-\frac{\partial P_{\kappa}}{\partial p_{i}}=\frac{1}{2} \frac{d}{d t}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right) . \tag{9}
\end{equation*}
$$

Therefore, in order for a function $H$ that fulfills the requirements (1) to exist, it is, in any case, necessary that $P_{1}, \ldots, P_{n}$ must be linear functions in the $p^{\prime \prime}$ between which the:

$$
\frac{1}{2} n(n-1)+n^{2}+\frac{1}{2} n(n-1)
$$

relations (7), (8), (9) exist identically.
Conversely, as soon one appends equations (3) and (5.b) to equations (7), (8), (9), equations (4), (5.a), and (6) will then, in turn, emerge from that, so just the original integrability conditions (4), (5), (6).

Those necessary conditions will then be likewise sufficient when, as long as they are fulfilled, the $n$ functions $\psi_{1}, \ldots, \psi_{n}$ can always be determined such that the equations (3) and (5.b), or what amounts to the same thing, according to (7), that the $n^{2}+\frac{1}{2} n(n-1)$ equations:

$$
\begin{align*}
& \frac{\partial \psi_{i}}{\partial p_{\kappa}^{\prime}}=\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime \prime}}  \tag{10}\\
& \frac{\partial \psi_{i}}{\partial p_{\kappa}}-\frac{\partial \psi_{\kappa}}{\partial p_{i}}=\frac{1}{2} \frac{d}{d t}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right) \tag{11}
\end{align*}
$$

are satisfied identically.
In order to show that, I shall now assume that the $n$ functions $P_{1}, \ldots, P_{n}$ fulfill the necessary conditions that were given above, and above all, infer some further identities from the identities (9).

Since third differential quotients of the $p$ do not occur on the left-hand sides of those identities, they will next demand that:

$$
\begin{equation*}
\frac{\partial}{\partial p_{\lambda}^{\prime \prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right) \equiv 0 \tag{12}
\end{equation*}
$$

An application of formulas (C) and (B) to them will then imply that:

$$
\begin{equation*}
\frac{\partial}{\partial p_{\lambda}^{\prime \prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}}-\frac{\partial P_{\kappa}}{\partial p_{i}}\right) \equiv \frac{1}{2} \frac{\partial}{\partial p_{\lambda}^{\prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}}-\frac{\partial P_{\kappa}}{\partial p_{i}}\right) \tag{13}
\end{equation*}
$$

and

$$
\frac{\partial}{\partial p_{\lambda}^{\prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}}-\frac{\partial P_{\kappa}}{\partial p_{i}}\right) \equiv \frac{1}{2} \frac{\partial}{\partial p_{\lambda}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right)+\frac{1}{2} \frac{d}{d t} \frac{\partial}{\partial p_{\lambda}^{\prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right) .
$$

If one successively permutes the indices $\lambda, i, \kappa$ with $i, \kappa, \lambda$ and $\kappa, \lambda, i$, adds the three formulas that thus arise, and considers that on the one hand, one has:

$$
\begin{aligned}
& \frac{\partial}{\partial p_{\lambda}^{\prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}}-\frac{\partial P_{\kappa}}{\partial p_{i}}\right)+\frac{\partial}{\partial p_{i}^{\prime}}\left(\frac{\partial P_{\kappa}}{\partial p_{\lambda}}-\frac{\partial P_{\lambda}}{\partial p_{\kappa}}\right)+\frac{\partial}{\partial p_{\kappa}^{\prime}}\left(\frac{\partial P_{\lambda}}{\partial p_{i}}-\frac{\partial P_{i}}{\partial p_{\lambda}}\right) \\
\equiv- & \left\{\frac{\partial}{\partial p_{\lambda}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right)+\frac{\partial}{\partial p_{i}}\left(\frac{\partial P_{\kappa}}{\partial p_{\lambda}^{\prime}}-\frac{\partial P_{\lambda}}{\partial p_{\kappa}^{\prime}}\right)+\frac{\partial}{\partial p_{\kappa}}\left(\frac{\partial P_{\lambda}}{\partial p_{i}^{\prime}}-\frac{\partial P_{i}}{\partial p_{\lambda}^{\prime}}\right)\right\},
\end{aligned}
$$

and that on the other hand, that sum will vanish identically when one replaces the partial differential quotients with respect to $p_{\kappa}, p_{i}, p_{\lambda}$ in it with ones with respect to $p_{\kappa}^{\prime}, p_{i}^{\prime}, p_{\lambda}^{\prime}$, and one will then find that:

$$
\begin{equation*}
\frac{\partial}{\partial p_{\lambda}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right)+\frac{\partial}{\partial p_{i}}\left(\frac{\partial P_{\kappa}}{\partial p_{\lambda}^{\prime}}-\frac{\partial P_{\lambda}}{\partial p_{\kappa}^{\prime}}\right)+\frac{\partial}{\partial p_{\kappa}}\left(\frac{\partial P_{\lambda}}{\partial p_{i}^{\prime}}-\frac{\partial P_{i}}{\partial p_{\lambda}^{\prime}}\right) \equiv 0 . \tag{14}
\end{equation*}
$$

Now, equations (10) next demand that the function $\psi_{\lambda}$ must be a common solution to the $n$ partial differential equations:

$$
\frac{\partial \psi_{\lambda}}{\partial p_{1}^{\prime}}=\frac{\partial P_{1}}{\partial p_{\lambda}^{\prime \prime}}, \quad \frac{\partial \psi_{\lambda}}{\partial p_{2}^{\prime}}=\frac{\partial P_{2}}{\partial p_{\lambda}^{\prime \prime}}, \ldots, \quad \frac{\partial \psi_{\lambda}}{\partial p_{n}^{\prime}}=\frac{\partial P_{n}}{\partial p_{\lambda}^{\prime \prime}}
$$

They are free of $p^{\prime \prime}$, and from (12) one has:

$$
\frac{\partial \frac{\partial P_{i}}{\partial p_{\lambda}^{\prime \prime}}}{\partial p_{\kappa}^{\prime}} \equiv \frac{\partial \frac{\partial P_{\kappa}}{\partial p_{\lambda}^{\prime \prime}}}{\partial p_{i}^{\prime}} .
$$

Equations ( $10^{\prime}$ ) will then fulfill the integrability conditions. One can then determine $\psi \lambda$ from them by a mere quadrature, and when that yields:

$$
\psi_{\lambda}=\chi_{\lambda}\left(t, p_{1}, \ldots, p_{n}, p_{1}^{\prime}, \ldots, p_{n}^{\prime}\right)
$$

as a common solution to the $n$ equations (10), one will have:

$$
\begin{equation*}
\psi \lambda=\chi_{\lambda}\left(t, p_{1}, \ldots, p_{n}, p_{1}^{\prime}, \ldots, p_{n}^{\prime}\right)+\omega_{\lambda}\left(t, p_{1}, \ldots, p_{n}\right), \tag{15}
\end{equation*}
$$

in which $\omega \lambda$ is an arbitrary function, as its general solution.
If one substitutes those values for the $\psi \lambda$ in the equations (11) and introduces the abbreviations:

$$
\begin{equation*}
\frac{1}{2}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right)-\left(\frac{\partial \chi_{i}}{\partial p_{\kappa}}-\frac{\partial \chi_{\kappa}}{\partial p_{i}}\right) \equiv \Omega_{i \kappa} \tag{16}
\end{equation*}
$$

such that $\Omega_{\kappa i} \equiv-\Omega_{i \kappa}$, then those equations will go to:

$$
\begin{equation*}
\frac{\partial \omega_{i}}{\partial p_{\kappa}}-\frac{\partial \omega_{\kappa}}{\partial p_{i}} \equiv \Omega_{i \kappa} \tag{17}
\end{equation*}
$$

From (12), the right-hand side of that is free of the $p^{\prime \prime}$, because from (10) and (7):

$$
\frac{\partial \chi_{\mu}}{\partial p_{\lambda}^{\prime}} \equiv \frac{\partial P_{\lambda}}{\partial p_{\mu}^{\prime \prime}} \equiv \frac{\partial P_{\mu}}{\partial p_{\lambda}^{\prime \prime}}
$$

As a result:

$$
\begin{aligned}
\frac{\partial \Omega_{i \kappa}}{\partial p_{\lambda}^{\prime}} & \equiv \frac{1}{2} \frac{\partial}{\partial p_{\lambda}^{\prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right)-\left(\frac{\partial^{2} \chi_{i}}{\partial p_{\kappa} \partial p_{\lambda}^{\prime}}-\frac{\partial \chi_{\kappa}}{\partial p_{i} \partial p_{\lambda}^{\prime}}\right) \\
& \equiv \frac{1}{2} \frac{\partial}{\partial p_{\lambda}^{\prime}}\left(\frac{\partial P_{i}}{\partial p_{\kappa}^{\prime}}-\frac{\partial P_{\kappa}}{\partial p_{i}^{\prime}}\right)-\left(\frac{\partial^{2} \chi_{i}}{\partial p_{\kappa} \partial p_{\lambda}^{\prime}}-\frac{\partial \chi_{\kappa}}{\partial p_{i} \partial p_{\lambda}^{\prime}}\right),
\end{aligned}
$$

and that is zero, from (13).
The $\frac{1}{2} n(n-1)$ equations (17) then include only the variables $p_{1}, \ldots, p_{n}$ themselves, along with possibly the parameter $t$.

Moreover, from (16) and (14), the following relations will exist between their right-hand sides:

$$
\begin{equation*}
\frac{\partial \Omega_{i \kappa}}{\partial p_{\lambda}}+\frac{\partial \Omega_{\kappa \lambda}}{\partial p_{i}}+\frac{\partial \Omega_{\lambda i}}{\partial p_{\kappa}} \equiv 0 . \tag{18}
\end{equation*}
$$

If one now arranges equations (17) into:

$$
\begin{array}{rlrll}
\frac{\partial \omega_{1}}{\partial p_{2}}=\frac{\partial \omega_{2}}{\partial p_{1}}+\Omega_{12}, & \frac{\partial \omega_{1}}{\partial p_{3}}=\frac{\partial \omega_{3}}{\partial p_{1}}+\Omega_{13}, & \cdots & \frac{\partial \omega_{1}}{\partial p_{n}} & =\frac{\partial \omega_{n}}{\partial p_{1}}+\Omega_{1 n}, \\
\frac{\partial \omega_{2}}{\partial p_{3}}=\frac{\partial \omega_{3}}{\partial p_{2}}+\Omega_{23}, & \cdots & \frac{\partial \omega_{2}}{\partial p_{n}} & =\frac{\partial \omega_{n}}{\partial p_{2}}+\Omega_{2 n},  \tag{17'}\\
& \ldots & \cdots & & \cdots \\
& & & \frac{\partial \omega_{n-1}}{\partial p_{n}} & =\frac{\partial \omega_{n}}{\partial p_{n-1}}+\Omega_{n-1, n},
\end{array}
$$

and assumes that one can already find functions $\omega_{n,} \omega_{n-1}, \ldots, \omega_{\lambda+1}$ that fulfill the last $n-\lambda-1$ rows in (17') then one will get the following equations for determining $\omega_{\lambda}$ :
(17") $\quad \frac{\partial \omega_{\lambda}}{\partial p_{\lambda+1}}=\frac{\partial \omega_{\lambda+1}}{\partial p_{\lambda}}+\Omega_{\lambda, \lambda+1}, \quad \frac{\partial \omega_{\lambda}}{\partial p_{\lambda+2}}=\frac{\partial \omega_{\lambda+2}}{\partial p_{\lambda}}+\Omega_{\lambda, \lambda+2}, \quad \ldots, \quad \frac{\partial \omega_{\lambda}}{\partial p_{n}}=\frac{\partial \omega_{n}}{\partial p_{\lambda}}+\Omega_{\lambda_{n}}$.

In order for those $n-\lambda$ equations to be fulfillable, it is necessary and sufficient that one must have:

$$
\frac{\partial^{2} \omega_{\mu}}{\partial p_{\lambda} \partial p_{v}}+\frac{\partial \Omega_{\lambda \mu}}{\partial p_{v}} \equiv \frac{\partial^{2} \omega_{v}}{\partial p_{\lambda} \partial p_{\mu}}+\frac{\partial \Omega_{\lambda v}}{\partial p_{\mu}}
$$

for $\lambda+1 \leq \mu<v \leq n$. However, as a result of equations (17'), which are already satisfied, by assumption, one will have:

$$
\frac{\partial \omega_{\mu}}{\partial p_{v}}=\frac{\partial \omega_{v}}{\partial p_{\mu}}+\Omega_{\mu v}
$$

for all of those values of $\mu$ and $\nu$, and therefore one will also have:

$$
\frac{\partial^{2} \omega_{\mu}}{\partial p_{\lambda} \partial p_{v}} \equiv \frac{\partial^{2} \omega_{v}}{\partial p_{\mu} \partial p_{\lambda}}+\frac{\partial \Omega_{\mu \nu}}{\partial p_{\lambda}} .
$$

Since $\Omega_{\lambda \nu} \equiv-\Omega_{\nu \lambda}$, the integrability conditions for equations (17") can then be written:

$$
\frac{\partial \Omega_{\lambda \mu}}{\partial p_{v}}+\frac{\partial \Omega_{\mu v}}{\partial p_{\lambda}}+\frac{\partial \Omega_{\nu \lambda}}{\partial p_{\mu}} \equiv 0
$$

and from (18), they are, in fact, fulfilled.
Once one has chosen $\omega_{n}$ as a function of $t, p_{1}, \ldots, p_{n}$, one can then first determine $\omega_{n-1}$ from the last equation in (17'), and then $\omega_{n-1}$ from the two penultimate equations. etc., and finally $\omega_{1}$ from the first $n-1$ equations (17'), and indeed, one will always get each new function $\omega$ from the ones that were obtained already by a mere quadrature.

The substitution of the values of $\omega_{\lambda}$ thus-obtained in the equations (15) that were found before will yield $n$ functions $\psi_{1}, \ldots, \psi_{n}$ that satisfy all equations (10) and (11) identically. As a result of our assumptions on the $P$, the integrability conditions (4), (5), (6) for equations (2) will be fulfilled identically by those functions, and one will then ultimately get $H$ itself, as well, from those $2 n$ equations by a mere quadrature.

The question that was raised will be answered by that, and that will show that:

In order for a function $H$ to exist that satisfies the $n$ demands (1), it is not only necessary, but also sufficient, that $P_{1}, \ldots, P_{n}$ must be functions of the $p, p^{\prime}, p^{\prime \prime}$ that are linear in $p^{\prime \prime}$ that satisfy the conditions (7), (8), (9), regardless of whether they do or do not include t itself.

However, in order to avoid any ambiguity, it is perhaps not entirely superfluous to remark that one obviously assumes that the functions $P$ must obey the given conditions.

If $H=H_{1}$ is any solution of the $n$ equations (1) then with the substitution:

$$
H=H_{1}+H_{2},
$$

they will go to the equations:

$$
-\frac{\partial H_{2}}{\partial p_{i}}+\frac{d}{d t} \frac{\partial H_{2}}{\partial p_{i}^{\prime}}=0
$$

However, those $n$ partial differential equations cannot be satisfied in any other way than under the assumption that:

$$
H_{2}=\frac{d \Phi\left(t, p_{1}, \ldots, p_{n}\right)}{d t}
$$

in which the function $\Phi$ remains arbitrary. The general solution of equations (1) is then necessarily of the form:

$$
\begin{equation*}
H=H_{1}+\frac{d \Phi\left(t, p_{1}, \ldots, p_{n}\right)}{d t} \tag{19}
\end{equation*}
$$

There must also be arbitrary functions then that enter into the values of the individual functions $\omega$ under the general integration of the system of partial differential equations (17) and finally combine into a single arbitrary function.

However, when one performs the integration of that system in the way that was given above by successive quadratures, one can next choose the function $\omega_{n}$ to be entirely arbitrary, and then use (17") to also add an arbitrary function of $t, p_{1}, \ldots, p_{\lambda}$ alone to any other function $\omega_{\lambda}$, and one must not immediately overlook the fact that all of the $n$ arbitrary functions can be combined into a single arbitrary function.

It is then very pleasing that one no longer needs to worry about those individual arbitrary functions, because equations (17) are also once more of an entirely analogous nature to equations (1) in their own right. Namely, if the $n$ equations:

$$
\omega_{\lambda}=u_{\lambda}\left(t, p_{1}, \ldots, p_{n}\right)
$$

yield any well-defined system of functions $\omega$ that satisfy the $\frac{1}{2} n(n-1)$ equations (17), and one then sets each:

$$
\omega_{\lambda}=u \lambda+v \lambda
$$

then those equations will reduce to:

$$
\frac{\partial v_{i}}{\partial p_{\kappa}}-\frac{\partial v_{\kappa}}{\partial p_{i}}=0
$$

and that will then demand that every $\nu_{\lambda}$ must have the form:

$$
v \lambda=\frac{\partial \Phi\left(t, p_{1}, \ldots, p_{n}\right)}{\partial p_{\lambda}}
$$

in which $\Phi$ is one and the same arbitrary function for each $\lambda$.
The most-general values of the functions $\omega_{\lambda}$ that satisfy equations (17) will then have the form:

$$
\omega_{\lambda}=u_{\lambda}\left(t, p_{1}, \ldots, p_{n}\right)+\frac{\partial \Phi\left(t, p_{1}, \ldots, p_{n}\right)}{\partial p_{\lambda}}
$$

As a result, from (15), the general solutions $\psi \lambda$ of equations (10) and (11) will have the form:

$$
\psi_{\lambda}=\chi_{\lambda}\left(t, p_{1}, \ldots, p_{n}, p_{1}^{\prime}, \ldots, p_{n}^{\prime}\right)+u_{\lambda}+\frac{\partial \Phi}{\partial p_{\lambda}} .
$$

For those solutions, equations (2) will become:

$$
\frac{\partial H}{\partial p_{i}^{\prime}}=\chi_{i}+u_{i}+\frac{\partial \Phi}{\partial p_{i}}, \quad \frac{\partial H}{\partial p_{i}}=\frac{d\left(\chi_{i}+u_{i}\right)}{d t}+\frac{d}{d t} \frac{\partial \Phi}{\partial p_{i}}-P_{i} .
$$

Therefore, under the assumption that $\Phi \equiv 0$, if $H=H_{1}$ is any well-defined solution of those $2 n$ equations then that will imply that the most general value of $H$ that fulfills the $n$ equations (1) is:

$$
H=H_{1}+\frac{d \Phi\left(t, p_{1}, \ldots, p_{n}\right)}{d t}
$$

which agrees completely with the result (19) that was known to begin with.
In particular, if the $P_{i}$ are functions of the $p, p^{\prime}, p^{\prime \prime}$ that are free of $t$ and linear in the $p^{\prime \prime}$ and fulfill the conditions (7), (8), (9) identically then $t$ will not enter into equations (10) and (11), either, so there will always be solutions to those equations:

$$
\psi \lambda=\chi \lambda+u \lambda
$$

that are likewise free of $t$, and therefore there will also exist functions $H$ in this case that are also always free of $t$ and satisfy the $n$ equations (1).


[^0]:    $\left({ }^{1}\right)$ Journal für die reine und angewandte Mathematik, Bd. 100, pps. 165 and 166.
    ( ${ }^{2}$ ) Sitzungsberichte der Kgl. Preuss. Akademie d. Wiss. zu Berlin, 30 July 1896, pp. 932-935.

