"Zur Pfaff'schen Lösung des Pfaff'schen Problems," Math. Ann. **17** (1880), 523-530.

On Pfaff's solution of the Pfaff problem

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The way that Pfaff adopted for the integration of an equation of the form:

$$X_1\,dx_1 + X_2\,dx_2 + \ldots + X_m\,dx_m = 0$$

rests upon the repeated application of one and the same transformation, which can be expressed by the following problem:

Determine $x_1, x_2, \ldots, x_m$ *as mutually independent functions of m new variables* $t, \alpha_2, \ldots, \alpha_m$ *that give identically:*

I. $$\sum_{h=1}^{m} X_h\,dx_h = \frac{1}{N}\sum_{i=2}^{m} \mathrm{A}_i\,d\alpha_i\,,$$

where N is a function of $t, \alpha_2, \ldots, \alpha_m$ *, but the quantities* $\mathrm{A}_2, \ldots, \mathrm{A}_m$ *are functions of just* $\alpha_2, \ldots, \alpha_m$ *,*

and it is necessary for the entire construction of the Pfaff method, as well as the general applicability of the Pfaff method, that this problem is always soluble for an even *m*, while, in general, no solution is allowed when *m* is odd. Neither the one case nor the other one has actually been proved up to now.

Namely, as far as I can see, it will always be assumed that the so-called *first Pfaff system* of ordinary differential equations by itself already suffices to solve the problem. However, this is correct only as long as the skew determinant that is defined from the elements:

$$\alpha_{i\chi} = \frac{\partial X_i}{\partial x_\chi} - \frac{\partial X_\chi}{\partial x_i}$$

does not vanish (a case that will generally be considered exclusively, as a rule). The goal of the present note is to show this, as well as to place the Pfaff method on firm foundations, and everything that does not seem immediately necessary for this purpose has been left aside, while in line with the desire to found the method in an absolutely clear manner, perhaps one might excuse the presence of entirely too much rigor in regard to some other points, namely, the ones that one ordinarily considers to be self-explanatory. –

The requirement (I) first leads to the condition:

$$\text{(1)} \qquad \mathsf{B} \equiv \sum_{h=1}^{m} X_h \frac{\partial x_h}{\partial t} = 0,$$

and likewise gives the following values for the A_i :

$$\text{(2)} \qquad \mathsf{A}_i = \sum_{h=1}^{m} NX_h \frac{\partial x_h}{\partial \alpha_i}.$$

These should be free of *t*. Therefore, equation (1) and the *m* equations:

$$\text{(3)} \qquad \frac{\partial \mathsf{A}_i}{\partial t} - \sum_{h=1}^{m} \frac{\partial NX_h}{\partial t} \frac{\partial x_h}{\partial \alpha_i} + \sum_{h=1}^{m} NX_h \frac{\partial^2 x_h}{\partial \alpha_i \, \partial t} = 0$$

are the necessary and sufficient conditions that the problem is supposed to satisfy by way of independent functions $x_1, x_2, \ldots, x_m$ of the variables $t, \alpha_2, \ldots, \alpha_m$.

One now has:

$$\text{(4)} \qquad \frac{\partial N}{\partial t} \mathsf{B} \equiv \sum_{h=1}^{m} \frac{\partial NX_h}{\partial t} \frac{\partial x_h}{\partial t} - N \sum_{h=1}^{m} \frac{\partial X_h}{\partial t} \frac{\partial x_h}{\partial t},$$

$$\frac{\partial N\mathsf{B}}{\partial \alpha_i} = \mathsf{B} \frac{\partial N}{\partial t} + N \sum_{h=1}^{m} \frac{\partial X_h}{\partial t} \frac{\partial x_h}{\partial \alpha_i} + \sum_{h=1}^{m} NX_h \frac{\partial^2 x_h}{\partial t \, \partial \alpha_i}.$$

One then has:

$$\text{(5)} \qquad \frac{\partial \mathsf{A}_i}{\partial t} - \frac{\partial N\mathsf{B}}{\partial \alpha_i} \equiv \sum_{h=1}^{m} \frac{\partial NX_h}{\partial t} \frac{\partial x_h}{\partial \alpha_i} - N \sum_{h=1}^{m} \frac{\partial X_h}{\partial \alpha_i} \frac{\partial x_h}{\partial t} - B \frac{\partial N}{\partial \alpha_i}.$$

As a result of conditions (1) and (3), one must then have for $\alpha_i = t, \alpha_2, \ldots, \alpha_m$:

$$\text{(6)} \qquad \sum_{h=1}^{m} \frac{\partial NX_h}{\partial t} \frac{\partial x_h}{\partial \alpha_i} - N \sum_{\chi=1}^{m} \frac{\partial X_\chi}{\partial \alpha_i} \frac{\partial x_\chi}{\partial t} = 0,$$

and likewise formulas (4) and (5) yield:

II. *One can, conversely, replace the original conditions* (1) *and* (3) *with the m conditions* (6), *as long as it is possible to satisfy the latter ones without making* $\partial N / \partial t = 0$. *By comparison, if it then follows from equations* (6) *that* $\partial N / \partial t = 0$ *then one must add the condition* (1) *to these equations.*

Conditions (6), however, may be deduced in another, simpler way.

Since:

$$\frac{\partial X_\chi}{\partial \alpha_i} \equiv \sum_{h=1}^{m} \frac{\partial X_\chi}{\partial x_h} \frac{\partial x_h}{\partial \alpha_i},$$

one can, in fact, next write these equations as:

$$\sum_{h=1}^{m}\left[\frac{\partial NX_h}{\partial t} - N\sum_{\chi=1}^{m}\frac{\partial X_\chi}{\partial x_h}\frac{\partial x_\chi}{\partial t}\right]\frac{\partial x_h}{\partial \alpha_i} = 0.$$

Now, should $x_1, \ldots, x_m$ be independent functions of $t, \alpha_2, \ldots, \alpha_m$ then:

$$\sum \pm \frac{\partial x_1}{\partial t}\frac{\partial x_2}{\partial \alpha_2}\cdots\frac{\partial x_m}{\partial \alpha_m}$$

would be non-zero. The conditions (6) thus decompose into the following ones:

$$\frac{\partial NX_h}{\partial t} - N\sum_{\chi=1}^{m}\frac{\partial X_\chi}{\partial x_h}\frac{\partial x_\chi}{\partial t} = 0.$$

However, by means of the formula:

$$\frac{\partial NX_h}{\partial t} \equiv \frac{\partial N}{\partial t}X_h + N\sum_{\chi=1}^{m}\frac{\partial X_\chi}{\partial x_h}\frac{\partial x_\chi}{\partial t},$$

it can be converted into:

$$N\sum_{\chi=1}^{m}\left(\frac{\partial X_\chi}{\partial x_h} - \frac{\partial X_h}{\partial x_\chi}\right)\frac{\partial x_\chi}{\partial t} = \frac{\partial N}{\partial t}X_h.$$

If one then sets:

(7) $$\frac{\partial X_\chi}{\partial x_h} - \frac{\partial X_h}{\partial x_\chi} = \alpha_{\chi h}$$

then conditions (6) finally reduce to the following m equations:

(8) $$N\sum_{\chi=1}^{m}\alpha_{\chi h}\frac{\partial x_\chi}{\partial t} = X_h\frac{\partial \log N}{\partial t}.$$

The present problem will then be solved in all situations by the $m + 1$ equations (1) and (8), assuming that one can satisfy them by independent functions $x_1, \ldots, x_m$.

The latter is, however, always the case, as long as the determinant:

$$\Delta = \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{m1} & -X_1 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{1m} & \cdots & \alpha_{mm} & -X_m \\ X_1 & \cdots & X_m & 0 \end{vmatrix}$$

of the $m + 1$ equations (8) and (1) is zero.

Therefore, if $\Delta = 0$ then one can always satisfy these equations by values of $\partial x_\chi / \partial t$, which are not all zero. However, if (1) and (8) were fulfilled by the assumptions:

$$\frac{\partial x_1}{\partial t} : \frac{\partial x_2}{\partial t} : \ldots : \frac{\partial x_m}{\partial t} : \frac{\partial \log N}{\partial t} = u_1 : u_2 : \ldots : u_m : M,$$

where $u_1, u_2, \ldots, u_m$ are not all zero, then, since t itself does not appear at all in equations (1) and (8), $u_1, u_2, \ldots, u_m, M$ are functions of only $x_1, x_2, \ldots, x_m$, and when, say, u_1 is non-zero, the values:

(9) $$\frac{\partial x_2}{\partial x_1} = \frac{u_2}{u_1}, \ldots, \frac{\partial x_m}{\partial x_1} = \frac{u_m}{u_1},$$

when coupled with the value:

(10) $$\frac{\partial \log N}{\partial x_1} = \frac{M}{u_1},$$

identically satisfy the equations:

$$\sum_{\chi=1}^{m} X_\chi \frac{\partial x_\chi}{\partial x_1} = 0,$$

$$\sum_{\chi=1}^{m} \alpha_{\chi h} \frac{\partial x_\chi}{\partial x_1} = X_h \frac{\partial \log N}{\partial x_1},$$

i.e., the $m + 1$ equations that emerge from (1) and (8) when one assumes that $t = x_1$.

Equations (9), however, define a system of $m - 1$ ordinary differential equations between $x_2, \ldots, x_m$, and x_1. Therefore, if:

$$x_\chi = \varphi_\chi(x_1, \alpha_1, \ldots, \alpha_m)$$

are its complete solutions then the values:

$$x_1 = x_1, \qquad x_2 = \varphi_2, \ldots, \qquad x_m = \varphi_m,$$

in which they all fulfill the condition equations of the transformation, and likewise:

$$\sum \pm \frac{\partial x_1}{\partial x_1} \frac{\partial x_2}{\partial \alpha_2} \cdots \frac{\partial x_m}{\partial \alpha_m} = \sum \pm \frac{\partial \varphi_2}{\partial \alpha_2} \cdots \frac{\partial \varphi_m}{\partial \alpha_m},$$

which are then non-zero, yield solutions of the given problem *).

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*) If one wishes to also find the associated values of the coefficients A_i themselves then one must only, after substituting any complete solution, compute from (10) by a simple quadrature:

$$N = \gamma e^{\int \frac{M}{u_1} dx_1},$$

The problem is, in fact, always soluble, as long as the determinant Δ is zero, and indeed its solution will then be obtained by complete integration of a system of $m - 1$ ordinary differential equations.

By contrast, it possesses no solution when Δ is non-vanishing. Equations (1) and (8) can then be satisfied only for the values:

$$\frac{\partial x_1}{\partial t} = \frac{\partial x_2}{\partial t} = \ldots = \frac{\partial x_m}{\partial t} = \frac{\partial \log N}{\partial t} = 0,$$

and this contradicts the demand that $x_1, x_2, \ldots, x_m$ should be independent functions of t, $\alpha_2, \ldots, \alpha_m$.

However, as a skew determinant of degree $m + 1$, Δ is zero when m is even, and generally non-zero when m is odd. Therefore, one has the theorem:

III. *The given transformation problem is soluble when and only when m is an even number, and indeed in this case its solution requires the complete integration of a system of* $m - 1$ *ordinary differential equations.*

How one can construct the Pfaff method for the integration of any linear total differential equation simply and naturally from this theorem was set down by Gauss in his announcement of the Pfaff treatise *) in such classic brevity and clarity that it would be completely superfluous to go into it here.

By contrast, the validity in regard to the first Pfaff system emerges from the foregoing alone, or the statement made by equations (8) would still not be clear. We thus now fix our attention on the case:

$$m = 2n$$

more closely!

In general, the determinant:

$$\mathsf{A} = \sum \pm\ \alpha_{11}\,\alpha_{22}\,\ldots\,\alpha_{mm}$$

is non-zero here. However, if this is the case then equations (8) are mutually independent. Therefore, equation (1), when it, together with equations (8), defines a system of vanishing determinant Δ, must necessarily a mere consequence of these m equations.

In fact, when A is not zero, none of the sums:

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and substituting this value for N in the formula that arises from (2):

$$\mathsf{A}_i = N \sum_{h=2}^{m} X_h \frac{\partial \varphi_h}{\partial \alpha_i},$$

where one can give the arbitrary constant γ the value 1, since it can have no influence upon the demand that A_i must be free of x_1, and is then removed from the transformation I itself. This latter computation, however, can be completely spared, when one has introduced the initial values of $x_2, \ldots, x_m$ as arbitrary constants in the solutions of system (9) using the Jacobi method (Crelle J., **17**, pp. 156).

*) Gauss, Werke III, pp. 231.

$$\sum_{h=1}^{m} \mathsf{A}_{\chi h} X_h$$

vanish, where one understands $\mathsf{A}_{\chi h}$ to mean the coefficient of the element $\alpha_{\chi h}$ in the determinant A. However, one must then have each $X_h = 0$. Therefore, let, say:

$$\sum_{h=1}^{m} \mathsf{A}_{1h} X_h$$

be non-zero. If we then take $t = x_1$ then equations (8) become:

$$\sum_{\chi=1}^{m} \alpha_{\chi h} \frac{\partial x_\chi}{\partial x_1} = X_h \frac{\partial \log N}{\partial x_1}$$

and yield:

$$\mathsf{A} = \frac{\partial \log N}{\partial x_1} \sum_{h=1}^{m} \mathsf{A}_{1h} X_h ,$$

so $\partial \log N / \partial x_1$ has a finite and non-zero value. From II, equations (8) alone therefore suffice for a solution to this problem in the case considered.

On the contrary, if $\mathsf{A} = 0$ for even m then, for that reason, the skew determinant of degree m:

$$\Delta_{11} = \begin{vmatrix} \alpha_{22} & \cdots & \alpha_{m2} & -X_2 \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{2m} & \cdots & \alpha_{mm} & -X_m \\ X_2 & \cdots & X_m & 0 \end{vmatrix}$$

does not need to be zero. It is then independent of X_1, while for a given $X_2, \ldots, X_m$ the equation $\mathsf{A} = 0$ is a partial differential equation of first order for X_1. However, when Δ_{11} is non-zero then not all sub-determinants of m^{th} degree vanish in the determinant Δ, and the $m + 1$ equations (8) and (1) then reduce to only m equations, but no fewer. They then uniquely determine their unknowns, which are the ratios:

$$\frac{\partial x_1}{\partial t} : \frac{\partial x_2}{\partial t} : \ldots : \frac{\partial x_m}{\partial t} : \frac{\partial \log N}{\partial t} .$$

As long as one can then give values to these unknowns that satisfy equations (1) and (8), these are the only values that satisfy these equations in the case we assumed. However, one obtains such values when one sets $\partial \log N / \partial t = 0$ and then determines the ratios of the $\partial x_\chi / \partial t$ from the $m + 1$ equations:

$$(11)\qquad \begin{cases} \sum_{\chi=1}^{m} \alpha_{\chi h} \dfrac{\partial x_\chi}{\partial t} = 0, \\ \sum_{\chi=1}^{m} X_\chi \dfrac{\partial x_\chi}{\partial t} = 0. \end{cases}$$

Such a determination is possible. Then since, by assumption, the skew determinant A of degree $m = 2n$ vanishes, all of its sub-determinants of degree $m - 1$ are also = 0, and the first m equations (11) then reduce to $m - 2$ equations. In the case in question then, equations (1) and (8) necessarily yield $\partial \log N / \partial t = 0$ and then, from II, system (8) alone no longer suffices as a solution of the problem, but this will instead first occur by means of equations (11).

When illuminated in this light, the problem does not, by any means, need to be indeterminate in the case where the determinant becomes A = 0 for a given m *). Rather, such an indeterminacy will first enter the picture when the determinant A vanishes, as well as all of its sub-determinants of order $m - 2$ or even lower order, and indeed there always exists an even indeterminancy in a singular way; i.e., in such a case, one can always choose an *even* number of the $m - 1$ ratios:

$$\frac{\partial x_1}{\partial t} : \frac{\partial x_2}{\partial t} : \ldots : \frac{\partial x_m}{\partial t}$$

arbitrarily, or, what will always be most convenient, set them equal to zero. Then, when all of the sub-determinants of degree $2r$ in a skew determinant vanish, then, as Frobenius had proved **), all sub-determinants of degree $2r - 1$ also vanish. Since $m = 2n$, the vanishing of all sub-determinants of degree $m - 2p$ in the determinant A necessarily implies the vanishing of all sub-determinants of degree $m - 2p - 1$, and this then reduces the $m + 1$ equations (11) to $m - 2p - 1$ equations, such that one can also therefore satisfy

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*) For the total differential equation:

$$f_1\, dx_1 + f_2\, dx_2 + \ldots + p_n\, dx_n - dz = 0,$$

e.g., the one that is equivalent to the partial differential equation of first order that is free of z:

$$p_1 = f_1(x_1, \ldots, x_n, p_2, \ldots, p_n), \qquad \left(p_1 = \frac{\partial z}{\partial x_1}\right),$$

equations (8) and (1) reduce to the known $2n - 1$ equations:

$$\frac{dx_h}{dx_1} = -\frac{\partial f_1}{\partial p_h}, \qquad \frac{dp_h}{dx_1} = \frac{\partial f_1}{\partial x_h},$$
$$\frac{dz}{dx_1} = f_1 - \sum_{h=2}^{n} p_h \frac{\partial f_1}{\partial p_h},$$

and the transformation problem remains completely determined here.

) Borchardt's J., **82, pp. 242.

them by values of $\partial x_\chi / \partial t$, which are not all zero, after one has set a certain $2p$ of the variables x equal to arbitrary constants.

If one recalls the Jacobi method for presenting the various total differential equations by the introduction of the initial values from the outset, which is what the integration of the given equation will successively come down to in the Pfaff method, then one easily overlooks how this advantage is handed down by the first $p - 1$ reductions step-by-step, with each step decreasing by two units.