# On unrestricted integrable systems of linear total differential equations and the simultaneous integration of linear partial differential equations 

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Any single linear first-order partial differential equation is equivalent to a certain system of ordinary differential equations. Similarly, an easily-recognizable reciprocal relationship exists between systems of first-order linear partial differential equations that admit a common solution and certain systems of linear total differential equations that has already been pointed out many times and employed in the individual cases, moreover; e.g., in Ampère's method for integrating second-order partial differential equations that possess an intermediate integral.

In fact, if the $m-1$ simultaneous partial differential equations:

$$
\begin{equation*}
A_{1}(f)=0, \quad A_{2}(f)=0, \quad \ldots, \quad A_{m-1}(f)=0 \tag{I}
\end{equation*}
$$

in which one generally has:

$$
A_{i}(f)=\frac{\partial f}{\partial x_{i}}+a_{m}^{i} \frac{\partial f}{\partial x_{m}}+\cdots+a_{n}^{i} \frac{\partial f}{\partial x_{n}}
$$

and the coefficients $a_{k}^{i}$ are given functions of $x_{1}, x_{2}, \ldots, x_{n}$, possess a common solution $f$ then that solution will always be a solution of the linear partial differential equation:

$$
\lambda_{1} A_{1}(f)+\lambda_{2} A_{2}(f)+\ldots+\lambda_{m-1} A_{m-1}(f)=0
$$

as well, no matter what arbitrary functions of $x_{1}, x_{2}, \ldots, x_{n}$ one might set $\lambda_{1}, \lambda_{2}, \ldots, \lambda_{m-1}$ equal to, so when that solution $f$ is set equal to an arbitrary constant, it will be an integral of the $n-1$ ordinary differential equations:

$$
d x_{1}: d x_{2}: \ldots: d x_{n}: d x_{m}: \ldots: d x_{n}=\lambda_{1}: \lambda_{2}: \ldots: \lambda_{m-1}: \sum_{h=1}^{m-1} \lambda_{h} a_{m}^{h}: \ldots: \sum_{h=1}^{m-1} \lambda_{h} a_{n}^{h},
$$

and as a result, it will also be an integral of the $n-m+1$ linear total differential equations:

$$
\begin{equation*}
d x_{k}=\sum_{h=1}^{m-1} a_{k}^{h} d x_{h}, \quad k=m, m+1, \ldots, n, \tag{II}
\end{equation*}
$$

which emerge from the foregoing by eliminating the $\lambda_{1}: \lambda_{2}: \ldots: \lambda_{m-1}$, and one will see immediately that conversely when equation (II) possesses an integral $f=$ const. - i.e., when there is a function $f$ of $x_{1}, x_{2}, \ldots, x_{n}$ whose differentials are identically zero as a result of equations (II) alone - that function will be a common solution of equations (I).

The problem of finding a common solution to the $m-1$ linear partial differential equations (I) is then identical to the problem of discovering an integral to the $n-m+1$ linear total differential equations (II). From that, one might expect that any method that shows us how to integrate equations (II) must, at the same time, also contain the germ of a method for integrating equations (I). That thought gives rise to the following investigations, whose main goal is to find a way by which one can arrive at a common solution to several simultaneous linear first-order partial differential equations with the same unknown function by as few integrations as possible.

However, a very essential simplification can be introduced into that from the outset. Namely, as Clebsch showed $\left({ }^{1}\right)$, any system of partial differential equations of that kind that possesses a common solution at all can be reduced to a Jacobi system - i.e., to a system of the form (I), in particular, in which the $(m-2)(m-1) / 2$ identities exist between the operators $A$ :

$$
\begin{equation*}
A_{i}\left(A_{k}(f)\right)-A_{k}\left(A_{i}(f)\right)=0, \tag{III}
\end{equation*}
$$

so it would only be necessary to direct one's attention to those systems of total differential equations whose coefficients fulfill the conditions that follow from (III).

Those systems possess the property that they will be satisfied by $n-m+1$ integrals, and one will next show that their integration comes down to the complete integration of $m-1$ systems of $n-m+1$ first-order ordinary differential equations, as Natani remarked before ( ${ }^{*}$ ), under the assumption of a system of total differential equations that possess the given number of integrals. However, by a transformation of the given equations that is constructed from them, and with the help of which, P. du BoisReymond showed how to reduce the linear total differential equations that are integrable by an equation to just one second-order ordinary differential equation in two variables ${ }^{* *}$ ), one can arrange that the integration of the first of those $m-1$ systems will already suffice to completely integrate the given equations, which will simultaneously convert the complete solution of the equivalent Jacobi system to the complete integration of a single system of $n-m+1$ first-order ordinary differential equations. Finally (and this is much more important in the applications), that will imply that in order to ascertain a common solution of the Jacobi system, it is only necessary to know a single integral of that system of ordinary differential equations, by which, e.g., the number of integrals that one will need for the complete solution of a first-order nonlinear partial differential equation when one ignores the first one, can be reduced by precisely one-half in comparison to the

[^0]number of integrals that one was required to know for the most preferable of the earlier methods, namely, the Weiler-Clebsch method ( ${ }^{*}$ ).

## § 1. - Conditions for unrestricted integrability.

One will arrive at the systems of linear total differential equations that shall be considered exclusively in what follows, when one sets $n-m+1$ arbitrary, mutuallyindependent functions of $n$ variables $x_{1}, x_{2}, \ldots, x_{n}$ equal to arbitrary constants and completely differentiates the equations that thus arise. When one solves the derived equations for $n-m+1$ of the $n$ differentials, one will get $n-m+1$ simultaneous differential equations of the form:

$$
\begin{equation*}
d x_{k}=\sum_{h=1}^{m-1} a_{k}^{h} d x_{h}, \quad k=m, m+1, \ldots, n, \tag{1}
\end{equation*}
$$

in which the $a_{k}^{h}$ are given functions of all $n$ variables, and which can be satisfied as a consequence of the way that they arose in such a way that one sets $x_{m}, x_{m+1}, \ldots, x_{n}$ equal to suitable functions of the independent variables $x_{1}, x_{2}, \ldots, x_{m-1}$, which are functions that include $n-m+1$ arbitrary constants, moreover. In order to be able to refer to them briefly, I would like to call such a system of linear total differential equations (1) an unrestricted integrable system.

Conversely, when a system of linear total differential equations of the form (1) is given, one next asks: Under what conditions is it unrestricted integrable, and furthermore, when those conditions are fulfilled, how can one integrate the system?

Should $n-m+1$ functions $x_{m}, x_{m+1}, \ldots, x_{n}$ of the independent variables $x_{1}, x_{2}, \ldots, x_{m-1}$ be given that satisfy the given equations (1) identically, if $h$ and $i$ are any two distinct numbers from $1,2, \ldots, m-1$ then it must follow for the functions:

$$
\begin{equation*}
\frac{\partial x_{k}}{\partial x_{h}}=a_{k}^{h}, \quad \frac{\partial x_{k}}{\partial x_{i}}=a_{k}^{i} \tag{2}
\end{equation*}
$$

that when one lets the characteristic $d$ suggest that in the differentiation $x_{m}, \ldots, x_{n}$ are to be regarded as functions of $x_{h}$ and $x_{i}$ for which the relations (2) are valid, one will have:

$$
\frac{d a_{k}^{h}}{d x_{i}}-\frac{d a_{k}^{i}}{d x_{h}}=0
$$

or that those functions must satisfy the equations:

$$
\begin{equation*}
\frac{\partial a_{k}^{h}}{\partial x_{i}}-\frac{\partial a_{k}^{i}}{\partial x_{h}}+\sum_{\lambda=m}^{n}\left(a_{\lambda}^{i} \frac{\partial a_{k}^{h}}{\partial x_{\lambda}}-a_{\lambda}^{h} \frac{\partial a_{k}^{i}}{\partial x_{\lambda}}\right)=0 . \tag{3}
\end{equation*}
$$

(*) Crelle's J., 65, pp. 263.

If one introduces the general notation $A_{i}(f)$ for the operation:

$$
\begin{equation*}
A_{i}(f)=\frac{\partial f}{\partial x_{i}}+\sum_{\lambda=m}^{n} a_{\lambda}^{i} \frac{\partial f}{\partial x_{\lambda}} \tag{4}
\end{equation*}
$$

then equations (3) can be written more briefly as:

$$
\begin{equation*}
A_{i}\left(a_{k}^{h}\right)-A_{h}\left(a_{k}^{i}\right)=0 . \tag{5}
\end{equation*}
$$

Those conditions, whose number is equal to:

$$
(n-m+1) \frac{(m-1)(m-2)}{2},
$$

must be satisfied by those functions $x_{m}, \ldots, x_{n}$ of the independent variables $x_{1}, \ldots, x_{m-1}$ that solve equations (1). However, should those functions include $n-m+1$ constants, as will be assumed here, that will be possible only when those conditions are already satisfied identically.

The existence of the relations (3) or (5) as identities will then be necessary whenever equations (1) are supposed to be unrestricted integrable. That this is also sufficient will be shown in the following §, which will show how one can determine $x_{m}, \ldots, x_{n}$ as functions of $x_{1}, \ldots, x_{m-1}$ and $n-m+1$ arbitrary constants in such a way that equations (1) will be satisfied identically.

First, I shall point out that since, from (4):

$$
A_{i}\left(A_{k}(f)\right)-A_{k}\left(A_{i}(f)\right)=\sum_{\lambda=m}^{n}\left\{A_{i}\left(a_{\lambda}^{\lambda}\right)-A_{\lambda}\left(a_{\lambda}^{i}\right)\right\} \frac{\partial f}{\partial x_{\lambda}},
$$

the identities (5) also imply the following:

$$
\begin{equation*}
A_{i}\left(A_{k}(f)\right)=A_{k}\left(A_{i}(f)\right), \tag{6}
\end{equation*}
$$

which will be true for an arbitrary function $f$, and can conversely replace the conditions (5).

## § 2. - Reducing the system (1) to $m-1$ systems of $n-m+1$ first-order ordinary differential equations when the relations (3) are true identically.

If $x_{m}, \ldots, x_{n}$ are $n-m+1$ functions of the independent variables $x_{1}, \ldots, x_{m-1}$ that fulfill equations (1) then they must next satisfy the $n-m+1$ equations:

$$
\begin{equation*}
\frac{\partial x_{m}}{\partial x_{1}}=a_{m}^{1}, \quad \frac{\partial x_{m+1}}{\partial x_{1}}=a_{m+1}^{1}, \ldots, \quad \frac{\partial x_{n}}{\partial x_{1}}=a_{n}^{1} \tag{7}
\end{equation*}
$$

Those equations, in which then $x_{2}, \ldots, x_{m-1}$ enter only as constants, define a system of $n-$ $m+1$ ordinary differential equations between $x_{m}, \ldots, x_{n}$ and $x_{1}$.

Therefore, if:

$$
\begin{equation*}
\varphi_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{m-1}, x_{m}, \ldots, x_{n}\right)=c_{\lambda}, \quad \lambda=m, m+1, \ldots, n \tag{8}
\end{equation*}
$$

are $n-m+1$ mutually-independent integrals of that system then the solutions $x_{m}, \ldots, x_{n}$ of the equations (1) must be included in equations (8), whose integration constants $c_{\lambda}$ can depend upon only $x_{2}, \ldots, x_{m-1}$.

Equations (8), as complete integral equations of the system (7), are always soluble for $x_{m}, \ldots, x_{n}$. One can then employ those equations in order to introduce the $x_{m}, \ldots, x_{n}$ in place of $c_{m}, \ldots, c_{n}$ as new dependent variables.

Completely differentiating (8) and substituting equations (1) will yield:

$$
d c_{\lambda}=\sum_{h=1}^{m-1}\left(\frac{\partial \varphi_{\lambda}}{\partial x_{h}}+\sum_{k=m}^{n} a_{k}^{h} \frac{\partial \varphi_{\lambda}}{\partial x_{h}}\right) d x_{h} .
$$

However, the coefficient of $d x_{1}$ in this is identically zero, because equations (8) are integrals of the system (7), by assumption. When one introduces the notation (4), all that will remain is:

$$
\begin{equation*}
d c_{\lambda}=\sum_{h=2}^{m-1} A_{h}\left(\varphi_{\lambda}\right) d x_{h} . \tag{9}
\end{equation*}
$$

The newly-introduced quantities $c_{m}, \ldots, c_{n}$ will then be determined from those $n-m+1$ equations, in which one replaces the $x_{m}, \ldots, x_{n}$ on the right-hand side with the values that follow from equations (8).

However, should equations (8) remain integrals of the system (7), then the $c_{\lambda}$ would have to be independent of $x_{1}$; hence, $x_{1}$ cannot occur in equations (9).

That is in fact the case. Namely, since:

$$
A_{1}\left(A_{h}(f)\right)=A_{h}\left(A_{1}(f)\right),
$$

from (6), and $A_{1}\left(\varphi_{\lambda}\right)=0$, one will also have that:

$$
f=A_{h}\left(\varphi_{\lambda}\right)
$$

is a solution of the equation $A_{1}(f)=0$, or that $A_{h}\left(\varphi_{\lambda}\right)=$ const. is an integral of the system (7). Therefore, after substituting the values of $x_{m}, \ldots, x_{n}$ that one gets from the integrals (8) of that system, the expressions $A_{h}\left(\varphi_{\lambda}\right)$ will be independent of $x_{1}$.

Equations (9) will all be free of $x_{1}$ then, and therefore nothing can change when one assigns any values to those variables.

The given system (1) has now been reduced to the system (9), which includes only $m$ 2 independent variables. Obviously, the latter system can be first exhibited, in general (i.e., as long as one cannot establish the system of integrals of equations (7) by which the
$c_{\lambda}$ are introduced as integration constants more precisely), only after one has found those integrals. However, it will be most advantageous if one takes the $c_{\lambda}$ to be a well-defined system of integration constants of equations (7), namely, the initial values of the dependent variables, if one is to be able to define equations (9) before all integrations.

One can see that directly from the system (9), but it will be clearer when one starts from another system that is equivalent to equations (9).

If one lets:

$$
\begin{equation*}
x_{k}=\psi_{k}\left(x_{1}, x_{2}, \ldots, x_{m-1}, c_{m}, \ldots, c_{n}\right) \tag{10}
\end{equation*}
$$

denote the solutions of the integrals (8) for $x_{m}, \ldots, x_{n}$ or the complete solutions of the differential equations (7) and introduces the $c_{\lambda}$ directly as new dependent variables in equations (1) then one will now get the equations:

$$
\begin{equation*}
\sum_{\lambda=m}^{n} \frac{\partial \psi_{k}}{\partial c_{\lambda}} d c_{\lambda}=\sum_{\lambda=2}^{m-1}\left(a_{k}^{h}-\frac{\partial \psi_{k}}{\partial x_{\lambda}}\right) d x_{h} \tag{11}
\end{equation*}
$$

for the determination of the $c_{\lambda}$, in which the $a_{k}^{h}$ are also expressed in terms of the $x_{1}, x_{2}$, $\ldots, x_{m-1}, c_{m}, \ldots, c_{n}$ by the substitutions (10) and in exhibiting them, one uses the fact that one will have:

$$
a_{k}^{h}-\frac{\partial \psi_{k}}{\partial x_{\lambda}}=0
$$

identically by those substitutions.
When one solves those $n-m+1$ equations for the $d c_{m}, \ldots, d c_{n}$, one must, in turn, arrive at equations (9). One can then replace the latter with equations (11), and since from the foregoing, equations (9) are free of $x_{1}$, it will be permissible to also assign any arbitrary value to the variable $x_{1}$ for which those equations still remain soluble directly in equations (11) before solving them.

Having assumed that, now let:

$$
\begin{equation*}
x_{k}=\chi_{k}\left(x_{1}, x_{2}, \ldots, x_{m-1}, x_{m}^{0}, \ldots, x_{n}^{0}\right) \tag{12}
\end{equation*}
$$

be the complete solutions of equations (7) when expressed in terms of $x_{1}$ and the values $x_{m}^{0}, \ldots, x_{n}^{0}$ of the dependent variables $x_{m}, \ldots, x_{n}$ that belong to the constant initial values of $x_{1}$. The initial value $x_{1}^{0}$ can be chosen arbitrarily, but only when the associated values of the dependent variables remain arbitrary, so none of the quantities $a_{k}^{1}$ become infinite or undetermined. The expressions $\chi_{k}$, which are the values that one gets from solving the equations:

$$
\varphi_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{m-1}, x_{m}, \ldots, x_{n}\right)=\varphi_{\lambda}\left(x_{1}^{0}, x_{2}, \ldots, x_{m-1}, x_{m}^{0}, \ldots, x_{n}^{0}\right)
$$

that follow from the integrals (8) for the variables $x_{k}$, then have the property that they reduce to $x_{k}^{0}$ for $x_{1}=x_{1}^{0}$.

If one then sets $x_{1}=x_{1}^{0}$ in the system:

$$
\sum_{\lambda=m}^{n} \frac{\partial \chi_{k}}{\partial x_{\lambda}^{0}} d x_{\lambda}^{0}=\sum_{h=2}^{m-1}\left(a_{k}^{h}-\frac{\partial \chi_{k}}{\partial x_{h}}\right) d x_{h},
$$

which is implied by (1) when one introduces the initial values $x_{m}^{0}, \ldots, x_{n}^{0}$ in place of $x_{m}$, $\ldots, x_{n}$ as new variables, and in which, from the foregoing, $x_{1}$ must take on an arbitrary value, $x_{1}=x_{1}^{0}$, then that system will reduce to:

$$
\begin{equation*}
d x_{k}^{0}=\sum_{h=2}^{m-1} a_{k}^{h 0} d x_{h} \tag{13}
\end{equation*}
$$

in which $a_{k}^{h 0}$ generally denotes the value that $a_{k}^{h}$ assumes under the substitution:

$$
x_{1}=x_{1}^{0}, \quad x_{m}=x_{m}^{0}, \quad \ldots, \quad x_{n}=x_{n}^{0} .
$$

The initial values can be determined as functions of $x_{2}, \ldots, x_{m-1}$ from the $n-m+1$ equations (13), which can be exhibited before any integration of the system (7), as one sees.

However, equations (13) define a system that is just the same as the given equation (1), but with one less independent variable $x_{1}$. Since one has, by assumption:

$$
\frac{\partial a_{k}^{h}}{\partial x_{i}}-\frac{\partial a_{k}^{i}}{\partial x_{h}}+\sum_{\lambda=m}^{n}\left(a_{\lambda}^{i} \frac{\partial a_{k}^{h}}{\partial x_{\lambda}}-a_{\lambda}^{h} \frac{\partial a_{k}^{i}}{\partial x_{\lambda}}\right)=0
$$

identically in them, one will also have:

$$
\frac{\partial a_{\lambda}^{h 0}}{\partial x_{i}}-\frac{\partial a_{\lambda}^{i 0}}{\partial x_{h}}+\sum_{\lambda=m}^{n}\left(a_{\lambda}^{i 0} \frac{\partial a_{k}^{h 0}}{\partial x_{\lambda}^{0}}-a_{\lambda}^{h 0} \frac{\partial a_{k}^{i 0}}{\partial x_{\lambda}^{0}}\right)=0
$$

identically, so the system (13) will also fulfill the conditions of unrestricted integrability. One can then treat this system in detail in the same way that one treated the system that was given before, namely, by reducing it to an unrestricted integrable system with only $m$ -3 independent variables by integrating a second system of $n-m+1$ first-order ordinary differential equations, etc., such that one ultimately arrives at the complete integration of the given system (1) after integrating $m-1$ systems of $n-m+1$ first-order ordinary differential equations that are each presented independently of the others and can then be treated and will be obtained by means of a recurrent system of formulas that express $x_{m}$, $\ldots, x_{n}$ in terms of the $x_{1}, x_{2}, \ldots, x_{m-1}$ and the $n-m+1$ arbitrary constants of the last of those $m-1$ systems.

## § 3. - Reducing the unrestricted-integrable system (1) to a single system of $n-m+1$ first-order ordinary differential equations.

The integration of the given unrestricted-integrable system (1) will be reduced to the integration of $m-1$ systems of $n-m+1$ ordinary differential equations by the method that was given in the previous §. However, when the special case occurs in which one can choose the constant $x_{1}^{0}$ in such a way that all of the $(m-2)(n-m+1)$ quantities:

$$
a_{k}^{2}, a_{k}^{3}, \ldots, a_{k}^{m-1}
$$

take the value zero for $x_{1}=x_{1}^{0}$, equations (13), to which the given system (1) will reduce upon integrating equations (7), will then reduce to:

$$
d x_{k}^{0}=0,
$$

and will then immediately give:

$$
x_{m}^{0}=\text { const. }, \quad x_{m+1}^{0}=\text { const } ., \quad . ., \quad x_{n}^{0}=\text { const. }
$$

One will then immediately have the complete solutions to the first of those $n-m+1$ systems of ordinary differential equations expressed in terms of $x_{1}$ and the initial values of $x_{m}, \ldots, x_{n}$ for $x_{1}=x_{1}^{0}$, as long as the initial values in them can be regarded as arbitrary constants that are independent of $x_{2}, \ldots, x_{m-1}$ and yield complete solutions to the system (1).

Now, this seemingly quite special case can always be arranged by a suitable transformation of equations (1).

If one introduces $m-1$ other quantities $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ as new variables in place of $x_{1}, x_{2}, \ldots, x_{m-1}$ by means of $m-1$ arbitrary mutually-independent equations:

$$
\begin{equation*}
x_{h}=x_{h}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}\right) \tag{14}
\end{equation*}
$$

then that will convert equations (1) into:

$$
\begin{equation*}
d x_{k}=\sum_{i=1}^{m-1} b_{k}^{i} d \alpha_{i}, \tag{15}
\end{equation*}
$$

in which:

$$
\begin{equation*}
b_{k}^{i}=\sum_{h=1}^{m-1} a_{k}^{h} \frac{\partial x_{h}}{\partial \alpha_{i}} . \tag{16}
\end{equation*}
$$

At the same time, when one makes the substitutions (14) in an arbitrary function $f$ of $x_{1}$, $x_{2}, \ldots, x_{n}$, one will get:

$$
\frac{\partial f}{\partial \alpha_{i}}=\sum_{h=1}^{m-1} \frac{\partial f}{\partial x_{h}} \frac{\partial x_{h}}{\partial \alpha_{i}},
$$

and as a result:

$$
\begin{equation*}
\frac{\partial f}{\partial \alpha_{i}}+\sum_{h=1}^{m-1} b_{k}^{i} \frac{\partial f}{\partial x_{k}}=\sum_{h=1}^{m-1} \frac{\partial x_{h}}{\partial \alpha_{i}}\left(\frac{\partial f}{\partial x_{h}}+\sum_{k=1}^{n} a_{k}^{h} \frac{\partial f}{\partial x_{k}}\right) \tag{17}
\end{equation*}
$$

Since we know from the foregoing that the original system (1) is unrestricted integrable, as long as the identities (3) exist, it will follow immediately that with that assumption the transformed system (15) will also possess that same property and therefore the relations:

$$
\begin{equation*}
\frac{\partial b_{k}^{\rho}}{\partial \alpha_{\sigma}}-\frac{\partial b_{k}^{\sigma}}{\partial \alpha_{\rho}}+\sum_{\lambda=m}^{n}\left(b_{\lambda}^{\rho} \frac{\partial b_{k}^{\sigma}}{\partial x_{\lambda}}-b_{\lambda}^{\sigma} \frac{\partial b_{k}^{\rho}}{\partial x_{\lambda}}\right)=0 \tag{18}
\end{equation*}
$$

must exist identically between the coefficients $b_{k}^{i}$ in it, in which $k=m, m+1, \ldots, n$, and $\rho$ and $\sigma$ are any two of the numbers $1,2, \ldots, m-1$, and when we set:

$$
\begin{equation*}
B_{\rho}(f)=\frac{\partial f}{\partial \alpha_{\rho}}+\sum_{\lambda=m}^{n} b_{\lambda}^{\rho} \frac{\partial f}{\partial x_{\lambda}} \tag{19}
\end{equation*}
$$

in general, that will imply the following:

$$
\begin{equation*}
B_{\rho}\left(B_{\sigma}(f)\right)=B_{\sigma}\left(B_{\rho}(f)\right) \tag{20}
\end{equation*}
$$

That can be easily verified by calculation.
Namely, from (16), one has:

$$
\frac{\partial b_{k}^{\sigma}}{\partial \alpha_{\rho}}-\frac{\partial b_{k}^{\rho}}{\partial \alpha_{\sigma}}=\sum_{h=1}^{m-1}\left(\frac{\partial a_{k}^{h}}{\partial \alpha_{\rho}} \frac{\partial x_{h}}{\partial \alpha_{\sigma}}-\frac{\partial a_{k}^{h}}{\partial \alpha_{\sigma}} \frac{\partial x_{h}}{\partial \alpha_{\rho}}\right)=\sum_{h=1}^{m-1} \sum_{\mu=1}^{m-1} \frac{\partial a_{k}^{h}}{\partial x_{\mu}}\left(\frac{\partial x_{h}}{\partial \alpha_{\sigma}} \frac{\partial x_{\mu}}{\partial \alpha_{\rho}}-\frac{\partial x_{h}}{\partial \alpha_{\rho}} \frac{\partial x_{\mu}}{\partial \alpha_{\sigma}}\right)
$$

and

$$
\begin{gathered}
\sum_{\lambda=m}^{n}\left(b_{\lambda}^{\rho} \frac{\partial b_{k}^{\sigma}}{\partial x_{\lambda}}-b_{\lambda}^{\sigma} \frac{\partial b_{k}^{\rho}}{\partial x_{\lambda}}\right)=\sum_{\lambda=1}^{n} \sum_{\mu=1}^{m-1} a_{\lambda}^{h}\left(\frac{\partial b_{k}^{\sigma}}{\partial x_{\lambda}} \frac{\partial x_{\mu}}{\partial \alpha_{\rho}}-\frac{\partial b_{k}^{\rho}}{\partial x_{\lambda}} \frac{\partial x_{\mu}}{\partial \alpha_{\sigma}}\right) \\
=\sum_{h=1}^{m-1} \sum_{\mu=1}^{m-1} \sum_{h=1}^{m-1} a_{\lambda}^{\mu} \frac{\partial a_{k}^{h}}{\partial x_{\lambda}}\left(\frac{\partial x_{h}}{\partial \alpha_{\sigma}} \frac{\partial x_{\mu}}{\partial \alpha_{\rho}}-\frac{\partial x_{h}}{\partial \alpha_{\rho}} \frac{\partial x_{\mu}}{\partial \alpha_{\sigma}}\right) .
\end{gathered}
$$

If one then forms the left-hand side of equation (18) and switches the two summation indices $h$ and $\mu$ in the negative terms then one will get:

$$
\frac{\partial b_{k}^{\sigma}}{\partial \alpha_{\rho}}-\frac{\partial b_{k}^{\rho}}{\partial \alpha_{\sigma}}+\sum_{\lambda=m}^{n}\left(b_{\lambda}^{\rho} \frac{\partial b_{k}^{\sigma}}{\partial x_{\lambda}}-b_{\lambda}^{\sigma} \frac{\partial b_{k}^{\rho}}{\partial x_{\lambda}}\right)
$$

$$
=\sum_{h=1}^{m-1} \sum_{\mu=1}^{m-1} \frac{\partial x_{h}}{\partial \alpha_{\sigma}} \frac{\partial x_{\mu}}{\partial \alpha_{\rho}}\left\{\frac{\partial a_{k}^{h}}{\partial x_{\mu}}-\frac{\partial a_{k}^{\mu}}{\partial x_{h}}+\sum_{\lambda=m}^{n}\left(a_{\lambda}^{\mu} \frac{\partial a_{k}^{h}}{\partial x_{\lambda}}-a_{\lambda}^{h} \frac{\partial a_{k}^{\mu}}{\partial x_{\lambda}}\right)\right\},
$$

which is a formula that shows directly that either of the two systems of identity relations (3) and (18) will always imply the other.

One can then employ precisely the same method for the integration of the system (15) that emerges from the given unrestricted-integrable system (1) by the substitutions (14) that one obtained in the previous § for the integration of (1).

After that, we will first have to integrate the $n-m+1$ ordinary differential equations:

$$
\begin{equation*}
\frac{\partial x_{m}}{\partial \alpha_{1}}=b_{m}^{1}, \quad \frac{\partial x_{m+1}}{\partial \alpha_{1}}=b_{m+1}^{1}, \ldots, \quad \frac{\partial x_{n}}{\partial \alpha_{1}}=b_{n}^{1} \tag{21}
\end{equation*}
$$

completely and express the integration constants in terms of the values $x_{m}^{0}, \ldots, x_{n}^{0}$ of the variables $x_{m}, \ldots, x_{n}$ that correspond to the constant initial value $\alpha_{1}^{0}$ of $\alpha_{1}$. The complete solutions of equations (21) that are thus obtained will then give us the complete solutions of the system (15), as well, when we set the $x_{m}^{0}, \ldots, x_{n}^{0}$ in them equal to those functions of $\alpha_{2}, \ldots, \alpha_{m-1}$ that one gets by completely integrating the system:

$$
\begin{equation*}
d x_{k}^{0}=\sum_{i=2}^{m-1} b_{k}^{i 0} d \alpha_{i} \tag{22}
\end{equation*}
$$

whose coefficients $b_{k}^{i 0}$ will emerge from the coefficients:

$$
b_{k}^{i}=\sum_{h=1}^{m-1} a_{k}^{h} \frac{\partial x_{h}}{\partial \alpha_{i}}
$$

when one assumes that:

$$
\alpha_{1}=\alpha_{1}^{0}, \quad x_{m}=x_{m}^{0}, \quad \ldots, \quad x_{n}=x_{n}^{0} .
$$

However, the choice of substitutions (14) is entirely open for us, and that easily explains the fact that we can always make that choice in such a way that all of the coefficients $b_{k}^{i 0}$ in equations (22) vanish. In fact, at the end of the substitutions (14), we need only to take the form:

$$
\begin{equation*}
x_{h}=x_{h}^{0}+\left(\alpha_{1}-\alpha_{1}^{0}\right) f_{h}, \tag{23}
\end{equation*}
$$

in which $\alpha_{1}^{0}$ and the $x_{h}^{0}$ are constants, while $f_{1}, f_{2}, \ldots, f_{m-1}$ are $m-1$ arbitrary functions of $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$, which must obviously always be chosen in such a way that equations (23) will be mutually independent relative to $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$.

From that, one will have:

$$
b_{k}^{1}=\sum_{h=1}^{m-1} a_{k}^{h}\left(f_{h}+\left(\alpha_{1}-\alpha_{1}^{0}\right) \frac{\partial f_{h}}{\partial \alpha_{1}}\right),
$$

and for $i>1$ :

$$
b_{k}^{i}=\left(\alpha_{1}-\alpha_{1}^{0}\right) \sum_{h=1}^{m-1} a_{k}^{h} \frac{\partial f_{h}}{\partial \alpha_{i}} .
$$

When we then assign any constant values to the quantities $\alpha_{1}^{0}$ such that none of the $m$ 1 functions $f_{h}$ become infinite or undetermined for $\alpha_{1}=\alpha_{1}^{0}$, so when we assume that the constants:

$$
x_{1}^{0}, x_{2}^{0}, \ldots, x_{m-1}^{0}
$$

are such that all of the $a_{k}^{h}$ remain finite and well-defined for:

$$
x_{1}=x_{1}^{0}, \quad x_{2}=x_{2}^{0}, \quad \ldots, \quad x_{m-1}=x_{m-1}^{0},
$$

moreover, then each $b_{k}^{i}=0$ when $i>1$, while the quantities $b_{k}^{1}$ will keep finite and welldefined values for $\alpha_{1}=\alpha_{1}^{0}$.

With that choice of substitutions (14), the complete solutions of the $n-m+1$ ordinary differential equations (21), when expressed in terms of $\alpha_{1}$ and the initial values of $x_{m}, \ldots, x_{n}$ for $\alpha_{1}=\alpha_{1}^{0}$, when one considers the initial values in them be arbitrary constants that are independent of $\alpha_{2}, \alpha_{3}, \ldots, \alpha_{m-1}$, will also represent the solution of the system of total differential equations (15). However, one will get the solutions of the given system (1) from that when one replaces the $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ with the values that follow from the substitutions (23).

The integration of the given system of $n-m$ linear total differential equations:

$$
\begin{equation*}
d x_{k}=\sum_{h=1}^{m-1} a_{k}^{h} d x_{h}, \quad k=m, m+1, \ldots, n \tag{1}
\end{equation*}
$$

can then be reduced to the integration of a single system of $n-m+1$ ordinary differential equations under the assumption that the identity relations:

$$
\frac{\partial a_{k}^{h}}{\partial x_{i}}-\frac{\partial a_{k}^{i}}{\partial x_{h}}+\sum_{\lambda=m}^{n}\left(a_{\lambda}^{i} \frac{\partial a_{k}^{h}}{\partial x_{\lambda}}-a_{\lambda}^{h} \frac{\partial a_{k}^{i}}{\partial x_{\lambda}}\right)=0
$$

exist between the coefficients:
One introduces the quantities $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ as new independent variables in place of $x_{1}, x_{2}, \ldots, x_{m-1}$, under the restrictions that were just given that the substitutions:

$$
\begin{equation*}
x_{h}=x_{h}^{0}+\left(\alpha_{1}-\alpha_{1}^{0}\right) f_{h} \tag{23}
\end{equation*}
$$

are chosen arbitrarily. With that, the given system (1) will go to the following one:

$$
\begin{equation*}
d x_{k}=\sum_{i=1}^{m-1} b_{k}^{i} d \alpha_{i}, \tag{15}
\end{equation*}
$$

from whose coefficients:

$$
\begin{cases}b_{k}^{1}=\sum_{h=1}^{m-1} a_{k}^{h}\left(f_{h}+\left(\alpha_{1}-\alpha_{1}^{0}\right) \frac{\partial f_{h}}{\partial \alpha_{1}}\right), &  \tag{24}\\ b_{k}^{i}=\left(\alpha_{1}-\alpha_{1}^{0}\right) \sum_{h=1}^{m-1} a_{k}^{h} \frac{\partial f_{h}}{\partial \alpha_{1}}, & i>1\end{cases}
$$

one eliminates $x_{1}, x_{2}, \ldots, x_{m-1}$ by the substitutions (23). If one has then completely integrated the first-order ordinary differential equations:

$$
\begin{equation*}
\frac{\partial x_{m}}{\partial \alpha_{1}}=b_{m}^{1}, \quad \frac{\partial x_{m+1}}{\partial \alpha_{1}}=b_{m+1}^{1}, \ldots, \quad \frac{\partial x_{n}}{\partial \alpha_{1}}=b_{n}^{1}, \tag{25}
\end{equation*}
$$

which follow from (15) and which express the integration constants in terms of the initial values $x_{m}^{0}, \ldots, x_{n}^{0}$ for $\alpha_{1}=\alpha_{1}^{0}$ then the equations in the:

$$
\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}
$$

and the arbitrary constants $x_{m}^{0}, \ldots, x_{n}^{0}$ that one obtains in that way will be the complete integral equations for the ordinary differential equations (25), as well as the total differential equations (15), and one then needs only to eliminate $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ from those equations with the help of formulas (23) in order to obtain the complete integral equations of the given system (1).

The simplest way of satisfying the requirements that are imposed upon the substitutions (23) in all cases is for one to set:

$$
x_{1}=\alpha_{1}
$$

and

$$
x_{h}=x_{h}^{0}+\left(\alpha_{1}-\alpha_{1}^{0}\right) \alpha_{h}
$$

for $h=2,3, \ldots, m-1$, in which the constants $\alpha_{1}^{0}, x_{2}^{0}, \ldots, x_{m-1}^{0}$ only need to be chosen in such a way that none of the quantities will become infinite or undetermined when:

$$
x_{1}=\alpha_{1}^{0}, \quad x_{2}=x_{2}^{0}, \ldots, \quad x_{m-1}=x_{m-1}^{0} .
$$

One will then get:

$$
\begin{aligned}
& b_{k}^{1}=a_{k}^{1}+\alpha_{2} a_{k}^{2}+\cdots+\alpha_{m-1} a_{k}^{m-1}, \\
& b_{k}^{i}=\left(\alpha_{1}-\alpha_{1}^{0}\right) a_{k}^{i}, \quad i>1 .
\end{aligned}
$$

In the derivation of the foregoing theorem, no use was made of the equivalence of the unrestricted-integrable systems of linear total differential equations and the Jacobi systems of linear partial differential equations for the purpose of inferring the integration the latter from merely examining the former. However, if one would like to draw upon the well-known properties of the Jacobi system then one could also convince oneself of the reducibility of the unrestricted-integrable system (1) to the system of ordinary differential equations (25) in a completely-different way without calculation. In order to not block the path of investigation, I shall postpone to the conclusion of this article (§ 7) the second derivation of the theorem above, which connects with the line of reasoning by which P. du Bois-Reymond proved that reducibility for a special case of a single linear total differential equation even more than the foregoing derivation does.

## § 4. - Integrating Jacobi’s system $A_{k}(f)=0$.

From the previous §, the complete integral equations for the ordinary differential equations (25), when expressed in terms of $\alpha_{1}$ and the constant initial values of $x_{m}, \ldots, x_{n}$ for $\alpha_{1}=\alpha_{1}^{0}$, are simultaneously the complete integral equations for the system of total differential equations (15) that emerges from the given one (1) by the substitutions (23), as well.

However, the complete integral equations for a system of first-order differential equations possess the property that they must be soluble for its dependent variables, as well as its initial values. One can then employ the complete integral equations for the system (25) to determine $x_{m}, \ldots, x_{n}$ from them, in one case, and $x_{m}^{0}, \ldots, x_{n}^{0}$, in the other.
Let:

$$
\begin{equation*}
x_{k}=\psi_{k}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}^{0}, \ldots, x_{n}^{0}\right) \tag{26}
\end{equation*}
$$

and

$$
\begin{equation*}
x_{k}^{0}=\chi_{k}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}\right) \tag{27}
\end{equation*}
$$

be the values of the $x_{k}$ and the $x_{k}^{0}$, resp., that are obtained in that way. Equations (27) must then be fulfilled identically by the substitutions (26), and as a result, the expressions:

$$
\frac{\partial \chi_{k}}{\partial \alpha_{h}}+\sum_{\lambda=m}^{n} \frac{\partial \chi_{k}}{\partial x_{\lambda}} \frac{\partial x_{\lambda}}{\partial \alpha_{h}}
$$

must vanish identically. However, since from the foregoing, those substitutions likewise satisfy the system (15) or the equations:

$$
\frac{\partial x_{\lambda}}{\partial \alpha_{h}}=b_{\lambda}^{h}
$$

the same thing must also be true for the expressions:

$$
B_{h}\left(\chi_{k}\right)=\frac{\partial \chi_{k}}{\partial \alpha_{h}}+\sum_{\lambda=m}^{n} b_{\lambda}^{h} \frac{\partial \chi_{k}}{\partial x_{\lambda}}
$$

which also emerges directly from the fact that the expressions $B_{h}\left(\chi_{k}\right)$ must be independent of $\alpha_{1}$ by the substitutions (26), since $B_{1}\left(\chi_{k}\right)=0$ by our assumption, and therefore due to the fact that:

$$
B_{1}\left(B_{h}(f)\right)=B_{h}\left(B_{1}(f)\right),
$$

we will also have $B_{1}\left(B_{h}\left(\chi_{k}\right)\right)=0$, but those expressions will vanish when one sets $\alpha_{1}=$ $\alpha_{1}^{0}$, since $\chi_{k}$ must reduce to $x_{k}$ in that way, and every $b_{k}^{h}=0$ when $h>1$, moreover.

However, the zero result of the substitution of the values (26) in the expressions $B_{h}$ $\left(\chi_{k}\right)$ cannot change when one back-substitutes the values (27) for the $x_{k}^{0}$ in them, which will reverse the substitution itself. Hence, one must already have:

$$
B_{h}\left(\chi_{k}\right)=0
$$

before the substitution, or:

$$
f=\chi_{m}, \chi_{m+1}, \ldots, \chi_{n}
$$

must be solutions of the Jacobi system of $m-1$ linear partial differential equations:

$$
B_{h}(f)=\frac{\partial f}{\partial \alpha_{h}}+\sum_{\lambda=m}^{n} b_{\lambda}^{h} \frac{\partial f}{\partial x_{\lambda}}=0
$$

However, as formula (17) shows, that Jacobi system emerges from the other one:

$$
A_{h}(f)=\frac{\partial f}{\partial x_{h}}+\sum_{\lambda=m}^{n} a_{\lambda}^{h} \frac{\partial f}{\partial x_{\lambda}}=0
$$

in such a way that one must introduce $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ in place of $x_{1}, x_{2}, \ldots, x_{m-1}$ by the substitutions (23), and one must conversely convert them into the latter when one expresses the $\alpha$ in terms of the $x$. The solutions $f=\chi_{m}, \chi_{m+1}, \ldots, \chi_{n}$ of the first system then likewise give us the solutions of the second one, as well, which is the Jacobi system that is equivalent to the given system (1), as long as we set the $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ in them equal to the values that follow from the substitutions (23).

That implies the following method for the complete integration of the given Jacobi system of $m-1$ linear partial differential equations:

$$
\begin{equation*}
A_{h}(f)=\frac{\partial f}{\partial x_{h}}+\sum_{\lambda=m}^{n} a_{\lambda}^{h} \frac{\partial f}{\partial x_{\lambda}}=0, \quad h=1,2, \ldots, m-1 . \tag{28}
\end{equation*}
$$

One can use the $m-1$ substitutions:

$$
\begin{equation*}
x_{h}=x_{h}^{0}+\left(\alpha_{1}-\alpha_{1}^{0}\right) f_{h}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}\right) \tag{23}
\end{equation*}
$$

that are chosen arbitrarily from among the restrictions that were given in the previous § in order to express the quantities:

$$
b_{k}^{1}=\sum_{h=1}^{m-1} a_{k}^{h}\left(f_{h}+\left(\alpha_{1}-\alpha_{1}^{0}\right) \frac{\partial f_{h}}{\partial \alpha_{1}}\right)
$$

and

$$
b_{k}^{i}=\left(\alpha_{1}-\alpha_{1}^{0}\right) \sum_{h=1}^{m-1} a_{k}^{h} \frac{\partial f_{h}}{\partial \alpha_{i}}, \quad i>1
$$

in terms of $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}$ and construct the $n-m+1$ first-order ordinary differential equations:

$$
\frac{\partial x_{m}}{\partial \alpha_{1}}=b_{m}^{1}, \quad \frac{\partial x_{m+1}}{\partial \alpha_{1}}=b_{m+1}^{1}, \ldots, \quad \frac{\partial x_{n}}{\partial \alpha_{1}}=b_{n}^{1}
$$

with the former. If one has integrated those equations completely and expressed the integration constant in terms of the initial values of the dependent variables for $\alpha_{1}=\alpha_{1}^{0}$ then the solution of the integral equations that one obtains for those initial values will yield $n-m+1$ functions:

$$
x_{k}^{0}=\chi_{k}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}\right)
$$

that are the $n-m+1$ solutions of the Jacobi system:

$$
\begin{equation*}
B_{h}(f)=\frac{\partial f}{\partial \alpha_{h}}+\sum_{\lambda=m}^{n} b_{\lambda}^{h} \frac{\partial f}{\partial x_{\lambda}}=0 \tag{29}
\end{equation*}
$$

and which will go to the $n-m+1$ solutions of the given Jacobi system when one eliminates the $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ with the help of equations (23).

## § 5. - In order to find a solution to the given Jacobi system (28), one is only required to know an arbitrary integral of the ordinary differential equations (25).

From the last theorem, the search for all solutions to a Jacobi system of the form (28) will be reduced to the complete integration of a single system of $n-m+1$ first-order
ordinary differential equations. For the most important applications of the Jacobi systems, in the integration of first-order partial differential equations, and in the Pfaff problem, however, one does not address the general solution to the Jacobi system that appears at all, but one always comes down to ascertaining one solution of it. Therefore, it would be of greatest importance to examine whether or not one can find a solution to the Jacobi systems (28) or (29) without having to integrate the system (25) completely.

Assuming that, one must find any integral:

$$
F\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}\right)=\text { const. }
$$

of the differential equations (25). The complete solutions of those differential equations, when expressed in terms of $\alpha_{1}$ and the initial values of $x_{m}, \ldots, x_{n}$ for $\alpha_{1}=\alpha_{1}^{0}$, will then satisfy the equation:

$$
\begin{equation*}
U=F\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}\right)-F\left(\alpha_{1}^{0}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}^{0}, \ldots, x_{n}^{0}\right)=0 \tag{30}
\end{equation*}
$$

However, from § 3 , when one regards the $x_{m}^{0}, \ldots, x_{n}^{0}$ in those solutions as independent of $\alpha_{2}, \ldots, \alpha_{m-1}$, they will also satisfy the total differential equations (15) or the equations:

$$
\begin{equation*}
\frac{\partial x_{k}}{\partial \alpha_{h}}=b_{k}^{h} \tag{31}
\end{equation*}
$$

As a result, they must also satisfy the equations:

$$
\begin{equation*}
B_{h}(U)=\frac{\partial U}{\partial \alpha_{h}}+\sum_{k=m}^{n} b_{k}^{h} \frac{\partial U}{\partial x_{k}}=0 \tag{32}
\end{equation*}
$$

identically, which one obtains by differentiating the equation $U=0$ with respect to $\alpha_{h}$ while considering the relations (31) ( ${ }^{*}$ ). In that way, the form of the equation is entirely equivalent to $U=0$. Precisely the same thing will also be true for every equation $V=0$, which emerges from equation (30) by any sort of algebraic operations.

The first of the $m-1$ equations (32) is always an identity, or in case one has not formed that equation directly from equation (30), but from another arbitrary equation that is equivalent to it, since it is a mere algebraic consequence of the equation $U=0$. In some situations, part of the remaining ones can also be an identity or a mere algebraic
(*) One can also, in turn, see that directly. Namely, by assumption, one has $B_{1}(F)=0$, so as a result, one will also have $B_{1}(U)=0$, and since:

$$
B_{1}\left(B_{h}(F)\right)=B_{h}\left(B_{1}(F)\right),
$$

one will also have $B_{1}\left(B_{h}(U)\right)=0$. The value that $B_{h}(U)$ takes for the complete solutions of the differential equations (25) will then be independent of $\alpha_{1}$. However, that value will vanish when one sets $\alpha_{1}=\alpha_{1}^{0}$, since in that way, one will have $x_{m}=x_{m}^{0}, \ldots, x_{n}=x_{n}^{0}$, so from (30), one will have $\partial U / \partial \alpha_{h}=0$, and likewise every $b_{k}^{\lambda}$ will vanish.
consequence of the equation $U=0$. However, those of equations (32) that do not possess that property are new integral equations of the system (25). One can process each new integral equation of that sort in exactly the same way that one does with the equation $U=$ 0 , and thus recognize the possibility of deriving a whole series of new integral equations from a single integral of the ordinary differential equations (25) by merely differentiating it, which are integral equations that will all belong to that system of complete integral equations for the differential equations by which one determined the dependent variables in terms of $\alpha_{1}$ and the initial values that are taken when $\alpha_{1}=\alpha_{1}{ }^{0}$.

That is connected with the remark (which also could have been used before in the previous § in order to show that the expressions $B_{h}\left(\chi_{k}\right)$ that were obtained there must be identically zero) that equations that belong to such a system of complete integral equations can never yield an equation that is completely free of the initial values of the dependent variables, or that when one has obtained such an equation, it must necessarily be an identity, so one will be led in the following way in order to arrive at a solution to the Jacobi system (19) from the given integral $F=$ const. or $U=0$ of equations (25).

We bring that equation into the form:

$$
\begin{equation*}
x_{m}^{0}=U_{m}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}, x_{m+1}^{0}, \ldots, x_{n}^{0}\right) \tag{33}
\end{equation*}
$$

by solving it for any of the initial values of the dependent variables that enter into the integral $U=0-$ e.g., $x_{m}^{0}$, and then define the $m-1$ equations:

$$
\begin{equation*}
B_{h}\left(U_{m}\right)=\frac{\partial U_{m}}{\partial \alpha_{h}}+\sum_{k=m}^{n} b_{k}^{h} \frac{\partial U_{m}}{\partial x_{k}}=0, \tag{34}
\end{equation*}
$$

the first of which is an identity. None of those equations can be merely an algebraic consequence of equation (33), since $x_{m}^{0}$ does not enter into them at all. If they are all identities, like the first one, then the value $U_{m}$ of $x_{m}^{0}$ that is obtained from $U=0$ will immediately be a common solution to the $m-1$ linear partial differential equations (29). However, if that is not the case then one must always be able to determine some part of the remaining initial values $x_{m+1}^{0}, \ldots, x_{n}^{0}$ from equations (34), since it is impossible to eliminate those initial values completely, from the foregoing. If we assume that $x_{m+1}^{0}, \ldots$, $x_{m+h-1}^{0}$ can be determined from equations (34) then we can now operate with each of the values that are thus obtained:

$$
\begin{gathered}
x_{m+1}^{0}=U_{m+1}^{1}\left(\alpha_{1}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}, x_{m+h}^{0}, \ldots, x_{n}^{0}\right), \\
\left.\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots, x_{m-1}, x_{m}, \ldots, x_{n}, x_{m+h}^{0}, \ldots, x_{n}^{0}\right)
\end{gathered}
$$

in just the same way as we did before with equation (33), and in that way, we will arrive at equations that cannot be merely an algebraic consequence of the foregoing ones, since they do not include the quantities:

$$
x_{m}^{0}, \ldots, x_{m+h-1}^{0}
$$

at all, which must either be identities or part of the remaining ones must determine the initial values. In that way, in the event that one has not already obtained a common solution to the $m-1$ equations (29), one must ultimately succeed in expressing all of the initial values of the dependent variables that are contained in the given integral $U=0$ in terms of $\alpha_{1}, \ldots, \alpha_{m-1}, x_{m}, \ldots, x_{n}$ alone. However, if one now defines the $m-1$ equations $B_{h}(f)=0$ with any of those expressions then they will be free of all initial values and must then be identities. Each of those expressions is then (which would also follow directly from the previous §) a solution of the Jacobi system (29), and as a result also a solution of the given Jacobi system (28), once one has back-substituted their values for:

$$
\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}
$$

from the substitutions (23).
Therefore, in order to find a solution to this Jacobi system of $m-1$ linear partial differential equations with $n$ independent variables, it is only necessary to know an integral of the $n-m+1$ ordinary differential equations (25). By contrast, in the most preferable of the previous methods (*), the search for such a solution required that one had to know an integral of each of $m-1$ systems, the first of which consisted of $n-m+$ 1 first-order ordinary differential equations, while each of the remaining ones consisted of at most that many.

## § 6. - The integration of first-order partial differential equations and the Pfaff problem.

As is known, Jacobi had reduced the integration of first-order partial differential equations to the problem of finding a solution to each of a series of Jacobi systems of linear partial differential equations of the form (28) in succession. If $n$ independent variables enter into the given partial differential equation, which one can assume is free from the unknown function itself, then that Jacobi system will consist of:

$$
1, \quad 2, \ldots, m-1, \ldots, n-1
$$

linear partial differential equations with:

$$
2 n-1,2 n-2, \ldots, 2 n-m+1, \ldots, n+1
$$

independent variables, resp.
From the method that was described in the previous §, the complete solution of the given equation will then require only that one must ascertain an integral of each system of:

[^1]$$
2(n-1), 2(n-2), \ldots, 2(n-m+1), \ldots, 2
$$
first-order ordinary differential equations, whereas previously ( ${ }^{*}$ ), one was required to know an integral for a system of $2(n-1)$ ordinary differential equations and one for two systems of $2(n-2), \ldots, 2(n+m+1), \ldots, 2$ ordinary differential equations, and in the worst-possible cases, that number of integrations might still be insufficient.

When one chooses the simplest form for the substitutions (23), the integration will take the following form:

In general, the $(m-1)^{\text {th }} \mathbf{J a c o b i}$ system will have the form $\left(^{* *}\right)$ :

$$
\begin{equation*}
A_{h}(f)=\frac{\partial f}{\partial q_{h}}+\sum_{\lambda=m}^{n}\left(\frac{\partial p_{h}}{\partial q_{\lambda}} \frac{\partial f}{\partial p_{\lambda}}-\frac{\partial p_{h}}{\partial p_{\lambda}} \frac{\partial f}{\partial q_{\lambda}}\right)=0, \quad h=1,2, \ldots, m-1, \tag{35}
\end{equation*}
$$

in which $p_{1}, p_{2}, \ldots, p_{m-1}$ are functions of $q_{1}, q_{2}, \ldots, q_{n}, p_{m}, \ldots, p_{n}$ that are determined from the foregoing Jacobi system and for which the expressions:

$$
A_{h}\left(A_{i}(f)\right)-A_{i}\left(A_{h}(f)\right)
$$

vanish identically.
If one now sets:

$$
\begin{equation*}
q_{1}=\alpha_{1}, \quad q_{l}=q_{l}^{0}+\left(\alpha_{1}-\alpha_{1}^{0}\right) \alpha_{2}, \quad \ldots, \quad q_{m-1}=q_{m-1}^{0}+\left(\alpha_{1}-\alpha_{1}^{0}\right) \alpha_{m-1}, \tag{36}
\end{equation*}
$$

in which $\alpha_{1}^{0}, q_{2}^{0}, \ldots, q_{m-1}^{0}$ are arbitrarily-chosen constants, under the assumption that the functions $p_{1}, p_{2}, \ldots, p_{m-1}$ preserve well-defined, finite values for:

$$
q_{1}=\alpha_{1}^{0}, \quad q_{2}=q_{2}^{0}, \quad \ldots, \quad q_{m-1}=q_{m-1}^{0},
$$

and then eliminates $q_{1}, q_{2}, \ldots, q_{m-1}$ from the expressions:

$$
\left\{\begin{array}{l}
w_{1}=p_{1}+\alpha_{2} p_{2}+\cdots+\alpha_{m-1} p_{m-1},  \tag{37}\\
w_{i}=\left(\alpha_{1}-\alpha_{1}^{0}\right) p_{i}
\end{array} \quad i>1\right.
$$

then that will convert the given Jacobi system (35) into the following one:

$$
\begin{equation*}
B_{h}(f)=\frac{\partial f}{\partial \alpha_{h}}+\sum_{\lambda=m}^{n}\left(\frac{\partial w_{h}}{\partial q_{\lambda}} \frac{\partial f}{\partial p_{\lambda}}-\frac{\partial w_{h}}{\partial p_{\lambda}} \frac{\partial f}{\partial q_{\lambda}}\right)=0 . \tag{38}
\end{equation*}
$$

From what was discussed in the previous §, one can find a solution of this as long as one knows an integral of the system of $2(n-m+1)$ ordinary differential equations:

[^2]\[

$$
\begin{equation*}
\frac{\partial q_{\lambda}}{\partial \alpha_{1}}=-\frac{\partial w_{1}}{\partial p_{\lambda}}, \quad \frac{\partial p_{\lambda}}{\partial \alpha_{1}}=\frac{\partial w_{1}}{\partial q_{\lambda}}, \quad \lambda=m, m+1, \ldots, n \tag{39}
\end{equation*}
$$

\]

and one only needs to set:

$$
\alpha_{1}=q_{1}, \quad \alpha_{2}=\frac{q_{2}-q_{2}^{0}}{q_{1}-\alpha_{1}^{0}}, \ldots, \quad \alpha_{m-1}=\frac{q_{m-1}-q_{m-1}^{0}}{q_{1}-\alpha_{1}^{0}}
$$

in that solution in order to obtain a solution to the given system (35).
In a manner that is completely analogous to how one integrated first-order partial differential equations, the number of integrations that are required in the Pfaff problem i.e., the problem of integrating the given linear differential equation:

$$
\chi_{1} d x_{1}+\chi_{2} d x_{2}+\ldots+\chi_{2 n} d x_{2 n}=0
$$

by $n$ equations - will be diminished by almost one-half by using the procedure that was given. Namely, it is not difficult to see from the method that Clebsch has prescribed for the solution of that problem (*), that it can be reduced to the search for solutions to $n$ Jacobi systems of the form (28) that consist of $1,2, \ldots, n$, resp., linear partial differential equations that each have $2 n$ independent variables. From the foregoing, the search for a solution to the $i^{\text {th }}$ one of these systems depends upon finding an integral to $2 n-i$ firstorder ordinary differential equations. However, that $i^{\text {th }}$ system, which can first be exhibited once one has found a solution to each of the foregoing ones, will itself possess $i$ -1 known solutions, as a result of the way that it came about. None of those solutions is the one that one actually needs (since it must be independent of them), but each of them will provide us with an integral of those $2 n-i$ ordinary differential equations when we set it (as expressed in the new variables $\alpha$ ) equal to a constant. One then knows $i-1$ integrals of those equations from the outset and can reduce the $2 n-i$ differential equations to:

$$
2 n-i-(i-1)=2 n-2 i+1
$$

by means of them. The search for a useful solution of the $i$ th Jacobi system then requires only that one know an integral of:

$$
2 n-2 i+1
$$

first-order ordinary differential equations. As a result, for the complete solution of the Pfaff problem, it is sufficient to know an integral for each system of:

$$
2 n-1,2 n-3,2 n-5, \ldots, 1
$$

first-order ordinary differential equations.

[^3]
## § 7. - A different proof of the theorem in § 3.

Under the assumption that the identities:

$$
\begin{equation*}
A_{i}\left(a_{k}^{h}\right)-A_{h}\left(a_{k}^{i}\right)=0 \tag{40}
\end{equation*}
$$

are valid, as Clebsch showed ( ${ }^{*}$ ), the $m-1$ linear partial differential equations:

$$
\begin{equation*}
A_{h}(f)=\frac{\partial f}{\partial x_{h}}+\sum_{\lambda=m}^{n} a_{\lambda}^{h} \frac{\partial f}{\partial x_{\lambda}}=0 \tag{41}
\end{equation*}
$$

possess $n-m+1$ mutually-independent solutions, which might be denoted by:

$$
f_{m}, \quad f_{m+1}, \quad \ldots, \quad f_{n} .
$$

Due to the fact that when one sets:

$$
f=\varphi\left(x_{1}, x_{2}, \ldots, x_{m-1}, f_{m}, f_{m+1}, \ldots, f_{n}\right),
$$

one will have:

$$
A_{h}(f)=\frac{\partial \varphi}{\partial x_{h}}+\sum_{k=m}^{n} A_{h}\left(f_{k}\right) \frac{\partial \varphi}{\partial x_{k}}=\frac{\partial \varphi}{\partial x_{h}}
$$

one will see that these solutions must be mutually-independent with respect to $x_{m}, x_{m+1}$, $\ldots, x_{n}$.

If one then sets:

$$
\begin{equation*}
f_{m}=c_{m}, \quad f_{m+1}=c_{m+1}, \quad \ldots, \quad f_{n}=c_{n} \tag{42}
\end{equation*}
$$

then one must be able to determine $x_{m}, x_{m+1}, \ldots, x_{n}$ as functions of the variables $x_{1}, x_{2}$, $\ldots, x_{m-1}$ and the quantities $c_{m}, c_{m+1}, \ldots, c_{n}$ from those equations.

If one considers the latter to be arbitrary constants then the values of $x_{m}, \ldots, x_{n}$ that follow from (42) will satisfy the $n-m+1$ equations:

$$
\sum_{\lambda=m}^{n} \frac{\partial f_{k}}{\partial x_{\lambda}}\left(d x_{\lambda}-\sum_{h=1}^{m-1} a_{\lambda}^{h} d x_{h}\right)=0
$$

that are obtained by completely differentiating equations (42) with the use of the identities $A_{h}\left(f_{k}\right)=0$. However, since the determinant of those equations:

$$
\sum \pm \frac{\partial f_{m}}{\partial x_{m}} \frac{\partial f_{m+1}}{\partial x_{m+1}} \cdots \frac{\partial f_{n}}{\partial x_{n}}
$$

[^4]is itself non-zero, so that value cannot vanish identically under substitution either, it will then follow that it must satisfy the $n-m+1$ linear total differential equations:
\[

$$
\begin{equation*}
d x_{\lambda}=\sum_{h=1}^{m-1} a_{\lambda}^{h} d x_{h} . \tag{43}
\end{equation*}
$$

\]

If the identities (40) exist then there will always be $n-m+1$ functions of $x_{1}, x_{2}, \ldots, x_{m-1}$ and $n-m+1$ arbitrary constants $c_{m}, c_{m+1}, \ldots, c_{n}$ that are mutually-independent relative to the latter that will satisfy equations (43) identically when they are set equal to $x_{m}, x_{m+1}$, $\ldots, x_{n}$.

If we denote those solutions to the system (43) by:

$$
\begin{equation*}
f_{\lambda}=\varphi_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{m-1}, c_{m}, \ldots, c_{n}\right) \tag{44}
\end{equation*}
$$

and understand $x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}$ to mean undetermined constants then the $n-m+1$ equations:

$$
x_{\lambda}^{0}=\varphi_{\lambda}\left(x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}, c_{m}, \ldots, c_{n}\right)
$$

must always be soluble for $c_{m}, \ldots, c_{n}$. By substituting those values, the solutions (44) will assume the form:

$$
\begin{equation*}
x_{\lambda}=\psi_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{m-1}, x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}\right), \tag{45}
\end{equation*}
$$

where the functions $\psi_{\lambda}$ must reduce to $x_{\lambda}^{0}$ for:

$$
x_{1}=x_{1}^{0}, \quad x_{2}=x_{2}^{0}, \quad \ldots, \quad x_{m-1}=x_{m-1}^{0},
$$

as a result of the way that they came about.
If one now introduces the new variables $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$ for the $x_{1}, x_{2}, \ldots, x_{m-1}$, resp., by way of the $m-1$ equations:

$$
\begin{equation*}
x_{h}=x_{h}^{0}+\left(\alpha_{1}-\alpha_{1}^{0}\right) f_{h}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}\right), \tag{46}
\end{equation*}
$$

which might make:

$$
\begin{equation*}
\psi_{\lambda}\left(x_{1}, x_{2}, \ldots, x_{m-1}, x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}\right)=\Psi_{\lambda}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, \alpha_{1}^{0}, x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}\right) \tag{47}
\end{equation*}
$$

then one will obtain the solutions:

$$
\begin{equation*}
x_{\lambda}=\Psi_{\lambda}\left(\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}, \alpha_{1}^{0}, x_{1}^{0}, x_{2}^{0}, \ldots, x_{n}^{0}\right) \tag{48}
\end{equation*}
$$

as in (45) to the system of linear total differential equations in $x_{m}, \ldots, x_{n}$ and $\alpha_{1}, \alpha_{2}, \ldots$, $\alpha_{m-1}$ :

$$
\begin{equation*}
d x_{\lambda}=\sum_{h=1}^{m-1} b_{\lambda}^{h} d \alpha_{h} \tag{49}
\end{equation*}
$$

to which the system (43) goes under the substitutions (46).
Therefore, equations (48) then also satisfy the $n-m+1$ ordinary differential equations:

$$
\begin{equation*}
\frac{\partial x_{\lambda}}{\partial \alpha_{1}}=b_{\lambda}^{1}, \tag{50}
\end{equation*}
$$

in particular. However, if one has chosen the constants $\alpha_{1}{ }^{0}$ in such a way that none of the $m-1$ functions $f_{h}$ becomes infinite or undetermined for $\alpha_{1}=\alpha_{1}^{0}$ then each $x_{h}=x_{h}^{0}$ for $\alpha_{1}=\alpha_{1}^{0}$ from (46), and therefore each $\Psi_{\lambda}=x_{\lambda}^{0}$ from (47).

As a result of equations (48), they will be those solutions of the ordinary differential equations (50) that reduce to the values of the dependent variables $x_{\lambda}$ that belong to the initial value $\alpha_{1}^{0}$ of $\alpha_{1}$ for $\alpha_{1}=\alpha_{1}{ }^{0}$.

Conversely, it must then be always possible to determine the integration constants in a system of complete solutions to the $n-m+1$ ordinary differential equations (50) in such a way that those solutions will assume the values $x_{m}^{0}, x_{m-1}^{0}, \ldots, x_{n}^{0}$, which remain arbitrary for $\alpha_{1}=\alpha_{1}^{0}$, and the solutions that are thus obtained must likewise fulfill the total differential equations (49) when one regards the initial values in them as independent of $\alpha_{1}, \alpha_{2}, \ldots, \alpha_{m-1}$, so once one has back-substituted the values for $\alpha_{1}, \alpha_{2}$, $\ldots, \alpha_{m-1}$ that they obtain from the substitutions (46) in those equations, the total differential equations (43) must also be satisfied.

Leipzig, February 1872.


[^0]:    ( ${ }^{1}$ ) Crelle's J. 65, pp. 257.
    (*) Crelle's J. 58, pp. 303.
    (*) Crelle's J. 70, pp. 312.

[^1]:    (") Cf., Clebsch, Crelle’s J., 65, pp. 261.

[^2]:    (*) Cf., Clebsch, Crelle's J. 65, pp. 265.
    ${ }^{(* *)}$ Cf., Jacobi, Vorlesungen über Dynamik, pp. 292. Crelle's J., 60, pp. 23.

[^3]:    (") Cf., namely, Crelle's J. 61, pp. 153 and 65, pp. 266.

[^4]:    ( ${ }^{*}$ ) Crelle's J., 65, pp. 260.

