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On the theory of infinite continuous groups

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INTRODUCTION

As is well-known, the first work that was done on the theory of infinite groups (*) discussed only groups of infinitesimal transformations. These groups are defined by certain systems of partial differential equations that were called *equations of definition of the infinitesimal transformations.*

Back in 1885 (**), Engel presented a method for defining this system of equations for any group in n variables, and at that point, a noteworthy correspondence appeared between these groups and certain *finite* groups of particular compositions. These compositions were also determined by ENGEL in the case for which the equations of definitions were of first and second order. More recent considerations (***) of ENGEL, in which he demonstrated the generality of his method, made it easy for me to also determine this composition in the general case. This determination, along with an exposition of the method of ENGEL, will be found in § 2 of this paper.

The other paragraphs are intended to show another aspect of the correspondence between the groups thus found (which I call the *groups* γ_{sn}) and groups in n variables. The considerations all refer to the equations of finite transformations.

Note that in the memoir (****) in which Lie posed the fundamentals of the theory of infinite groups, the notion of group of infinitesimal transformations essentially coincides with that of infinite group. The finite transformations are defined by systems of partial differential equations that were called *equations of definition of the finite transformations*, and which were deduced from those of the infinitesimal transformations *by integrating a complete system.*

LIE found that the equations of the finite transformations had the following form:

$$I_k\left(y_1,\cdots,y_n,\frac{\partial y_1}{\partial x_1},\cdots\right) = \alpha_k(x_1, \ldots, x_n), \qquad k = 1, \ldots, m. \tag{1}$$

(*) LIE, *Ueber unendliche contiuirliche Gruppen* (Christiania, Videnskab. Forh., 1882). ENGEL, *Ueber die definitiongleichungen der continuirlichen Tr. gruppen* (Math. Ann., Bd. 27, 1886).

(**) ENGEL, op. cit.

(***) *Kleinere Beitrage IX* (Berichte der kgl. Sachs. Ges. der Wissenschaft, 1894).

(****) *Grundlagen für die Theorie der unendl. cont. Tr. gruppen* (Berichte der Sachs, G. d. W., 1891).

I shall not go into the more precise nature of the functions I_k of $y_1, \ldots, y_n$, $\partial y_1 / \partial x_1, \ldots$ I will prove that any system (1) may be obtained from the finite equations:

$$z'_k = f_k(z_1, \ldots, z_m, a_1, \ldots, a_r), \qquad k = 1, \ldots, m,$$

of a group γ_{sn} by replacing the $z_1, \ldots, z_m$ with appropriately chosen functions $\varpi_1(y_1, \ldots, y_n), \ldots, \varpi_m(y_1, \ldots, y_n)$ of the $\varpi_1(y_1, \ldots, y_n)$; in place of the $z'_1, \ldots, z'_m$, one puts the same functions of the $x_1, \ldots, x_n$, respectively, and finally, one replaces the $a_1, \ldots, a_r$ with certain functions of the $\partial y_1 / \partial x_1, \ldots$, that have the property of being independent invariants of certain special groups γ_{sn}.

This theorem can lead to many consequences; some of them are pointed out in the last paragraphs. We shall touch upon only the most important of them.

Given the equations of definition of the infinitesimal transformations of an infinite group Γ, one may immediately write down the infinitesimal transformations of the corresponding group γ_{sn}. The problem of finding the equations of the finite transformations then reduces to the problem of finding the finite equations of the group γ_{sn} and determining the functions $a\left(\frac{\partial y_1}{\partial x_1}, \cdots\right)$. All of these problems then reduce to the determination of the finite equations of a group γ_{sn}, and more precisely, a simply transitive group.

It is noteworthy that, conversely, given the finite equations:

$$z'_\mu = f_\mu(z_1, \ldots, z_\mu, a_1, \ldots, a_\rho) \tag{2}$$

of a transitive group with the composition γ_{sn}, one may *always* determine the $a_1, \ldots, a_\rho$ as functions of the $\partial y_1 / \partial x_1, \ldots$ in such a way that (2) represents a group for *any* system of functions $z_i = \varpi_i(y)$, $z'_i = \varpi_i(x)$.

The correspondence between groups – both finite and infinite – in n variables and finite groups with the composition γ_{sn} may be applied to great advantage to the study of certain problems of integration. It was this fact that led me to the research that is presented in the present memoir. I will reserve the task of making these applications to a second memoir and for that reason I will limit myself in this introduction to pointing out a problem that was studied by LIE (*):

Suppose that one knows the equations of definition of an infinite group Γ:

$$I_k\left(y_1, \cdots, y_n, \frac{\partial y_1}{\partial x_1}, \cdots\right) = B_k(x_1, \ldots, x_n),$$

as well as the general infinitesimal transformation:

(*) *Berichte der kgl. Sachs. Ges. der Wissenschaft 1895* (Verwerthung des Gruppenbegriffs).

$$Xf = \xi_1 \frac{\partial f}{\partial x_1} + \ldots + \xi_n \frac{\partial f}{\partial x_n},$$

of the group. Convert the integration of Xf = 0 *to simpler auxiliary equations.*

It would be interesting to study the groups γ_{sn} *rather than the equations* $I_k = B_k$ *of this integration problem.*

§ 1. Generalities on infinite groups.

In this paragraph, we summarize the notations and fundamental theorems of the theory of infinite groups.

A set of transformations:

$$y_i = F_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n \tag{1}$$

is called an *infinite group* when the $F_1, \ldots, F_n$ are the most general solutions of a system of partial differential equations:

$$W_k\left(x_1, \cdots, x_n, y_1, \cdots, y_n, \frac{\partial y_1}{\partial x_1}, \cdots\right) = 0, \qquad k = 1, 2, \ldots \tag{2}$$

that satisfies the two following conditions:

First: The most general system of solutions of the system (2) does not depend upon a finite number of arbitrary constants.

Second: Along with the systems of solutions:

$$y_i = F_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n,$$
$$y_i = \Phi_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n,$$

the system:

$$y_i = \Phi_i(F_1(x), \ldots, F_n(x)), \qquad i = 1, \ldots, n$$

must also be a system of solutions.

Equations (2) are then called the *equation of definition of the finite transformations* of the group. The system (2) contains the derivatives of the y with respect to x up to order s. One imagines that it has been converted into a form such that all of the equations of order less than or equal to s that may be deduced from the other ones by derivation and elimination are already deduced by just eliminations.

Any infinite group contains an infinitude of independent infinitesimal transformations. If Xf and Yf are two of them then all of the transformations $a\,Xf + b\,Yf$ belong to the group, no matter what the constants a, b, as well as $(XY)f$.

The infinitesimal transformations contained in the group may be defined by a finite number of differential equations of the form:

$$\sum_{i=1}^{n} \alpha_{ki}(x)\xi_i + \sum_i \sum_r \alpha_{ki\nu}(x)\frac{\partial \xi_i}{\partial x_\nu} + \ldots = 0, \qquad k = 1, 2, \ldots, \tag{3}$$

which are linear and homogeneous in the ξ and their derivatives, and which enjoy the following two properties:

First: The most general system of solutions does not depend upon only a finite number of arbitrary constants.

Second: If $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$ are two systems of solutions then $\sum_{r=1}^{n}\left(\xi_\nu \frac{\partial \eta_1}{\partial x_\nu} - \eta_\nu \frac{\partial \xi_1}{\partial x_\nu}\right), \ldots, \sum_{r=1}^{n}\left(\xi_\nu \frac{\partial \eta_n}{\partial x_\nu} - \eta_\nu \frac{\partial \xi_n}{\partial x_\nu}\right)$ is a system of solutions.

Equations (3) are called the *equations of definition of the infinitesimal transformations* of the group. Any system (3) is imagined as being converted into a form such that the equations that are deducible from it by derivations and eliminations and whose order does not exceed the order of equations (3) may be deduced from (3) by only eliminations.

This will be the case for all of the systems of equations of definition that will occur, without it being necessary for us to repeat that fact.

Equations (3) have the same order as equations (2), and are equal in number. One may give them the form:

$$\sum_i \xi_i(x)\left[\frac{\partial W_k}{\partial y_i}\right] + \sum_i\sum_\nu \frac{\partial \xi_i}{\partial x_\nu}\left[\frac{\partial W_k}{\partial y_{i\nu}}\right] + \sum_i\sum_\mu\sum_\nu \frac{\partial^2 \xi_i}{\partial x_\mu \partial x_\nu}\left[\frac{\partial W_k}{\partial y_{i\mu\nu}}\right] + \ldots = 0, \quad k = 1, 2, \ldots \tag{3′}$$

in which, we have set, for the sake of convenience:

$$y_{i\nu} = \frac{\partial y_i}{\partial x_\nu}, \qquad y_{i\mu\nu} = \frac{\partial^2 y_i}{\partial x_\nu \partial x_\mu}, \ldots$$

and in which the parentheses $[f]$ indicate what the function f of $x_1, \ldots, x_n, y_1, \ldots, y_n, y_{11}, \ldots, y_{nn}, y_{111}, \ldots$ becomes, after the substitution:

$$y_i = x_i, \qquad y_{i\nu} = \varepsilon_{i\nu}, \qquad \varepsilon_{i\mu\nu} = 0 \qquad (\varepsilon_{i\nu} = 0 \text{ if } i \neq n;\ \varepsilon_{ii} = 1). \tag{4}$$

In the meantime, we observe that equations (2) can be converted into equations (3) with only derivations. We might make another observation on the form of equations (3).

It is not necessary to recall that the equations $W_k = 0$ can always be put into the more convenient form:

$$I_k(y_1, \ldots, y_n, y_{11}, \ldots, y_{nn}, y_{111}, \ldots) - \alpha_k(x_1, \ldots, x_n) = 0,$$

in which the functions I_k depend upon the $y_1, \ldots, y_n, y_{11}, \ldots$, but not on the $x_1, \ldots, x_n$. In addition, this has the property of reducing all the corresponding functions $\alpha_k(x_1, \ldots, x_n)$ for the substitution (4) identically.

If one then puts the functions $I_k - \alpha_k$ in place of W_k in (3′) then, instead of (3′) one has:

$$\sum_i \xi_i(x)\left[\frac{\partial \alpha_k}{\partial x_i}\right] + \sum_i\sum_\nu \frac{\partial \xi_i}{\partial x_\nu}\left[\frac{\partial I_k}{\partial y_{i\nu}}\right] + \ldots = 0, \quad k = 1, 2, \ldots \tag{3″}$$

One can conclude nothing directly about the functions of $x_1, \dots, x_n$ that the symbols $\left[\frac{\partial I_k}{\partial y_{i\nu}}\right]$, … represent. The method of ENGEL, which I shall exhibit in § 2, shows that the $\left[\frac{\partial I_k}{\partial y_{i\nu}}\right]$, … are expressible in terms of only the $\alpha_1, \alpha_2, \dots$

Observe that for any infinite group there exist systems of equations (3) that define the infinitesimal transformations that are contained in the group. Conversely, one might demand that, given a system of equations (3), with the *first* and *second* properties, there always exists an infinite group with those equations of definition for the infinitesimal transformations. The response to this demand is the following fundamental theorem:

Theorem 1. *Let there be given a system of linear, homogeneous partial differential equations:*

$$\sum_{i=1}^{n} \alpha_{ki}(x)\xi_i + \sum_i \sum_r \alpha_{ki\nu}(x)\frac{\partial \xi_i}{\partial x_\nu} + \dots = 0, \qquad k = 1, 2, \dots, m \tag{3}$$

that has the following properties:

If equations (3) *are of order s then they cannot be put into the form of equations of order < s by means of derivations and eliminations, and independently of* (3).

The most general system of solutions cannot depend upon only a finite number of arbitrary constants.

If the $\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n$ *are two systems of solutions then any:*

$$\sum_{r=1}^{n}\left(\xi_\nu \frac{\partial \eta_i}{\partial x_\nu} - \eta_\nu \frac{\partial \xi_i}{\partial x_\nu}\right), \qquad i = 1, \dots, n$$

is a system of solutions.

The system (3) *then defines the most general infinitesimal transformation:*

$$Xf = \sum_{i=1}^{n} \xi_i(x)\frac{\partial f}{\partial x_i},$$

of some infinite group. The finite transformations of this group are determined by m independent partial differential equations of order s of the form:

$$I_k(y_1, \dots, y_n, y_{11}, \dots) = \alpha_k(x_1, \dots, x_n), \quad k = 1, \dots, m \tag{2}$$

that reduce to the identity under the substitution:

$$y_i = x_i, \qquad y_{i\mu} = \varepsilon_{i\mu}, \qquad y_{i\nu\mu} = 0, \dots$$

The functions $I_1, \dots, I_\mu$ *have the property of remaining invariant under any transformation:*

$$y_i = F_i(y_1, \dots, y_n),$$

that belongs to the infinite group. Given equations (3), *the functions* $I_1, \dots, I_m$ *may be found by integrating a complete system.*

Here is how one forms this complete system:

Imagine that the transformation $Xf = \sum_{i=1}^{n} \xi_i(y) \frac{\partial f}{\partial y_i}$ is performed s times, which gives the first, second, …, s^{th} derivative of the y with respect to x. One will have:

$$X^{(s)}f = \sum_i \xi_i(y) \frac{\partial f}{\partial y_i} + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial y_\nu} B_{i\nu} f + \sum_i \sum_\nu \sum_\mu \frac{\partial^2 \xi_i}{\partial y_\mu \partial y_\nu} B_{i\mu\nu} f + \dots$$

in which the $B_{i\nu} f$, $B_{i\mu\nu} f$, … are transformations of the $y_1, \dots, y_{nn}, y_{111}, \dots$

Imagine that (3) has been written with the variables $y_1, \dots, y_n$ in place of the $x_1, \dots, x_n$, respectively. The equations thus obtained can be derived from m of the quantities:

$$\xi_1(y), \dots, \xi_n(y), \frac{\partial \xi_1}{\partial y_1}, \dots, \frac{\partial \xi_n}{\partial y_n}, \frac{\partial^2 \xi_1}{\partial y_1^2}, \dots$$

expressed as functions of the remaining ones. The expressions thus obtained are put into the symbol $X^{(s)}f$, which becomes, when we exhibit what remains in the $\xi_1, \dots, \xi_n, \partial\xi_1 / \partial y_1, \dots$:

$$X^{(s)}f = \sum_i \xi_i(y) \bar{B}_i f + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial y_\nu} \bar{B}_{i\nu} f + \dots,$$

in which the $\bar{B}_i f$, $\bar{B}_{i\nu} f$, … are transformations of the variables:

$$y_1, \dots, y_n, y_{11}, \dots, y_{nn}, y_{111}, \dots$$

The equations:

$$\bar{B}_i f = 0, \qquad \bar{B}_{i\nu} f = 0, \dots$$

define precisely the desired complete system. This system, as one can see from the way by which it was formed, *depends essentially* on the nature of the functions $\alpha_1(y), \dots, \alpha_m(y)$ that present themselves when (3) is written in the form (3′). We note that this circumstance is the reason why we will point out a method later on that makes the determination of the functions $I_1, \dots, I_m$ depend upon the integration of a complete system that is *independent* of the nature of the functions $\alpha_1(y), \dots, \alpha_m(y)$.

§ 2. Engel's method for the construction of the equations of definition.

In the preceding paragraph, we considered equations of the form:

$$\sum_{i=1}^{n} \alpha_{ki}\xi_i + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_\nu} \alpha_{ki\nu} + \ldots = 0, \qquad k = 1, 2, \ldots \tag{1}$$

Suppose that the properties that are attributed to this system all remain, except that one has, *firstly:* Suppose that the most general system of solutions depends only upon a finite of arbitrary constants.

Thus, as was pointed out (*), the (1) are the equations of definition of a *finite* group.

The problem of determining all of the systems of equations of definition for groups, whether finite or infinite, was solved by ENGEL (**). The demonstration of ENGEL's method was given by both LIE and ENGEL (***). I will outline the demonstration of ENGEL, which leads to a noteworthy form for equations (1) and permits us to generalize its primary results.

Let there given an infinite group:

$$I_k(y_1, \ldots, y_n, y_{11}, \ldots) = \alpha_k(x_1, \ldots, x_n), \qquad k = 1, \ldots, m, \tag{2}$$

in which the I_k contain all of the derivatives of the y with respect to the x up to the s^{th}.

Let $Xf = \sum_i \xi_i(y) \frac{\partial f}{\partial y_i}$ be an infinitesimal transformation of the group. We denote the s-fold repetition of the transformation Xf by $X^{(s)}f$. From theorem 1, one will then have:

$$X^{(s)}I_1 = 0, \qquad X^{(s)}I_m = 0.$$

For the moment, let $Zf = \sum_i \zeta_i(x) \frac{\partial f}{\partial x_i}$ be an arbitrary infinitesimal transformation in the variables $x_1, \ldots, x_n$ and let $Z^{(s)}f$ be the corresponding repeated transformation. ENGEL proved that:

$$Z^{(s)}I_k = \sum_i \sum_{\nu_1 \ldots \nu_n} \zeta_{i,\nu_1 \ldots \nu_n}(x)\, \alpha^k_{i,\nu_1 \ldots \nu_n}(I_1, \ldots, I_m), \quad k = 1, \ldots, m,$$

in which the $\alpha^k_{i,\nu_1,\ldots,\nu_n}$ are functions of only the $I_1, \ldots, I_m$, and in which:

$$\zeta_{i,\nu_1,\ldots,\nu_n} = \frac{\partial^{\nu_1+\cdots+\nu_n} \zeta_i(x)}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}} \qquad (\nu_1 + \ldots + \nu_n < s).$$

If we then set:

$$\sum_{k=1}^{m} \alpha^k_{i,\nu_1,\ldots,\nu_n}(I_1, \ldots, I_m) \frac{\partial f}{\partial I_k} = \bar{A}_{i,\nu_1,\ldots,\nu_n} f \tag{3}$$

then the expression:

(*) LIE, *Theorie der Transf. gruppen*, vol. I, page 47, theorem 28.
(**) *Math. Ann.*, Bd. 27.
(***) *Kleinere Beitrage IX* (Berichte der Sachs. Ges. d. W., 1894).

$$Zf + \sum_{i=1}^{n} \sum_{\nu_1,\dots,\nu_n} \zeta_{i,\nu_1,\dots,\nu_n}(x)\, \bar{A}_{i,\nu_1,\dots,\nu_n} f \tag{4}$$

will obviously represent the general infinitesimal transformation of an infinite group in the variables $x_1, \dots, x_n, I_1, \dots, I_m$.

Now assume that the functions $\varpi_1(x_1, \dots, x_n), \dots, \varpi_m(x_1, \dots, x_n)$ are arbitrary and write down the equations:

$$I_1 = \varpi_1(x_1, \dots, x_n), \dots, I_m = \varpi_m(x_1, \dots, x_n). \tag{5}$$

The totality of all transformations (4) that leave the system (5) invariant is again a group: On the other hand, the idea that a transformation (4) leaves the system of equations (5) invariant is expressed by:

$$\sum_{i=1}^{n} \sum_{\nu_1,\dots,\nu_n} \zeta_{i,\nu_1,\dots,\nu_n}(x)\, \alpha^{k}_{i,\nu_1,\dots,\nu_n}(\varpi_1, \cdots, \varpi_m) = \sum_{\mu} \zeta_\mu \frac{\partial \varpi_k}{\partial x_\mu}, \qquad k = 1, \dots, m. \tag{6}$$

Therefore, if:

$$\zeta_1^{(1)}, \dots, \zeta_n^{(1)}; \qquad \zeta_1^{(s)}, \dots, \zeta_n^{(s)},$$

are two systems of functions ζ that satisfy the relations (6) then so does the system of functions:

$$\sum_{\nu} \left(\zeta_\nu^{(1)} \frac{\partial \zeta_i^{(2)}}{\partial x_\nu} - \zeta_\nu^{(2)} \frac{\partial \zeta_i^{(1)}}{\partial x_\nu} \right), \qquad i = 1, \dots, n.$$

One then concludes that *equations* (6) *define a group in the arbitrary variables* $x_1, \dots, x_n$ *by the functions* $\varpi_1(x_1, \dots, x_n), \dots, \varpi_m(x_1, \dots, x_n)$.

In particular, if the functions $\alpha_1, \dots, \alpha_m$ of x that enter into the right-hand side of (2) take the place of $\varpi_1, \dots, \varpi_m$ then (6) will represent a group from which we started and whose finite transformations are defined by (2).

One then has the theorem (*):

Theorem 2: *Let:*

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} + \sum_{x=1}^{m} U_x(\alpha, \xi) \frac{\partial f}{\partial \alpha_x} \tag{7}$$

be the symbol of an infinitesimal transformation in the variables $x_1, \dots, x_n, \alpha_1, \dots, \alpha_m$ *, and the functions* $U_x(\alpha, \xi)$*, which are linear and homogeneous in the* ξ *and in their derivatives that is chosen in such a manner that the set of transformations thus obtained that leave the* ξ *arbitrary functions of the* $x_1, \dots, x_n$ *is a group.*

Take the $\alpha_1, \dots, \alpha_m$ *to be arbitrary functions of the* x *and write down the equations:*

(*) ENGEL, *Math. Ann.*, Bd. 27, page 31.

$$U_x(\alpha, x) - \sum_{\nu=1}^{n} \xi_\nu \frac{\partial \alpha_x}{\partial x_\nu} = 0, \qquad x = 1, \ldots, m. \tag{8}$$

This system of equations defines a group (finite or infinite) in the variables $x_1, \ldots, x_n$.

Take all possible infinite groups of the form (7) *and each time form the corresponding equations* (8), *and one then has all of the groups (finite or infinite) of variables* $x_1, \ldots, x_n$.

The functions $\alpha_1(x), \ldots, \alpha_m(x)$ of the preceding theorem are not entirely arbitrary. Indeed, they must be chosen in such a manner that equations (8) cannot produce new equations by derivation. This condition translates analytically into a certain number of relations between the α and their derivatives:

$$\chi_x\left(\alpha_1, \ldots, \alpha_m, \frac{\partial \alpha_1}{\partial x_1}, \ldots\right) = 0. \quad x = 1, 2, \ldots$$

When we say, in what follows, that the α in (8) are arbitrary, we always intend this to mean that this arbitrariness is limited by the equations $\chi_\alpha = 0$, if there are any (*).

Theorem 2 makes the determination of the group in the $x_1, \ldots, x_n$ depend upon that of certain groups in the variables $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_m$, and more precisely, on that of certain functions $U_x(\alpha, \xi)$. These functions are ones that contain the ξ and their derivatives in a linear and homogeneous way:

$$U_x(\alpha, \xi) = \sum_{i=1}^{n} \xi_i \bar{A}_i^x(\alpha) + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_\nu} \bar{A}_{i\nu}^x(\alpha) + \ldots$$

Now set:

$$\sum_{x=1}^{m} \bar{A}_i^\alpha(\alpha) \frac{\partial f}{\partial \alpha_x} = \bar{A}_i f, \qquad \sum_{x=1}^{m} \bar{A}_{i\nu}^\alpha(\alpha) \frac{\partial f}{\partial \alpha_x} = \bar{A}_{i\nu} f, \ldots$$

so the symbol (7) may then be written:

$$\sum_{i=1}^{m} \xi_i \frac{\partial f}{\partial \alpha_x} + \sum_{i=1}^{m} \xi_i \bar{A}_i f + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_\nu} \bar{A}_{i\nu} f + \ldots,$$

and on comparing this symbol with the one in (4), one meanwhile has that *with no loss of generality in the result one may suppose that:*

$$\bar{A}_i^x(\alpha) = 0, \qquad i = 1, \ldots, n, \quad x = 1, \ldots, m,$$

so the symbol (7) then becomes:

(*) ENGEL, *Math. Ann.*, Bd. 27, page 38.

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_\nu} \bar{A}_{i\nu} f + \sum_i \sum_\mu \sum_\nu \frac{\partial^2 \xi_i}{\partial x_\mu \partial x_\nu} \bar{A}_{i\mu\nu} f + \ldots, \tag{9}$$

which has a form that is identical to the symbol (4). The $\bar{A}_{i\nu} f$, $\bar{A}_{i\mu\nu} f$, … represent transformations in the variables $\alpha_1, \ldots, \alpha_m$. This poses the question:

What conditions must the transformations $\bar{A}_{i\nu} f$, $\bar{A}_{i\mu\nu} f$, … *be subjected to in order for* (9) *to be the general infinitesimal transformation of a group in the* $x_1, \ldots, x_m$, $\alpha_1, \ldots, \alpha_m$ *when the* $\xi_1, \ldots, \xi_m$ *are taken to be arbitrary functions of the x?*

ENGEL, in the paper in *Math. Ann.* that has been cited several times, has resolved the question in the two simplest cases. However, it is also easy to resolve it in the most general case. We first examine the results of ENGEL.

(*A*) In order for the symbol:

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_\nu} \bar{A}_{i\nu} f$$

to represent the general infinitesimal transformation of a group it is necessary and sufficient that transformations $A_{i\nu} f$ in the $\alpha_1, \ldots, \alpha_m$ form a group under the composition:

$$(\bar{A}_{ix}, \bar{A}_{\mu\nu}) = \varepsilon_{i\nu} \bar{A}_{\mu x} - \varepsilon_{\mu x} \bar{A}_{i\nu}.$$

(*B*) In order for the symbol:

$$\sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i} + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_\nu} \bar{A}_{i\nu} f + \sum_i \sum_\nu \sum_\mu \frac{\partial^2 \xi_i}{\partial x_\nu \partial x_\mu} \bar{A}_{i\nu\mu} f$$

to represent the general infinitesimal transformation of a group it is necessary and sufficient that the $\bar{A}_{i\nu} f$, $\bar{A}_{i\nu\mu} f$ form a group in the $\alpha_1, \ldots, \alpha_m$ with the composition:

$$\begin{aligned}
(\bar{A}_{ix}, \bar{A}_{\mu\nu}) &= \varepsilon_{i\nu} \bar{A}_{\mu x} - \varepsilon_{\mu x} \bar{A}_{i\nu}, \\
(\bar{A}_{ixj}, \bar{A}_{\mu\nu\pi}) &= 0, \\
(\bar{A}_{ix}, \bar{A}_{\mu\nu\pi}) &= \varepsilon_{i\nu} \bar{A}_{\mu x\pi} + \varepsilon_{i\mu} \bar{A}_{\mu\nu x} - \varepsilon_{x\mu} \bar{A}_{i\nu\pi}.
\end{aligned}$$

In order to extend this result to the case in which all derivatives of the ξ up to the s^{th} ($s > 2$) enter into the symbol (9), there are two different paths to follow. A first path would be this one: One first proves that in order for (9) to be the general transformation of a group it is necessary and sufficient that the $\bar{A}_{i\nu} f$, $\bar{A}_{i\nu\mu} f$, … form a group in the α_1, …, α_m with a composition that is the same for all of them. In order to then find it, it is enough to consider a special group (9) – e.g., the group of all point transformations in x_1, …, x_n performed up to s times, when one considers the derivatives of the y with respect to the x.

A more convenient process takes into account the manner by which one arrives at the symbol (9), or, what amounts to the same thing, the symbol (4).

One considers an arbitrary transformation $Zf = \sum \zeta_i(x) \frac{\partial f}{\partial x_i}$ in the $x_1, \ldots, x_n$, and if it is applied s times, one considers the derivatives of the $y_1, \ldots, y_n$ (the untransformed variables of Zf) with respect to the $x_1, \ldots, x_n$. The transformation $Z^{(s)}f$ thus obtained may be represented as follows:

$$Z^{(s)}f = \sum_{i=1}^{n} \zeta_i(x) \frac{\partial f}{\partial x_i} + \sum_i \sum_\nu \frac{\partial \zeta_i}{\partial x_\nu} A_{i\nu} f + \sum_i \sum_{\mu\nu} \frac{\partial^2 \zeta_i}{\partial x_\mu \, \partial x_\nu} A_{i\mu\nu} f + \ldots, \tag{10}$$

in which the $A_{i\nu} f$, $A_{i\mu\nu} f$, …are transformations in the variables:

$$y_1, \ldots, y_{nn}, y_{111}, \ldots, y_{i,\nu_1\cdots\nu_n}, \ldots$$

One will then find that:

$$Z^{(s)} I_k = \sum_i \sum_{\nu_1\cdots\nu_n} \zeta_{i,\nu_1\cdots\nu_n} \alpha^k_{i,\nu_1\cdots\nu_n}(I_1, \ldots, I_m), \qquad k = 1, \ldots, m.$$

Since I_k is a function of only the $y_1, \ldots, y_n, y_{11}, \ldots, y_{nn}, y_{111}, \ldots$, it follows that:

$$A_{i,\nu_1\ldots\nu_n}(I_k) = \alpha^k_{i,\nu_1\cdots\nu_n}(I_1, \ldots, I_m), \tag{11}$$

in which the symbol:

$$A_{i,\nu_1\ldots\nu_n} f$$

represents the transformation that has the coefficients $\zeta_{i,\nu_1\ldots\nu_n}$ in the symbol (10).

Equations (11) say that the equations:

$$I_1 = \text{const.}, \ldots, \qquad I_m = \text{const.}$$

represent an invariant division of the space of $y_1, \ldots, y_{nn}, y_{111}, \ldots$ into groups of:

$$A_{i,\nu_1\ldots\nu_n} f, \qquad i = 1, \ldots, n, \qquad \nu_1 + \ldots + \nu_n < s.$$

Then, from a theorem in the theory of finite groups (*), the infinitesimal transformation:

$$\sum_{k=1}^{m} A_{i,\nu_1\ldots\nu_n}(I_k) \frac{\partial f}{\partial I_k} = \sum_{k=1}^{m} \alpha^k_{i,\nu_1\ldots\nu_n}(I_1, \ldots, I_m) \frac{\partial f}{\partial I_k} = \bar{A}_{i,\nu_1\ldots\nu_n} f$$

(*) *Theorie der Tr. gruppen.* vol. I, page 307.

generates a group in the variables $I_1, \ldots, I_m$ that is *isomorphic* to the group of the $A_{i,\nu_1\ldots\nu_n} f$.

We thus have the theorem:

Theorem 3. *In order for the transformation:*

$$\sum_i \xi_i \frac{\partial f}{\partial x_i} + \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_i} \bar{A}_{i\nu} f \; + \ldots$$

that contains all derivatives of the ξ *up to the* s^{th} *to be the general infinitesimal transformation of a group in the* $x_1, \ldots, x_m$, $\alpha_1, \ldots, \alpha_m$, *it is necessary and sufficient that the transformation:*

$$\bar{A}_{i,\nu_1\ldots\nu_n} f$$

in the $\alpha_1, \ldots, \alpha_m$ *form a group that is isomorphic to the group of* $A_{i,\nu_1\ldots\nu_n} f$ *in the* $y_1, \ldots, y_{nn}, y_{111}, \ldots$ *that extends the* $A_{i,\nu_1\ldots\nu_n} f$ *and is defined by the symbol:*

$$Z^{(s)} = Zf + \sum_i \sum_{\nu_1\ldots\nu_n} \zeta_{i,\nu_1\ldots\nu_n}(x) A_{i,\nu_1\ldots\nu_n} f \, .$$

If one sets $s = 1, 2$ then one finds the compositions (A), (B) precisely.

The composition of the group $A_{i,\nu_1\ldots\nu_n} f$ will be briefly denoted by the symbol $\gamma_{s\mu}$.

Combining theorem 3 with theorem 2, one has:

Theorem 4. *The equations of definition of the infinitesimal transformations for any group in the* $x_1, \ldots, x_m$, *may be put into the form:*

$$\sum_{i=1}^{n} \sum_{\nu_1\ldots\nu_n} \zeta_{i,\nu_1\ldots\nu_n}(x) \alpha^k_{i,\nu_1\ldots\nu_n}(\varpi_1, \ldots, \varpi_m) = \sum_{i=1}^{n} \zeta_i(x) \frac{\partial \varpi_x}{\partial x_i}, \qquad x = 1, \ldots, m, \tag{6}$$

in which:

$$\sum_{x=1}^{m} \alpha^x_{i,\nu_1\ldots\nu_n}(\varpi_1, \ldots, \varpi_m) \frac{\partial f}{\partial \varpi_x} = \bar{A}_{i,\nu_1\ldots\nu_n} f$$

are transformations that form a group with the composition γ_{sn}.

Conversely, any group with this composition that leads to equations (6) *for any system of functions* $\varpi_1, \ldots, \varpi_m$ *defines a group in the* $x_1, \ldots, x_n$.

Let $N_{(s)}$ be the number of derivatives of the y with respect to the x up to order s. In order for the (6) to effectively define a group, it is necessary that the group that is formed by the $\bar{A}_{i,\nu_1\ldots\nu_n} f$ have a number of parameters that is not greater than $N_{(s)}$. The problem of determining all of the groups in $x_1, \ldots, x_n$ thus comes down to the following one:

For any value of s, determine all of the groups that have the composition γ_{sn} *and a number of parameters that is not greater than* $N_{(s)}$.

§ 3. Determination of the groups with the composition γ_{sn}.

The problem that we were led to in the preceding paragraph can be solved for any value of s by the method that was used by LIE in his work on the theory of finite groups. It is therefore necessary to consider this problem with more care.

Let there given two arbitrary infinitesimal transformations, one of which is in the variables $y_1, \ldots, y_n$, and the other of which is in the variables $x_1, \ldots, x_n$:

$$Yf = \sum_{i=1}^{n} \eta_i(y) \frac{\partial f}{\partial y_i}, \qquad Xf = \sum_{i=1}^{n} \xi_i(x) \frac{\partial f}{\partial x_i}.$$

Consider, along with the variables $y_1, \ldots, y_n, x_1, \ldots, x_n$, the variables:

$$y_{i,\nu_1,\ldots,\nu_n} = \frac{\partial^{\nu_1+\cdots+\nu_n} y_i}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}}, \qquad i = 1, \ldots, n, \quad \nu_1 + \ldots + \nu_s \leq s,$$

and imagine that Yf and Xf are extended with respect to these new variables. If one sets, e.g., $s = 1$ then one will have:

$$\left.\begin{aligned} Y^{(1)} f &= \sum_i \eta_i(y) \frac{\partial f}{\partial y_i} + \sum_i \sum_\nu \frac{\partial \eta_\nu}{\partial y_i} \left(\sum_\mu y_{\nu\mu} \frac{\partial f}{\partial y_{i\mu}} \right), \\ X^{(1)} f &= \sum_i \xi_i(x) \frac{\partial f}{\partial x_i} - \sum_i \sum_\nu \frac{\partial \xi_i}{\partial x_\nu} \left(\sum_\mu y_{\mu\nu} \frac{\partial f}{\partial y_{\mu i}} \right). \end{aligned}\right\} \tag{1}$$

Since one has:

$$(Y^{(1)} X^{(1)}) = (Y X)^{(1)},$$

and on the other hand $(Y\ X) = 0$, so one has $(Y^{(1)}\ X^{(1)}) = 0$ for any arbitrary transformations Yf and Xf. Form the parentheses $(Y^{(1)}\ X^{(1)})$ with the expressions (1) for $Y^{(1)}$ and $X^{(1)}$, and write that these parentheses must be equal to zero identically for any arbitrary functions $\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n$. One then finds that the transformations:

$$\sum_\mu y_{\nu\mu} \frac{\partial f}{\partial y_{i\mu}}, \qquad \nu, i = 1, \ldots, n \tag{2}$$

are all permutable with the transformations:

$$\sum_\mu y_{\mu\nu} \frac{\partial f}{\partial y_{\mu i}}, \qquad \nu, i = 1, \ldots, n,$$

which is easily verified in this special case of $s = 1$.

The transformations (2) form a simply transitive group, as well as the transformations (3). We may summarize all of this by saying:

The groups (2) *and* (3) *are simply transitive and reciprocal to each other.*

As one sees, they are two parametric groups for the general linear homogeneous group in n variables.

We turn to the case of arbitrary s. Let:

$$\begin{aligned} Y^{(s)} f &= \sum_i \eta_i(y) \frac{\partial f}{\partial y_i} + \sum_i \sum_{\nu_1 \cdots \nu_n} \eta_{i,\nu_1 \cdots \nu_n}(y) B_{i,\nu_1 \cdots \nu_n} f \\ X^{(s)} f &= \sum_i \xi_i(x) \frac{\partial f}{\partial x_i} + \sum_i \sum_{\nu_1 \cdots \nu_n} \xi_{i,\nu_1 \cdots \nu_n}(x) A_{i,\nu_1 \cdots \nu_n} f \end{aligned} \qquad (\nu_1 + \ldots + \nu_n < s).$$

A rationale that is analogous to the one that we made in the case of $s = 1$ leads to this theorem:

The transformations:

$$B_{i,\nu_1 \cdots \nu_n} f, \qquad i = 1, \ldots, n, \qquad \nu_1 + \ldots + \nu_n < s$$

in the $N_{(s)}$ *variables* $y_{11}, \ldots, y_{i,\nu_1 \cdots \nu_n}, \ldots$ *form a simply transitive group.*

The transformations

$$A_{i,\nu_1 \cdots \nu_n} f, \qquad i = 1, \ldots, n, \qquad \nu_1 + \ldots + \nu_n < s$$

in the same variables form a simply transitive group.

The two groups are reciprocal to each other.

They are thus similar to each other; their common composition is what we have called γ_{sn}. This group has a great importance that has not been noted up to now in the theory of groups in n variables and with equations of definition of order s. This notion will recur several times in this paper: For the sake of simplicity, I will call the group of $B_{i,\nu_1 \cdots \nu_n} f$ the *group B* and the group of $A_{i,\nu_1 \cdots \nu_n} f$ the *group A*.

Imagine three series of variables:

$$z_1, \ldots, z_n; \qquad y_1, \ldots, y_n; \qquad x_1, \ldots, x_n,$$

and think of the $y_1, \ldots, y_n$ as functions of the $x_1, \ldots, x_n$, while the $z_1, \ldots, z_n$ are functions of the $y_1, \ldots, y_n$. The $z_1, \ldots, z_n$ are then implicit functions of the $x_1, \ldots, x_n$; we write down the formula that expresses the derivatives of the z with respect to the x as functions of the derivatives of the z with respect to the y and the derivatives of the y with respect to the x.

These formulas for the first, second, …, s^{th} derivatives are:

$$\left.\begin{aligned}
&\frac{\partial z_i}{\partial x_\mu} = \sum_{\nu=1}^{n} \frac{\partial z_i}{\partial y_\nu}\frac{\partial y_\nu}{\partial x_\mu}, \qquad i, \mu = 1, \cdots, n,\\
&\frac{\partial^2 z_i}{\partial x_\mu\, \partial x_\nu} = \sum_\rho \sum_\sigma \frac{\partial^2 z_i}{\partial y_\sigma\, \partial y_\rho}\frac{\partial y_\sigma}{\partial x_\mu}\frac{\partial y_\rho}{\partial x_\nu} + \sum_\sigma \frac{\partial z_i}{\partial y_\sigma}\frac{\partial^2 y_\sigma}{\partial x_\mu\, \partial x_\nu},\\
&\qquad\qquad \cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3}$$

In these equations, consider the derivatives of the z with respect to the x as new variables, the derivatives of the z with respect to the y as old variables, and the derivatives of the y with respect to the x as parameters. *They then represent an $N_{(s)}$-fold simply transitive group with the composition γ_{sn} that is precisely the group A.*

If we now consider the derivatives of the $z_1, \ldots, z_n$ with respect to the $y_1, \ldots, y_n$ to be the parameters, instead, while considering the derivatives of the $y_1, \ldots, y_n$ with respect to the $x_1, \ldots, x_n$ as the old variables and considering the derivatives of the z with respect to the x as the new variables, then we have in (3) the finite equations of another $N_{(s)}$-fold simply transitive group with the composition γ_{sn} that is precisely the group B. Therefore:

Equations (3) *are the finite equations of two groups A, B.*

If one has a transitive group for which the finite equations are known then one may find all of the invariant divisions for that group without integrations. One may therefore find all of the invariant divisions of the space $y_{11}, \ldots, y_{nn}, y_{111}, \ldots, y_{i,\nu_1\cdots\nu_n}, \ldots$ ($\nu_1 + \ldots + \nu_n < s$) for the group A *without integration.*

In particular, if the group A is simply transitive then one may follow this path for the determination of its invariant divisions, or, since it amounts to the same thing, its isomorphic groups (*):

First of all, we determine the subgroups of B. Let:

$$B_1 f, \ldots, B_{N-m} f$$

be an $(N - m)$-fold subgroup and let:

$$\varpi_1(y_{11}, \ldots), \ldots, \varpi_m(y_{11}, \ldots)$$

be independent invariants of this subgroup. The transformations:

$$\sum_{x=1}^{m} A_{i,\nu_1\cdots\nu_n}(\varpi_x)\frac{\partial f}{\partial \varpi_x} = \sum_{x=1}^{m} \alpha^x_{i,\nu_1\cdots\nu_n}(\varpi_1, \cdots, \varpi_m)\frac{\partial f}{\partial \varpi_x} = \bar{A}_{i,\nu_1\cdots\nu_n} f$$

(*) *Theorie der Tr. gruppen*, vol. I, theorem 78, page 439.

then generate a transitive group in the m variables $\varpi_1, \ldots, \varpi_m$ that is isomorphic A and is endowed with the composition γ_{sn}. This $(N - l)$-fold group in the group $B_1, \ldots, B_{N-m}$ gives an l-fold subgroup, but no larger subgroup that is invariant in the group $B_{i,\nu_1\cdots\nu_n} f$.

It is therefore N-fold with the composition γ_{sn} when and only when the group $B_1, \ldots, B_{N-m}$ is not invariant and not contained in any invariant subgroup of the group B.

One obtains all of the transitive groups with the composition γ_{sn} by this process. After that, it is easy to obtain all of the intransitive groups with that composition and a number of parameters that is $\leq N$. However, I will not go further into that problem.

§ 4. Search for the equations of definition of finite transformations.

Let there given an arbitrary group Γ, along with its equations of definition for the infinitesimal transformations. The problem that we will now study is that of finding the equations of definition of the finite transformations.

Theorem 1 (§ 1) reduces this problem to the integration of a complete $N_{(s)}$-fold system in $N_{(s)} + m$ variables, if m is the number of equations and s is their order. It is therefore this problem of integration that I will now occupy myself with.

The equations of definition of the group Γ can *always* (Theorem 3, § 2) be put into the form:

$$\sum_i \sum_{\nu_1\cdots\nu_n} \zeta_{i,\nu_1\cdots\nu_n}(x)\,\alpha^x_{i,\nu_1\cdots\nu_n}(\varpi_1,\cdots,\varpi_m) = \sum_i \zeta_i \frac{\partial \varpi_x}{\partial x_i}, \tag{1}$$

in which the $\varpi_1, \ldots, \varpi_m$ are well-defined functions of the $x_1, \ldots, x_n$.

However, one can also make the $\varpi_1, \ldots, \varpi_m$ arbitrary functions of the x, as long as these functions satisfy certain relations:

$$\chi_x\left(\varpi_1,\ldots,\varpi_m,\frac{\partial \varpi_1}{\partial x_1},\cdots\right) = 0, \qquad x = 1, 2, \ldots$$

so that (1) define a group. Therefore, if one writes the equations of the group Γ in the form (1) then one will associate it with a series of groups in the $x_1, \ldots, x_n$. Now, since the complete system upon which the determination of the finite equations depends forms the group Γ with the equations (1), and this has many analogies with the groups that are associated with Γ, it is natural to think that it is possible to obtain the equations of definition of the finite transformations of all the groups (1) by the integration of *just one* complete system in m independent solutions.

This is, in fact, the case, and I intend to prove that fact in this paragraph.

The equations of the finite transformations of the group Γ are of the form:

$$I_k(y_1, \ldots, y_n, y_{11}, \ldots) = \varpi_k(x_1, \ldots, x_n), \qquad k = 1, \ldots, m,$$

in which the $I_1, \ldots, I_m$ are the functions that one would like to determine. These functions have the following two properties:

α) One must have:

$$A_{i,\nu_1\cdots\nu_n}(I_k) = \alpha^k_{i,\nu_1\cdots\nu_n}(I_1,\ldots,I_m), \qquad k = 1, \ldots, m,$$

β) and one must have that $I_k(y_1, \ldots, y_n, y_{11}, \ldots) = \varpi_k(x_1, \ldots, x_n)$ identically under the substitution:

$$y_i = x_i, \qquad y_{i\nu} = \varepsilon_{i\nu}, \qquad y_{i\mu\nu} = 0, \ldots$$

As one sees, these properties define the functions I_k completely. Thus, I propose to look for the most general system of functions $I_1, \ldots, I_m$ that satisfies all of the conditions α), β).

Since the $A_{i,\nu_1\cdots\nu_n} f$ are transformations in only the derivatives of y with respect to x, and not the y (which contain none of the coefficients), one may say:

Find the most general system of functions $I_1, \ldots, I_m$ *in the* $y_1, \ldots, y_n, y_{ir}, \ldots, y_{i,\nu_1\ldots\nu_n}, \ldots$ *such that one has:*

$$A_{i,\nu_1\cdots\nu_n}(I_k) = \alpha^k_{i,\nu_1\cdots\nu_n}(I_1,\ldots,I_m), \qquad k = 1, \ldots, m,$$

and they reduce to $\varpi_1(y), \ldots, \varpi_m(y)$*, respectively, under the substitution:*

$$y_{i\nu} = \varepsilon_{i\nu}, \qquad y_{i\mu\nu} = 0, \ldots \tag{2}$$

Let:

$$I_k = I_k(y_1, \ldots, y_n, y_{11}, \ldots) \tag{3}$$

be the desired system. To say that the functions I_k satisfy the relations:

$$A_{i,\nu_1\cdots\nu_n}(I_k) = \alpha^k_{i,\nu_1\cdots\nu_n}(I_1,\ldots,I_m) \tag{4}$$

is equivalent to saying that the *system of equations:*

$$I_k = I_k(y_1, \ldots, y_n, y_{11}, \ldots)$$

between the variables $I_1, \ldots, I_m, y_1, \ldots, y_n, y_{11}, \ldots$ must be *invariant* with respect to all of the transformations:

$$A_{i,\nu_1\cdots\nu_n} f + \bar{A}_{i,\nu_1\cdots\nu_n} f = U_{i,\nu_1\cdots\nu_n} f \tag{5}$$

in the variables $I_1, \ldots, I_m, y_1, \ldots, y_n, y_{11}, \ldots$

The symbol $\bar{A}_{i,\nu_1\cdots\nu_n} f$ represents, as it always has up to now, the infinitesimal transformation $\sum_{x=1}^{m} \alpha^x_{i,\nu_1\cdots\nu_n} \frac{\partial f}{\partial I_x}$.

Conversely, the most general system of equations of the form (3) that admits the $U_{i,\nu_1\cdots\nu_n} f$ will show us the most general system of functions $I_1, \ldots, I_m$ that satisfy (4).

The transformations $U_{i,\nu_1\cdots\nu_n} f$ obviously form a group with the composition γ_{sn} ; in order to find the system of invariant equations for this group, consider the matrix formed with the coefficients of the transformations $U_{i,\nu_1\cdots\nu_n} f$. In this matrix, not all of the determinants of order $N_{(s)}$ are annulled, and none of them can be annulled by virtue of a system of the form (3), since those determinants are formed from only the $y_{11}, \ldots, y_{nn}, \ldots$, and are not identically zero. Any system of the form (3) thus represents relations between m independent solutions of the equations:

$$U_{i,\nu_1\cdots\nu_n} f = 0, \qquad i = 1, \ldots, n, \qquad \nu_1 + \ldots + \nu_n \leq s. \tag{6}$$

Let:

$$\Phi_1(I_1, \ldots, I_m, y_{11}, \ldots, y_{nn}, y_{111}, \ldots), \ldots, \Phi_m(I_1, \ldots, y_{11}, \ldots)$$

be m independent solutions of the complete system (6).

Since this complete system is soluble with respect to the $\dfrac{\partial f}{\partial y_{i,\nu_1\cdots\nu_n}}$, the functions Φ will be soluble with respect to the $I_1, \ldots, I_m$.

One then concludes that the most general system (3) will have the form:

$$\Phi_1 = c_1, \ldots, \Phi_m = c_m , \tag{7}$$

in which the $c_1, \ldots, c_m$ are arbitrary constants or arbitrary functions of the $y_1, \ldots, y_n$. Solving the system (7) with respect to the $I_1, \ldots, I_m$, one has:

$$I_\mu = \Omega_\mu\{c_1(y), \ldots, c_m(y), y_{11}, \ldots, y_{nn}, y_{111}, \ldots\}, \qquad \mu = 1, \ldots, m, \tag{7′}$$

and these are the most general functions of the $y_1, \ldots, y_n$, $y_{11}, \ldots$ that satisfy the relations (4).

In order for the functions $\Omega_1, \ldots, \Omega_m$ to also be reducible to $\varpi_1(y), \ldots, \varpi_m(y)$ under the substitution $y_{i\nu} = \varepsilon_{i\nu}$, $y_{i\mu\nu} = 0, \ldots$, it is enough to suppose that the $\Phi_1, \ldots, \Phi_m$ in relations (7) are *principal solutions* of equations (5) with respect to the system of values $y_{i\nu} = \varepsilon_{i\nu}$, $y_{i\mu\nu} = 0, \ldots$ The $\Phi_1, \ldots, \Phi_m$ then indeed reduce to $I_1, \ldots, I_m$, respectively, for that system of values, and relations (7) thus become:

$$I_1 = \varpi_1(y), \ldots, I_m = \varpi_m(y) \tag{8}$$

for that system of values, in which the $\varpi_1(y), \ldots, \varpi_m(y)$ are taken to be arbitrary functions of the $y_1, \ldots, y_n$.

Now, since the system (7′) is also equivalent to the system (7) under the substitution $y_{i\nu} = \varepsilon_{i\nu}$, $y_{i\mu\nu} = 0, \ldots$, it must reduce to the identity (8).

One thus has the theorem:

Theorem 5. *There is a unique system of functions:*

$$I_1(y_1, \ldots, y_n, y_{11}, \ldots, y_{nn}, \ldots), \ldots, I_m(y_1, \ldots, y_n, \ldots)$$

that satisfies the relations:

$$A_{i,\nu_1\ldots\nu_n}(I_k) = \alpha^k_{i,\nu_1,\ldots,\nu_n}(I_1,\ldots,I_m),$$

and for the substitution:

$$y_{i\nu} = \varepsilon_{i\nu},\ y_{i\mu\nu} = 0, \ldots, \tag{2}$$

it reduces to the system of functions:

$$\varpi_1(y), \ldots, \varpi_m(y),$$

in which the ϖ *are however, given functions of the* $y_1, \ldots, y_n$.

In order to find this system of functions, it is enough to determine the principal solutions of the complete system:

$$A_{i,\nu_1\ldots\nu_n} f + \bar{A}_{i,\nu_1\ldots\nu_n} f = 0 \tag{6}$$

with respect to the system of values (2). *If:*

$$\Phi_1\{I_1, \ldots, I_m, y_{11}, \ldots, y_{nn}, \ldots\}, \ldots, \Phi_m\{I_1, \ldots, I_m, y_{11}, \ldots\}$$

are these solutions then it is enough to solve the equations:

$$\Phi_1 = \varpi_1(y), \Phi_m = \varpi_m(y)$$

with respect to the $I_1, \ldots, I_m$. *The expressions thus obtained:*

$$I_1 = \Omega_1\{\varpi_1(y), \ldots, \varpi_m(y), y_{11}, \ldots\}, \ldots, I_m = \Omega_\mu\{\varpi_1(y), \ldots, y_{11}, \ldots\}$$

are the desired ones.

One must add that the equations:

$$I_1 = \varpi_1(x), \ldots, I_m = \varpi_m(x) \tag{9}$$

are the equations of definition of the finite transformations – for the group Γ, one replaces the $\varpi_1, \ldots, \varpi_m$ in equations (1) with the characteristic functions of Γ, and for the associated group to Γ, one successively substitutes all systems of functions that satisfy the relations $\chi_x\left(\varpi_1,\cdots,\varpi_m,\frac{\partial\varpi_1}{\partial x_i},\cdots\right) = 0$. The study of equations (9) makes everything depend upon the integration of a complete system, viz., the system (6), whose form depends only upon the nature of the functions $\alpha_{i,\nu_1,\ldots,\nu_n}(\varpi_1,\ldots,\varpi_m)$, but not on the nature of the functions $\varpi_1, \ldots, \varpi_m$.

We have thus proved what we wished to prove, and we have also found that the equations of definition of any group are of the form:

$$\Omega_\mu\{\varpi_1(y), \ldots, \varpi_m(y), y_{11}, \ldots\} = \varpi_\mu(x), \qquad \mu = 1, \ldots, m.$$

This result is important for the manner in which it makes the functions $I_k(y_1, \ldots, y_n, y_{11}, \ldots)$ depend upon the variables $y_1, \ldots, y_n$. This result will be completed in the paragraphs that follow.

§ 5. On intransitive groups.

Before we proceed with the study of the equations of the finite transformations, we shall briefly occupy ourselves with the intransitive groups so we can them limit that study to the transitive ones.

The group γ_{sn} has the infinitesimal transformation:

$$\bar{A}_{i,\nu_1,\ldots,\nu_n} f = \sum_x \alpha^x_{i,\nu_1,\ldots,\nu_n}(\varpi_1,\cdots,\varpi_m)\frac{\partial f}{\partial \varpi_x}. \tag{1}$$

The equations in the corresponding groups in $x_1, \ldots, x_n$ are then:

$$\sum_x \sum_{\nu_1,\ldots,\nu_n} \zeta_{i,\nu_1,\ldots,\nu_n} \alpha^x_{i,\nu_1,\ldots,\nu_n}(\varpi_1,\cdots,\varpi_m) = \sum_\mu \zeta_\mu \frac{\partial \varpi_x}{\partial x_\mu}. \tag{2}$$

A change of variables $\varpi_1, \ldots, \varpi_m$ transforms the system (2) into an equivalent system and then the series of groups (2) into itself.

Now, suppose that the group γ_{sn} is intransitive, and let:

$$\varphi_1(\varpi_1, \ldots, \varpi_m), \ldots, \varphi_\kappa(\varpi_1, \ldots, \varpi_m)$$

be its independent invariants. Introduce the new variables $\varphi_1, \ldots, \varphi_\kappa, \psi_1, \ldots, \psi_{m-k}$ in place of the old variables $\varpi_1, \ldots, \varpi_\mu$, in which the ψ are independent of each other and the φ.

Since $\bar{A}_{i,\nu_1,\ldots,\nu_n}(\varphi_\nu) = 0$, the system (2) takes on the form:

$$\left.\begin{aligned} &\sum_{\mu=1}^m \zeta_\mu \frac{\partial \varphi_\nu}{\partial x_\mu} = 0, && \nu = 1,\cdots,k, \\ &\sum_i \sum_{\nu_1,\cdots,\nu_n} \zeta_{i,\nu_1,\cdots,\nu_n}(x)\beta^\sigma_{i,\nu_1,\cdots,\nu_n}(\varphi_1,\cdots,\varphi_k,\psi_1,\cdots,\psi_{m-k}) = \sum_{\mu=1}^m \zeta_\mu \frac{\partial \psi_\nu}{\partial x_\mu}, && \sigma = 1,\cdots,m-k, \end{aligned}\right\} \tag{3}$$

so one can say:

Theorem 6. *If the group γ_{sn} is intransitive and has k independent invariants:*

$$\varphi_1(\varpi_1, \ldots, \varpi_m), \ldots, \varphi_k(\varpi_1, \ldots, \varpi_m)$$

then any corresponding group in the $x_1, \ldots, x_n$ is intransitive and has k independent invariants. For a particular group with the system of functions $\varpi_1(x), \ldots, \varpi_m(x)$, these invariants are:

$$\varphi_1[\varpi_1(x), \ldots, \varpi_m(x)], \ldots, \varphi_k[\varpi_1(x), \ldots, \varpi_m(x)].$$

We can express the same thing in this way: There are k equations of the form:

$$\varphi_\nu(\varpi_1', \cdots, \varpi_m') = \varphi_k(\varpi_1, \ldots, \varpi_m), \qquad n = 1, \ldots, k$$

between the finite equations of the group γ_{sn} and k equations of the form:

$$\varphi_\nu\{\varpi_1(y), \ldots, \varpi_m(y)\} = \varphi_\nu\{\varpi_1(x), \ldots, \varpi_m(x)\}, \qquad n = 1, \ldots, k.$$

Example:

The equations:

$$\left.\begin{array}{l} \displaystyle\sum_{i=1}^{n} \xi_i \frac{\partial \alpha_1}{\partial x_i} + A_1(\alpha_1, \cdots, \alpha_m) \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_i} = 0, \\ \cdots\cdots\cdots \\ \displaystyle\sum_{i=1}^{n} \xi_i \frac{\partial \alpha_m}{\partial x_i} + A_m(\alpha_1, \cdots, \alpha_m) \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_i} = 0 \end{array}\right\} \tag{4}$$

define an infinite group in the $x_1, \ldots, x_n$, no matter what the functions $\alpha_1, \ldots, \alpha_m$ of the x and the functions $A_1, \ldots, A_m$ of the α, as long as $m < n$. The corresponding group γ_{sn} is simply infinite; its infinitesimal transformations are:

$$Af = A_1(\alpha_1, \ldots, \alpha_m) \frac{\partial f}{\partial \alpha_1} + \ldots + A_m(\alpha_1, \ldots, \alpha_m) \frac{\partial f}{\partial \alpha_m}.$$

Let:

$$\bar{\alpha}_x = \varphi_x(\alpha_1, \ldots, \alpha_m), \qquad x = 1, \ldots, m \tag{5}$$

be a transformation of the α that takes Af to the canonical form $\dfrac{\partial f}{\partial \bar{\alpha}_m}$.

Then:

$$A(\varphi_1) = 0, \ldots, A(\varphi_{m-1}) = 0, \quad A(\varphi_{m-1}) = 1.$$

With the substitution (5), equations (4) become:

$$\left.\begin{aligned} \sum_{\mu=1}^{n} \xi_\mu \frac{\partial \varphi_i}{\partial x_\mu} = 0, \qquad i = 1, \cdots, m-1 \\ \sum_{\mu=1}^{n} \xi_\mu \frac{\partial \varphi_m}{\partial x_m} + \sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_i} = 0. \end{aligned}\right\} \tag{4'}$$

The first m – 1 of these equations say that any group (4) leaves m – 1 functions invariant. The last of (4′) says that any group (4) leaves an n-fold integral invariant.

More simply, one may say:

Any group (4) *is characterized by leaving m n-fold integrals invariant.*

The determination of these integrals and then the equations of definition of the finite transformations of any group (4) is a problem that is *equivalent* to that of reducing Af to the canonical form $\frac{\partial f}{\partial \bar{\alpha}_m}$.

§ 6. Properties of the equations of definition of the finite transformations. Special case.

Now, consider only transitive groups γ_{sn}; they have a number of parameters that does not exceed $N_{(s)}$. In this paragraph, I will limit myself to the consideration of those groups γ_{sn} that have precisely $N_{(s)}$ parameters.

Let there be given the equations of definition of the infinitesimal transformations of a group Γ in $x_1, \dots, x_n$. Let the corresponding group γ_{sn} have m variables $I_1, \dots, I_m$ and $N_{(s)}$ parameters. Its infinitesimal transformations will be denoted, as usual, by $\bar{A}_{i,\nu_1,\dots,\nu_n} f$.

Recall the following theorem from the theory of finite groups (*):

Theorem 7. *The r independent infinitesimal transformations:*

$$X'_k f = \sum_{i=1}^{n} \xi_{ki}(x'_1, \dots, x'_n) \frac{\partial f}{\partial x'_i}, \qquad k = 1, \dots, r$$

in the n variables $x'_1, \dots, x'_n$ *satisfy relations of the form:*

$$(X'_k, X'_j) = \sum_{s=1}^{r} c_{kjs} X'_s .$$

The infinitesimal transformations in the variables $\alpha_1, \dots, \alpha_r$:

(*) *Theor. der Tr. gruppen*, vol. I, page 154, theorem 23.

$$A_h f = \sum_{\mu=1}^{r} \alpha_{k\mu}(a_1, \dots, a_r) \frac{\partial f}{\partial a_\mu}, \qquad k = 1, \dots, r$$

that satisfy:

$$(A_k, A_j) = \sum_{s=1}^{r} c_{kjs} A_s ,$$

and do not identically annul the determinant of the functions $\alpha_{k\mu}$ *form the complete r-fold system:*

$$X'_k f + A_k f = 0, \qquad k = 1, \dots, r,$$

and determine the principal solution with respect to a certain system of values $a_k = a_k^0$*. The* $x_i = F_i(x'_1, \dots, x'_n, a_1, \dots, a_r)$ *are these principal solutions of the equations that have been solved with respect to the* x'*, in which the:*

$$x'_i = f_i(x_1, \dots, x_n, a_1, \dots, a_r)$$

represent a continuous r-fold group. This group contains the identity transformation and is generated by the infinitesimal transformations:

$$\lambda_1 X'_1 f + \dots + \lambda_r X'_r f .$$

In the preceding theorem, set $r = N_{(s)}$, $n = m$, and take:

The variables $a_1, \dots, a_r$ to be the N variables $y_{i,\nu_1 \cdots \nu_n}$,
The variables $x'_1, \dots, x'_n$ to be the m variables $I_1, \dots, I_m$,
The transformations $A_k f$ to be the N transformations $A_{i,\nu_1 \dots \nu_n} f$,
The transformations $X'_k f$ to be the N transformations $\bar{A}_{i,\nu_1 \dots \nu_n} f$,

and finally take the system of values $a_k = a_k^0$ to be the system of values:

$$y_{i\nu} = \varepsilon_{i\nu}, \qquad y_{i\mu\nu} = 0, \dots \tag{1}$$

for the $y_{i,\nu_1,\dots,\nu_n}$. Thus:

$$\Phi_\mu(I_1, \dots, I_m, y_{i,\nu_1,\dots,\nu_n}, \dots), \qquad \mu = 1, \dots, m$$

are the principal solutions of the complete system:

$$A_{i,\nu_1,\dots,\nu_n} f + \bar{A}_{i,\nu_1,\dots,\nu_n} f = 0$$

with respect to the system of values (1), and when $I'_\mu = \Phi_\mu(I, y_{i,\nu_1,\dots,\nu_n})$ are solved with respect to $I_1, \dots, I_m$ this gives:

$$I_\mu = f_\mu\,(I'_1, \dots, I'_m, \dots, y_{i,\nu_1,\dots,\nu_n}, \dots), \quad \mu = 1, \dots, m, \tag{2}$$

so (2) are the finite equations of the group γ_{sn} ; its first parametric group is the group $A_{i,\nu_1\dots\nu_n} f$. The parameters of the identity transformation are given by (1).

If one sets $I_\mu = \varpi_\mu(x)$, $I'_\mu = \varpi_\mu(y)$ ($\mu = 1, \dots, m$) in (2), and one thinks along the lines described in § 4 to form the equations of definition of the finite transformations of the group that corresponds to the group γ_{sn} then one notices that:

$$\varpi_\mu(x) = f_\mu\,[\varpi_1(y), \dots, \varpi_m(y), \dots, y_{i,\nu_1,\dots,\nu_n}, \dots]$$

are the equations of the finite transformations of a group in $x_1, \dots, x_n$ for any system of functions $\varpi_1(y), \dots, \varpi_m(y)$.

The problem of determining these equations for a given group in $x_1, \dots, x_n$ thus coincides with the problem of finding the finite equations of the corresponding group γ_{sn} in such a manner that its first parametric group is precisely the group A.

Conversely, let:

$$z'_\lambda = f_\lambda\,[z_1, \dots, z_n, a_1, \dots, a_n, a_{i,\nu_1,\dots,\nu_n}, \dots, a_N], \qquad \lambda = 1, \dots, m \tag{3}$$

be an N-fold group in m variables with the composition γ_{sn} .

It is always possible to the determine the $a_{i,\nu_1,\dots,\nu_n}$ as functions of the $y_{i,\nu_1,\dots,\nu_n}$:

$$a_{i',\nu'_1,\dots,\nu'_n} = \sigma_{i',\nu'_1,\dots,\nu'_n}(\dots, y_{i,\nu_1,\dots,\nu_n}, \dots) \tag{4}$$

in such a manner that the first parametric group is the group A.

If one makes the substitution (4) in (3) and replacing the $z_1, \dots, z_n$ with $\varpi_1(y), \dots, \varpi_m(y)$, respectively, and replacing the $z'_1, \dots, z'_m$ with $\varpi_1(x), \dots, \varpi_m(x)$ then one obtains:

$$\varpi_\mu(x) = F_\mu\{\varpi_1(y), \dots, \varpi_m(y), \dots, y_{i,\nu_1,\dots,\nu_n}, \dots\}, \qquad \mu = 1, \dots, m. \tag{5}$$

Indeed, now write, along with (5):

$$\varpi_\mu(y) = F_\mu\{\varpi_1(z), \dots, \varpi_m(z), \dots, \frac{\partial^{\nu_1+\dots+\nu_n} z_i}{\partial y_1^{\nu_1} \cdots \partial y_n^{\nu_n}}, \dots\}, \quad \mu = 1, \dots, m. \tag{5′}$$

Combining these last equations with (5) and recalling that the finite equations of the group A are (3) from § 3, in which we considered the derivatives of the z with respect to the y to be variables, and those of y with respect to the x to be parameters, we find:

$$\varpi_\mu(x) = F_\mu\{\varpi_1(z), \ldots, \varpi_m(z), \ldots, \frac{\partial^{\nu_1+\cdots+\nu_n} z_i}{\partial x_1^{\nu_1}\cdots\partial x_n^{\nu_n}}, \ldots\}, \quad \mu = 1, \ldots, m. \qquad (5'')$$

The set of transformations defined by the system (5) is thus such that along with:

$$y_i = F_i(x_1, \ldots, x_n), \qquad i = 1, \ldots, n,$$
$$z_i = \Phi_i(y_1, \ldots, y_n), \qquad i = 1, \ldots, n,$$

the transformations:

$$z_i = \Phi_i[F_1(x), \ldots, F_1(x)], \qquad i = 1, \ldots, n$$

belong to the set. One thus has the theorem:

Theorem 8. *If a group* Γ *in the variables* $x_1, \ldots, x_n$ *is such that its group* γ_{sn} *has* $N(s)$ *parameters then the equations of definition of the finite transformations of* Γ *are of the form:*

$$\varpi_\mu(x) = f_\mu\{\varpi_1(y), \ldots, \varpi_m(y), \ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots\}, \qquad \mu = 1, \ldots, m$$

in which the equations $z'_\lambda = f_\lambda(z_1, \ldots, z_n, \ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots)$ *are the finite equations of the group* γ_{sn} *that is associated with* Γ.

Conversely, if $z'_\mu = f_\mu(z_1, \ldots, z_n, \ldots, a_{i,\nu_1,\ldots,\nu_n}, \ldots)$ *are the finite equations of the group* γ_{sn} *in* $N_{(s)}$ *parameters, and if:*

$$y_{i,\nu_1,\ldots,\nu_n} = \rho_{i,\nu_1,\ldots,\nu_n}(\ldots, a_{i',\nu'_1,\ldots,\nu'_n}, \ldots)$$

is the substitution that takes the first parametric group to the group A, a substitution whose inverse is:

$$a_{i',\nu'_1,\ldots,\nu'_n} = \sigma_{i',\nu'_1,\ldots,\nu'_n}(\ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots)$$

then the equations:

$$\varpi_\mu(x) = f_\mu\{\varpi_1(y), \ldots, \varpi_m(y), \ldots, \sigma_{i',\nu'_1,\ldots,\nu'_n}(\ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots) \ldots\}, \qquad \mu = 1, \ldots, m \qquad (1)$$

define a group in n variables for any system of functions $\varpi_1, \ldots, \varpi_m$.

In order it to not be possible to obtain new equations of order not greater than s from (1) by derivations and eliminations, it is necessary and sufficient that the functions $\varpi_1, \ldots, \varpi_m$ satisfy certain relations:

$$\chi_x\left(\varpi_1, \cdots, \varpi_m, \frac{\partial \varpi_i}{\partial x_1}, \cdots\right) = 0, \qquad x = 1, 2, \ldots,$$

which are the same as for the equations of the infinitesimal transformations, so the number of them is equal to the number of (1).

It is easy to show that (1) reduces to the identity under the substitution:

$$y_i = x_i, \qquad y_{i\nu} = \varepsilon_{i\nu}, \qquad y_{i\mu\nu} = 0, \ldots$$

Indeed, it is enough to observe that the finite equations of the parametric group reduce to the identity for this substitution.

Examples. Let $s = 1$, so $N_{(s)} = n^2$.

I.) For the group γ_{sn}, take the one that is defined by the transformations:

$$A_{ix} = -\alpha_i \frac{\partial f}{\partial \alpha_x}, \tag{1′}$$

and to this form of linear, homogeneous group there correspond groups in $x_1, \ldots, x_n$ whose infinitesimal transformations are defined by the equations:

$$\sum_{i=1}^{n} \alpha_i \frac{\partial \xi_i}{\partial x_\nu} + \sum_{i=1}^{n} \xi_i \frac{\partial \alpha_\nu}{\partial \xi_i} = 0, \qquad \nu = 1, \ldots, n. \tag{2′}$$

The finite transformations of these groups have the equations of definition:

$$\sum_{i=1}^{n} \alpha_i(y) \frac{\partial y_i}{\partial x_\mu} = \alpha_\mu(x), \qquad \mu = 1, \ldots, \mathrm{n}, \tag{3′}$$

in which the α are the same functions as in (2′).

For any system of functions α, (3′) [or, what amounts to the same thing, (2′)] define a group. All such groups have the property of leaving a PFAFF expression invariant, such as:

$$\sum_{i=1}^{n} \alpha_i(x)\, dx_i \,.$$

II.) For the group γ_{sn}, now take:

$$A_{ix} = \alpha_x \frac{\partial f}{\partial \alpha_i}, \tag{1″}$$

so the corresponding groups in $x_1, \ldots, x_n$, along with their infinitesimal transformations, satisfy a system of the form:

$$\sum_{i=1}^{n} \alpha_i \frac{\partial \xi_i}{\partial x_\nu} - \sum_{x=1}^{n} \xi_x \frac{\partial \alpha_i}{\partial x_x} = 0, \qquad i = 1, \ldots, n. \tag{2″}$$

The equations of definition of the finite transformations are:

$$\sum_{i=1}^{n} \alpha_i(y) \frac{\partial \Omega}{\partial y_{i\lambda}} = \alpha_\lambda(x), \qquad \lambda = 1, \dots, n, \tag{3''}$$

in which:

$$\Omega = \sum \pm y_{11} \dots y_{nn}, \qquad y_{ix} = \frac{\partial y_i}{\partial x_x}.$$

The groups (3″) are characterized by leaving the system of $n - 1$ PFAFF equations invariant:

$$dx_1 : dx_2 : \dots : dx_n = \alpha_1(x) : \alpha_2(x) : \dots : \alpha_n(x),$$

or by the fact that the infinitesimal transformation $Af = \sum_i \alpha_i(x) \frac{\partial f}{\partial x_i}$ leaves it invariant, as well, in such a way that if Xf is the generic infinitesimal transformation of a group (3″) then one has $(X, A) = 0$.

III.) Finally, take the group γ_{sn} to be the group:

$$Z_{ix} f = -\varepsilon_{in} z_x \sum_{\tau=1}^{n-1} z_\tau \frac{\partial f}{\partial z_\tau} + z_x \frac{\partial f}{\partial z_i} + \varepsilon_{ix} v \frac{\partial f}{\partial v}, \tag{1'''}$$

in which we agree to let:

$$\varepsilon_{ix} = 0 \quad \text{if} \quad i \neq x, \quad \varepsilon_{ix} = 1, \qquad z_n = 1.$$

This group γ_{sn} will correspond to a group in $x_1, \dots, x_n$ that is defined by:

$$\left.\begin{array}{r} \displaystyle\sum_{i=1}^{n} \frac{\partial \xi_i}{\partial x_i} = \sum_{i=1}^{n} \xi_i \frac{\partial v}{\partial x_i} \\ \displaystyle\sum_{x=1}^{n} z_x \frac{\partial \xi_\sigma}{\partial x_x} - z_\sigma \sum_{x=1}^{n} z_x \frac{\partial \xi_n}{\partial x_x} = \sum_{i=1}^{n} \xi_i \frac{\partial z_\sigma}{\partial x_i}, \quad \sigma = 1, \dots, n-1, \end{array}\right\} \tag{2'''}$$

where the first of these equations defines a group in itself; and the remaining $n - 1$ equations define another group by themselves.

These two groups we sought and their two corresponding groups γ_{sn} are found to have one and $n^2 - 1$ parameters, respectively. The equations of the finite transformations of these two types of groups will be found in the following paragraph.

§ 7. General case.

I propose to extend the considerations of the preceding paragraph to groups in n variables, as well, for which the group γ_{sn} has a number of parameters that is less than $N_{(s)}$.

We base any considerations upon finite groups.

Let there be given a simply transitive group in r variables $x_1, \ldots, x_r$:

$$X_1, \ldots, X_r ,$$

and let its reciprocal group be $Y_1, \ldots, Y_r$.

Let $A_1, \ldots, A_r$ be a simply transitive group that is isomorphic to the group X in m variables $\alpha_1, \ldots, \alpha_m$. There then exists an $(r - m)$-fold subgroup in the group $Y_1, \ldots, Y_r$:

$$\bar{Y}_1, \ldots, \bar{Y}_{r-m},$$

and a system of independent variables $\alpha_1(x), \ldots, \alpha_m(x)$ of this group such that the transformations:

$$\sum_{\mu=1}^{m} X_k(\alpha_\mu)\frac{\partial f}{\partial \alpha_\mu} = \sum_{\mu=1}^{m} \varpi_{k\mu}(\alpha_\mu)\frac{\partial f}{\partial \alpha_\mu}, \qquad x = 1, \ldots, r$$

are precisely the proposed $A_1, \ldots, A_r$.

If the group $A_1, \ldots, A_r$ is $(r - l)$-fold and if $A_1, \ldots, A_r$ are independent transformations then there exists an l-fold subgroup of the group of $\bar{Y}_1, \ldots, \bar{Y}_{r-m}$ that is invariant in the group $Y_1, \ldots, Y_r$. One may determine a system of independent invariants of this l-fold subgroup:

$$\varphi_1, \ldots, \varphi_{r-l}$$

such that:

$$\sum_{\nu=1}^{r-l} X_k(\varphi_\nu)\frac{\partial f}{\partial \varphi_\nu} = U_k f, \qquad k = 1, \ldots, r - l$$

are transformations in the $\varphi_1, \ldots, \varphi_{r-l}$ that are independent of each other; the group $U_1, \ldots, U_r$ is then simply transitive. We would like to demonstrate that this is holomorphically isomorphic to the group $A_1, \ldots, A_{r-l}$.

Meanwhile, if there exist relations:

$$\begin{aligned}(A_i, A_x) &= \sum_{s=1}^{r} c_{ixs} A_s ,\\ & \qquad\qquad\qquad i, x = 1, \ldots, r\\ (U_i, U_x) &= \sum_{s=1}^{r} c_{ixs} U_s ,\end{aligned}$$

then the group $A_1, \ldots, A_r$, as well as the group $U_1, \ldots, U_r$, is isomorphic to the group $X_1, \ldots, X_r$. It is then enough to prove that the $A_{r-l+1}, \ldots, A_r$ are expressed as functions of the $A_1, \ldots, A_{r-l}$ in the following way:

$$A_{r-l+\mu} = \sum_{\sigma=1}^{r-l} d_{\mu\sigma} A_\sigma , \tag{1}$$

and there are relations between the $U_1, \dots, U_r$ of the form:

$$U_{r-l+\mu} = \sum_{\sigma=1}^{r-l} d_{\mu\sigma} U_\sigma \,. \tag{1}$$

For this purpose, it is enough to observe that the $\varphi_1, \dots, \varphi_{r-l}$ are functions of the $\alpha_1, \dots, \alpha_r$. The relations (1) resolve into the following $n \cdot l$:

$$X_{r-l+\mu}(\alpha_x) = \sum_\mu d_{\mu\sigma} X_\sigma(\alpha_x)\,, \qquad x = 1, \dots, m, \quad \mu = 1, \dots, l,$$

from which, one deduces that:

$$\sum_x \frac{\partial \varphi_\mu}{\partial \alpha_x} X_{r-l+\mu}(\alpha_x) = \sum_\sigma d_{\mu\sigma} \sum_x \frac{\partial \varphi_\mu}{\partial \alpha_x} X_\sigma(\alpha_x)\,,$$

i.e.:

$$X_{r-l+\mu}(\varphi_\nu) = \sum_\sigma d_{\mu\sigma} X_\sigma(\varphi_\nu)\,, \qquad \nu = 1, \dots, r-l, \qquad \mu = 1, \dots, l.$$

Q.E.D.

We then write down the equations:

$$U_k f + A_k f = 0, \qquad k = 1, \dots, r-l,$$

which form a complete $(r - l)$-fold system in the $r - l + m$ variables:

$$\alpha_1, \dots, \alpha_m, \varphi_1, \dots, \varphi_{r-l}\,.$$

Let:

$$\psi_\nu(\varphi_1, \dots, \varphi_{r-l}, \alpha_1, \dots, \alpha_m), \qquad \nu = 1, \dots, m \tag{3}$$

be the principal solutions of this complete system with respect to the system of values $\varphi_1 = \varphi_1^0, \dots, \varphi_{r-l}^0$, so:

$$\psi_\nu(\varphi_1^0, \dots, \varphi_{r-l}^0, \alpha_1, \dots, \alpha_m) = \alpha_\nu\,.$$

In (3), replace the $\varphi_1, \dots, \varphi_{r-l}$ with their expressions in the $x_1, \dots, x_r$. The ψ become independent functions of the $x_1, \dots, x_r, \alpha_1, \dots, \alpha_m$ that are solutions of the complete system:

$$X_k f + A_k f = 0, \qquad k = 1, \dots, r.$$

If one thus desires to determine the principal solutions of this system for the values $x_1 = x_1^0, \dots, x_r = x_r^0$ then it is enough to first calculate the values of the $\varphi_1, \dots, \varphi_{r-l}$ for $x_1 = x_1^0, \dots, x_r = x_r^0$ and after giving $\varphi_1^0, \dots, \varphi_{r-l}^0$ their values thus found, determine the principal solutions of the system $U_k + A_k = 0$ with respect to:

$$\varphi_1 = \varphi_1^0\,,\ \ldots,\ \varphi_{r-l} = \varphi_{r-l}^0\,.$$

Assuming this, suppose that one has the equations of definition of the infinitesimal transformations of a group in $x_1, \ldots, x_n$ and one then recognizes that the corresponding group $\bar{A}_{i,\nu_1\ldots\nu_n} f$ in m variables $\alpha_1, \ldots, \alpha_m$ is $\{N_{(s)} - l\}$-fold. The group $\bar{A}_{i,\nu_1\ldots\nu_n} f$ then corresponds to an $(N - m)$-fold subgroup of B that contains an l-fold invariant subgroup; this latter group is denoted by γ.

Let $\bar{C}_{i,\nu_1\ldots\nu_n} f$ by $N - l$ independent transformations from the $\bar{A}$. One sees that they then determine a system of independent invariants $\varphi_1, \ldots, \varphi_{N-l}$ of the group γ such that the $N - l$ transformations $C_{i,\nu_1\ldots\nu_n} f$ [this symbol denotes the transformation that is formed from the A in the same way that the $\bar{C}_{i,\nu_1\ldots\nu_n} f$ are formed from the $\bar{A}$] form an $(N - l)$-fold simply transitive group in the variables $\varphi_1, \ldots, \varphi_{N-l}$.

One also sees that the group of the Cf is isomorphic to the group of the $\bar{C}f$. In order to exhibit the isomorphism of the two groups, it is enough to define the correspondence between their transformations with the same indices.

In § 4, in order to find the equations of the finite transformations of the group in question, one considered the complete system:

$$\bar{A}_{i,\nu_1,\ldots,\nu_n} f + A_{i,\nu_1,\ldots,\nu_n} f = 0,$$

and determined the principal solutions with respect to a certain system of values for the $y_{i,\nu_1,\ldots,\nu_n}$. Now, one takes this other path:

One calculates the values $\varphi_1^0, \ldots, \varphi_{N-l}^0$ of the $\varphi_1, \ldots, \varphi_{N-l}$ for:

$$y_{ir} = \varepsilon_{ir}, \qquad y_{i\mu\nu} = 0, \ldots$$

One then determines the principal solutions of the complete system:

$$C_{i,\nu_1,\ldots,\nu_n} f + \bar{C}_{i,\nu_1,\ldots,\nu_n} f = 0$$

with respect to the system of values $\varphi = \varphi^0$. If $\Phi_\mu(\varphi_1, \ldots, \varphi_{N-l}, \alpha_1, \ldots, \alpha_m)$ are these solutions then one solves the:

$$\alpha'_\mu = \Phi_\mu(\varphi_1, \ldots, \varphi_{N-l}, \alpha_1, \ldots, \alpha_m), \qquad \mu = 1, \ldots, m$$

with respect to the $\alpha_1, \ldots, \alpha_m$. One then obtains:

$$\alpha_\mu = f_\mu(\alpha'_1, \ldots, \alpha'_m, \varphi_1, \ldots, \varphi_{N-l}), \qquad \mu = 1, \ldots, m, \tag{4}$$

so the equations:

$$\varpi_\mu(x) = f_\mu\{\varpi_1(x), \ldots, \varpi_\mu(x), \varphi_1(\ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots), \ldots, \varphi_{N-l}(\ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots)\},$$

$$\mu = 1, \ldots, m \tag{5}$$

define, for a certain system of functions ϖ, the group that we started out with, and for any other system of functions ϖ will be a group.

Applying the theorem 7 of § 6, one sees that (4) are the finite equations of the group $A_{i,\nu_1\ldots\nu_n} f$.

Therefore:

Theorem 9. *For any group in the variables $x_1, \ldots, x_n$ the equations of definition of the finite transformations give the finite equations of the corresponding group γ_{sn}:*

$$z'_\mu = f_\mu(z_1, \ldots, z_m, a_1, \ldots, a_{N-l}), \qquad \mu = 1, \ldots, m$$

when the $z_1, \ldots, z_m$ are taken to be well-defined functions of the $y_1, \ldots, y_n$, the z'_μ are the same functions of the $x_1, \ldots, x_n$, and the $a_1, \ldots, a_{N-l}$ are a system of conveniently determined invariants of an invariant l-fold subgroup in B.

This theorem does not prove that for any group the corresponding γ_{sn} is transitive. If the group γ_{sn} is intransitive then the group in the $x_1, \ldots, x_n$ is certainly intransitive, as well. One may then say that the preceding theorem is valid for any transitive group in the $x_1, \ldots, x_n$.

The considerations of § 5 give us a hint that an analogous theorem also subsists for intransitive groups.

We would now like to prove:

Theorem 10. *If:*

$$z'_\mu = f_\mu(z_1, \ldots, z_m, a_1, \ldots, a_{N-l}), \qquad \mu = 1, \ldots, m \tag{6}$$

are the finite equations of an arbitrary transitive group with the composition γ_{sn} then one may determine functions:

$$a_x = a_x(\ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots),$$

such that the equations:

$$\varpi_\mu(x) = f_\mu\{\varpi_1(y), \ldots, \varpi_\mu(y), \ldots, a_x(\ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots), \ldots), \qquad \mu = 1, \ldots, m \tag{7}$$

determine a group in the $x_1, \ldots, x_n$ for any system of functions $\varpi_1, \ldots, \varpi_m$.

Let:

$$\bar{C}_1 f, \ldots, \bar{C}_{N-l} f$$

be independent infinitesimal transformations of the group (6), and let:

$$C_1 f, \ldots, C_{N-l} f$$

be independent transformations of the first parameter group (6). If one makes the Cf correspond to a $\bar{C}f$ with the same index then the two groups will relate to each other in an isomorphic manner. If one makes:

$$\bar{A}_{i,\nu_1,\ldots,\nu_n} = \sum_{\sigma=1}^{N-l} d_{i,\nu_1,\ldots,\nu_n,\sigma} \bar{C}_\sigma f \ ,$$

correspond to $A_{i,\nu_1,\ldots,\nu_n}$ then the two groups $\bar{C}$ and A will relate to each other in an isomorphic way, and if one makes:

$$\Gamma_{i,\nu_1,\ldots,\nu_n} f = \sum_{\sigma=1}^{N-l} d_{i,\nu_1,\ldots,\nu_n,\sigma} C_\sigma f$$

correspond to $A_{i,\nu_1,\ldots,\nu_n}$ then the two groups C and A will relate to each other in an isomorphic way.

Suppose that we let:

$$\Gamma_{i,\nu_1,\ldots,\nu_n} f = \sum_{x=1}^{N-l} \alpha^x_{i,\nu_1,\ldots,\nu_n} (a_1, \cdots, a_{N-l}) \frac{\partial f}{\partial a_x} .$$

There is then one and only one system of functions:

$$a_x = a_x(\ldots,\ y_{i,\nu_1,\ldots,\nu_n}\ , \ldots), \qquad x = 1, \ldots, N - l, \tag{8}$$

such that:

$$A_{i,\nu_1,\ldots,\nu_n}(a_x) = \alpha^x_{i,\nu_1,\ldots,\nu_n}(a_1, \cdots, a_{N-l}) ,$$

and for:

$$\gamma_{i\nu} = \varepsilon_{i\nu}, \qquad \gamma_{i\mu\nu} = 0, \ \ldots,$$

they assume the values $a_1 = a_1^0, \ldots, a_{N-l} = a_{N-l}^0$ (*).

The functions (8) thus determined also have the following property: If one determines the principal solutions of:

$$C_\sigma + \bar{C}_\sigma = 0$$

with respect to $a_1 = a_1^0, \ldots, a_{N-l} = a_{N-l}^0$, and makes the substitution (8) in the functions thus found then one has the principal solutions of:

$$A_{i,\nu_1,\ldots,\nu_n} f + \bar{A}_{i,\nu_1,\ldots,\nu_n} f = 0$$

(*) If the group $\bar{C}_1 f, \ldots, \bar{C}_{N-l} f$ relates to itself in an isomorphic manner, without it being a change of variables that makes one pass from one form to another, then if one applies the preceding rationale to any form one with obtain *two* systems of functions $a_x(\ldots,\ y_{i_1,\nu_1,\ldots,\nu_n}\ , \ldots)$. See examples 2 and 3 at the end of the paragraph.

with respect to $y_{i\nu} = \varepsilon_{i\nu}$, $y_{i\mu\nu} = 0$, …

For the system of values $a_1 = a_1^0$, …, $a_{N-l} = a_{N-l}^0$, take the one that makes the group (6) correspond to the identity transformation. Recalling the way in which we obtained (6) (Theorem 7 of § 6), one sees that equations (7) thus obtained determine the principal solutions $\Phi_\mu(\alpha_1, \ldots, \alpha_m, \ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots)$ of $A_{i,\nu_1,\ldots,\nu_n} f + \bar{A}_{i,\nu_1,\ldots,\nu_n} f = 0$ with respect to $y_{i\nu} = \varepsilon_{i\nu}$, …, so one takes $\alpha'_\mu = \Phi_\mu(\alpha_1, \ldots)$ and solves these equations with respect to α_1, …, α_m :

$$\alpha_\mu = f_\mu(\alpha'_1, \ldots, \alpha'_m, \ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots),$$

and finally take the α'_1, …, α'_m to be arbitrary functions of the y and take the α_1, …, α_m to be the same functions of the x, respectively. However, this path is precisely the one for obtaining the equations of the finite transformations of the group that corresponds to the group $\bar{A}_{i,\nu_1,\ldots,\nu_n} f$ that was pointed out in § 4.

Therefore, (7) are the equations of the finite transformations for the group in x_1, …, x_n that corresponds to the group (6). Q. E. D.

The problem of determining the functions $a_x(\ldots, y_{i,\nu_1,\ldots,\nu_n}, \ldots)$ depends upon the integration of a complete system. It is possible to show that this problem reduces to the problem of determining the finite equations of the group $\Gamma_{i,\nu_1,\ldots,\nu_n} f$ by quadrature; however, the proof of this is perhaps a bit too long.

Example. Suppose that $s = 1$. The composition γ_{sn} is then that of the general linear, homogeneous group.

As is well-known, there are only two invariant subgroups in this group. The number l may then assume only two values (other than the value 0), which are precisely the values $l = 1$, $l = n^2 - 1$.

If, for any value of l, one takes the *minimum* possible value for m then one finds, in all, three categories of groups in x_1, …, x_n . These three categories were found by ENGEL in his paper that we already cited (Math. Ann., Bd. 27) in precisely this manner.

I recall the equations of definition of the infinitesimal transformations, *along with those of the finite transformations.* It is enough to imagine the form of the latter in order to illustrate theorem 9.

First category:

The groups of this category are the first ones that one encounters in the space of x_1, …, x_n .

They are defined by the *unique* equation:

$$\alpha(x)\sum_{i=1}^{n}\frac{\partial \xi_i}{\partial x_i} + \sum_{i=1}^{n}\xi_i\frac{\partial \alpha}{\partial x_i} = 0, \tag{1'}$$

so the finite equations are defined by:

$$\alpha(y)\,\Omega = \alpha(x), \qquad \text{setting } \Omega = \sum \pm y_{11}, \ldots, y_{nn}, \tag{2′}$$

so the group (1′) is therefore generated by the transformations that leave an n-fold integral invariant.

The equations (2′), when written in the form $za = z'$, show that the corresponding group γ_{sn} is an ∞^1 group of the one-dimensional variety.

Second category:

It is composed of the groups that leave an equation of the form $\frac{\partial f}{\partial x_n} + \sum_{i=1}^{n-1} \alpha_i \frac{\partial f}{\partial x_i} = 0$ invariant, so they are defined by:

$$\frac{\partial \xi_i}{\partial x_n} + \sum_{x=1}^{n-1} \alpha_x \frac{\partial \xi_i}{\partial x_x} - \alpha_i \left(\frac{\partial \xi_i}{\partial x_n} + \sum_{x=1}^{n-1} \alpha_x \frac{\partial \xi_i}{\partial x_x} \right) - \sum_{\nu=1}^{n} \alpha_{i\nu} \xi_\nu = 0, \qquad i = 1, \ldots, n-1, \tag{1″}$$

which are equations that may be represented symbolically in a simpler way. It is enough to put $Af = \frac{\partial f}{\partial x_n} + \sum_{i=1}^{n-1} \alpha_i \frac{\partial f}{\partial x_i}$, $Xf = \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i}$. (1″) then say that one must have:

$$(A, X) = \lambda\, A\, f,$$

in which the $\lambda(x)$ is an undetermined function of the $x_1, \ldots, x_n$.

The equations of the finite transformations are:

$$\alpha_\mu(x) = \frac{\sum_{i=1}^{n-1} \alpha_i(y)\Omega_{i\mu} + \Omega_{n\mu}}{\sum_{i=1}^{n-1} \alpha_i(y)\Omega_{in} + \Omega_{nn}}, \qquad \text{setting } \Omega_{i\mu} = -\frac{\partial \Omega}{\partial y_{i\mu}}, \quad \mu = 1, \ldots, n-1, \tag{2″}$$

so the corresponding group γ_{sn} is presented in the form:

$$\left.\begin{aligned} A_{in} &= \frac{\partial f}{\partial \alpha_i}, & A_{ni} &= -\alpha_i \sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j}, \\ A_{ix} &= \alpha_x \frac{\partial f}{\partial \alpha_i}, & A_{nn} &= -\sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j}, \end{aligned}\right\} \qquad i, x = 1, \ldots, n-1.$$

Third category.

This is composed of the groups that leave invariant a complete system of the form:

$$\frac{\partial f}{\partial x_n} + \alpha_i \frac{\partial f}{\partial x_i} = 0, \qquad i = 1, \dots, n-1.$$

The equations of the infinitesimal transformations are:

$$\frac{\partial \xi_n}{\partial x_i} + \sum_{x=1}^{n-1} \alpha_x \frac{\partial \xi_x}{\partial x_i} - \alpha_i \left(\frac{\partial \xi_n}{\partial x_n} + \sum_{x=1}^{n-1} \alpha_x \frac{\partial \xi_x}{\partial x_n} \right) + \sum_{\nu=1}^{n} \alpha_{i\nu} \xi_\nu = 0, \qquad i = 1, \dots, n-1. \tag{1'''}$$

The finite transformations satisfy relations:

$$\alpha_\mu(x) = \frac{\sum_{i=1}^{n-1} \alpha_i(y) y_{i\mu} + y_{n\mu}}{\sum_{i=1}^{n-1} \alpha_i(y) y_{in} + y_{nn}}, \qquad \mu = 1, \dots, n-1, \tag{2'''}$$

so the corresponding group γ_{sn} is presented in the form:

$$\left.\begin{array}{ll} A_{in} = \alpha_i \sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j}, & A_{ni} = -\frac{\partial f}{\partial \alpha_i}, \\ A_{ix} = -\alpha_i \frac{\partial f}{\partial \alpha_x}, & A_{nn} = \sum_{j=1}^{n-1} \alpha_j \frac{\partial f}{\partial \alpha_j}, \end{array}\right\} \qquad i, x = 1, \dots, n-1.$$

§ 8. Some applications.

The search for the equations of definition for the finite transformations of a transitive group in $x_1, \dots, x_n$ comes down to these two problems:

Determine the finite equations of a finite transitive group from those of the infinitesimal transformations.

Determine the functions $a_x = a_x(\dots, y_{i,\nu_1,\dots,\nu_n}, \dots)$.

Along with this second problem, one may recall the determination of the finite equations of a finite simply transitive group.

The search for the equations of the finite transformations of transitive groups may therefore lead to that of the finite equations of a finite group (with a given composition).

The problem of determining *all* of the transitive groups in n variables (finite or infinite) reduces to the problem of determining all of the finite transitive groups with the composition γ_{sn} for any value of s.

The problem of determining the finite equations of all the transitive groups with a given composition is soluble by performable operations, so the problem of determining the equations of definition of the finite transformations of all groups in n variables whose equations are of order s is also soluble by performable operations.

We would now like to point out an application of the theorem that was found in § 2.

Assume that one has a group Γ in $x_1, \ldots, x_n$ and the that infinitesimal transformations of the corresponding γ_{sn} are denoted by $\bar{A}_{i,\nu_1,\ldots,\nu_n} f$ ($i = 1, \ldots, n$, $\nu_1 + \ldots + \nu_n \leq s$).

Let the group $\bar{A}_{i,\nu_1,\ldots,\nu_n} f$ be imprimitive (although it is assumed to be transitive) and let:

$$\varphi_1(\alpha_1, \ldots, \alpha_m) = \text{const.}, \ldots, \varphi_{m_1}(\alpha_1, \ldots, \alpha_m) = \text{const.}$$

be an invariant division of the space of $\alpha_1, \ldots, \alpha_m$.

Introduce the new variables $\varphi_1, \ldots, \varphi_{m_1}, \psi_1, \ldots, \psi_{m-m_1}$ in place of the $\alpha_1, \ldots, \alpha_m$ .In the new variables, let:

$$\bar{A}_{i,\nu_1,\ldots,\nu_n} f = \sum_{x=1}^{m_1} \bar{A}_{i,\nu_1,\ldots,\nu_n}(\varphi_x) \frac{\partial f}{\partial \varphi_x} + \sum_{\sigma=1}^{m-m_1} \bar{A}_{i,\nu_1,\ldots,\nu_n}(\psi_\sigma) \frac{\partial f}{\partial \psi_\sigma}, \tag{1}$$

so the transformations:

$$\bar{\bar{A}}_{i,\nu_1,\ldots,\nu_n} f = \sum_{x=1}^{m_1} \bar{A}_{i,\nu_1,\ldots,\nu_n}(\varphi_x) \frac{\partial f}{\partial \varphi_x}$$

in the $\varphi_1, \ldots, \varphi_{m_1}$ define a group that is isomorphic to the group $\bar{A}_{i,\nu_1,\ldots,\nu_n} f$.

If one introduces the expressions (1) in the $\bar{A}_{i,\nu_1,\ldots,\nu_n} f$ in the equations of the infinitesimal transformations of the group Γ then one sees that among these m equations there are m_1 of them that contain just the symbols $\varphi_1, \ldots, \varphi_{m_1}$. These equations in themselves define a group in n variables that has m_1 equations of definition.

One has the following theorem:

Theorem 11. *In order for the m equations of definition for the infinitesimal (or finite) transformations of a group* Γ *to produce* $m_1 < m$ *of them that define a group in themselves by eliminations and combinations it is necessary and sufficient that the group* γ_{sn} *in the* $\alpha_1, \ldots, \alpha_m$ *that corresponds to* Γ *be imprimitive and admit an invariant division of the space of* $\alpha_1, \ldots, \alpha_m$ *into* ∞^m *varieties of* $m - m_1$ *dimensions.*

If the finite equations for a finite transitive group are known then one may find its invariant divisions without integrations. Therefore, given a group Γ in $x_1, \ldots, x_n$ the problem of determining all of the groups with a lower number of equations of the same or lower order, and which contain the group Γ is solved without integrations when one takes into account the finite equations of the group γ_{sn} that is associated with Γ.

Rome, January 1897.