STUDY OF PHYSICAL STATICS

GENERAL PRINCIPLE

FOR

DETERMINING PRESSURES AND TENSIONS

IN AN ELASTIC SYSTEM

BY

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INTRODUCTION

Consider a system of material points that are connected to each other by some links and to which one applies external forces that are kept in mutual equilibrium by the intermediary of the system that they act upon. The usual rules of statics indicate the relationships that must exist between those forces in order for equilibrium to actually exist. However, if one would like to study how the external efforts and forces are distributed between those links, no matter what pressures are exerted on the fixed points that can exist in the system, then the problem (except in a small number of cases) will be generally indeterminate as long as one considers the system links to be rigid, or in other words, as having invariable length. Indeed, the efforts that are supported by the links that connect at a given point must be in equilibrium with the external forces that are applied to that point, so when the number of those links exceeds a restricting limit, one can imagine an infinitude of different distributions of efforts that are all suitable to maintain the equilibrium between the external forces.

However, that indeterminacy ceases at the moment when one considers the elasticity of the body, or more precisely, the elasticity of the links that couple the various points of the system to each other.

Elasticity is a general principle of all bodies, by virtue of which, if some external effort changes the respective positions of the molecules of matter that comprise the body slightly then the molecular actions that determine those new positions will tend to revert to their original positions when the action of the external forces ceases ([1]).

Nonetheless, the determination of the efforts that are supported by the links – which are efforts that I shall call *internal forces* – present great difficulties, and in each particular case, one will generally appeal to some hypotheses that are more or less based in reality in order to simplify the problem.

Hence, upon taking one of the simplest cases as an example, namely, the example of an elastic prism that is curved by a force that is applied at its extremity, one will suppose, in general, that two consecutive sections of the prism that are normal to its direction before the flexure will remain normal to the curve that is affected by the prism after flexure.

That is the geometric condition on the problem. However, that hypothesis is not always very exact, and in practice, one is often obliged to correct the results that one deduces.

In general, the determination of the internal forces is obtained, as one can see, by considering the geometric conditions that the system must satisfy before and after the deformation that is caused by the external forces. Hence, upon considering a certain number of elastic cords that are each fixed at one extremity and which have the other extremity connected to the same point, one applies a force to that point that will cause it to displace, and that displacement will rise to variations in the lengths of the various cords.

The geometric condition of existence for the system is that the cords will again be connected at the same point after the displacement that results from the extension that

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([1]) See the *Leçons sur la Théorie mathématique de l'élasticité des corps solides* by G. LAMÉ.

cord experiences as the effect of the external force. Upon expressing that condition analytically, one will obtain equations that will provide the necessary and sufficient elements for the determination of those tensions when they are combined with the equation of equilibrium between the *external* force that is applied to the point of concurrence and the tensions that are developed in the cords.

We once more examine the case of a planar surface of invariable form (in other words, it is rigid) that is supported at some points by compressible elements and loaded by weights that are distributed in various ways. If one takes into account only the equations of equilibrium between the weights or external forces and the pressures that are exerted at the support points then the equations will be insufficient to determine those pressures when the number of support points exceeds three (or even two, if those points are situated along the same line). However, the geometric condition of the system is that after the compression that the support points experience, the remaining ones will again be all situated on the same plane after displacement of the original position of that plane. Upon expressing that condition analytically, one will get some new equations that will suffice to solve the problem completely when they are combined with the equilibrium equations. That is the question that Euler treated in which paper that was entitled "De pressione ponderis in planum cui incombit," Nov. Comment. Acad. Petrop., v. XVIII. It was presented and developed very elegantly by BRESSE in his beautiful *Traité de Mécanique appliquée.*

In the case that we just pointed out, the geometric conditions on the system are established very easily. However, as the system becomes more complicated, the determination of those conditions will become increasingly difficult and can even become impossible *in practice.*

However, those obstacles will disappear by means of the new principle that defines the goal of this paper and whose statement is this:

When an arbitrary elastic system is in equilibrium under the action of external forces, the total work developed under the extension and compression of the links as a result of the relative displacements of the points of the system, or in other words, the work developed by the internal forces, is a MINIMUM.

I shall apply this theorem to just the case of *small* relative displacements of the points of the system, and one will see, as a result, how it leads to some new equations that will be sufficient to determine the tensions in the various links that couple the points of the system to each other when the new equations are combined with those of equilibrium. In addition, one will recognize that the subsidiary equations are nothing but the same ones that express the geometric conditions that the system must satisfy before the deformation and are linked with the effect of external forces.

I presented the statement of that new principle to the Turin Academy of Sciences in the year 1857. Then, in 1858 (session on 31 May), I made it the subject of a communication to the Institut de France (Académie des Sciences). In the proof that I gave, I appealed to a consideration of the transmission of work throughout the body. Although that proof seemed sufficiently rigorous to me, it seemed to some geometers to be too subtle to be accepted without criticism. On the other hand, the significance of the equations that are deduced from that theorem is not pointed out nearly enough. That is why I believe that I must once more address that study, which was interrupted more than once by a series of events that my position obliged me to take part in. Today, I shall present the new research that has had the result of leading to a proof of the theorem in question that is entirely simple and rigorous and that I can call elementary, and to establish the significance of the equations that one deduces in an obvious manner, which, as I have said, express the geometric conditions that the system must satisfy after the deformation that it experiences as the effect of external forces.

To abbreviate, I will use the term PRINCIPLE OF ELASTICITY to refer to the new principle or theorem that expresses it. That term seems sufficiently justified to me, because it applies to a general property of all bodies or elastic systems. It is a property that provides a general method for determining the distribution of pressures and tensions. One can also call it the *principle of least work*. Hence, VÉNE (Chef de Bataillon du Génie), in a writing entitled "Mémoire sur les lois que suivent les pressions," which was published in 1836, declared that in the year 1818, he stated that *the sum of the squares of the pressures that are produced by weights must be a minimum*. However, in order to prove that proposition, the author appealed to some philosophical considerations that undermine the geometric rigor of that proof.

PAGANI, who taught mathematics with distinction at the University of *Louvain*, proved the latter proposition in several papers, and extended it to the case of weights suspended at the point of convergence of several *homogeneous cords* whose other extremities are held by fixed points (see volume I, series II of the Mémoires de l'Académie des Sciences de Turin and volume VIII of the Mémoires de l'Académie des Science de Bruxelles).

MOSSOTTI likewise gave a proof of VÉNE's theorem, and also announced it in the BULLETIN de FÉRUSSAC in 1828. However, as one will see, that theorem and one that is analogous to PAGANI's are included in the *general principle of elasticity* as only special cases when the support points and cords are homogeneous.

Here is the order of ideas that I have followed in this paper: After presenting some considerations on the stated principle and before giving its proof, I will treat a series of problems on the distribution of pressure and tensions in which I shall have occasion to show the coincidence of the results that one obtains by the direct method, which appeals to the geometric conditions that must be satisfied in each particular case with the ones that one deduces from the *principle of elasticity*. I will then give the proof of that principle and present the general method that one must follow in order to deduce the subsidiary equations for the determinations of the efforts in the elastic links, which are equations that are nothing but the same ones that express the geometric conditions of the system.

In particular, I shall examine the case of a system that is partially rigid and partially elastic, and I will prove that the stated principle applies to it just as it does to a system that contains fixed points. That examination has a special importance, because it refers to the usual hypotheses that one assumes in practice in order to simplify the solution of problems such as the flexure and torsion of prisms, which are hypotheses that generally amount to assuming that everything in the prism under consideration is rigid in these two special cases, with the exception of the fibers that contained in two consecutive sections in which one must determine the tensions.

The principle of elasticity can even show one how such hypotheses can be defective and will provide the means to correct them.

In order to give the question of the distribution of the tension all of the scope that it deserves in the context of physics, one must take into account some *thermodynamic* phenomena that manifest themselves in the act of changing the form of elastic bodies or systems, but I will consider the body at the moment where equilibrium is established between the *internal* forces and the external ones by supposing that the temperature does not vary. One can then assume that the work that is developed can be summarized in the work that is found to be concentrated in the *latent state* in the elastic system by the effect of external forces.

I do not know if I am delusional, but it seems to be that this study will serve to fill a lacuna that still exists in *physical statics* by presenting a general method that is, due to its simplicity, suitable for being introduced into the teaching of how to solve problems that relate to the distribution of pressures and tensions.

I believe that method will also be particularly useful to engineers who have frequently needed to calculate the efforts that support the various construction pieces in their constructions (and above all, in the ones of the present era) in order to determine the dimensions and establish their conditions of stability.

I. – Preliminary considerations.

Consider a system of material points that are coupled to each other by elastic links and remain in equilibrium under the action of external forces. That equilibrium cannot be established without some of the system *links* being lengthened, while the other ones are shortened, and consequently, without the positions of the various points having varied. The variations of length of the links will develop *internal forces* of tension or compression that will bring equilibrium to the forces that are applied to them.

From now on, we shall assume that the changes of form that the system experiences are very small and that as a result the *internal forces* are reasonably proportional to the variations of the distances between the different points. Experiments justify that viewpoint in the usual applications. Having said that, we shall the following notations in this paper: The various points will be indicated by the indices 1, 2, 3, …, n, etc.

x, y, z are the rectangular coordinates of an arbitrary point.

l is the distance between two points, in general, after the external forces have acted upon them. $l_{(i,k)}$ denotes the distance between the two points i and k.

λ is the variation of the distance l, which can be positive or negative.

T is the tension in the link between two given points that correspond to a variation λ.

ε is a coefficient that depends upon the nature of the link, and which we call the *coefficient of resistance.*

The tension will be expressed by:

$$T = \varepsilon \lambda . \tag{A}$$

When λ is negative, T will express a compression.

When the link is a homogeneous prism of section ω, if E represents the modulus of elasticity that corresponds to the matter that the prism is composed of, then one will have:

$$\varepsilon = \frac{E\omega}{l},$$

and in that case, the tension is expressed by:

$$T = E\omega \cdot \frac{\lambda}{l}.$$

If the product $E\omega$ varies from one section of the link to the other then one will have:

$$\varepsilon = \frac{1}{\int_0^l \frac{dl}{E\omega}}.$$

In order to lengthen or shorten a link, one must overcome a resistance that corresponds to a work. Hence, for an elementary variation $d\alpha$ that corresponds to an absolute variation of the link α, the work that is done will be:

$$\varepsilon\, \alpha\, d\alpha.$$

Consequently, the total work that is done in order to produce the total variation λ will be:

$$\text{Work} = \tfrac{1}{2}\,\varepsilon\,\lambda^2. \tag{B}$$

That is the expression for the internal work that is concentrated into the link when it is in equilibrium with external forces, and which will remain in the *latent* state as long as that equilibrium persists. We shall let Θ denote the latent work that corresponds to an arbitrary link *l*.

Upon substituting the value of λ that is deduced from equation (A) into equation (B), one will have this new expression:

$$\Theta = \frac{1}{2} \cdot \frac{1}{\varepsilon}\, T^2. \tag{C}$$

If one lets the symbol Σ denote the sum of the quantities that it pertains to when it is extended over all the system then the total latent work that is developed under the action of the external forces will be:

$$\sum \Theta = \tfrac{1}{2} \sum \varepsilon\,\lambda^2 = \tfrac{1}{2} \sum \frac{1}{\varepsilon} T^2. \tag{D}$$

The principle of elasticity whose proof we shall give in due course expresses the idea that the preceding sum is a *minimum*. When all of the links are homogeneous, ε being constant, equation (D) will become:

$$\sum \Theta = \frac{1}{2} \cdot \frac{1}{\varepsilon} \sum T^2,$$

which signifies that in this case the *square* of the tensions will be a *minimum*, which conforms to what VÉNE, PAGANI, and MOSSOTTI said in the case of a distribution of pressures that are produced by a weight.

When a system is in equilibrium under the action of external forces, if one considers (as one will see later) only the relations that must exist between those forces in order for them to be in equilibrium then, except in a number of very restrictive cases, there will generally be an infinitude of ways of imagining the distribution of internal forces that all satisfy the equilibrium conditions with the external forces. However, if the constitution of the system is given then only one well-defined distribution of the internal forces can exist. It is the one that corresponds to a minimum of latent work. In order to express that condition, one observes that the variations of the distribution of the internal forces are the result of the corresponding variations in the values of λ. Hence, if one denotes the variations of λ, T, and Θ by the letter δ then one will have:

$$\delta\sum\Theta = \sum\varepsilon\lambda\,\delta\lambda = \sum\frac{1}{\varepsilon}T\,\delta T\,.$$

In order to express the minimum, one poses the equation:

$$\sum\varepsilon\lambda\,\delta\lambda = \sum\frac{1}{\varepsilon}T\,\delta T = 0, \tag{E}$$

which is the expression for the principle of elasticity. By analogy, I call equation (E) the *equation of elasticity*.

Before giving the general proof, we shall verify it in several special cases that will, at the same time, show one how it must be applied and the significance that the subsidiary equation that one deduces from them must have.

II. – First problem.

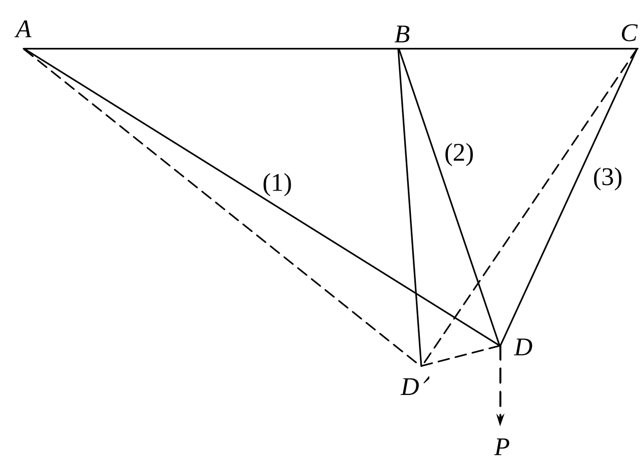

Figure 1.

Three elastic cords (1), (2), (3) (Fig. 1) *are fixed at one of their extremities at the points A, B, C, respectively, which are located along a straight line, and those cords converge at a point D. Determine the tensions* T_1, T_2, T_3 *in the three cords that result from applying a force P that lies in the plane ABCD to the point D.*

SOLUTION. – Let α be the angle between the direction of the force P and the line $\overline{AC}$. Let φ_1, φ_2, φ_3 be the angles that the cords (1), (2), (3) make with that same line. The equilibrium equations that relate to that point will be:

$$\text{(1)} \qquad \begin{cases} P\cos\alpha = T_1\cos\varphi_1 + T_2\cos\varphi_2 + T_3\cos\varphi_3, \\ P\sin\alpha = T_1\sin\varphi_1 + T_2\sin\varphi_2 + T_3\sin\varphi_3, \end{cases}$$

or rather, since $T = \varepsilon\lambda$:

$$\text{(2)} \qquad \begin{cases} P\cos\alpha = \varepsilon_1\lambda_1\cos\varphi_1 + \varepsilon_2\lambda_2\cos\varphi_2 + \varepsilon_3\lambda_3\cos\varphi_3, \\ P\sin\alpha = \varepsilon_1\lambda_1\sin\varphi_1 + \varepsilon_2\lambda_2\sin\varphi_2 + \varepsilon_3\lambda_3\sin\varphi_3 . \end{cases}$$

The quantities to determine – viz., λ_1, λ_2, λ_3 – are three in number, while we have only two equations. In order to find a third equation, consider the point D', which we assume is the one at which the three cords concur after the force P is applied. The point D' will be transported to D by the effect of the latter, and the cords will have consequently changed direction. Let k denote the distance that separates the two points D' and D before and after the displacement that we suppose is always very small. Let θ be the angle between the line $\overline{DD'}$ and the line $\overline{AC}$. One will reasonably have:

$$\text{(3)} \qquad \lambda_1 = k\cos(\varphi_1 - \theta), \qquad \lambda_2 = k\cos(\varphi_2 - \theta), \qquad \lambda_3 = k\cos(\varphi_3 - \theta).$$

The elimination of k and θ from these three equations will give:

$$\text{(4)} \qquad \lambda_1\sin(\varphi_1 - \theta) + \lambda_2\sin(\varphi_2 - \theta) + \lambda_3\sin(\varphi_3 - \theta) = 0,$$

which is an equation that will serve to determine the elongations λ_1, λ_2, λ_3 when it is combined with the preceding ones, and as a result, the corresponding tensions.

An analogous process will lead to the solution of the problem in the case of a larger number of cords. If the strings are not in the same plane, but form the edges of a pyramid, then one will have three equilibrium equations for the point D. If one then lets (α, β, γ) denote the angles that one of the makes cords with the three orthogonal axes and lets (φ, ψ, θ) denote the angles that the line $\overline{DD'}$ makes with those axes then one will have:

$$\lambda_1 = k\cdot\{\cos\alpha_1\cos\varphi + \cos\beta_1\cos\psi + \cos\gamma_1\cos\theta\},$$
$$\lambda_1 = \text{etc. } \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Upon eliminating k, φ, ψ, θ from the latter equations, while keeping in mind the relation $\cos^2 \varphi + \cos^2 \psi + \cos^2 \theta = 1$, one will have a number of new relations between λ_1, λ_2, … that is equal to the number of *cords, minus three.* That will provide the complete solution to the problem.

Now apply the *principle of elasticity:* In order to do that, recall equations (1). Since there are only two equations between the given force P and the three tensions T_1, T_2, T_3, that will amount to saying that there are an infinitude of ways of distributing those tensions in order to be in equilibrium with the force P. One must then express the idea that one can vary the tensions without perturbing the equilibrium while P keeps the same value, which is expressed by means of the equations:

$$(5) \qquad \begin{cases} 0 = \delta T_1 \cos\varphi_1 + \delta T_2 \cos\varphi_2 + \delta T_3 \cos\varphi_3, \\ 0 = \delta T_1 \sin\varphi_1 + \delta T_2 \sin\varphi_2 + \delta T_3 \sin\varphi_3, \end{cases}$$

or rather:

$$(6) \qquad \begin{cases} 0 = \varepsilon_1\, \delta\lambda_1 \cos\varphi_1 + \varepsilon_2\, \delta\lambda_2 \cos\varphi_2 + \varepsilon_3\, \delta\lambda_3 \cos\varphi_3, \\ 0 = \varepsilon_1\, \delta\lambda_1 \sin\varphi_1 + \varepsilon_2\, \delta\lambda_2 \sin\varphi_2 + \varepsilon_3\, \delta\lambda_3 \sin\varphi_3\,. \end{cases}$$

The elasticity equation will be:

$$(7) \qquad \varepsilon_1\, \lambda_1\, \delta\lambda_1 + \varepsilon_2\, \lambda_2\, \delta\lambda_2 + \varepsilon_3\, \lambda_3\, \delta\lambda_3 = 0.$$

Those three equations must be true simultaneously. They serve to eliminate *two* of the three variations $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$. The coefficient of the remaining variation must be zero, so one will then have a third equation for the determination of the various values of λ.

We remark that in equations (6) and (7), the coefficients of resistance ε_1, ε_2, ε_3 multiply the variations $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$, in such a way that the final equation that one will obtain upon eliminating the latter quantities from equations (6) and (7) will not contain the coefficients ε, and consequently, it will give only one geometric relation between the elongations λ.

In order to carry out that elimination by a process that indicates more clearly the identity of the two methods, multiply the two equations (6) by the indeterminate coefficients A and B, respectively, add the result to equation (7), and then equate the terms that multiply $\varepsilon_1\, \delta\lambda_1$, $\varepsilon_2\, \delta\lambda_2$, $\varepsilon_3\, \delta\lambda_3$ to zero separately; one will have:

$$(8) \qquad \begin{cases} \lambda_1 + A\cos\varphi_1 + B\sin\varphi_1 = 0, \\ \lambda_2 + A\cos\varphi_2 + B\sin\varphi_2 = 0, \\ \lambda_3 + A\cos\varphi_3 + B\sin\varphi_3 = 0. \end{cases}$$

However, if one represents $k \cos \theta$ by $-A$ and $k \sin \theta$ by $-B$ then equations (3) and (8) will be identities and will consequently lead to the same final equation that was obtained before:

$$(4) \qquad \lambda_1 \sin(\varphi_1 - \theta) + \lambda_2 \sin(\varphi_2 - \theta) + \lambda_3 \sin(\varphi_3 - \theta) = 0.$$

Conversely, one can deduce the elasticity equation (7) from equations (6) and (3). In order to do that, multiply the first equation (6) by k cos θ and the second one by k sin θ and add them together; one will get:

$$0 = \varepsilon_1\, \delta\lambda_1 \cdot k \cos(\varphi_1 - \theta) + \varepsilon_2\, \delta\lambda_2 \cdot k \cos(\varphi_2 - \theta) + \varepsilon_3\, \delta\lambda_3 \cdot k \cos(\varphi_3 - \theta) = 0,$$

and by virtue of equations (3), that will reduce to:

$$\varepsilon_1\, \lambda_1\, \delta\lambda_1 + \varepsilon_2\, \lambda_2\, \delta\lambda_2 + \varepsilon_2\, \lambda_3\, \delta\lambda_3 = 0,$$

which is the elasticity equation.

One proceeds in an analogous manner for an arbitrary number of cords.

III. – Second problem.

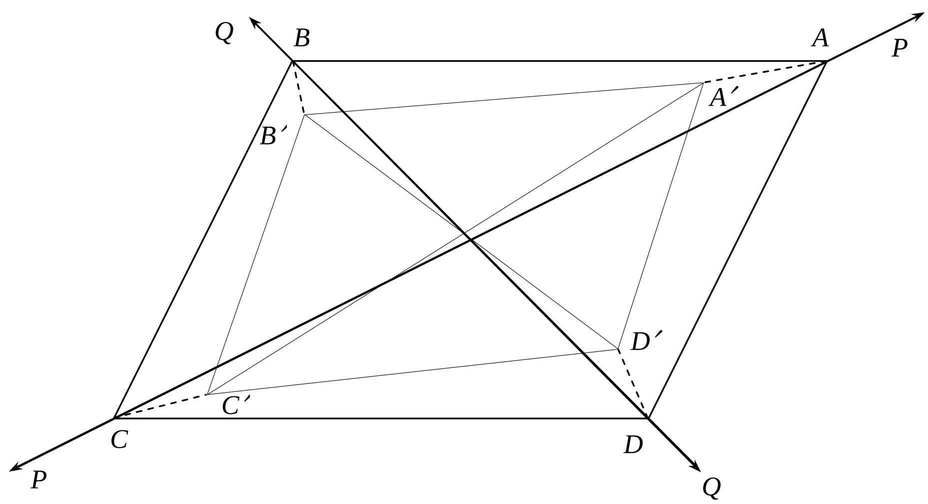

Figure 2.

Let the parallelogram ABCD have sides and diagonals that are composed of elastic rods. Some pair-wise-equal forces with opposite directions are applied to the summits and in the directions of each diagonal. P corresponds to the diagonal $\overline{AC}$*, and Q corresponds to the diagonal* $\overline{BD}$*. Determine the tensions and changes of form that the elements of the quadrilateral suffer under the action of the forces P and Q.*

SOLUTION. – Use the following notations: α, β are the angles that the diagonal $\overline{AC}$ makes with the sides $\overline{AB}$ and $\overline{AD}$, φ, θ are the angles that the diagonal $\overline{BD}$ makes with the sides $\overline{AB}$ and $\overline{BC}$, l_1, l_2 are the lengths of the sides $\overline{AB} = \overline{CD}$ and $\overline{AD} = \overline{BC}$, respectively, l_3, l_4 are the lengths of the sides $\overline{AC}$ and $\overline{BD}$, respectively, and ε_1, ε_2, ε_3, ε_4 are the coefficients of resistance that correspond to l_1, l_2, l_3, l_4, respectively.

Since the forces P and Q are applied simultaneously, and the parallel sides are homogeneous, so the tensions that correspond to the parallel sides will be pair-wise-equal.

Having said that, the equilibrium equations between the external forces and the internal forces will be:

$$(1)\qquad \begin{cases} P = T_1\cos\alpha + T_2\cos\beta + T_3 = \varepsilon_1\lambda_1\cos\alpha + \varepsilon_2\lambda_2\cos\alpha + \varepsilon_3\lambda_3, \\ Q = T_1\cos\varphi + T_2\cos\theta + T_3 = \varepsilon_1\lambda_1\cos\varphi + \varepsilon_2\lambda_2\cos\theta + \varepsilon_4\lambda_4, \end{cases}$$

$$(2)\qquad \begin{cases} T_1\sin\alpha - T_2\sin\beta = \varepsilon_1\lambda_1\sin\alpha - \varepsilon_2\lambda_2\sin\beta = 0, \\ T_1\sin\varphi - T_2\sin\theta = \varepsilon_1\lambda_1\sin\varphi - \varepsilon_2\lambda_2\sin\theta = 0. \end{cases}$$

If one observes that:

$$l_1 \sin\alpha = l_2 \sin\beta \qquad \text{and} \qquad l_1 \sin\varphi = l_2 \sin\theta$$

then the two equations (2) will reduce to just one, namely:

$$(3)\qquad l_1\, T_1 - l_2\, T_2 = \varepsilon_1\, \lambda_1\, l_2 - \varepsilon_1\, \lambda_2\, l_1 = 0.$$

In order to find a fourth equation, examine the displacements that take place in the system. Let: $\overline{A'B'C'D'}$ (Fig. 2) be the original form of the parallelogram after the forces P, Q are applied, let $\overline{AA'} = \overline{CC'} = k_1$, $\overline{BB'} = \overline{DD'} = k_2$ be the lengths of the lines of displacement of the summits, and let ψ_1, ψ_2 be the angles that those lines make with the side $\overline{AB}$. One will have the following relations:

$$(4)\qquad \begin{cases} \lambda_1 = k_1\cos\psi_1 + k_2\cos\psi_2, \\ \lambda_2 = k_1\cos(\alpha+\beta-\psi_1) + k_2\cos(\varphi+\theta-\psi_2), \\ \lambda_3 = 2k_1\cos(\psi_1-\alpha), \\ \lambda_4 = 2k_2\cos(\psi_2-\alpha). \end{cases}$$

Upon observing that one has the relations:

$$(5)\qquad \begin{cases} l_1\sin\alpha = l_2\sin\beta, & l_1\sin\varphi = l_2\sin\theta, \\ l_3\sin\alpha = l_2\sin(\alpha+\beta), & l_4\sin\varphi = l_2\sin(\varphi+\theta), \\ l_3 = l_1\cos\alpha + l_2\cos\beta, & l_4 = l_1\cos\varphi + l_2\cos\theta, \\ \varphi+\theta = \varpi - (\alpha+\beta); & \end{cases}$$

ϖ is the ratio of the circumference to the diameter. One easily eliminates k_1, k_2, ψ_1, ψ_2 from equations (4), and the resultant equation will be:

$$(6)\qquad \lambda_3\, l_3 + \lambda_4\, l_4 - 2\,(\lambda_1\, l_1 + \lambda_2\, l_2) = 0.$$

Combined with equations (1) and (2), it will give the complete solution of the problem.

If the parallelogram is rectangular then one will have:

$$l_3 = l_4\,, \qquad l_1 = l_3 \cos \alpha, \qquad l_2 = l_4 \sin \alpha,$$

and equation (6) will become:

$$\lambda_3 + \lambda_4 - 2\,(\lambda_1 \cos \alpha + \lambda_2 \sin \alpha) = 0;$$

for the square, one will have:

$$\lambda_3 + \lambda_4 - \sqrt{2} \cdot (\lambda_1 + \lambda_2) = 0.$$

In order to apply the principle of elasticity to this problem, we begin by expressing the idea that the distribution of tensions can vary while the equilibrium with the forces P,Q is maintained. Hence, one will deduce from equations (1) and (3) that:

$$(7) \qquad \left\{ \begin{array}{l} \varepsilon_1\,\delta\lambda_1 \cdot \cos\alpha + \varepsilon_2\,\delta\lambda_2 \cdot \cos\beta + \varepsilon_3\,\delta\lambda_3 = 0, \\ \varepsilon_1\,\delta\lambda_1 \cdot \cos\varphi + \varepsilon_2\,\delta\lambda_2 \cdot \cos\theta + \varepsilon_4\,\delta\lambda_4 = 0, \\ \varepsilon_1\,\delta\lambda_1 \cdot l_2 - \varepsilon_2\,\delta\lambda_2 \cdot l_1 = 0. \end{array} \right.$$

The elasticity equation gives:

$$(8) \qquad 2\,\varepsilon_1\,\lambda_1\,\delta\lambda_1 + 2\,\varepsilon_2\,\lambda_2\,\delta\lambda_2 + \varepsilon_3\,\lambda_3\,\delta\lambda_3 + \varepsilon_4\,\lambda_4\,\delta\lambda_4 = 0.$$

Multiply each of equations (7) by A, B, C, respectively, add them together with equation (8), and equate the coefficients of the variations $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$, $\delta\lambda_4$ to zero separately. One will have:

$$(9) \qquad \left\{ \begin{array}{l} A\cos\alpha + B\cos\varphi + C l_2 + 2\lambda_1 = 0, \\ A\cos\beta + B\cos\theta - C l_1 + 2\lambda_2 = 0, \\ \quad A + \lambda_2 = 0, \qquad B + \lambda_4 = 0, \end{array} \right.$$

and easily deduces that:

$$\lambda_3\,(l_1 \cos \alpha + l_2 \cos \beta) + \lambda_4\,(l_1 \cos \varphi + l_1 \cos \theta) - 2\,(\lambda_1\,l_1 + \lambda_2\,l_2) = 0.$$

Upon appealing to the relations (5), which gives:

$$l_3 = l_1 \cos \alpha + l_2 \cos \beta, \qquad l_4 = l_1 \cos \varphi + l_1 \cos \theta,$$

the preceding equation will become:

$$\lambda_3\;l_3 + \lambda_4\,l_4 - 2\,(\lambda_1\,l_1 + \lambda_2\,l_2) = 0,$$

which is identical to equation (6), which was obtained by the direct route.

Upon multiplying equations (4) by $\delta\lambda_1$, $\delta\lambda_2$, $\delta\lambda_3$, $\delta\lambda_4$, one will deduce:

$$(10)\qquad \left\{\begin{aligned} &2\varepsilon_1\,\lambda_1\,\delta\lambda_1+2\varepsilon_2\,\lambda_2\,\delta\lambda_2+\varepsilon_3\,\delta\lambda_3+\varepsilon_4\,\delta\lambda_4 = \\ &2k_1\cos\psi_1\cdot\{\varepsilon_1\,\delta\lambda_1+\varepsilon_2\,\delta\lambda_2\cos(\alpha+\beta)+\varepsilon_3\,\delta\lambda_3\cos\alpha\} \\ &+2k_1\sin\psi_1\cdot\{\varepsilon_2\,\delta\lambda_2\sin(\alpha+\beta)+\varepsilon_3\,\delta\lambda_3\sin\alpha\} \\ &+2k_2\sin\psi_2\cdot\{\varepsilon_1\,\delta\lambda_1+\varepsilon_2\,\delta\lambda_2\cos(\varphi+\theta)+\varepsilon_4\,\delta\lambda_4\cos\varphi\} \\ &+2k_2\sin\psi_2\cdot\{\varepsilon_2\,\delta\lambda_2\sin(\varphi+\theta)+\varepsilon_4\,\delta\lambda_4\sin\varphi\}. \end{aligned}\right.$$

Now, upon combining the first two equations (7) with the following ones:

$$\varepsilon_1\;\delta\lambda_1\cdot\sin\alpha-\varepsilon_2\;\delta\lambda_2\cdot\sin\beta=0,$$
$$\varepsilon_1\;\delta\lambda_1\cdot\sin\varphi-\varepsilon_2\;\delta\lambda_2\cdot\sin\theta=0,$$

which one deduces from equations (2), it will be easy to see that the coefficient of k_1 and k_2 in the left-hand side of equation (10) are zero. Consequently, that equation will reduce to:

$$2\varepsilon_1\,\lambda_1\delta\lambda_1+2\varepsilon_2\,\lambda_2\delta\lambda_2+\varepsilon_3\,\delta\lambda_3+\varepsilon_4\,\delta\lambda_4=0,$$

which is the elasticity equation.

One determines the values of λ by means of equations (1), (3), and (6), which have degree one. Upon letting l' denote the original values of l before the forces P, Q are applied, one will have:

$$l'=l-\lambda,$$

in general.

In the case where the parallelogram becomes a square that is composed of homogeneous rods, one will have:

$$\varepsilon_1=\varepsilon_2=\varepsilon_3=\varepsilon_4\,,\qquad l_1=l_2\,,\qquad l_3=l_4\,,$$

and as a result, one will find that:

$$\lambda_1=\lambda_2=\frac{1}{\varepsilon}\cdot\frac{P+Q}{4\cdot\sqrt{2}}\,,$$

$$\lambda_3=\frac{1}{\varepsilon}\cdot\frac{3P-Q}{4}\,,$$

$$\lambda_4=\frac{1}{\varepsilon}\cdot\frac{3Q-P}{4}\,.$$

These values of λ tell one about the tensions and changes in form of the elements of the square, including the diagonals.

When $P = Q$, one has:

$$\lambda_1 = \lambda_2 = \frac{1}{\varepsilon} \cdot \frac{P}{2 \cdot \sqrt{2}},$$

$$\lambda_3 = \lambda_4 = \frac{1}{\varepsilon} \cdot \frac{P}{2}.$$

If $P = 3Q$ then the tension in the diagonal l_4 is zero then one will have:

$$\lambda_1 = \lambda_2 = \frac{1}{\varepsilon} \cdot \frac{P}{3 \cdot \sqrt{2}},$$

$$\lambda_3 = \frac{1}{\varepsilon} \cdot \frac{2 \cdot P}{3},$$

$$\lambda_4 = 0, \ldots$$

IV. – Third problem.

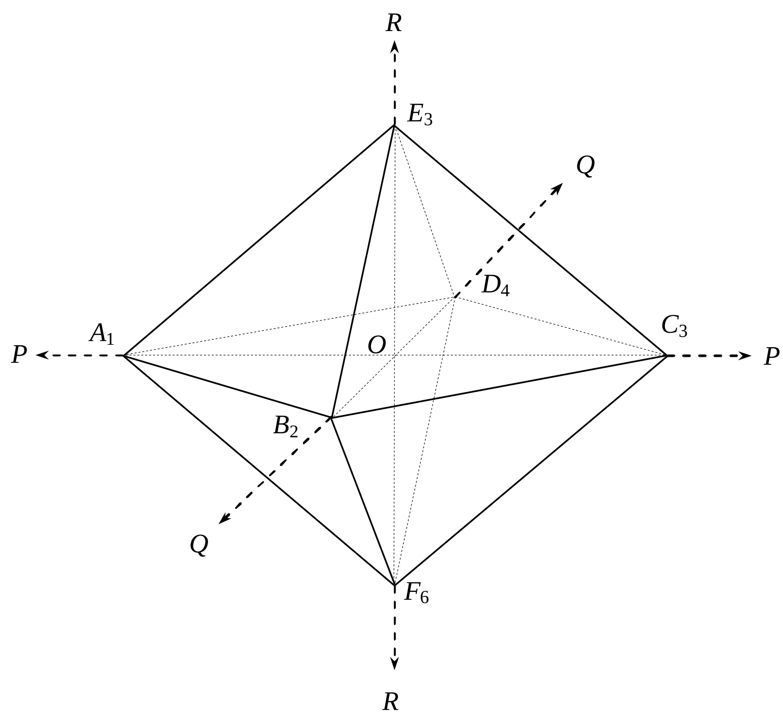

Figure 3.

If one is given a regular octahedron whose edges and diagonals are composed of elastic rods then determine the tensions that come about under the action of forces P, Q, R that are applied pair-wise to each summit and in opposite sense along the directions of the respective diagonals.

SOLUTION. – We denote the summits of the octahedron by the numbers 1, 2, 3, 4, 5, 6. The forces will be applied as follows: *P*, to the summits 1, 3; *Q*, to 2, 4; *R*, to 5, 6. The diagonals that correspond to those summits are taken pair-wise. The tensions in the sides of each square, which are the resultants of the diagonal sections of the octahedron, are obviously equal, so one will have the following three equations for equilibrium:

$$
(1) \qquad \left\{
\begin{aligned}
P &= T_{(1,3)} + \sqrt{2}\,(T_{(1,2)} + T_{(1,5)}) \\
&= \varepsilon_{(1,3)}\,\lambda_{(1,3)} + \sqrt{2}\,(\varepsilon_{(1,2)}\,\lambda_{(1,3)} + \varepsilon_{(1,5)}\,\lambda_{(1,5)}), \\
Q &= T_{(2,4)} + \sqrt{2}\,(T_{(1,2)} + T_{(2,5)}) \\
&= \varepsilon_{(2,4)}\,\lambda_{(2,4)} + \sqrt{2}\,(\varepsilon_{(1,2)}\,\lambda_{(1,3)} + \varepsilon_{(2,5)}\,\lambda_{(2,5)}), \\
R &= T_{(5,6)} + \sqrt{2}\,(T_{(1,5)} + T_{(2,5)}) \\
&= \varepsilon_{(5,6)}\,\lambda_{(5,6)} + \sqrt{2}\,(\varepsilon_{(1,5)}\,\lambda_{(1,5)} + \varepsilon_{(2,5)}\,\lambda_{(2,5)}).
\end{aligned}
\right.
$$

One has taken the preceding observation into account in these equations, and by virtue of that observation, one will have:

$$
(2) \qquad \left\{
\begin{aligned}
T_{(1,2)} &= T_{(2,3)} = T_{(3,4)} = T_{(1,4)}, \\
T_{(1,5)} &= T_{(3,5)} = T_{(3,6)} = T_{(1,6)}, \\
T_{(2,5)} &= T_{(5,4)} = T_{(4,6)} = T_{(2,6)}.
\end{aligned}
\right.
$$

Upon following the path that was traced out in § III, one will then get the following equations for the geometric conditions that relate to each of the squares that result from the diagonal sections:

$$
(3) \qquad \left\{
\begin{aligned}
\lambda_{(1,3)} + \lambda_{(2,4)} &= \frac{1}{\sqrt{2}} \cdot (\lambda_{(1,2)} + \lambda_{(2,3)} + \lambda_{(3,4)} + \lambda_{(4,1)}), \\
\lambda_{(1,3)} + \lambda_{(5,6)} &= \frac{1}{\sqrt{2}} \cdot (\lambda_{(1,5)} + \lambda_{(5,3)} + \lambda_{(3,6)} + \lambda_{(6,1)}), \\
\lambda_{(5,6)} + \lambda_{(2,4)} &= \frac{1}{\sqrt{2}} \cdot (\lambda_{(6,2)} + \lambda_{(2,5)} + \lambda_{(5,4)} + \lambda_{(4,5)}).
\end{aligned}
\right.
$$

One transforms the preceding equations by replacing the values of λ with their corresponding expressions T / ε. One will then have three equations, and when they are combined with the other three (1), that will suffice to determine the six unknown tensions, namely, the ones in the three diagonals and the ones in the sides of each of the diagonal squares.

The principle of elasticity likewise leads one to equations (3). The equation of elasticity in the present case can be written in this form:

(4)
$$\begin{cases} T_{(1,2)}\,\delta T_{(1,2)}\cdot\left(\dfrac{1}{\varepsilon_{(1,2)}}+\dfrac{1}{\varepsilon_{(2,3)}}+\dfrac{1}{\varepsilon_{(3,4)}}+\dfrac{1}{\varepsilon_{(4,1)}}\right) \\ +T_{(1,5)}\,\delta T_{(1,5)}\cdot\left(\dfrac{1}{\varepsilon_{(1,3)}}+\dfrac{1}{\varepsilon_{(5,3)}}+\dfrac{1}{\varepsilon_{(5,6)}}+\dfrac{1}{\varepsilon_{(6,1)}}\right) \\ +T_{(2,5)}\,\delta T_{(2,5)}\cdot\left(\dfrac{1}{\varepsilon_{(2,5)}}+\dfrac{1}{\varepsilon_{(5,4)}}+\dfrac{1}{\varepsilon_{(4,6)}}+\dfrac{1}{\varepsilon_{(6,2)}}\right) \\ +\dfrac{1}{\varepsilon_{(1,3)}}\cdot T_{(1,3)}\,\delta T_{(1,3)}+\dfrac{1}{\varepsilon_{(2,4)}}\cdot T_{(2,4)}\,\delta T_{(2,4)}+\dfrac{1}{\varepsilon_{(5,6)}}\cdot T_{(3,6)}\,\delta T_{(5,6)}=0. \end{cases}$$

One will deduce from equations (1) that:

(5)
$$\begin{cases} 0=\delta T_{(1,3)}+\sqrt{2}\,(\delta T_{(1,2)}+\delta T_{(1,5)}), \\ 0=\delta T_{(2,4)}+\sqrt{2}\,(\delta T_{(1,2)}+\delta T_{(2,5)}), \\ 0=\delta T_{(5,6)}+\sqrt{2}\,(\delta T_{(1,5)}+\delta T_{(2,5)}). \end{cases}$$

If one multiplies the last three equations by the undetermined coefficients *A*, *B*, *C*, respectively, and then sums them, along with equation (4), and equates the factors of the independent variations of *T* to zero then one will have:

(6)
$$\begin{cases} \dfrac{1}{\varepsilon_{(1,3)}}\cdot T_{(1,3)}+A=0, \\ \dfrac{1}{\varepsilon_{(2,4)}}\cdot T_{(2,4)}+B=0, \\ \dfrac{1}{\varepsilon_{(5,6)}}\cdot T_{(5,6)}+C=0, \\ T_{(1,2)}\cdot\left(\dfrac{1}{\varepsilon_{(1,5)}}+\dfrac{1}{\varepsilon_{(2,3)}}+\dfrac{1}{\varepsilon_{(3,4)}}+\dfrac{1}{\varepsilon_{(4,1)}}\right)+\sqrt{2}\cdot(A+B)=0, \\ T_{(1,5)}\cdot\left(\dfrac{1}{\varepsilon_{(1,5)}}+\dfrac{1}{\varepsilon_{(5,3)}}+\dfrac{1}{\varepsilon_{(3,4)}}+\dfrac{1}{\varepsilon_{(4,1)}}\right)+\sqrt{2}\cdot(A+C)=0, \\ T_{(2,5)}\cdot\left(\dfrac{1}{\varepsilon_{(2,5)}}+\dfrac{1}{\varepsilon_{(5,4)}}+\dfrac{1}{\varepsilon_{(4,6)}}+\dfrac{1}{\varepsilon_{(6,2)}}\right)+\sqrt{2}\cdot(B+C)=0. \end{cases}$$

Upon substituting the values of *A*, *B*, *C* that are deduced from the first three equations in the last three and replacing the *T*’s with the expressions $\varepsilon\lambda$, one will recover the three equations (3) that are obtained by the direction geometric method.

V. – Fourth problem.

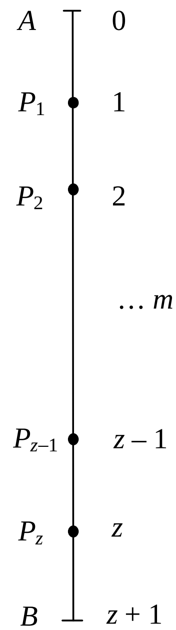

Figure 4.

Being given an elastic rod $\overline{AB}$ (Fig. 4) *that is fixed vertically at its two extremities and loaded with weights* P_1, P_2, …, P_{z-1}, P_z *that are distributed at the various points* 1, 2, …, z *along its length, determine the tensions that exist in each of the elements* $l_{(0,1)}$, $l_{(1,2)}$, $l_{(2,3)}$, etc.

SOLUTION. – The equilibrium conditions for each elements of the rod are expressed by the following equations:

$$(1) \qquad \left\{ \begin{array}{l} T_{0,1} = \varepsilon_{0,1}\,\lambda_{0,1} = P_1 + T_{1,2}, \quad T_{1,2} = \varepsilon_{1,2}\,\lambda_{1,2} = P_2 + T_{2,3}, \\ T_{m,m+1} = \varepsilon_{m,m+1}\,\lambda_{m,m+1} = P_{m+1} + T_{(m+1),(m+2)}, \cdots\cdots\cdots\cdots \\ T_{z-1,z} = \varepsilon_{z-1,z}\,\lambda_{z-1,z} = P_z + T_{z,z+1}. \end{array} \right.$$

The geometric condition of constraint on the system is that the total length *AB* of the rod should not vary; hence, one must have:

$$(2) \qquad \lambda_{0,1} + \lambda_{1,2} + \ldots + \lambda_{z,z+1} = 0.$$

When that equation is combined with the z equations (1), that will suffice to determine the tensions. The principle of elasticity leads immediately to the same result. Indeed, the equation of elasticity, as it applies to the present case is:

$$(3) \qquad \varepsilon_{0,1}\,\lambda_{0,1}\,\delta\lambda_{0,1} + \varepsilon_{1,2}\,\lambda_{1,2}\,\delta\lambda_{1,2} + \ldots + \varepsilon_{z,z+1}\,\lambda_{z,z+1}\,\delta\lambda_{z,z+1} = 0.$$

However, equations (1) give:

(4) $$\varepsilon_{0,1}\ \delta\lambda_{0,1} = \varepsilon_{1,2}\ \delta\lambda_{1,2} = \ldots = \varepsilon_{z,z+1}\ \delta\lambda_{z,z+1}\ ,$$

and when one combines this with equation (3), one will deduce that:

$$\lambda_{0,1} + \lambda_{1,2} + \ldots + \lambda_{z,z+1} = 0,$$

which is a result that is identical to (2).

Upon combining equations (1) and (2), one will easily obtain:

(5) $$\begin{cases} T_{0,1}\cdot\sum_{0,1}^{z,z+1}\cdot\frac{1}{\varepsilon} = P_1\cdot\sum_{0,1}^{z,z+1}\cdot\frac{1}{\varepsilon} + P_2\cdot\sum_{0,1}^{z,z+1}\cdot\frac{1}{\varepsilon} + \cdots + \frac{1}{\varepsilon_{z,z+1}}P_z, \\ T_{1,2}\cdot\sum_{0,1}^{z,z+1}\cdot\frac{1}{\varepsilon} = -\frac{1}{\varepsilon_{0,1}}P_1 + P_2\cdot\sum_{2,3}^{z,z+1}\cdot\frac{1}{\varepsilon} + \cdots + \frac{1}{\varepsilon_{z,z+1}}P_z, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ T_{z,z+1}\cdot\sum_{0,1}^{z,z+1}\cdot\frac{1}{\varepsilon} = -\frac{1}{\varepsilon_{0,1}}P_1 - \left(\frac{1}{\varepsilon_{0,1}} + \frac{1}{\varepsilon_{1,2}}\right)P_2 + \cdots - P_z\left(\frac{1}{\varepsilon_{0,1}} + \frac{1}{\varepsilon_{1,2}} + \cdots + \frac{1}{\varepsilon_{z,z+1}}\right). \end{cases}$$

When the coefficients of resistance and the weights P_1, P_2, …, etc., are respectively equal, the preceding equations will become:

(6) $$\begin{cases} T_{0,1} = \tfrac{1}{2}(z+1)P, \\ T_{1,2} = \tfrac{1}{2}zP, \\ \ldots\ldots\ldots \\ T_{z,z+1} = -\tfrac{1}{2}(z+1)P. \end{cases}$$

If the rod is homogeneous and the weights are distributed uniformly and in a continuous manner then, upon denoting the weight per unit length by p and the total length of the rod by L, one will have:

(7) $$T = \tfrac{1}{2}pL - px$$

for the tension that corresponds to a point at a distance x from the origin A; for:

$$x = \tfrac{1}{2}L, \qquad \text{one will have} \quad T = 0.$$

In the most general case, where the distribution of weights along an inhomogeneous rod is done in a continuous, but non-uniform, manner, the equation of equilibrium that corresponds to a point x will be:

(8) $$dT = -\,p\,dx.$$

Equation (2) gives (*):

(*) See § I.

(9)
$$\sum \cdot \frac{1}{\varepsilon} T = \int_0^L \frac{dx}{E\omega} T \;= 0.$$

Upon integrating equation (8), one will have:

(10)
$$T = -\int_0^z p\, dx \;+ \text{const.}$$

Upon substituting that in equation (9), one will get:

(11)
$$\text{const.} \int_0^L \frac{dx}{E\omega} - \int_0^L \frac{dx}{E\omega} \cdot \int_0^x p\, dx \;= 0.$$

Hence, one deduces the value of the constant, which when substituted in equation (10), will give:

(12)
$$T = -\int_0^x p\, dx + \frac{\displaystyle\int_0^L \frac{dx}{E\omega} \cdot \int_0^x p\, dx}{\displaystyle\int_0^L \frac{dx}{E\omega}}.$$

VI. – Fifth problem.

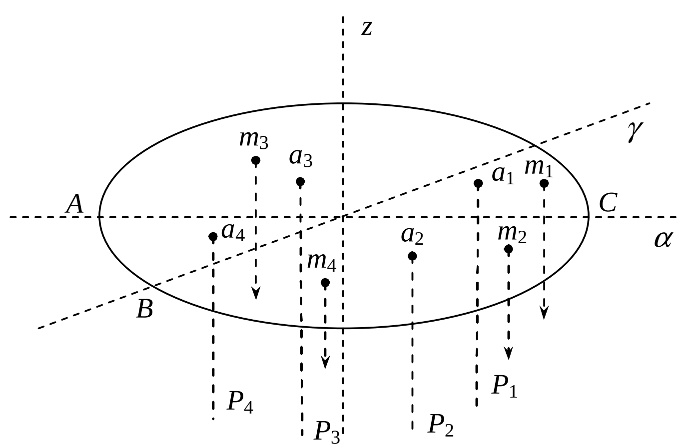

Figure 5.

Given a rigid material plane ABCD (Fig. 5) *that is supported by k vertical elastic columns and loaded with n weights* P_1, P_2, …, P_n *that are distributed in an arbitrary manner, determine the pressures that bear upon the support columns.*

Draw two rectangular axes *ox*, *oy* in the plane and a third orthogonal one *oz*. Let (α_1, β_1), (α_2, β_2), ... be the coordinates of the points of application a_1, b_1, … of the weights P_1,

P_2, …, and let (x_1, y_1), (x_2, y_2), … be the coordinates of the summits of the support columns. The equilibrium conditions give the following equations:

$$(1)\qquad \begin{cases} \sum P \;\; = \sum T \;\; = \sum \varepsilon\lambda, \\ \sum P\alpha = \sum Tx = \sum \varepsilon\lambda x, \\ \sum P\beta = \sum Ty = \sum \varepsilon\lambda y, \end{cases}$$

in which the Σ sign denotes the sum of terms of the same type.

We would like to start with a consideration of the geometric conditions of constraint on the system in order to obtain the other equations that are necessary for the determination of the pressures. We observe that the necessary condition is that the extremities of the columns must be found in the same plane before and after compression. That being the case, suppose that the plane is originally horizontal. λ represents the amount by which the length of one of the columns varies, and will, at the same time, be the ordinate of the corresponding point of the plane in its new position, whose equation will be represented by:

$$(2)\qquad \lambda = Ax + By + C.$$

A, B, C are the constants to be determined. In order to do that, substitute that expression for λ in equations (1) and get:

$$(3)\qquad \begin{cases} \sum P \;\; = A\sum \varepsilon x \;\; + B\sum \varepsilon y \;\; + C\sum \varepsilon, \\ \sum P\alpha = A\sum \varepsilon x^2 + B\sum \varepsilon xy + C\sum \varepsilon x, \\ \sum P\beta = A\sum \varepsilon xy + B\sum \varepsilon y^2 + C\sum \varepsilon y. \end{cases}$$

Those three equations serve to determine A, B, C, and as a result, equation (2) will give the values of λ, so one can then deduce the values of T that correspond to the various columns. In order to simplify the determination of A, B, C, one supposes that weights that are proportional to ε_1, ε_2, … are applied to each of the points (x_1, y_1), (x_2, y_2), … and that one takes the center of gravity of those weights to be the coordinate origin, and one will have:

$$\sum \varepsilon x = 0, \qquad \sum \varepsilon y = 0, \qquad \sum \varepsilon xy = 0.$$

Consequently, upon reducing equations (3), one will deduce that:

$$(4)\qquad A = \frac{\sum P\alpha}{\sum \varepsilon x^2}, \quad B = \frac{\sum P\beta}{\sum \varepsilon y^2}, \quad C = \frac{P}{\sum \varepsilon},$$

and as a result:

$$(5)\qquad \varepsilon\lambda = T = \frac{\sum P\alpha}{\sum \varepsilon x^2} \cdot x + \frac{\sum P\beta}{\sum \varepsilon y^2} \cdot x + \frac{\varepsilon \sum P}{\sum \varepsilon},$$

which is an equation that will give the various values of the pressure by replacing (x, y) with their corresponding values at the various support columns.

That is, in essence, the solution that EULER gave in his paper "De pressione ponderis in planum cui incumbit," and which was then developed by BRESSE in his *Mécanique appliquée*, as I have said before.

The principle of elasticity leads to the same solution immediately. Indeed, one must first express the idea that, in view of the indeterminacy in the problem, if one has only the three equations (1) to determine the pressures then those pressures can vary without changing the external forces – i.e., the weights; that will lead to the following equations:

$$(6) \qquad \sum \delta T = \sum \varepsilon\, \delta\lambda = 0, \qquad \sum x\, \delta T = \sum x\varepsilon \cdot \delta\lambda = 0, \qquad \sum y\, \delta T = \sum y\varepsilon\, \delta\lambda = 0.$$

The principle of elasticity will then yield this further equation:

$$(7) \qquad \sum \cdot \frac{1}{\varepsilon} T\, \delta T = \sum \varepsilon \cdot \lambda\, \delta\lambda = 0.$$

Multiply the three equations (6) by C, A, B, respectively, and then subtract equation (7), and then equate the individual coefficients of the different variations $\delta\lambda$ to zero. One will have the general equation:

$$\lambda = Ax + By + C,$$

which is identical with equation (2), which expresses the geometric condition of constraint of the system.

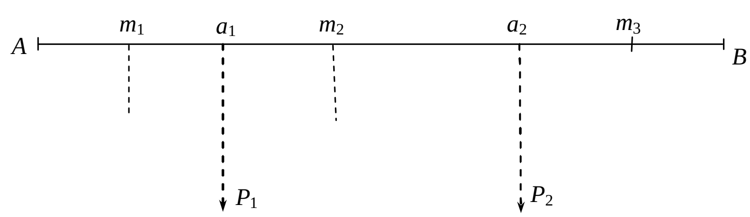

Figure 6.

If the plane reduces to a line AB (Fig. 6) that is loaded with weights P_1, P_2, …, and supported at the points m_1 , m_2 , m_3 , …, then if one sets $\beta = 0$, $y = 0$, formula (5) will give:

$$(8) \qquad T = \frac{\varepsilon \sum P\alpha}{\sum \varepsilon x^2} \cdot x + \frac{\varepsilon \sum P}{\sum \varepsilon},$$

which is an equation that one can use to determine the pressure that acts upon each of the supports.

It is important to not confuse that case with the case of an elastic rod that is supported by some rigid support points that are distributed along a horizontal line, since in the present case, one is, on the contrary, dealing with a line that is supposed to be *rigid* that is placed upon *compressible* support points.

VII. – General proof of the principle in the case of a free system.

Now that we have proved, in the preceding examples, the identity of the results to which one arrives for the determination of the tensions by either appealing directly to the geometric considerations of constraint in the system or by applying the principle of elasticity, we shall now give a direct proof of that principle.

In order to do that, consider the equations of equilibrium that relate to each point of the system. We denote those points by the indices 1, 2, 3, m, n, …, while the distance between two points m, n will be denoted by $l_{m,n}$, the variation of the length will be denoted by $\lambda_{m,n}$, x, y, z will be the coordinates of an arbitrary point, and X, Y, Z will be the components of the *external* forces that are applied to them. At each point, those forces must be in equilibrium with the internal forces that yield the tensions in the various links whose endpoints to which they are applied; one will then have:

$$(1)\quad \left\{\begin{array}{l} X_1 = \sum T_{1,m} \cdot \dfrac{x_m - x_1}{l_{1,m}} = \sum \varepsilon_{1,m}\, \lambda_{1,m} \cdot \dfrac{x_m - x_1}{l_{1,m}}, \\ Y_1 = \sum T_{1,m} \cdot \dfrac{y_m - y_1}{l_{1,m}} = \sum \varepsilon_{1,m}\, \lambda_{1,m} \cdot \dfrac{y_m - y_1}{l_{1,m}}, \\ Z_1 = \sum T_{1,m} \cdot \dfrac{z_m - z_1}{l_{1,m}} = \sum \varepsilon_{1,m}\, \lambda_{1,m} \cdot \dfrac{z_m - z_1}{l_{1,m}}, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ X_p = \sum T_{p,m} \cdot \dfrac{x_m - x_p}{l_{p,m}} = \sum \varepsilon_{p,m}\, \lambda_{p,m} \cdot \dfrac{x_m - x_p}{l_{p,m}}, \\ Y_p = \sum T_{p,m} \cdot \dfrac{y_m - y_p}{l_{p,m}} = \sum \varepsilon_{p,m}\, \lambda_{p,m} \cdot \dfrac{y_m - y_p}{l_{p,m}}, \\ Z_p = \sum T_{p,m} \cdot \dfrac{z_m - z_p}{l_{p,m}} = \sum \varepsilon_{p,m}\, \lambda_{p,m} \cdot \dfrac{z_m - z_p}{l_{p,m}}, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right.$$

If the number of points in the system is equal to n then the number of the preceding equations will be $3n$, which all contain a certain number of tensions that correspond to the links that terminate at each respective point.

Indeed, upon summing equations (1), which correspond to the components X, Y, Z, respectively, one will have:

$$(2)\qquad \sum X = 0, \qquad \sum Y = 0, \qquad \sum Z = 0.$$

Similarly, if one multiplies the X by the y and the Y by the x and then the X by the z and the Z by the x and then the Y by the z and the Z by the y and takes the indicated differences then one will get:

(3) $$\sum (Yx - Xy) = 0, \qquad \sum (Xz - Zx) = 0, \qquad \sum (Zy - Yz) = 0.$$

One will easily arrive at those results by observing that if in the expression for X_1, for example, there is a term $T_{1,3} \frac{x_3 - x_1}{Z_{1,0}}$ then one will find an analogous term $T_{1,3} \frac{x_1 - x_3}{Z_{1,3}}$ in the expression for X_3, and so on for the other ones.

As one knows, equations (2) and (3) represent the conditions that the external forces must satisfy in order to have equilibrium. It is thus proved that equations (1) reduce to $3n - 6$ equations in the external and internal forces.

If one supposes that each point is connected to all of the other points in the system then that number of *links*, and consequently that of the corresponding tensions, will be equal to:

$$\frac{n(n-1)}{2}.$$

Hence, when one has:

$$3n - 6 < \frac{n(n-1)}{2},$$

equations (1) will be insufficient to determine the tensions.

If one has:

$$3n - 6 = \frac{n(n-1)}{2}$$

then one will deduce that:

$$n = \frac{7 \pm 1}{2},$$

namely:

$$n' = 4, \qquad n'' = 3,$$

which correspond to the cases of a tetrahedron and three strings that are fixed at their extremities and converge to the same point, resp. In those cases, the tensions can be determined immediately by considering just the external forces; for more complicated figures, one appeals to other considerations. If all of the forces and links are in the same plane, which will be that of the coordinates, then equations (1) will be $2n$ in number, *three* of which will be independent of the tensions, which can be determined completely by those equations only in the case where one has:

$$2n - 3 < \frac{n(n-1)}{2}.$$

Upon taking:

$$2n - 3 = \frac{n(n-1)}{2},$$

as before, one will find that:

$$n = \frac{5 \pm 1}{2},$$

namely:

$$n' = 3, \qquad n'' = 2,$$

which corresponds to the case of a triangle and a line.

If one of the links – $l_{p,q}$, for example – does not exist then the corresponding tension will be *zero*. Hence, one sets $T_{p,q} = 0$ in the corresponding equations. Some new equilibrium conditions will generally exist between the external forces that are independent of (2) and (3).

In the general case, equations (1) are insufficient in number to determine the tensions in the system, which signifies that one can imagine an infinitude of ways to distribute those tensions, such that they can all satisfy the equilibrium conditions with the external forces.

In order to express that idea, it will suffice to write in equations (1) that the tensions T can vary without that having to be true of the corresponding variations in the components X, Y, Z. As one is dealing with only extremely small variations, one can assume that the directions of the links l will not vary appreciably, despite the variation in their lengths, which will be expressed by:

$$\delta\lambda.$$

Hence, one will deduce from equations (1) that:

$$(4) \qquad \left\{ \begin{array}{l} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots, \\ 0 = \sum \delta T_{pq} \cdot \dfrac{x_q - x_p}{l_{pq}} = \sum \varepsilon_{pq}\, \delta\lambda_{pq} \cdot \dfrac{x_q - x_p}{l_{pq}}, \\ 0 = \sum \delta T_{pq} \cdot \dfrac{y_q - y_p}{l_{pq}} = \sum \varepsilon_{pq}\, \delta\lambda_{pq} \cdot \dfrac{y_q - y_p}{l_{pq}}, \\ 0 = \sum \delta T_{pq} \cdot \dfrac{z_q - z_p}{l_{pq}} = \sum \varepsilon_{pq}\, \delta\lambda_{pq} \cdot \dfrac{z_q - z_p}{l_{pq}}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{array} \right.$$

If one lets α, β, γ denote the variations of the coordinates that come about under the action of external forces, in such a way that before those forces are applied, those coordinates will be:

$$x - \alpha, \qquad y - \beta, \qquad z - \gamma,$$

they one will have:

$$(5) \qquad l_{pq} - \lambda_{pq} = \sqrt{(x_q - \alpha_q - x_p + \alpha_q)^2 + (y_q - \beta_q - y_p + \beta_q)^2 + (z_q - \gamma_q - z_p + \gamma_q)^2}\,,$$

and since:

$$(6) \qquad l_{pq} = \sqrt{(x_q - x_p)^2 + (y_q - y_p)^2 + (z_q - z_p)^2}\,,$$

upon neglecting the powers of α, β, γ, that are greater than one, one will have:

(7) $$\lambda_{pq} = \frac{(\alpha_q - \alpha_p)(x_q - x_p)}{l_{pq}} + \frac{(\beta_q - \beta_p)(y_q - y_p)}{l_{pq}} + \frac{(\gamma_q - \gamma_p)(z_q - z_p)}{l_{pq}},$$

which is the general expression for the various values of λ, … Multiply those values by the corresponding $\varepsilon\,\delta\lambda$, respectively, and sum them. One will get:

(8) $$\sum \varepsilon_{pq}\,\lambda_{pq}\,\delta\lambda_{pq} = \sum \varepsilon_{pq}\,\delta\lambda_{pq}\left\{\frac{(\alpha_q - \alpha_p)(x_q - x_p)}{l_{pq}} + \frac{(\beta_q - \beta_p)(y_q - y_p)}{l_{pq}} + \frac{(\gamma_q - \gamma_p)(z_q - z_p)}{l_{pq}}\right\}.$$

On the other hand, if one adds equations (4) after multiplying them by the corresponding values of α, β, γ, respectively, then it is easy to see that one will have:

(9) $$\sum \varepsilon_{pq}\,\delta\lambda_{pq}\left\{\frac{(\alpha_q - \alpha_p)(x_q - x_p)}{l_{pq}} + \frac{(\beta_q - \beta_p)(y_q - y_p)}{l_{pq}} + \frac{(\gamma_q - \gamma_p)(z_q - z_p)}{l_{pq}}\right\} = 0.$$

That will reduce equation (8) to:

(10) $$\sum \varepsilon_{pq}\,\lambda_{pq}\,\delta\lambda_{pq} = \sum \cdot \frac{1}{\varepsilon_{pq}} T_{pq}\,\delta T_{pq} = 0,$$

which is the *equation of elasticity*, from which one concludes the theorem that we stated at the beginning of this paper, namely:

When an elastic system is in equilibrium under the action of external forces, the internal work that is done by the change of form that is derived from them will be a ***minimum***.

Equations (1) are $3n$ in number, and that will effectively reduce to $3n - 6$ equations between the external and internal forces, which will permit one to eliminate $(3n - 6)$ different values of the $\varepsilon\,\delta\lambda$ or δT in equation (10). Upon equating the remaining coefficients of the other variations $\varepsilon\,\delta\lambda$ to zero, one will get just as many equations in which the coefficients of resistance have disappeared and which will no longer contain the geometric relations between the values of λ and the constraints on the system. The number of those equations, when they are combined with equations (1), will be equal to that of the unknowns, and consequently, the problem of the determination of the tensions will solved completely.

In summary, the geometric relations between the elongations λ and the links express the idea that those links concur at the same points before and after the deformation of the system. One can deduce the change in form when one knows the values of λ, which the result of the action of external forces.

VIII. – Extension of the principle to the case in which the system contains fixed points or rigid components.

Let a, b, c, … be the indices of the fixed points in the system, while P, Q, R, … are the components of the pressures that are exerted on those points, respectively. The equilibrium equations will be:

$$(1)\qquad \left\{\begin{aligned} X_i &= \sum T_{im}\cdot\frac{x_m - x_i}{l_{im}}, \\ Y_i &= \cdots, \\ Z_i &= \cdots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right.$$

$$(2)\qquad \left\{\begin{aligned} X_a &= P_a + \sum T_{am}\cdot\frac{x_m - x_a}{l_{am}}, \\ Y_a &== Q_a + \sum T_{am}\cdot\frac{y_m - y_a}{l_{am}}, \\ Z_a &== R_a + \sum T_{am}\cdot\frac{z_m - z_a}{l_{am}}, \\ X_b &= P_b + \cdots, \\ Y_b &= Q_b + \cdots, \\ Z_b &= R_b + \cdots, \\ &\ldots\ldots\ldots\ldots \end{aligned}\right.$$

Upon proceeding as in § VII, one will have the following equations, which are independent of the internal tensions, but not the pressures on the fixed points:

$$(3)\qquad \left\{\begin{aligned} \sum X - \sum P_a &= 0, \\ \sum Y - \sum Q_a &= 0, \\ \sum Z - \sum R_a &= 0, \end{aligned}\right.$$

$$(4)\qquad \left\{\begin{aligned} \sum (Xy - Yx) - \sum (P_a\, y_a - Q_a\, x_a) &= 0, \\ \sum (Zx - Xz) - \sum (R_a\, x_a - P_a\, z_a) &= 0, \\ \sum (Yz - Zy) - \sum (Q_a\, z_a - R_a\, y_a) &= 0. \end{aligned}\right.$$

If the system contains only one fixed point a then if one takes that point to be the coordinate origin, one will have $x_a = 0$, $y_a = 0$, $z_a = 0$, and one will have the following equations for the condition of equilibrium between the external forces:

$$(5)\qquad \sum (Xy - Yx) = 0, \qquad \sum (Zx - Xz) = 0, \qquad \sum (Yz - Zy) = 0.$$

If the system contains two fixed points a and b then one can take the line that connects them to be the z-axis. x_a , x_b ; y_a , y_b will be zero for those points, and one will have only the following equation, which is independent of the pressures and the tensions:

$$\sum (Xy - Yx) = 0. \tag{6}$$

When the number of fixed points is greater than *two*, and they are not located along a straight line, one will not have any condition equation for equilibrium between the external forces that is independent of the pressures on the fixed points.

Recall the considerations of the preceding § VII and let α_a , β_a , γ_a , … denote the variations of the fixed points a, b, c, …, resp., that come about under the actions of external forces. That being the case, one would like to express the idea that for the given external forces, there will be an infinitude of internal forces that can put them into equilibrium, so one can write:

$$\tag{7} \left\{ \begin{aligned} &0 = \sum \delta T_{mi} \cdot \frac{x_m - x_i}{l_{im}} = \sum \varepsilon_{im}\, \delta\lambda_{im} \cdot \frac{x_m - x_i}{l_{im}}, \\ &0 = \sum \varepsilon_{im}\, \delta\lambda_{im} \cdot \frac{y_m - y_i}{l_{im}}, \\ &0 = \sum \varepsilon_{im}\, \delta\lambda_{im} \cdot \frac{z_m - z_i}{l_{im}}, \\ &\qquad \ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \right. \qquad \left\{ \begin{aligned} &\delta P_a + \sum \varepsilon_{am}\, \delta\lambda_{am} \cdot \frac{x_m - x_a}{l_{im}} = 0, \\ &\delta Q_a + \sum \varepsilon_{am}\, \delta\lambda_{am} \cdot \frac{y_m - y_a}{l_{im}} = 0, \\ &\delta R_a + \sum \varepsilon_{am}\, \delta\lambda_{am} \cdot \frac{z_m - z_a}{l_{im}} = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \right.$$

Upon applying the considerations that led to equations (8), (9), and (10) in § VII, one will have the following equations:

$$\sum \varepsilon_{pq}\, \lambda_{pq} \delta\lambda_{pq} + \sum \cdot \{\alpha_a\, \delta P_a + \beta_a\, \delta Q_a + \gamma_a\, \delta R_a\} = 0, \tag{8}$$

in place of the last of those equations.

If the positions of the fixed points are invariable then one will have $\alpha_a = 0$, $\beta_a = 0$, $\gamma_a = 0$, …, and the pressures will generally remain indeterminate. However, the same thing is not true in real bodies, and those points will effectively displace under the action of external forces. In order to refer to them in a manner that conforms to physical reality more closely, I will call them *stopping points*, in order to distinguish them from *fixed points*, which correspond to the case of rigid bodies.

In order to know the significance of the terms:

$$\alpha_a\, \delta P_a + \beta_a\, \delta Q_a + \gamma_a\, \delta R_a\,,$$

consider a point a at which various columns or elastic strings can end and to which the components P_a , Q_a , R_a are applied. Suppose that the extremities of those strings or columns that correspond to the point a vary only in position, while the other extremities remain fixed; one will have:

$$\lambda_{ai} = \alpha_a \frac{x_i - x_a}{l_{ia}} + \beta_a \frac{y_i - y_a}{l_{ia}} + \gamma_a \frac{z_i - z_a}{l_{ia}}\,, \tag{9}$$

and as a result:

$$\left\{\begin{aligned}
P_a &= \sum \varepsilon_{ia}\,\lambda_{ia}\cdot\frac{x_i - x_a}{l_{ia}} = \sum \varepsilon_{ia}\left\{\alpha_a\cdot\left(\frac{x_i - x_a}{l_{ia}}\right)^2 + \beta_a\cdot\frac{x_i - x_a}{l_{ia}}\frac{y_i - y_a}{l_{ia}} + \gamma_a\cdot\frac{x_i - x_a}{l_{ia}}\frac{z_i - z_a}{l_{ia}}\right\},\\
Q_a &= \sum \varepsilon_{ia}\,\lambda_{ia}\cdot\frac{y_i - y_a}{l_{ia}} = \sum \varepsilon_{ia}\left\{\alpha_a\cdot\frac{x_i - x_a}{l_{ia}}\cdot\frac{y_i - y_a}{l_{ia}} + \beta_a\cdot\left(\frac{y_i - y_a}{l_{ia}}\right)^2 + \gamma_a\cdot\frac{y_i - y_a}{l_{ia}}\frac{z_i - z_a}{l_{ia}}\right\},\\
R_a &= \sum \varepsilon_{ia}\,\lambda_{ia}\cdot\frac{z_i - z_a}{l_{ia}} = \sum \varepsilon_{ia}\left\{\alpha_a\cdot\frac{x_i - x_a}{l_{ia}}\cdot\frac{z_i - z_a}{l_{ia}} + \beta_a\cdot\frac{y_i - y_a}{l_{ia}}\frac{z_i - z_a}{l_{ia}} + \gamma_a\cdot\left(\frac{z_i - z_a}{l_{ia}}\right)^2\right\},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right. \tag{10}$$

which are equations that reduce to the following form:

$$\left\{\begin{aligned}
P_a &= \alpha_a A_a + \beta_a M_a + \gamma_a N_a\,,\\
Q_a &= \beta_a B_a + \alpha_a M_a + \gamma_a O_a\,,\\
R_a &= \gamma_a C_a + \beta_a O_a + \alpha_a N_a\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right. \tag{11}$$

In general, one will then have:

$$\alpha\, \delta P + \beta\, \delta Q + \gamma\, \delta R = \alpha\, \delta\alpha\, A + \beta\, \delta\beta\, B + \gamma\, \delta\gamma\, C + M\, \delta\cdot\alpha\beta + N\, \delta\cdot\, \alpha\gamma + O\, \delta\cdot\, \beta\gamma. \tag{12}$$

Having said that, if one lets S denote the resultant of the three forces P, Q, R, and lets φ, θ, ω denote the angles that the resultant form with the x, y, z axes, resp., then the work that is done by the displacement α, β, γ along the axes of the stopping points will be represented by:

$$\left\{\begin{aligned}
&\tfrac{1}{2} S(\alpha\cos\varphi + \beta\cos\theta + \gamma\cos\omega)\\
&= \tfrac{1}{2}(\alpha^2 A + \beta^2 B + \gamma^2 C + 2\alpha\beta M + 2\beta\gamma O),
\end{aligned}\right\} \tag{13}$$

so

$$\alpha\, \delta P + \beta\, \delta Q + \gamma\, \delta R = \delta\cdot\left[\tfrac{1}{2} S(\alpha\cos\varphi + \beta\cos\theta + \gamma\cos\omega)\right].$$

Hence, one concludes that equation (8) further expresses the idea that the totality of work that is developed by the effect of the external forces, either internally or at the stopping points, is a *minimum*. In order to determine the pressures at the stopping points and the internal tensions, one follows the process that was developed before.

We remark that the expression for the square of the resultant S takes the form:

$$S^2 = \alpha^2 E^2 + \beta^2 F^2 + \gamma^2 G^2 + 2\alpha\beta E^2 + 2\alpha\gamma K^2 + 2\beta\gamma L^2 . \tag{14}$$

Upon changing the direction of the coordinate axes, one can reduce that expression to the form:

$$S^2 = \alpha'^2 A'^2 + \beta'^2 B'^2 + \gamma^2 C'^2, \tag{15}$$

in which α', β', γ' are the new coordinates. If one lets P', Q', R' denote the new components of S then one will have:

$$P' = A'\alpha', \qquad Q' = B'\beta', \qquad R' = C'\gamma', \qquad \ldots \tag{16}$$

If φ', θ', ω' are the angles between the direction of the resultant and the axes then one will have:

$$\cos\varphi' = \frac{A'\alpha'}{S}, \qquad \cos\theta' = \frac{B'\beta'}{S}, \qquad \cos\omega' = \frac{C'\gamma'}{S}. \tag{17}$$

One concludes from this that equation (15) represents an ellipsoid that is referred to its axes, so the resultant of the elastic reaction that comes about from the effect of the displacement of the *stopping point* is directed along the normal to that ellipsoid.

When the displacement takes plane along one of the aces, the elastic reaction will be consequently directed along that axis. The ellipsoid in question is called the *elasticity ellipsoid*, and its axes will be the *elasticity axes.*

The elastic reaction force is the resultant of the three elastic reactions that are calculated along each axis and correspond to the projections of the displacement along the axes.

In my paper entitled "Études sur la théories des vibrations," [Mémoires de l'Académie des Sciences de Turin (2) **15** (1854)], I showed that a point that is restrained by elastic links and put into vibration will execute mutually-independent isochronous vibrations along each of the elasticity axes, and that as a consequence, the effective motion is the resultant of those three motion.

If one supposes that the system that is subject to the action of external forces is rigid and that the only *stopping* points are restrained by elastic links then it will be clear that the internal work will be zero for the rigid system, and that consequently, the term $\sum \varepsilon_{pq}\,\lambda_{pq}\,\delta\lambda_{pq}$ in equation (8) will be equal to zero. That being the case, equation (8) will reduce to (*):

(*) An analogous equation was given by DORNA for a case that corresponds to the one that was treated above in which the elasticity axes of the various *stopping points* have the same direction, respectively. See v. XVIII of the Mémoires de l'Académie des Sciences de Turin, series II.

(18) $$\sum(\alpha\,\delta P+\beta\,\delta Q+\gamma\,\delta R)=0,$$

where, the *indices* have been omitted, for simplicity. Equations (3) and (4), which are independent of the tensions T, which remain indeterminate, will give:

(19) $$\left\{\begin{array}{c}\sum\delta P=0,\quad \sum\delta Q=0,\quad \sum\delta R=0,\quad \sum(y\,\delta P-x\,\delta Q)=0,\\ \sum(x\,\delta R-z\,\delta P)=0,\quad \sum(z\,\delta Q-y\,\delta R)=0.\end{array}\right.$$

Multiply the latter equations (19) by the indeterminate coefficients $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$, $\underline{E}$, $\underline{F}$, sum them with equation (18), and equate the coefficients of δP, δQ, δR to *zero*; one will get:

(20) $$\left\{\begin{array}{l}\alpha+\underline{A}+\underline{D}\,y-\underline{E}\,z=0,\\ \beta+\underline{B}+\underline{F}\,z-\underline{D}\,x=0,\\ \gamma+\underline{C}+\underline{E}\,x-\underline{F}\,y=0.\end{array}\right.$$

One substitutes the expressions for α, β, γ in the expressions for P, Q, R in equation (11). The latter are then substituted in equations (3) and (4). One will then have six equations for the determination of the coefficients $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$, $\underline{E}$, $\underline{F}$, which, in turn, will yield α, β, γ, and as a result P, Q, R.

When the elasticity axes of the various stopping points are directed in the same sense, one takes them to be the directions of the coordinate axes, and the components P, Q, R will then have the form:

(21) $$P=\alpha I,\qquad Q=\beta K,\qquad R=\gamma L.$$

Upon making the substitution in equations (20), (3) and (4), one will have the following equations for the determination of the indeterminate coefficients:

(22) $$\left\{\begin{array}{l}\sum X+\underline{A}\sum I+\underline{D}\sum I\,y-\underline{E}\sum I\,z=0,\\ \sum Y+\underline{B}\sum K+\underline{F}\sum K\,z-\underline{D}\sum K\,x=0,\\ \sum Z+\underline{C}\sum L+\underline{E}\sum L\,x-\underline{F}\sum L\,y=0,\\ \sum(Xy-Yx)+\sum\{I(\underline{A}+\underline{D}y-\underline{E}z)y-K(\underline{B}+\underline{F}z-\underline{D}x)x\}=0,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{array}\right.$$

When one has $I=K=L$ from the constitution of the system, one can simplify the preceding equations by taking the direction of the coordinate axes in such a way that one will have:

(23) $$\left\{\begin{array}{lll}\sum I\,x\;=0, & \sum I\,y\;=0, & \sum I\,z\;=0,\\ \sum I\,xy=0, & \sum I\,zx=0, & \sum I\,yz=0.\end{array}\right.$$

Equations (22) will then reduce to the following ones:

$$
(24) \qquad \left\{
\begin{array}{l}
\sum X + \underline{A} \sum I = 0, \quad \sum Y + \underline{B} \sum I = 0, \quad \sum Z + \underline{C} \sum I = 0, \\
\sum (Xy - Yx) + \underline{D} \sum I(x^2 + y^2) = 0, \\
\sum (Zx - Xz) + \underline{E} \sum I(x^2 + z^2) = 0, \\
\sum (Yz - Zy) \ + \underline{F} \sum I(y^2 + z^2) = 0.
\end{array}
\right.
$$

One deduces from (20), (21), and (24) that:

$$
(25) \qquad \left\{
\begin{array}{l}
P = \dfrac{I \sum X}{\sum I} + \dfrac{I\, y \sum (X\ y - Y\ x)}{\sum I(x^2 + y^2)} - \dfrac{I\, z \sum (Z\ x - X\ z)}{\sum I(x^2 + z^2)}, \\[2ex]
Q = \dfrac{I \sum Y}{\sum I} + \dfrac{I\, z \sum (Y\ z - Z\ y)}{\sum I(x^2 + z^2)} - \dfrac{I\, x \sum (X\ y - Y\ x)}{\sum I(x^2 + y^2)}, \\[2ex]
R = \dfrac{I \sum Z}{\sum I} + \dfrac{I\, x \sum (Z\ x - X\ z)}{\sum I(x^2 + z^2)} - \dfrac{I\, y \sum (Y\ z - Z\ y)}{\sum I(y^2 + z^2)}.
\end{array}
\right.
$$

When all of the stopping points are situated on the same plane, which we take to be the *xy*-plane, and the external forces are all directed along the *z*-axis, equations (25) will reduce to the following ones:

$$
(26) \qquad P = 0, \qquad Q = 0, \qquad R = \frac{I \sum Z}{\sum I} + \frac{I\, x \sum Z\, x'}{\sum I\, x^2} + \frac{I\, y \sum Z\, y'}{\sum I\, y^2}.
$$

(x_1', y_1') is the point of application of the force *Z*.

The latter expression for *R* coincides, in essence, with formula (5), which was obtained as the solution of the fifth problem, § VI; one should note that all of the *z* are zero.

Take the case of an elastic prism with a symmetric section with one extremity that is acted upon by two forces, one of which is directed parallel to the side of the prism and the other of which is perpendicular to it, in such a manner that the plane that passes through those two directions, which will be the *yz*-plane, divides the prism into two symmetric parts in the sense of its length. Suppose, in addition, that a couple whose plane is perpendicular to the direction of the prism tends to produce torsion in it. According to the method that is generally adopted, one considers two consecutive parallel sections of the prism to be kept normal to the curve that the prism will take under the effect of the flexure, and that those same sections turn entirely through the same angle for all of their points under the effect of the torsion. That amounts to saying that one considers the fibers that are contained between two consecutive normal sections as being the only ones to bring equilibrium to the external forces, independently of the others. One then abstracts from the elasticity of the other parts of the prism and considers them to constitute a rigid system.

Having said that, that hypothesis will bring us back to the most general case in which only the *stopping points* are restrained by elastic links, which are the fibers that are found between two consecutive sections of the prism in question. Hence, equations (20), (21), and (22) will contain the solution to the problem that was posed.

If one considers the simplest case, which is the one in which the coefficients of resistance are the same for all directions, then one can apply equations (25) to it, in which the z-axis is taken to be parallel to the edge of the prism and the xy-plane is, consequently, parallel to its normal section. One will then have the following expressions for the components of the tension, in which the product Mm represents the moment of the couple that is expressed by:

$$\sum (Xy - Yx)$$

in equations (24). x', y', z' are the coordinates of the point of application of the forces Y and Z, upon taking the origin to be on the same section that one considers in the prism and such that equations (23) are verified; one further notes that the z are zero:

$$(27) \qquad \left\{ \begin{aligned} P &= \frac{I\,y\,M\,m}{\sum I(x^2+y^2)}, \\ Q &= \frac{I\,Y}{\sum I} - \frac{I\,x\,M\,m}{\sum I(x^2+y^2)}, \\ R &= \frac{I\,Z}{\sum I} - \frac{I\,y\,(Yz'-Zy')}{\sum I\,y^2} + \frac{I\,x\,Z\,x'}{\sum I\,x^2}. \end{aligned} \right.$$

Those formulas contain the solution of the cases that are usually treated in practice on the basis of the established hypothesis.

When the only external force is Y, one will have:

$$(28) \qquad P = 0, \qquad Q = \frac{I\,Y}{\sum I}, \qquad R = -\frac{I\,Y\,z'}{\sum I\,y^2} \cdot y\,.$$

If, in addition to the force Y, there is another force Z whose direction one supposes to pass through the coordinate origin, as it was established before, then equations (27) will give:

$$(29) \qquad P = 0, \qquad Q = \frac{I\,Y}{\sum I}, \qquad R = \frac{I\,Z}{\sum I} - \frac{I\,Y\,z'}{\sum I\,y^2} \cdot y\,.$$

The last expression is generally the one that one employs in practice to calculate the tension that a fiber supports. Indeed, if R' is the tension per unit area of the fiber that one considers then one will have:

$$R' = \frac{R}{\omega}.$$

ω is the section of the fiber whose length between the two consecutive sections of the prism is *l*, and *E* is the modulus of elasticity; hence:

$$I = E\frac{\omega}{l}.$$

Upon substituting and reducing the third equation (29), one will get:

$$R' = -\frac{E\,Y\,z'}{\sum E\omega\,y^2}\cdot y + \frac{E\,Z}{\sum E\omega}, \tag{30}$$

which coincides with the known formula.

Finally, if the external forces reduce to the simple couple *Mm* then equations (27) will give:

$$P = \frac{E\,\omega Mm}{\sum E\omega(x^2+y^2)}\cdot y, \qquad Q = -\frac{E\,\omega Mm}{\sum E\omega(x^2-y^2)}\cdot x, \qquad R = 0, \tag{31}$$

upon introducing the previous value of *I*. Upon denoting the effective tension by *T*, one will have:

$$T = \sqrt{P^2+Q^2} = \frac{E\,\omega Mm}{\sum E\omega(x^2+y^2)}\cdot\sqrt{x^2+y^2}\,. \tag{32}$$

Let *r* denote the distance from the point (*x*, *y*) to the origin. Let φ be the angle between the radius vector *r* and the axis of the origin. Substitute the polar coordinates for the rectangular ones. When one takes:

$$r = f(\varphi),$$

one will have:

$$\omega = r\,dr\cdot d\varphi, \quad r = \sqrt{x^2+y^2}\,.$$

Substitute this into equation (32), and take T' to be the tension per unit area. One will have:

$$T' = \frac{T}{\omega} = \frac{EMm}{\iint Er^3 dr\,d\varphi}\cdot r \tag{33}$$

for the fiber that corresponds to $r\varphi$, which is an expression that coincides with the usual formula that one employs in order to determine the resistance to rupture under torsion.

It is easy to deduce from equations (27) and the following ones the expressions that give the changes of form that the prism experiences under the effect of torsion and flexure, but I shall not stop to do that here. It will suffice that I have proved the generality of the new method that I propose by means of the preceding examples that were analyzed.

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