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Investigations into the problem of quantum electricity

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(First communication)

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Introduction

1. In this treatise, the search for a "wave equation" for electrodynamics will be undertaken. First of all, some long-known facts will be recalled in the first two sections.

In my papers on the theory of matter ([1]), it was shown that one must generally distinguish between two different groups of state quantities, namely, the group of so-called "extensive quantities" and the so-called "intensive quantities." One needs quantities of one or the other kind in order to completely describe the state of a physical system. If the quantities of the one kind are given completely – for example, the intensive quantities – then one can calculate the quantities of the other kind – viz., the extensive quantities – from them with the help of the "world function" Φ by certain partial differentiations ([2]). For example, if one lets the intensive quantities be denoted by the symbols $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, φ (viz., the potentials) and $\mathfrak{e}_1$, $\mathfrak{e}_2$, $\mathfrak{e}_3$, $\mathfrak{b}_1$, $\mathfrak{b}_2$, $\mathfrak{b}_3$ (viz., the field intensities) then Φ will mean a function that depends explicitly upon only those extensive quantities and is, at the same time, an invariant for Lorentz transformations (i.e., a four-dimensional scalar). One then calculates the extensive quantities $\mathfrak{v}_1$, $\mathfrak{v}_2$, $\mathfrak{v}_3$, ρ (current and charge density) and $\mathfrak{d}_1$, $\mathfrak{d}_2$, $\mathfrak{d}_3$, $\mathfrak{h}_1$, $\mathfrak{h}_2$, $\mathfrak{h}_3$ (the field excitations) by the following formulas:

$$(1) \qquad \mathfrak{v}_i = -\frac{\partial \Phi}{\partial \mathfrak{f}_i}, \qquad \rho = \frac{\partial \Phi}{\partial \varphi}, \qquad \mathfrak{d}_i = -\frac{\partial \Phi}{\partial \mathfrak{e}_i}, \qquad \mathfrak{h}_i = \frac{\partial \Phi}{\partial \mathfrak{b}_i}.$$

Conversely, one can also exhibit a "world function" of the extensive quantities, which I would like to denote by $\Psi(\mathfrak{v}, \rho, \mathfrak{d}, \mathfrak{h})$ ([3]), from which, one can arrive at partial

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([1]) G. Mie, Ann. d. Phys. **37** (1912), 515.

([2]) *Loc. cit.*, pp. 524.

([3]) In my previous papers [Ann. d. Phys. **37** (1912), pp. 523 and 524], I used the seemingly-impractical symbol *H*, in place of Φ. We must now reserve *H* for the **Hamilton**ian function, so in this paper, I shall write Ψ.

derivatives of the expression by an entirely analogous process that represents the intensive quantities as functions of the extensive quantities ([1]). The relation ([2]):

$$\Psi = \Phi + (\mathfrak{v} \cdot \mathfrak{f}) - \varphi \cdot \rho + (\mathfrak{e} \cdot \mathfrak{d}) - (\mathfrak{b} \cdot \mathfrak{h})$$

exists between the quantities Φ and Ψ.

2. One can distinguish the same two kinds of state quantities in the mechanics of discrete mass particles. Let the degrees of freedom of the system be denoted by the whole number *f*.

1. The "extensive quantities" are the *f* "coordinates" q_i that determine the configuration of the mechanical system, and the *f* velocity components that one ordinarily denotes by $\dot{q}_i$. The *f* differential equations:

$$\frac{dq_i}{dt} = \dot{q}_i \tag{2}$$

exist between these quantities.

2. The intensive quantities are the *f* forces P_i and the *f* impulses p_i. Likewise, *f* differential equations exist between them:

$$\frac{dp_i}{dt} = P_i \,. \tag{3}$$

The intensive quantities can be calculated as functions of the 2*f* extensive quantities; the causality principle is then satisfied by the 2*f* differential equations (2) and (3). In order to obtain the expressions for the intensive quantities, one must obtain the so-called **Lagrangian** function $L(q_i, \dot{q}_i)$. One calculates from it:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad P_i = \frac{\partial L}{\partial q_i}. \tag{4}$$

One now calculates the **Hamilton**ian function *F* from the **Lagrangian** function *L* as follows:

$$H(q_i, p_i) = -L + \sum_{k=1}^{f} p_k \cdot \dot{q}_k \,, \tag{5}$$

([1]) *Loc. cit.*, pp. 523, equation (9).
([2]) *Loc. cit.*, pp. 525, equation (15).

in which the $\dot{q}_k$ on the right-hand side are replaced with functions of the q_i, p_i – i. e., $\dot{q}_k\,(q_i,\, p_i)$ – that one can obtain from the first group of equations (4). When one notes equations (4), (5) will yield:

$$\begin{aligned} dH(q_i, p_i) &= -\sum P_i \cdot dq_i - \sum p_i \cdot d\dot{q}_i + \sum p_i \cdot d\dot{q}_i + \sum \dot{q}_i \cdot dq_i \\ &= -\sum P_i \cdot dq_i + \sum \dot{q}_i \cdot dp_i \,. \end{aligned}$$

One then has:

$$\frac{\partial H}{\partial q_i} = -P_i\,, \qquad \frac{\partial H}{\partial p_i} = \dot{q}_k\,, \tag{6}$$

and one can now also write the differential equations (2) and (3) as follows:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}\,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}\,. \tag{7}$$

These are **Hamilton**'s differential equations for a mechanical system of discrete massive particles. They are 2*f* differential equations for the 2*f* unknowns: p_i, q_i . The numerical value of the quantity *H* is invariant in the course of time *t*; it represents the "energy" of the system.

Hamilton's differential equations in the physics of the ether

3. In order to arrive at the **Hamilton**ian function for the continuum ether, we next define:

$$H\,(\mathfrak{d},\, \rho,\, \mathfrak{f},\, \mathfrak{d}) = \Phi + (\mathfrak{e} \cdot \mathfrak{d}) - \varphi \cdot \rho, \tag{8}$$

in which $\mathfrak{e}$ and φ are expressed as functions of $\mathfrak{d}$, ρ, $\mathfrak{f}$, $\mathfrak{d}$ on the right-hand side that one obtains:

$$\rho = \frac{\partial \Phi}{\partial \varphi}\,, \qquad \mathfrak{d}_i = -\frac{\partial \Phi}{\partial \mathfrak{e}_i}\,.$$

With consideration of equations (1), that will give:

$$\begin{aligned} dH &= -(\mathfrak{v} \cdot d\mathfrak{f}) + \rho \cdot d\varphi - (\mathfrak{d} \cdot d\mathfrak{e}) + (\mathfrak{h} \cdot d\mathfrak{b}) + (\mathfrak{d} \cdot d\mathfrak{e}) + (\mathfrak{e} \cdot d\mathfrak{d}) - \rho \cdot d\varphi - \varphi \cdot d\rho \\ &= -(\mathfrak{v} \cdot d\mathfrak{f}) + (\mathfrak{h} \cdot d\mathfrak{b}) + (\mathfrak{e} \cdot d\mathfrak{d}) - \varphi \cdot d\rho, \end{aligned}$$

so:

$$\frac{\partial H}{\partial \mathfrak{f}_i} = -\mathfrak{v}_i\,, \quad \frac{\partial H}{\partial \mathfrak{b}_i} = \mathfrak{h}_i\,, \quad \frac{\partial H}{\partial \mathfrak{d}_i} = \mathfrak{e}_i\,, \quad \frac{\partial H}{\partial \rho} = -\rho\,. \tag{9}$$

The fundamental equations of electrodynamics can now be written as follows:

(10a) $$\frac{\partial \mathfrak{d}_1}{\partial t} = \frac{\partial H}{\partial \mathfrak{f}_1} + \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial \mathfrak{b}_3}\right) - \frac{\partial}{\partial z}\left(\frac{\partial H}{\partial \mathfrak{b}_2}\right),$$

(10b) $$\frac{\partial \mathfrak{f}_1}{\partial t} = -\frac{\partial H}{\partial \mathfrak{d}_1} + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial \rho}\right).$$

If we set:

$$\rho = \frac{\partial \mathfrak{d}_1}{\partial x} + \frac{\partial \mathfrak{d}_2}{\partial y} + \frac{\partial \mathfrak{d}_3}{\partial z}, \qquad \mathfrak{b}_1 = \frac{\partial \mathfrak{f}_3}{\partial y} - \frac{\partial \mathfrak{f}_2}{\partial z}, \qquad \text{etc.,}$$

in the right-hand side then we will have six partial differential equations for the six state quantities $\mathfrak{d}_1$, $\mathfrak{d}_2$, $\mathfrak{d}_3$, $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$. The numerical value of H means the spatial density of the energy of the electromagnetic field ([1]).

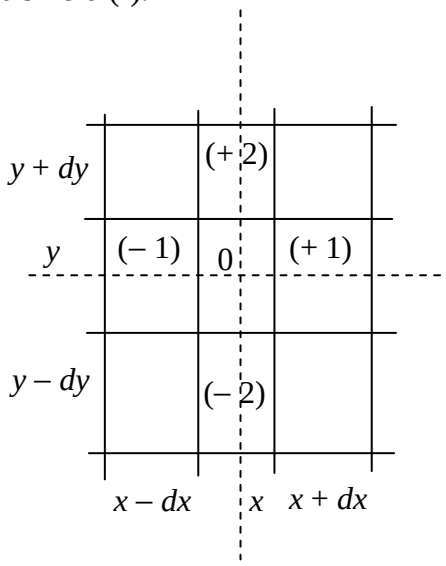

Figure 1.

4. In order to calculate the **Hamilton**ian function for the electromagnetic field itself, we must observe that the state of a continuum can only be described by infinitely many variables. We will first content ourselves with an approximation that involves very many variables, and it will then be easy to make the passage to the limit. We think of the space G that the field inhabits as being divided into very many (viz., infinitely many) parallelepipeds $dx \cdot dy \cdot dz = dG$, in which, we think of the x-axis as being divided into many small *equal* pieces of length dx, and likewise, the y-axis and z-axis into nothing but equal pieces dy (dz, resp.). We can briefly call the coordinate triple (x, y, z) of the midpoint of such a parallelepiped the *number* of the parallelepiped in question, and $\mathfrak{d}(x, y, z)$, $\rho(x, y, z)$, $\mathfrak{f}(x, y, z)$, $\mathfrak{b}(x, y, z)$ will be the state quantities that belong to this parallelepiped – i.e., to this "particle." Here, the coordinates are therefore not counted with the state quantities, as in the case of the mechanics of discrete particles, but they

([1]) **G. Mie**, Ann. d. Phys. **37** (1912), pp. 524, equation (16).

enter in place of the numbers i that are attached to the state quantities q and q: q_i, p_i . They are now nothing but f numbers, which are, however, infinite in number, corresponding to the number of particles in the continuum. For the sake of brevity, we would like to denote the particle (x, y, z) by 0, the particle $(x + dx, y, z)$ and $(x – dx, y, z)$ with 1 (– 1, resp.), $(x, y + dy, z)$ with 2 (– 2, resp.), $(x, y, z + dz)$ and $(x, y, z – dz)$ by 3 (– 3, resp.) (cf., the accompanying Fig. 1). We can then write:

$$\rho_0 = \frac{\mathfrak{d}_x^0 - \mathfrak{d}_x^{(-1)}}{dx} + \frac{\mathfrak{d}_y^0 - \mathfrak{d}_y^{(-2)}}{dy} + \frac{\mathfrak{d}_z^0 - \mathfrak{d}_z^{(-3)}}{dz}, \tag{11}$$

$$\begin{cases} \mathfrak{b}_x^0 = \dfrac{\mathfrak{f}_x^0 - \mathfrak{f}_x^{(-2)}}{dy} - \dfrac{\mathfrak{f}_y^0 - \mathfrak{f}_y^{(-3)}}{dz}, \\ \mathfrak{b}_y^0 = \dfrac{\mathfrak{f}_x^0 - \mathfrak{f}_x^{(-3)}}{dz} - \dfrac{\mathfrak{f}_z^0 - \mathfrak{f}_z^{(-1)}}{dx}, \\ \mathfrak{b}_z^0 = \dfrac{\mathfrak{f}_y^0 - \mathfrak{f}_y^{(-1)}}{dx} - \dfrac{\mathfrak{f}_z^0 - \mathfrak{f}_z^{(-3)}}{dx}, \end{cases} \tag{12}$$

and the **Hamilton**ian function $\mathfrak{H}$ of the system will now be defined by:

$$\int_G H(\mathfrak{d}, \rho, \mathfrak{f}, \mathfrak{b}) \cdot dG = \mathfrak{H}(\mathfrak{d}_l^i, \mathfrak{f}_l^i), \tag{13}$$

where $dG = dx \cdot dy \cdot dz$ and G means the domain that the field inhabits. The index l shall mean the three numbers 1, 2, 3, corresponding to the three coordinate directions x, y, z, while the index i shall run through all of the "particle numbers" (x, y, z), so $\mathfrak{H}$ is actually a function of infinitely many variables. If H depends upon only $\mathfrak{d}$ and $\mathfrak{f}$, but not upon ρ and $\mathfrak{b}$, then $\mathfrak{H}$ will be composed of infinitely many summands, each of which always includes only quantities with a single index $i = (x, y, z)$. However, since ρ also appears as an argument, for example, $\mathfrak{d}_x^0$ will enter into H^0, as well as $H^{(+1)}$, and likewise $\mathfrak{d}_y^0$ will enter into H^0, as well as $H^{(+2)}$, and $\mathfrak{d}_z^0$ will enter into H^0, as well as $H^{(+3)}$, and one can make similar statements about $\mathfrak{f}_x^0$, $\mathfrak{f}_y^0$, $\mathfrak{f}_z^0$, since $\mathfrak{b}_x$, $\mathfrak{b}_y$, $\mathfrak{b}_z$ also appear as arguments. We would now like to define the partial differential quotients of $\mathfrak{H}$ with respect to $\mathfrak{d}_x^0$:

$$\begin{cases} \dfrac{\partial \mathfrak{H}}{\partial \mathfrak{d}_x^0} = \left(\dfrac{\partial H}{\partial \mathfrak{d}_x}\right)^0 \cdot dG + \left(\dfrac{\partial H}{\partial \rho}\right)^0 \cdot \dfrac{1}{dx} \cdot dG - \left(\dfrac{\partial H}{\partial \rho}\right)^{(+1)} \cdot \dfrac{1}{dx} \cdot dG, \\ \dfrac{\partial \mathfrak{H}}{\partial \mathfrak{d}_x^0} = \left\{ \left(\dfrac{\partial H}{\partial \mathfrak{d}_x}\right)^0 - \left(\dfrac{\partial}{\partial x}\left(\dfrac{\partial H}{\partial \rho}\right)\right)^{(+1)} \right\} \cdot dG, \end{cases} \tag{14}$$

in which 0 and (+1) denote the infinitely-close particles with the coordinates (x, y, z) and ($x + dx$, y, z), resp. We would now like to set i equal to more general values than 0 and allow i to assume all values (x, y, z), so one now sees that one can regard equations (10b) as the following system of total differential equations with infinitely many dependent variables:

$$\frac{d\mathfrak{f}_x^i}{dt} = -\frac{\partial \mathfrak{H}}{\partial(\mathfrak{d}_x^i \cdot dG)}, \quad \frac{d\mathfrak{f}_y^i}{dt} = -\frac{\partial \mathfrak{H}}{\partial(\mathfrak{d}_y^i \cdot dG)}, \quad \frac{d\mathfrak{f}_z^i}{dt} = -\frac{\partial \mathfrak{H}}{\partial(\mathfrak{d}_z^i \cdot dG)}. \tag{15}$$

One likewise gets:

$$\begin{cases} \dfrac{\partial \mathfrak{H}}{\partial \mathfrak{f}_x^0} = \left(\dfrac{\partial H}{\partial \mathfrak{f}_x}\right)^0 \cdot dG + \left(\dfrac{\partial H}{\partial \mathfrak{b}_y}\right)^0 \cdot \dfrac{1}{dz} \cdot dG \\ \qquad\qquad - \left(\dfrac{\partial H}{\partial \mathfrak{b}_z}\right)^0 \cdot \dfrac{1}{dy} \cdot dG - \left(\dfrac{\partial H}{\partial \mathfrak{b}_y}\right)^{(+3)} \cdot \dfrac{1}{dz} \cdot dG \\ \qquad\qquad + \left(\dfrac{\partial H}{\partial \mathfrak{b}_z}\right)^{(+2)} \cdot \dfrac{1}{dy} \cdot dG, \\ \dfrac{\partial \mathfrak{H}}{\partial \mathfrak{f}_x^0} = \left\{ \left(\dfrac{\partial H}{\partial \mathfrak{f}_x}\right)^0 + \left(\dfrac{\partial}{\partial y}\left(\dfrac{\partial H}{\partial \mathfrak{b}_y}\right)\right)^{(+2)} - \left(\dfrac{\partial}{\partial z}\left(\dfrac{\partial H}{\partial \mathfrak{b}_y}\right)\right)^{(+3)} \right\} \cdot dG. \end{cases} \tag{16}$$

One can then also write the differential equations (10a) as the following system of total differential equations with infinitely many dependent variables:

$$\frac{d(\mathfrak{d}_x^i \cdot dG)}{dt} = \frac{\partial \mathfrak{H}}{\partial \mathfrak{f}_x^i}, \quad \frac{d(\mathfrak{d}_y^i \cdot dG)}{dt} = \frac{\partial \mathfrak{H}}{\partial \mathfrak{f}_y^i}, \quad \frac{d(\mathfrak{d}_z^i \cdot dG)}{dt} = \frac{\partial \mathfrak{H}}{\partial \mathfrak{f}_z^i}. \tag{17}$$

We have thus brought the fundamental equations of electrodynamics into the form of **Hamilton**'s differential equations. The numerical value of the quantity $\mathfrak{H}$ is also invariant in the course of time here, since it is the total energy of the field. Here, the infinitude of pairs of quantities $(\mathfrak{d}_l^i \cdot dG, \mathfrak{f}_l^i)$ enter in place of the f pairs of associated quantities (q_i, p_i).

The differentiation of a function of a variable-continuum

5. The rules of differentiation that are derived here for a function that depends upon a continuum of variables can also be obtained from the known methods of the calculus of variations. Let F be a function that depends upon not only a certain number of state quantities u, v, w, …, but also upon their derivatives with respect to the variables x, y, z, so for example, upon:

(18) $$u^{hik} = \frac{\partial^h}{\partial x^h} \cdot \frac{\partial^i}{\partial y^i} \cdot \frac{\partial^k}{\partial z^k} u \,.$$

The function of the infinitely-many variables at all space elements of the domain G is then:

(19) $$\mathfrak{F} = \int_G F\,(u, v, w, \ldots, u^{hik}, \ldots, x, y, z) \cdot dG.$$

In order to now define the differential quotients of $\mathfrak{F}$ with respect to the value u_0 that u has at a well-defined point (x, y, z), we let u vary in a *very small* domain ΔG that surrounds the point (x, y, z) in such a way that δu is zero on the boundary of that domain, together with all derivatives with respect to x, y, z. I refer to the following mean value:

$$\delta u^0 = \frac{1}{\Delta G} \cdot \int_{\Delta G} \delta u \cdot dG$$

as the variation δu^0 of the variable u^0. We now calculate the corresponding variation of $\mathfrak{F}$:

$$\delta \mathfrak{F} = \int_G \left(\frac{\partial F}{\partial u} \cdot \delta u + \sum_{h,i,k} \frac{\partial F}{\partial u^{hik}} \cdot \delta u^{hik} \right) \cdot dG.$$

The well-known process of partial integration then yields:

$$\delta \mathfrak{F} = \int_{\Delta G} \left\{ \frac{\partial F}{\partial u} + \sum_{h,i,k} (-1)^{h+i+k} \cdot \frac{\partial}{\partial x^h} \cdot \frac{\partial}{\partial y^i} \cdot \frac{\partial}{\partial z^k} \left(\frac{\partial F}{\partial u^{hik}} \right) \right\}_0 \cdot \delta u \cdot dG.$$

We now let the domain ΔG become small enough that one can set the expression in curly brackets under the integral sign equal to its mean value, which is its value at the point (x, y, z) that is enveloped by the domain ΔG. That now yields:

$$\delta \mathfrak{F} = \left\{ \frac{\partial F}{\partial u} + \sum_{h,i,k} (-1)^{h+i+k} \cdot \frac{\partial}{\partial x^h} \cdot \frac{\partial}{\partial y^i} \cdot \frac{\partial}{\partial z^k} \left(\frac{\partial F}{\partial u^{hik}} \right) \right\}_0 \cdot \delta u_0 \cdot \Delta G.$$

The desired differential quotient with respect to u_0 is now nothing but the quotient $\delta \mathfrak{F} / \delta u_0$, and when we introduce the notation dG_0 for the infinitely-small domain ΔG, and thus set $\Delta G = dG_0$, we will then have:

(20) $$\frac{\partial F}{\partial u_0} = \left\{ \frac{\partial F}{\partial u} + \sum_{h,i,k} (-1)^{h+i+k} \cdot \frac{\partial}{\partial x^h} \cdot \frac{\partial}{\partial y^i} \cdot \frac{\partial}{\partial z^k} \left(\frac{\partial F}{\partial u^{hik}} \right) \right\}_0 \cdot dG_0 \,.$$

It is easy to see that equation (20) is only a generalization of formulas (14) and (16).

In the sequel, we will often have to deal with even more general forms for functions of infinitely-many variables. For example, such a function can be envisioned as follows: Let a sequence of quantities $\mathcal{S}_1$, $\mathcal{S}_2$, …, $\mathcal{S}_n$ be defined by the well-defined integrals:

$$(21)\qquad \begin{cases} \mathcal{S}_1 = \int_G S_1(u, v, \ldots, u^{hik}, \ldots, x, y, z)\cdot dG, \\ \mathcal{S}_2 = \int_G S_2(u, v, \ldots, u^{hik}, \ldots, x, y, z)\cdot dG, \\ \quad \ldots \\ \mathcal{S}_n = \int_G S_n(u, v, \ldots, u^{hik}, \ldots, x, y, z)\cdot dG. \end{cases}$$

The following quantity $\mathfrak{F}$ is then a function of infinitely-many quantities u, v, …, that belong to the individual spatial element dG:

$$(22)\qquad \mathfrak{F}(u, v, \ldots) = \mathrm{F}(\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n),$$

in which F should mean any function of a finite number n of variables. The first partial derivatives of $\mathfrak{F}$ with respect to the individual values u now prove to be:

$$(23)\qquad \frac{\partial \mathfrak{F}}{\partial u} = \frac{\partial \mathrm{F}}{\partial \mathcal{S}_1}\cdot\frac{\partial \mathcal{S}_1}{\partial u} + \frac{\partial \mathrm{F}}{\partial \mathcal{S}_2}\cdot\frac{\partial \mathcal{S}_2}{\partial u} + \ldots + \frac{\partial \mathrm{F}}{\partial \mathcal{S}_n}\cdot\frac{\partial \mathcal{S}_n}{\partial u},$$

in which the expressions $\partial\mathcal{S} / \partial u$ are all calculated using formula (20).

This is not the place to develop a general mathematical theory of functions of a variable-continuum. That theory exists already, and was treated thoroughly by **Vito Volterra**, to whose investigations ([1]) we shall only refer here.

The action function

6. We now once more consider a domain G that is extended in such a way that all physical quantities that are necessary for a complete description of a material system that is contained in G can simply be set equal to zero on its boundary in all computations. That condition shall be called the "boundary condition" in what follows, for brevity. Let the potentials $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, φ be given everywhere in the domain G during the time from t_1 to t_2 , and indeed in such a way that **Maxwell**'s equations are fulfilled at all points of the aforementioned four-dimensional continuum. Thus, since $\mathfrak{b} = \mathrm{rot}\,\mathfrak{f}$, $\mathfrak{e} = -\dot{\mathfrak{f}} - \nabla\varphi$, the "world function" $\Phi(\mathfrak{f}, \mathrm{rot}\,\mathfrak{f}, \dot{\mathfrak{f}}, \varphi, \nabla\varphi)$ will be given everywhere in the four-dimensional domain. We now denote:

([1]) **Vito Volterra**, *Leçons sur les Fonctions de Lignes*, Paris, 1913, and furthermore: **P. Lévy**, *Leçons d'analyse fonctionelle*, Paris, 1922.

(24) $$U = \int_{t_1}^{t_2}\int_{G} \Phi\,(\mathfrak{f},\ \mathrm{rot}\,\mathfrak{f},\ \dot{\mathfrak{f}}\ ,\ \varphi,\ \nabla\varphi)\cdot dG\cdot dt$$

as the "quantity of action" of the system and calculate its variation δU, which corresponds to an infinitely-small variation of the quantities $\mathfrak{f}$, φ at all points of the four-dimensional domain considered:

$$\delta U = \int_{t_1}^{t_2}\int_{G} \left\{\left(\frac{\partial\Phi}{\partial\mathfrak{f}} + \mathrm{rot}\left(\frac{\partial\Phi}{\partial(\mathrm{rot}\,\mathfrak{f})}\right) - \frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial\dot{\mathfrak{f}}}\right)\right)\cdot\delta\mathfrak{f} + \left(\frac{\partial\Phi}{\partial\varphi} - \mathrm{div}\left(\frac{\partial\Phi}{\partial(\nabla\varphi)}\right)\right)\cdot\delta\varphi\right\}\cdot dG\cdot dt$$

$$+ \int_{t_1}^{t_2}\int_{G}\left(\mathrm{div}\left[\delta\mathfrak{f}\cdot\frac{\partial\Phi}{\partial(\mathrm{rot}\,\mathfrak{f})}\right] - \frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial\dot{\mathfrak{f}}}\right) + \mathrm{div}\left(\delta\varphi\cdot\frac{\partial\Phi}{\partial(\nabla\varphi)}\right)\right)\cdot dG\cdot dt + \int_{t_1}^{t_2}\int_{G}\frac{\partial}{\partial t}\left(\frac{\partial\Phi}{\partial\dot{\mathfrak{f}}}\cdot\delta\mathfrak{f}\right)\cdot dG\cdot dt\,.$$

Here, and in what follows, a vector whose three components are defined by differentiating Φ with respect to the three components of a vector – e.g., $\mathfrak{f}$, $\mathfrak{rot}\ \mathfrak{f}$, etc. – shall be described as the differential quotient of Φ with respect to the vector – e.g., $\partial\Phi / \partial\mathfrak{f}$, $\partial\Phi / \partial(\mathfrak{rot}\ \mathfrak{f})$, etc. – for the sake of brevity.

Of the three terms that δU is composed of, however, the first one will now vanish, since the factors of $\delta\mathfrak{f}_1$, $\delta\mathfrak{f}_2$, $\delta\mathfrak{f}_3$, $\delta\varphi$ are nothing but the left-hand sides of the first group of **Maxwell**'s equations. Moreover, the second term vanishes as a result of the boundary condition, and in the third one, one can carry out the integration over time:

(25) $$\delta U = \int_{t_1}^{t_2}\int_{G}\delta\Phi\cdot dG\cdot dt \ = \left[\int_{G}\frac{\partial\Phi}{\partial\dot{\mathfrak{f}}}\cdot\delta\mathfrak{f}\cdot dG\right]_{t_2} - \left[\int_{G}\frac{\partial\Phi}{\partial\dot{\mathfrak{f}}}\cdot\delta\mathfrak{f}\cdot dG\right]_{t_1}.$$

We now think of the entire continuum of state quantities $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, φ as depending upon one or more parameters on the four-dimensional domain considered, by whose variation the state quantities will be varied in such a way that **Maxwell**'s equations will always remain fulfilled. Let $\delta\mathfrak{f}$, $\delta\varphi$, $\delta\Phi$ mean the differentials that correspond to an infinitely-small variation of these parameters. We start from a system of initial values for the $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, φ (which we would like to denote by the index 0) up to a system of final values whose symbols might bear no special index. By integrating along any path that leads from the system of initial values to the system of final values, one will get from (25) that:

(26) $$\int_{t_1}^{t_2}\int_{G}\delta\Phi\cdot dG\cdot dt - \int_{t_1}^{t_2}\int_{G}\delta\Phi_0\cdot dG\cdot dt \ = \left[\int_{G}\int_{\mathfrak{f}_0}^{\mathfrak{f}}\frac{\partial\Phi}{\partial\dot{\mathfrak{f}}}\cdot\delta\mathfrak{f}\cdot dG\right]_{t_2} - \left[\int_{G}\int_{\mathfrak{f}_0}^{\mathfrak{f}}\frac{\partial\Phi}{\partial\dot{\mathfrak{f}}}\cdot\delta\mathfrak{f}\cdot dG\right]_{t_1}.$$

Since the left-hand side of this equation is independent of the path of integration, the same thing will also be true for the right-hand side. The two bracketed expressions then depend upon on the initial-value and final-value continua of the ϝ, φ in the space G at the times t_2 (t_1, resp.). One will then have:

$$\left[\int_G\int_{\mathfrak{f}_0}^{\mathfrak{f}} \frac{\partial\Phi}{\partial\mathfrak{f}}\cdot\delta\mathfrak{f}\cdot dG\right]_{t_2} = \mathcal{W}(\mathfrak{f},\,\varphi,\,t_2) - \mathcal{W}(\mathfrak{f}_0,\,\varphi,\,t_2),$$

$$\left[\int_G\int_{\mathfrak{f}_0}^{\mathfrak{f}} \frac{\partial\Phi}{\partial\mathfrak{f}}\cdot\delta\mathfrak{f}\cdot dG\right]_{t_1} = \mathcal{W}(\mathfrak{f},\,\varphi,\,t_1) - \mathcal{W}(\mathfrak{f}_0,\,\varphi,\,t_1),$$

in which $\mathcal{W}$ means a function of the continuum of the ϝ, φ in the spatial domain G at a well-defined time t_2 (t_1, resp.), which can include the time t, in addition. If one goes from the value continuum ϝ, φ to an infinitely-close one ϝ + δϝ, $\varphi + \delta\varphi$ while keeping time t constant then that will yield:

$$\frac{\partial\mathcal{W}}{\partial\mathfrak{f}_i} = \frac{\partial\Phi}{\partial\mathfrak{f}_i}\cdot dG, \qquad\qquad \frac{\partial\mathcal{W}}{\partial\varphi} = 0.$$

It then follows: The value-continuum of φ follows from the function $\mathcal{W}$ in its own right.

In summary, we can write:

(27) $$\int_{t_1}^{t_2}\int_G \Phi\cdot dG\cdot dt = \mathcal{W}(\mathfrak{f},\,t_2) - \mathcal{W}(\mathfrak{f},\,t_1),$$

(28) $$\frac{\partial\mathcal{W}(\mathfrak{f},t)}{\partial\mathfrak{f}_i} = \frac{\partial\Phi}{\partial\mathfrak{f}_i}\cdot dG,$$

(29) $$\frac{d\mathcal{W}}{dt} = \int_G \frac{\partial\mathcal{W}}{\partial\mathfrak{f}}\cdot\frac{d\mathfrak{f}}{dt} + \frac{\partial\mathcal{W}}{\partial t} = \int_G \Phi\cdot dG.$$

Since $\mathfrak{e}_i = -\dfrac{\partial\mathfrak{f}_i}{\partial t} - \dfrac{\partial\varphi}{\partial x_i}$, it then follows from (1) that:

$$\mathfrak{d}_i = -\frac{\partial\Phi}{\partial\mathfrak{e}_i} = \frac{\partial\Phi}{\partial\mathfrak{f}_i},$$

and one can then also write equation (28) as follows:

(28a) $$\frac{\partial \mathcal{W}(\mathfrak{f},t)}{\partial \mathfrak{f}_i} = \mathfrak{d}_i \cdot dG.$$

We will call $\mathcal{W}$ the "action function."

Hamilton's partial differential equation

7. We can write the four components of the last column of the energy-impulse matrix ([1]) for the physics of the ether as follows:

(30) $$\begin{cases} T_{14} = i\cdot(\mathfrak{b}_y\cdot\mathfrak{d}_y - \mathfrak{b}_z\cdot\mathfrak{d}_y + \mathfrak{f}_x\cdot\rho) = i\cdot\left(\operatorname{div}(\mathfrak{f}_x\cdot\mathfrak{d}) - \sum_{a=1}^{3}\frac{\partial \mathfrak{f}_a}{\partial x}\cdot\mathfrak{d}_a\right), \\ T_{24} = i\cdot(\mathfrak{b}_z\cdot\mathfrak{d}_x - \mathfrak{b}_x\cdot\mathfrak{d}_z + \mathfrak{f}_y\cdot\rho) = i\cdot\left(\operatorname{div}(\mathfrak{f}_z\cdot\mathfrak{d}) - \sum_{a=1}^{3}\frac{\partial \mathfrak{f}_a}{\partial z}\cdot\mathfrak{d}_a\right), \\ T_{34} = i\cdot(\mathfrak{b}_x\cdot\mathfrak{d}_y - \mathfrak{b}_y\cdot\mathfrak{d}_x + \mathfrak{f}_x\cdot\rho) = i\cdot\left(\operatorname{div}(\mathfrak{f}_z\cdot\mathfrak{d}) - \sum_{a=1}^{3}\frac{\partial \mathfrak{f}_a}{\partial z}\cdot\mathfrak{d}_a\right), \\ T_{44} = \Phi + (\mathfrak{b}\cdot\mathfrak{d}) - \varphi\cdot\rho) = \Phi - \operatorname{div}(\varphi\cdot\mathfrak{d}) - \sum_{a=1}^{3}\frac{\partial \mathfrak{f}_a}{\partial t}\cdot\mathfrak{d}_a. \end{cases}$$

If we now form the integral over a spatial domain G in which the entire electromagnetic field is contained as a closed system then we will get:

(31) $$\begin{cases} \int_G T_{14}\cdot dG = -i\cdot J_1, \quad \int_G T_{24}\cdot dG = -i\cdot J_2, \\ \int_G T_{34}\cdot dG = -i\cdot J_3, \quad \int_G T_{44}\cdot dG = E, \end{cases}$$

in which J_1, J_2, J_3 mean the components of the total mechanical impulse, and E means the total energy of the field. If we transform to a coordinate system $(\overline{x},\overline{y},\overline{z},i\cdot\overline{t})$ in which $J_1 = J_2 = J_3 = 0$, then the fourth integral will give the rest energy E_0 of the field, and from the principle of relativity, we will now have:

(32) $$E^2 - (J_1^{\,2} + J_2^{\,2} + J_3^{\,2}) = E_0^{\,2}\ .$$

If we substitute the expressions (30) into formula (31) then with the use of the "boundary condition" we will obtain:

([1]) **G. Mie**, Ann. d. Phys. **37** (1912), pp. 525, eq. (16).

$$(33)\qquad \begin{cases} J_1 = \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial x} \cdot \mathfrak{d}_a \cdot dG, \qquad J_2 = \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial y} \cdot \mathfrak{d}_a \cdot dG, \\ J_3 = \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial z} \cdot \mathfrak{d}_a \cdot dG, \qquad E = \int_G \left(\Phi - \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial t} \cdot \mathfrak{d}_a \right) \cdot dG. \end{cases}$$

We substitute the value for $\mathfrak{d}_a \cdot dG$ from (28a) here:

$$(34)\qquad \begin{cases} J_1 = \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial x} \cdot \frac{\partial \mathcal{W}}{\partial \mathfrak{f}_a}, \qquad J_2 = \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial y} \cdot \frac{\partial \mathcal{W}}{\partial \mathfrak{f}_a}, \\ J_3 = \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial z} \cdot \frac{\partial \mathcal{W}}{\partial \mathfrak{f}_a}, \qquad E = \int_G \left(\Phi \cdot dG - \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial t} \cdot \frac{\partial \mathcal{W}}{\partial \mathfrak{f}_a} \right). \end{cases}$$

Here, one must note that, from (20) and (23), the expressions for the partial differential quotients of a function of a variable-continuum always have the factor *dG*, so the three expressions that were found for J_1, J_2, J_3 are also once more proper functions of a continuum of variables. We can evaluate the expression for *E* with the help of formula (29), which yields:

$$(34a)\qquad E = \frac{\partial \mathcal{W}}{\partial t}.$$

Equation (32) now gives:

$$(35)\qquad \left(\frac{\partial \mathcal{W}}{\partial t} \right)^2 - \left\{ \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial x} \cdot \frac{\partial \mathcal{W}}{\partial \mathfrak{f}_a} \right\}^2 - \left\{ \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial y} \cdot \frac{\partial \mathcal{W}}{\partial \mathfrak{f}_a} \right\}^2 - \left\{ \int_G \sum_{a=1}^{3} \frac{\partial \mathfrak{f}_a}{\partial z} \cdot \frac{\partial \mathcal{W}}{\partial \mathfrak{f}_a} \right\}^2 - E_0^{\,2} = 0.$$

This is *Hamilton's partial differential equation of ether physics*, which is a first-order partial differential equation of degree two for the action function $\mathcal{W}$ in the independent variable *t* and the continuum of $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$.

Schrödinger's differential equation

8. In order to have as good as solved the problem of finding the "wave equation" of ether physics, we must simply appeal to the prescription that **Schrödinger** gave ([1]). Without going into the deeper meaning, let it only be said that **Hamilton**'s principle, which has been the basis for the theory up to now, will be completely overturned by the decisive step that we shall now take. The precise formulation of the new principle that will now enter in place of **Hamilton**'s shall be the subject of a later examination.

([1]) **E. Schrödinger**, Ann. d. Phys. **81** (1926), pp. 133, eq. (35).

Following the **Schrödinger** prescription, we replace the symbol $\partial\mathcal{W} / \partial \mathfrak{f}_a$ with the operator $h / 2\pi i \cdot \partial / \partial \mathfrak{f}_a$, and likewise replace $\partial\mathcal{W} / \partial t$ with $h / 2\pi i \cdot \partial / \partial t$, and then apply the operator that thus arises on the left-hand side of (35) to the "probability amplitude" ψ. The quantity ψ, just like $\mathcal{W}$, is a function of time t and the variable-continuum $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$. We thus get the following "differential equation" for $\psi(\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3)$:

$$\frac{\partial^2\psi}{\partial t^2} - \sum_{\alpha=1}^{3} \int_G \sum_{\beta=1}^{3} \frac{\partial \mathfrak{f}_\beta}{\partial x_\alpha} \cdot \frac{\partial}{\partial \mathfrak{f}_\beta} \int_G \sum_{\gamma=1}^{3} \frac{\partial \mathfrak{f}_\gamma}{\partial x_\alpha} \cdot \frac{\partial \psi}{\partial \mathfrak{f}_\gamma} + \frac{4\pi^2}{h^2} \cdot E_0^2 \cdot \psi = 0. \tag{36}$$

This is Schrödinger's differential equation for electricity.

Equation (36) is a proper partial differential equation in a variable-continuum. From (20) and (23), the quantity $\partial\psi / \partial \mathfrak{f}_\gamma$ will then include the factor dG, and therefore $\int_G \sum_{\gamma=1}^{3} \frac{\partial \mathfrak{f}_\gamma}{\partial x_\alpha} \cdot \frac{\partial \psi}{\partial \mathfrak{f}_\gamma}$ will once more be a proper function of a variable-continuum that can be partially differentiated with respect to $\mathfrak{f}_\beta$. Since this differential quotient, in turn, has the factor dG, the entire second term on the left-hand side will also be a proper function of a variable-continuum. If we may then only assume that the necessary convergence conditions are fulfilled then the entire left-hand side of (36) will make reasonable sense. One might indeed say that (36) represents a homogeneous, linear differential equation for ψ that is of order two in the variable t and the continuum of $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$.

Electricity and mechanics

9. In the event that one thinks of the blurring of the field lines that is represented by the probability amplitude ψ as being restricted to the extent that only the location of the corpuscle that excites the field is indistinct (whether it be an electron or a light quantum, which will be regarded as a dipole), but everywhere else the field of that uncertain starting point is determined by the laws of **Maxwell**'s theory, (36) will go over to the mechanical "wave equation" of a single corpuscle that **Schrödinger** treated so thoroughly. Namely, if the location of the corpuscle is denoted by (ξ, η, ζ) then, as a result of the restriction that was made, the variable-continuum of $\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3$ will depend upon the parameters ξ, η, ζ in such a way that every individual $\mathfrak{f}$-quantity that belongs to a spatial element dG with the coordinates (x, y, z) is a function of the quantities $(x - \xi)$, $(y - \eta)$, $(z - \zeta)$ that does not contain time t explicitly. Thus, it is assumed that the velocity of the corpuscle is constant. Since ψ can also be assumed to be a function of the parameters ξ, η, ζ, one will then have:

(37)
$$\begin{cases} \int_G \sum_{\gamma=1}^{3} \frac{\partial \mathfrak{f}_\gamma}{\partial x} \cdot \frac{\partial \psi(\mathfrak{f},t)}{\partial \mathfrak{f}_\gamma} = -\frac{\partial \psi(\xi,\eta,\xi,t)}{\partial \xi}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

On the other hand, one can, as we saw in the explanation for (36), regard the left-hand side of (37), and thus, the quantity $-\partial\psi / \partial\xi$, as well, as a function of the variable-continuum of the ƒ-quantities and time t, so I will suggest that by writing:

(38a)
$$\int_G \sum_{\gamma=1}^{3} \frac{\partial \mathfrak{f}_\gamma}{\partial x} \cdot \frac{\partial \psi(\mathfrak{f},t)}{\partial \mathfrak{f}_\gamma} = -\frac{\partial \psi}{\partial \xi}(\mathfrak{f}, t).$$

In the same way, we further obtain:

(38b)
$$\int_G \sum_{\beta=1}^{3} \frac{\partial \mathfrak{f}_\beta}{\partial x} \cdot \frac{\partial}{\partial \mathfrak{f}_\beta}\left(-\frac{\partial \psi}{\partial \xi}(\mathfrak{f},t)\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial \psi}{\partial \xi}(\xi,\eta,\zeta,t)\right) = \frac{\partial^2 \psi}{\partial \xi^2}.$$

Thus, in the case of the field of a single corpuscle that moves with constant velocity, equation (36) will go to the following equation:

(39)
$$\frac{\partial^2 \psi}{\partial t^2} - \left(\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} + \frac{\partial^2 \psi}{\partial \zeta^2}\right) + \frac{4\pi^2}{h^2} \cdot E_0^{\,2} \cdot \psi = 0.$$

ψ is regarded as a function of (ξ, η, ζ, t) in this. If we substitute the rest mass of an electron for E_0 in (39) then we will get the equation for the "probability wave" that accompanies it; if we set $E_0 = 0$ then we will get the case of light quanta, which is a probability wave that advances at the speed of light.

The case of an electron that moves in a force field can be treated easily. In that case, we must only pose the "boundary condition" in a somewhat different way from what we did in the derivation of the expression (33). Here, we cannot set the state quantities ƒ, φ equal to zero on the boundary of the domain G that surrounds the electron, together with its entire field up to distances at which it can be counted as zero, but we must assign it the value of the "external field" on the boundary. Here, a second restriction upon the fuzziness of the field will be imposed, namely, that the *external* field should be sharp: In addition, as one always ordinarily does in mechanics, we establish that two assumptions are permissible, with no further comment:

1. The domain G that surrounds the corpuscle, together with its field, is small enough that one can regard the potential of the external field as constant on the entire boundary: ƒ(ξ, η, ζ, t), φ (ξ, η, ζ, t).

2. The acceleration of the corpuscle is infinitely small, such that here one can also regard the individual quantities ƒ of the variable continuum as pure functions of ($x - \xi$), (y

– η), (z – ζ) in the calculation of the differential quotients of ψ that contain time t explicitly.

If, for the time being, we denote the outer surface of the domain G by S then we will have:

$$\int_G \operatorname{div}(\mathfrak{f}_x \cdot \mathfrak{d}) \cdot dG = \mathfrak{f}_x(\xi, \eta, \zeta, t) \cdot \int_S (\mathfrak{d} \cdot d\mathfrak{S}) = \varepsilon \cdot \mathfrak{f}_x(\xi, \eta, \zeta, t),$$

in which ε means the charge of the corpuscle, and $\mathfrak{f}_x$ means the potential of the *external field*. The calculation of the other three expressions in (30) is performed similarly, in which, one simply replaces $\mathfrak{f}_x$ with $\mathfrak{f}_y$, $\mathfrak{f}_z$, φ, resp. We thus get the following values, instead of (33):

$$(40)\qquad \begin{cases} J_1 = \int_G \sum_{\alpha=1}^{3} \frac{\partial \mathfrak{f}_\alpha}{\partial x} \cdot \mathfrak{d}_\alpha \cdot dG - \varepsilon \cdot \mathfrak{f}_x(\xi,\eta,\zeta,t), \\ J_2 = \int_G \sum_{\alpha=1}^{3} \frac{\partial \mathfrak{f}_\alpha}{\partial y} \cdot \mathfrak{d}_\alpha \cdot dG - \varepsilon \cdot \mathfrak{f}_y(\xi,\eta,\zeta,t), \\ J_3 = \int_G \sum_{\alpha=1}^{3} \frac{\partial \mathfrak{f}_\alpha}{\partial z} \cdot \mathfrak{d}_\alpha \cdot dG - \varepsilon \cdot \mathfrak{f}_z(\xi,\eta,\zeta,t), \\ E = \int_G \left(\Phi - \sum_{\alpha=1}^{3} \frac{\partial \mathfrak{f}_\alpha}{\partial t} \cdot \mathfrak{d}_\alpha \right) - \varepsilon \cdot \varphi(\xi,\eta,\zeta,t). \end{cases}$$

One now gets the following wave equation for the corpuscle ([1]):

$$(41)\qquad \left(\frac{\partial}{\partial t} - \frac{2\pi i}{h} \cdot \varepsilon \cdot \varphi \right)^2 \psi - \sum_{\alpha=1}^{3} \left(\frac{\partial}{\partial \xi_\alpha} + \frac{2\pi i}{h} \cdot \varepsilon \cdot \mathfrak{f}_\alpha \right)^2 \psi + \frac{4\pi^2}{h^2} \cdot E_0^{\,2} \cdot \psi = 0,$$

in which, one replaces ξ, η, ζ with ξ_1, ξ_2, ξ_3. One gets the form of the wave equation for a single particle that **Schrödinger** employed from equation (41) by simple calculations by neglecting the "relativistic corrections." If, in the case of a complicated atom, one were to exhibit an equation (41) for each of the corpuscles from which it is composed in which the potentials $\mathfrak{f}$, φ would now generally include the coordinates of all the remaining corpuscles as parameters then one would arrive at a single wave equation for the complicated atom by a suitable combination of all of these equations.

([1]) Confer: **W. Gordon**, Zeit. Phys. **40** (1927), 119, eq. (8). In our equation (41), the imaginary unit has the opposite sign, which is naturally completely inconsequential.

Outlook. Gravitational field

10. One comes to a new, especially interesting, problem when one regards the probability amplitude ψ as truly a function of the variable-continuum $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, and posing the problem of finding the simplest integral of the partial differential equation (36) that satisfies certain boundary condition. These conditions are that ψ should remain finite for any field configuration with finite values of the field quantities, and that it should drop off to zero very quickly when the individual field quantities increase to very large values or the field spreads out to a very broad spatial domain. On might expect that such solutions to equation (36) would exist for only certain "eigenvalues" E_0 that would lead to light quanta ($E_0 = 0$), electrons, and atomic nuclei. The problem of the electron and the problem of the corpuscle thus emerge in an entirely new form. However, I do not believe that mathematics has the means on hand for treating such an enormously complicated equation as (36). It will probably be necessary to first give the problem a somewhat clearer form.

Perhaps it might be advisable to also introduce the concepts of general relativity that **A. Einstein** has developed in his theory of gravitation. For that reason, let it be remarked here that the expression (30) and equation (36) that it yields can be extended with no further conditions by adding the gravitational field quantities. One must then introduce the ten components of the gravitational potential $\omega^{\mu\nu}$ ([1]) as intensive quantities and the "excitation of the gravitational field," whose forty components I denoted by $\mathfrak{k}^{l}_{\mu\nu}$ (μ, ν, l = 1, 2, 3, 4) in my previous paper, as the extensive quantities. Of the latter, generally, only the ones with an upper index of 4 will come under consideration, so we would like to set: $\mathfrak{k}^{4}_{\mu\nu} = i \cdot \kappa_{\mu\nu}$. We will then get:

$$
(42)\qquad \left\{\begin{aligned}
T_{14} &= i\cdot\left(\operatorname{div}(\mathfrak{f}_x\cdot\mathfrak{d}) - \sum_{\alpha=1}^{3}\frac{\partial \mathfrak{f}_\alpha}{\partial x}\cdot\mathfrak{d}_\alpha - \sum_{\alpha=1}^{3}\frac{\partial \omega^{\mu\nu}}{\partial x}\cdot\kappa_{\mu\nu}\right),\\
T_{24} &= i\cdot\left(\operatorname{div}(\mathfrak{f}_y\cdot\mathfrak{d}) - \sum_{\alpha=1}^{3}\frac{\partial \mathfrak{f}_\alpha}{\partial y}\cdot\mathfrak{d}_\alpha - \sum_{\alpha=1}^{3}\frac{\partial \omega^{\mu\nu}}{\partial y}\cdot\kappa_{\mu\nu}\right),\\
T_{34} &= i\cdot\left(\operatorname{div}(\mathfrak{f}_z\cdot\mathfrak{d}) - \sum_{\alpha=1}^{3}\frac{\partial \mathfrak{f}_\alpha}{\partial z}\cdot\mathfrak{d}_\alpha - \sum_{\alpha=1}^{3}\frac{\partial \omega^{\mu\nu}}{\partial z}\cdot\kappa_{\mu\nu}\right)\\
T_{44} &= \Phi - \operatorname{div}(\varphi\cdot\mathfrak{d}) - \sum_{\alpha=1}^{3}\frac{\partial \mathfrak{f}_\alpha}{\partial t}\cdot\mathfrak{d}_\alpha - \sum_{\alpha=1}^{3}\frac{\partial \omega^{\mu\nu}}{\partial t}\cdot\kappa_{\mu\nu}
\end{aligned}\right.
$$

for the fourth column of the impulse-energy matrix.

The quantity ψ is now a function of the continuum of $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, the continuum of the $\omega^{\mu\nu}$, and the time t, in addition. If one replaces, not just the $\mathfrak{d}_\alpha$ with the operators $h / 2\pi i \cdot \partial / \partial \mathfrak{f}_\alpha$, but at the same time, the $\kappa_{\mu\nu}$ with the operators $h / 2\pi i \cdot \partial / \partial \omega^{\mu\nu}$, then that will

([1]) The notations used here were used by me in Ann. d. Phys. **69** (1922), 1, *et seq.*, where one can also find more precise definitions.

yield the “partial differential equation” for ψ, extended by the gravitational quantities, all by itself.

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