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Reciprocal figures in graphical statics

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In free translation from the Italian and with some introductory remarks regarding a brief derivation of the theory of reciprocal polyhedra.

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The theory of reciprocal polyhedra is based upon the consideration of a special kind of so-called dual or reciprocal relationship (*), whose most natural derivation is undoubtedly given by specializing the general reciprocal relationship that can, however, also be developed immediately from the consideration of forces in space.

It is known that a system of forces in space can be reduced to a system of two forces (that do not generally lie in a plane) in infinitely many ways; if one of them is given then the other one will be determined uniquely. Likewise: If any condition is given for one of the forces then such a condition will also exist for the other force. The connection between such restricting conditions shall be discussed here as briefly as possible, to the extent that the theory of reciprocal polyhedra demands.

Let a system of forces in space be reduced to two isolated forces P and P_1. They shall be replaced with two other forces Q and Q_1, while the force will be subject to the condition that it must always go through a given point M. The rotational effects of the first two forces P and P_1 and the other forces Q and Q_1 must be equal to each other relative to any point in space, and thus also relative to the point M. Since Q always goes through M, the rotational effect of the other force Q_1 by itself must be equal to that of the two forces P and P_1 . However, since this will always exert the same rotational effect relative to a point, the same thing must also be true for Q_1, which can be the case only when the latter force always lies in a *certain* plane that goes through M.

Therefore, if one of the two forces goes through a point then the other one will lie in a plane that goes through the point. Conversely, let the condition on the force Q be given that it has to lie in a certain plane, so the other force Q_1 will cut that plane in a point, in

(*) *Reye* gave this special kind of relationship the name of "null system."

general. Now, since the rotational effect of each of the forces that lie in the plane relative to each point in space – and therefore also relative to the point of intersection of the other corresponding force with the plane – must be equal to that of the forces P and P_1 relative to this respective point of intersection, although the former rotational effect always acts in the sense of the given plane, the point of intersection of the corresponding force Q_1 with the plane must coincide with a single point, since the rotation of P and P_1 in the sense of that plane can take place only relative to a single point.

Thus, if one force lies in a plane then the other one will go through a point that lies in the plane. If the one force lies in two planes at once – and thus, along a line – then the other force must go through the points that correspond to the two planes, and thus likewise lie on a certain line. Relative to a system of forces that cannot be reduced to two forces that lie in a plane, *any point will correspond to a plane that goes through it, any plane will correspond to a point that lies in it, and any line will correspond to another line.* One also easily sees that the plane that corresponds to a point is also the plane that corresponds to the point, and conversely.

Should a force go through a point and, at the same time, lie in a plane then the other force must lie in the plane that corresponds to the point, as well as also going through the point that corresponds to the plane. The latter must then be a point in the second plane, so the one plane that corresponds to a point will go through the point that corresponds to the (former) plane.

The point that corresponds to a plane will be called the *pole* of that plane, while the latter will be called the *polar plane* of the point. One calls two lines that correspond to each other *conjugate* or *reciprocal*.

A pencil of parallel planes corresponds to the points of a certain line as their poles, and that line will be the conjugate to the line at infinity that is the intersection of the planes. Its point I at infinity will be the pole of the plane at infinity. In this way, all pencils of parallel planes will correspond to the points of all lines that go through I, and will thus be mutually parallel. Among them, one will be perpendicular to the corresponding pencil of planes and will be called the *central axis* of the system.

Any line that is parallel to the central axis will thus have a conjugate at infinity. Any plane that is parallel to the central axis will have its pole at infinity.

If (as is always possible) one lays two parallel planes, one of which is regarded as being given, through any two forces that replace the system – or, what amounts to the same thing, any two conjugate lines – then one will see that its pole (as the point of intersection of the other force with the plane) will lie at infinity. Any pair of parallel planes that is laid through two conjugate lines will then be parallel to the central axis (*). It follows from this that when one projects two conjugate lines that are parallel to the central axis onto a plane that is perpendicular to the aforementioned axis (which is called an orthographic plane), the two projections must be parallel.

If one fixes one's attention upon this kind of projection then one will easily recognize that:

1. Several lines r in space whose projections coincide will correspond to other lines r' whose projection either coincide or are parallel according to whether the lines r (which

(*) This is a theorem whose validity will also emerge immediately from the fact that the lines of intersection of the pairs of parallel planes that are laid through any two mutually-corresponding forces will be parallel, which is not too hard to prove under closer consideration.

must necessarily be contained in a plane that is parallel to the central axis) are or are not parallel.

2. Several lines r in space whose projections are parallel will correspond to other lines r' whose projections will either coincide or be parallel according to whether the lines r (which must necessarily be parallel to one and the same plane that goes through the central axis) are or are not parallel.

One calls two polyhedra *reciprocal* when the vertices of one of them are the poles of the faces of the other one, such that from what was said the faces of the one polyhedron will be subsets of the polar planes to the vertices of the other polyhedron, and likewise, any two respective edges will be conjugate to each other.

Any physical n-vertex in the one will correspond – as a subset of the polar planes of the vertices – to a planar n-vertex in the other polyhedron, and conversely. However, any edge of the physical n-vertex will also belong to another physical multi-vertex. Therefore, any face of the planar n-vertex will belong to another planar multi-vertex. All of these multi-vertices are connected in this way as the face-surfaces of the second polyhedron, which is, by the way, simultaneously inscribed and circumscribed by the first one, since, on the one hand, its vertices are the poles of the faces of the other one, and thus lie in the latter, while on the other hand, its faces will have the vertices of the other polyhedron as poles, and will thus go through all of these vertices.

Now, if one projects two reciprocal polyhedra in the manner described above – i.e., parallel to the central axis onto a plane that is perpendicular to it, namely, an orthographic plane – then the projections will be orthographic figures that are provided with reciprocal properties. Any edge of the one figure will correspond to a parallel in the other one. If n rays emanate from a point in the one figure then the lines that are parallel to these rays will define the faces of a closed n-vertex. The number of nodes in the one figure will be equal to the number of polygons in the other one, and conversely. If the one polyhedron possesses a vertex at infinity then the other one will correspond to a face that is parallel to the central axis or perpendicular to the orthographic plane, and conversely; i.e., if the one orthographic figure has a vertex at infinity then the other one will contain a polygon whose sides fall along a line (which is parallel to the direction in which the vertex at infinity in the first figure lies), and conversely.

If the point at infinity of the central axis is the vertex of a physical n-vertex that belongs to the first polyhedron then the other polyhedron will have an n-vertex that lies in the plane at infinity. This will diminish the number of nodes in the first orthographic figure by one, the number of polygons by n, and likewise, the number of lines by n. In the second one, these numbers will diminish by n, 1, n, respectively, and one will see that in this case, as well, the number of nodes in the one figure will be equal to the number of polygons in the other figure, and conversely, and that the number of lines are likewise equal to each other.

Now, these orthographic figures – which are also called *reciprocal diagrams* – that one obtains as the orthographic projections of two reciprocal polyhedra are also found in graphical statics.

The mechanical property of reciprocal diagrams finds its expression in the following theorem of *Clerk Maxwell* (*):

If one let forces whose magnitudes are represented by lines act upon the endpoints of the corresponding lines of the reciprocal figure then all points of the reciprocal figure will be in equilibrium under the action of these forces.

The truth of this theorem emerges immediately when one remarks that the forces that are attached to any node of the second diagram (**) are parallel and proportional to the edges of a closed polygon in the first diagram. The theorem is useful when one applies it to the graphic determination of the internal forces in a framework of bars.

We will now demonstrate that *the force and funicular polygon can be reduced to two reciprocal diagrams.*

If n forces P_1, P_2, …, P_n are given in a plane (which is always considered to be an orthographic plane), and they are in equilibrium then one will imagine the force polygon to be a polygon whose edges 1, 2, …, *n* are equipollent (***) to the lines that represent the forces. If one takes a point *O* (in the same plane) that one calls the pole of the force polygon, projects the vertices of the polygon from it, and lets (*r s*) denote the ray that projects common vertex of the edges *r* and *s* then one will understand the funicular polygon (which corresponds to the pole *O*) to mean a polygon whose vertices fall upon the lines of action (†) of the forces, and whose edges will be parallel to the rays that go through *O*, respectively (††), such that the edge that lies between the lines of action of the forces P_r and P_s is parallel to the ray *O*(*r s*).

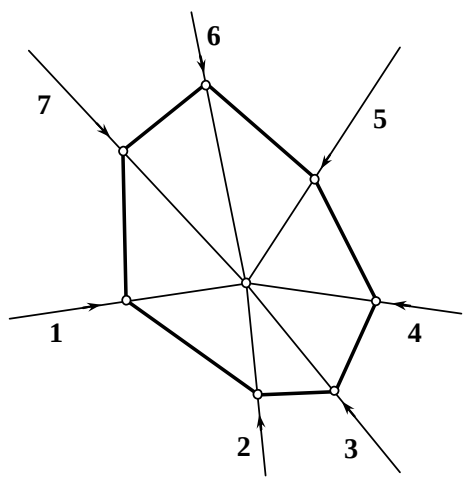

Figure 1.a.

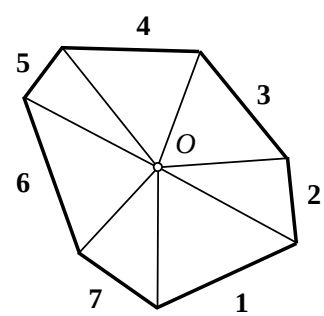

Figure 1.b.

(*) One must actually thank *Clerk Maxwell* for the geometric theory of reciprocal diagrams, which he first derived from the projection of two reciprocal polyhedra, even though these polyhedra were not reciprocal in the sense of *Möbius*'s theory.

(**) The lines that emanate from the nodes will be considered as such.

(***) I.e., equal in magnitude, direction, and sense.

(†) Which will be denoted by bold numerals **1**, **2**, **3**, …, *n*.

(††) The choice of the first vertex on the line of action in question is arbitrary in the construction of this polygon.

The funicular polygon, like the force polygon, must be a closed polygon, as a result of the equilibrium of the given forces.

Now, if the lines of action of the given forces intersect at a point (Fig. 1) then one will easily see that in that way one has already constructed two reciprocal diagrams that are obviously the orthographic projections of two *n*-sided pyramids.

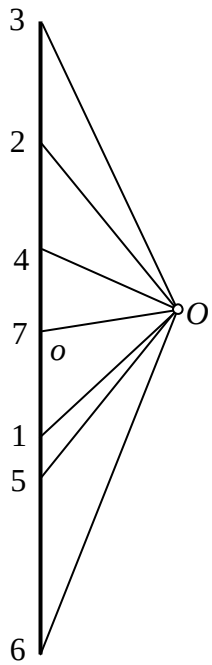

Figure 2.a

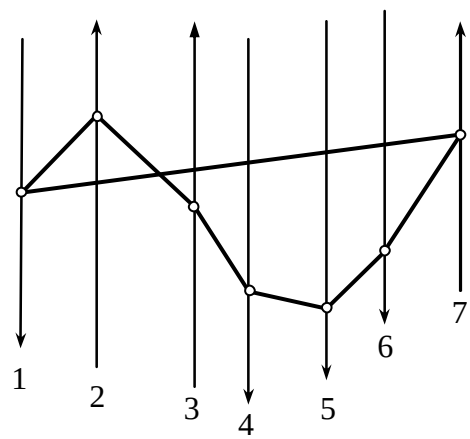

Figure 2.b

If the forces are parallel then the force polygon will reduce to a line. The base of the first pyramid will then be perpendicular to the orthographic plane, and the vertex of the second one will thus lie at infinity; i.e., the second polyhedron will be a prism with a base that is only at finite points. This case is represented by Fig. 2, in which the sides of the force polygon are not merely denoted by *one* numeral **1**, **2**, **3**, …, but by two numerals that are put at the ends of each section, such that sections 01, 12, 23, 34 will correspond to the lines **1**, **2**, **3**, **4**, resp., in the second diagram. (Here, as well as in what follows, two sequences of numbers 1, 2, 3, …, *r*, …, *s* and **1**, **2**, **3**, …, ***r***, …, ***s*** will be used in the text in order to distinguish the lines of one diagram from the corresponding lines in the other one.)

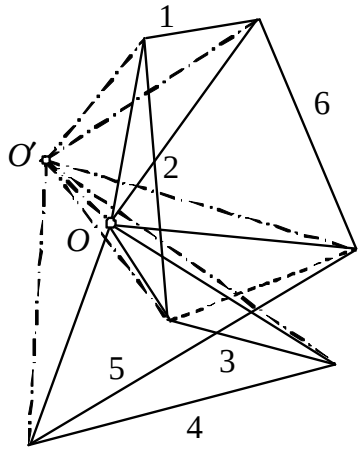

Figure 3.a

We now consider the general case, in which the forces do not intersect at a point. One chooses a second pole *O′*, links it with the vertices of the force polygon by

means of lines, and constructs a second funicular polygon that corresponds to the pole O' whose vertices fall upon the lines action of the forces, and whose sides are parallel to the rays that emanate from O', respectively.

(See Figs. 3 and 4, in which the rays that emanate from the second pole, as well as the corresponding funicular polygon, are drawn with dash-dotted lines.)

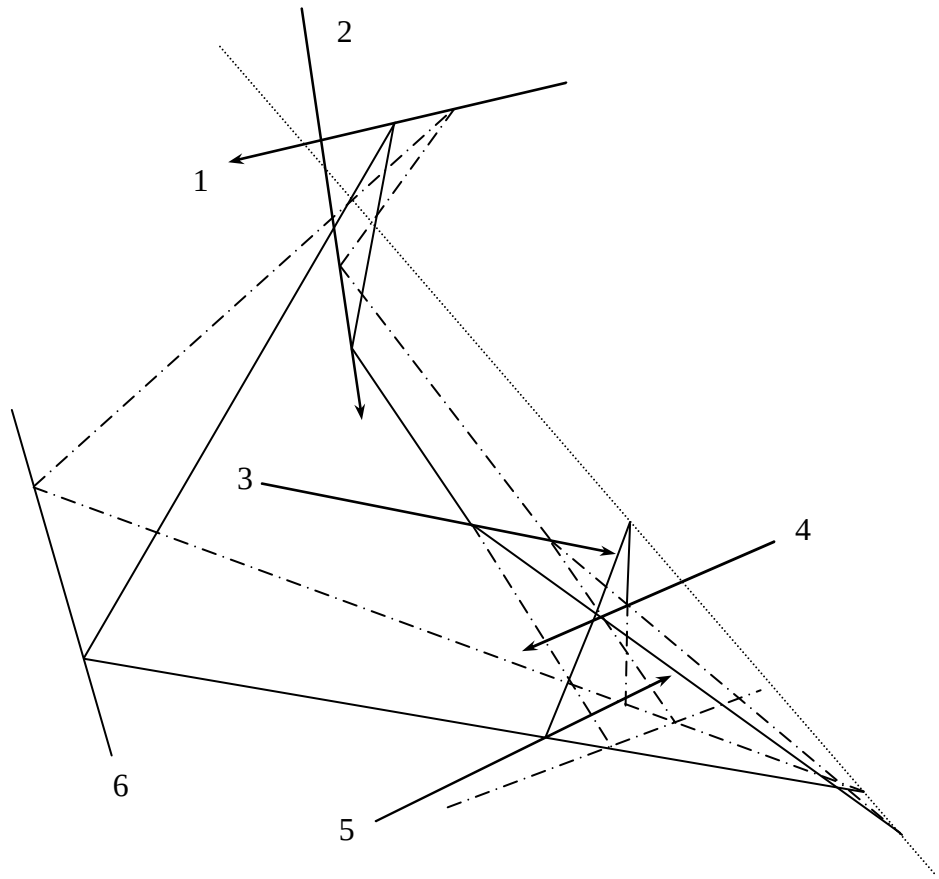

Figure 3.b

The diagram that is constructed in this way from force polygons and the rays that emanate from O and O', as well as the one that is constructed from the lines of action of the forces and the two funicular polygons are obviously reciprocal now. The former is the projection of one of two polyhedra (*) that are constructed from physical n-vertices whose corresponding faces intersect in an aplanar n-sided polygon. The second diagram is the projection of a polyhedron that is bounded by two n-sided planar polygons whose corresponding sides intersect.

The line that connects the vertices of the two physical n-vertices of the first polyhedron is conjugate to the line that is common to the two base planes of the second polyhedron. It follows from this that when one keeps in mind that any two conjugate lines will project as parallels, any two corresponding sides of the two funicular polygons will intersect in a line that is parallel to the connecting line of the two poles O and O'.

(*) This polyhedron has $3n$ edges, $2n$ three-vertex surfaces, two physical n-vertices, and n physical four-vertices. The other one has $3n$ edges, $2n$ physical three-vertices, two n-sided polygons as basic faces, and n four-sided polygons.

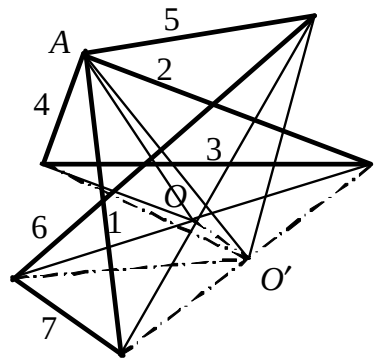

Figure 4.a

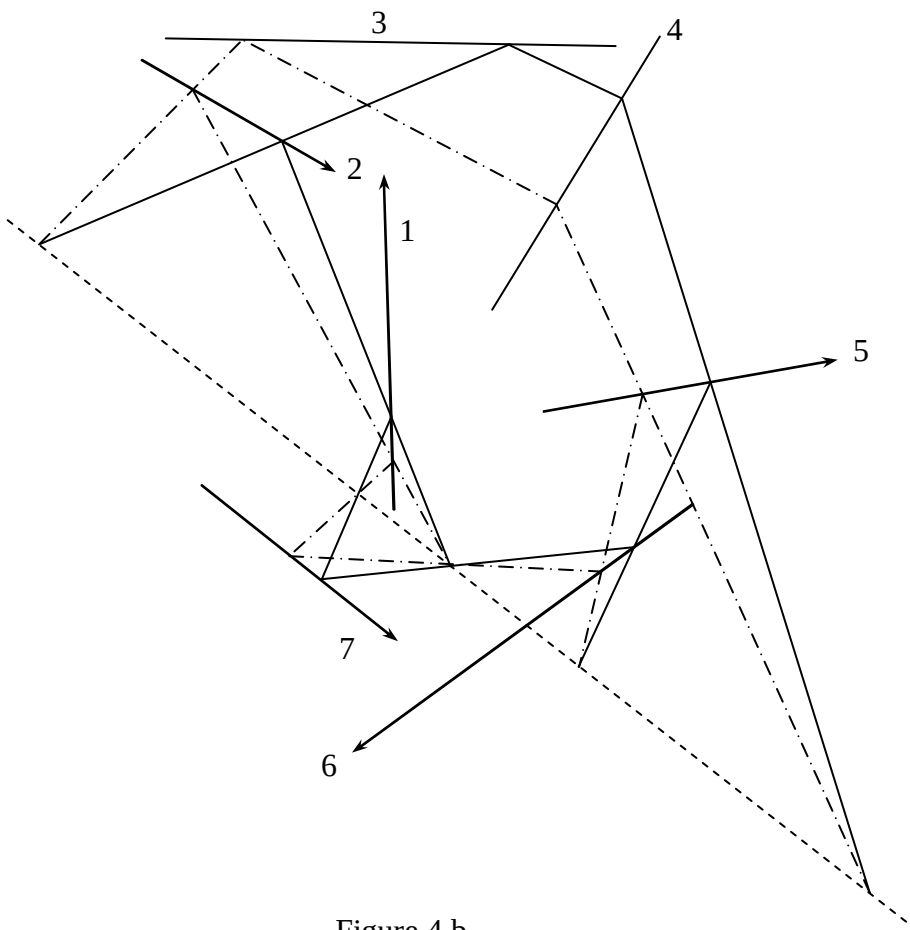

Figure 4.b

This is the most important fundamental theorem of graphical statics, and especially for Culmann's methods. If one lets the two poles O and O' coincide then the corresponding sides of the funicular polygon will prove to be parallel (Fig. 5). The line that connects the vertices of the two physical n-vertices will then be perpendicular to the

orthographic planes, while the bases of the two polyhedra will be parallel to each other (*).

The diagonal between two physical four-vertices of the first polyhedron (or the diagonals that connect the two edges of the aplanar polygon) is conjugate to the line of intersection of the corresponding four-sided face in the second polyhedron, which connects the two pairs of common points to the corresponding base sides with each other. In the orthographic projection, the first line is a diagonal that is drawn between two vertices (r, $r + 1$), (s, $s + 1$) of the force polygon – or a line that is equipollent to the resultant of the forces P_{r+1}, P_{r+2}, …, P_s ; the second line is the line of action of just that resultant. It follows from this that *the line of action of the resultants of an arbitrary number of successive forces* P_{r+1}, P_{r+2}, …, P_s *will go through the point that is common to the sides* (r, $r + 1$), (s, $s + 1$) *of the funicular polygon* – which is the second most important fundamental theorem of graphical statics. (See Fig. 3 for the resultant of the forces 6, 1, 2.).

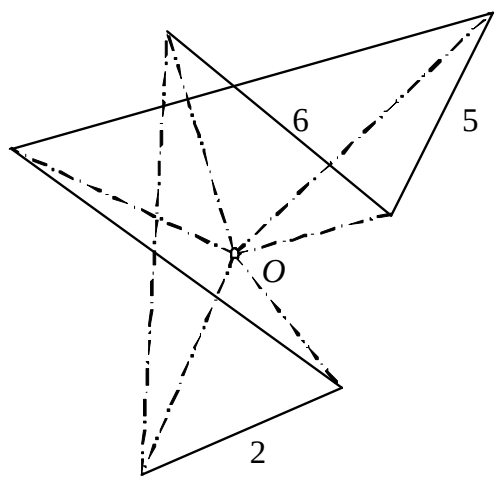

Figure 5.a

If the aforementioned diagonal of the first polyhedron is perpendicular to the orthographic plane then the conjugate line will be at infinity. The edges (r, $r + 1$), (s, $s + 1$) in the force polygon will then coincide in a point A, and the sides (r, $r + 1$), (s, $s + 1$) will be parallel in each of the two funicular polygons.

(*) As one sees, the first diagram proves to be as simple as possible in this way. However, if one would like to simplify the second one then one could shift the pole O' to infinity in an arbitrary direction. The polyhedron whose orthographic projection is the second diagram will then have the vertex of its two physical n-vertices at infinity, and since the polar plane of a point at infinity is parallel to the central axis (and thus perpendicular to the orthographic plane), the new funicular polygon that corresponds to the pole O' will have all of its sides on one line. Moreover, since the position of that line in the orthographic plane was completely arbitrary, one can place it at infinity.

One would arrive at an even simpler result in the following way: If one imagines that the polyhedron in question has the vertex of one of the two physical n-vertices at infinity on the central axis then the pole O' will vanish in the first diagram, since the edges of that physical n-vertex will project to the edges of the force polygon. The polar plane of the vertex of the physical n-vertex is, however, the plane at infinity then, so it will follow that the entire second funicular polygon lies at infinity.

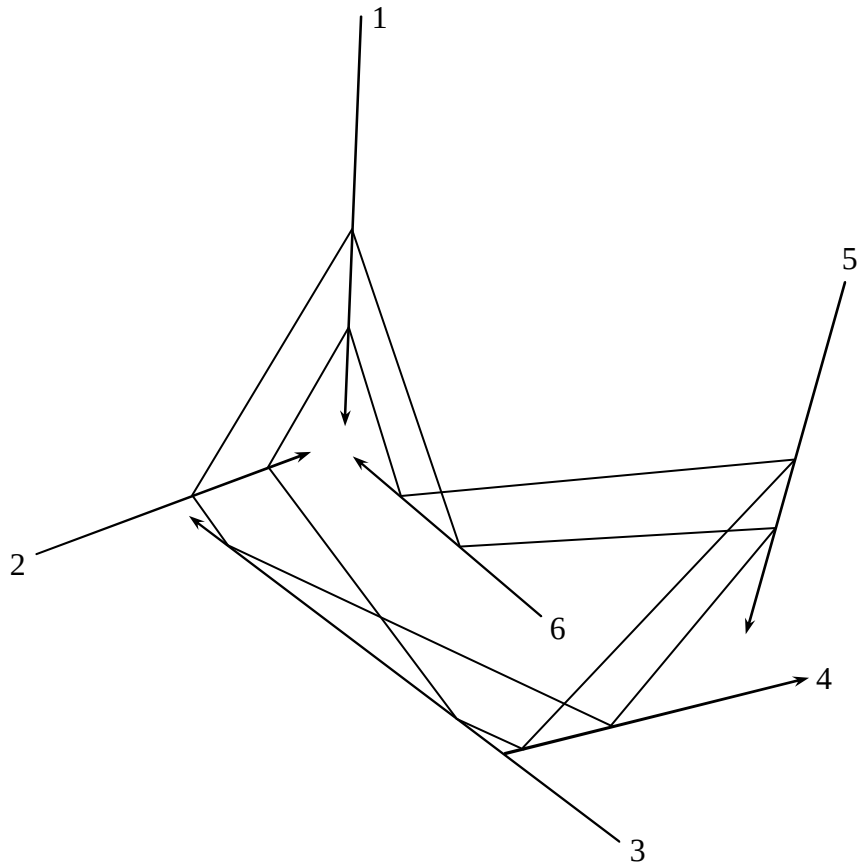

Figure 5.b

The resultant of the forces P_{r+1}, P_{r+2}, …, P_s then has a vanishingly small magnitude, and its line of action will fall along the line at infinity in the orthographic plane. It will be equivalent to a force-couple whose force acts along the aforementioned parallel funicular polygons, and its magnitude will be represented by the line that joins the point O to A. The sense of the force that acts along the side (r, $r + 1$) is from A to O, while the sense of the force that acts along the side (s, $s + 1$) is the opposite one from O to A.

If the forces P_1, P_2, P_3, …, P_{n-1} are given then the two polygons (viz., force and funicular) will serve to determine the resultant of all of these forces and the equal and opposite force P_n . (See Fig. 3, in which $n = 5$). In fact, if one constructs the broken line whose individual lines are equipollent to the given forces then the line that joins the two end points of the broken line to each other (which is directed from the end to the beginning) will be equipollent to the force P_n . Therefore, if one chooses a pole and one constructs a polygon (viz., a funicular polygon) whose first $n - 1$ vertices fall on the lines of action of the given forces and whose sides are parallel to the lines that join the vertices of the first polygon with the pole, respectively, then the line that goes through the last vertex of the funicular polygon (i.e., through the point of intersection of the first and last side attained) and runs parallel to the last side of the force polygon will be the line of action of the last force P_n .

It will follow, with no further analysis, from the fundamental theorem of graphical statics that when the pole moves along a line and one assumes that the first side of the

funicular polygon always goes through a fixed point that the other sides will also be rotated around just as many fixed points that all lie, along with the first ones, on a line that is parallel to the line in which the pole moves. This is a special case of a general theorem of modern geometry (*) that was already presented by the Roman geometry Pappus.

If one now chooses the pole, which can always occupy a position (in the plane of the force polygon) then the properties of the two polygons (viz., force and funicular) can be summarized in the geometric theorem below:

Let a planar n-sided polygon be given and let $n - 1$ lines in that plane be given that are parallel to the first $n - 1$ lines of polygon, respectively. One imagines the vertices of the given polygon as being projected from a point or pole – say, one that moves in the plane, with no loss of generality. One now imagines an n-sided variable polygon whose first $n - 1$ vertices should fall on the given $n - 1$ lines, in sequence, while the n sides should be parallel to the rays that go from the pole to the respective vertices of the given polygon. It will then be the point of intersection of whatever two sides of the variable polygon fall upon a certain line that runs parallel to the diagonal between the corresponding vertices of the given polygon.

This theorem (which is a repetition of the second fundamental theorem), which is not at all simple to prove with the help of only plane geometry, will become almost obvious when one considers the plane figures to be orthographic projections of reciprocal polyhedra.

We now go on to more complicated diagrams, such as one finds in the theory of frameworks. Let S and S' be two outer surface reciprocal polyhedra, the latter of which should be regarded as being simply closed and with a border (**). Let P be the polyhedron that is defined by S and that pyramidal outer surface whose vertex is an arbitrary fixed point O in space and whose guideline is the polygonal perimeter of P. If one now projects these two polyhedra orthographically in the known way then one will obtain two reciprocal diagrams that shall be considered here (***).

Now, if the projection of S' is the sketch for a framework with p nodes and m rectilinear members, for which the external forces will have the projections of the edges of the border of S' as their lines of action, and if the magnitudes of these forces are represented by the n sides of the polygon that is the projection of the border of S then the projection of that face of S' that lies in the plane O' (viz., the plane that is conjugate to the fixed point O), is the funicular polygon of the external forces, that corresponds to the

(*) *Reye*, first section, page 51.

(**) Here, we understand the term "border" to mean a polygon that is defined by the face-surfaces and encompasses the entire polyhedron, such that if, e.g., the border of S is a planar n-sided polygon that bounds the polyhedron then that of S' will be a point, and in particular, the vertex of a physical n-vertex. In order to make this easy to understand, it is best to represent S as a polyhedron that consists of two physical n-vertices whose edges intersect in a skew polygon; S' will then be a polyhedron that is bounded by two n-sided planar polygons and n four-vertices.

(***) If the border of S contains n sides, and if this polyhedron has m other edges (naturally, $m \geq u$) and p faces then the polyhedron P will have $n + p$ faces and $2n + m$ edges, and therefore $m + n - p + 2$ vertices. As a result, S will have $m - p + 1$ edges, in addition to the border (thus, m cannot be smaller than $p - 1$). Reciprocally, P' will have $m + n - p + 2$ faces, $n + p$ vertices, and $2n + m$ edges.

point $\mathfrak{O}$, which is the projection of O (*), and the projection of the m edges of S that do not belong to the border, will give the magnitudes of the internal forces that the elements of the framework are subjected to as a result of the given system of external forces (**).

If one lets the point O go to infinity along the direction that is perpendicular to the orthographic plane then O' will be the plane at infinity. The first diagram will then reduce to just the projection of S; i.e., to the lines that measure the external and internal forces. The second diagram will contain only the sketch of the framework, while the funicular polygon will vanish entirely.

In the accompanying figures, the first diagram will always be denoted by the symbol b, while the second one will always be denoted by a.

If the external forces – and thus the edges of the border of S' – are mutually parallel, which occurs frequently in practical applications, then the border of S will be a polygon that lies entirely in a plane that is perpendicular to the orthographic one, and for that reason, the sides of the polygon of external forces will fall upon one and the same line (***).

If the sketch of the framework and the system of external forces is given then, above all, it will be necessary to construct the polygon of these forces. In the figures, the external forces and the sides of the polygon are assigned the numerals 1, 2, 3, …, in such a way that when one goes through them in increasing order of the numbers then each side will be traversed in the sense of the force that it represents. One refers to this way of traversing the perimeter of the polygon by the name of a "cyclic ordering" of the polygonal perimeter.

If one is now dealing with the sketch for a framework of bars and the construction of the reciprocal diagram for the system of forces then the ordering in which one can arrange the sequence of these forces in order to construct the relevant polygon is not at all trivial. The ordering in which one then treats them will then be determined by considering the fact that in the polygon of external forces, which will make up part of diagram b, the two sides that are equipollent to forces must then follow in succession when the lines of action of these two forces belong to the perimeter of one and the same polygon in diagram a, while just that polygon will correspond to vertex that is common to those two sides.

One will then given the index 1 to any external force, and since the line of action of the chosen force is common to two polygons in diagram a, each of them will, however, contain the line of action of another external force in its perimeter, so one will have two

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(*) Since the sides of that face of S' that lies in the plane O' are conjugate to the edges of the pyramidal outer surface that is defined by O and the border of S.

(**) Since the m edges of S that do not belong to the border are conjugate to the edges of S' whose projections define the framework.

(***) Other degenerate polygons are provided by diagrams that are the projections of analogously degenerate figures in space. One thinks of, e.g., a physical four-vertex in space, two consecutive edges of which approach each other in their plane until they overlap, such that finally a physical four-vertex and a plane that goes through an edge of it will be present, in place of the four-vertex. Now, whereas previously the reciprocal figure was a four-sided surface, it will now appear to be a three-sided surface, namely, two sides of the previous surface will (without having lost their common vertex) will either lie in the same or opposite directions. If one now goes over to orthographic projections then one will have a point in the one diagram from which four lines will emanate that will have two equal directions. A four-vertex with three vertices on a line will appear in the other diagram, which will then be a triangle.

external forces in this way, which can be considered to be attached to the chosen force 1. It will then be irrelevant that we give any one of these two forces the index 2; naturally, the other one will then have the index *n* when *n* is, in fact, the number of external forces. Nothing will now remain arbitrary or uncertain in regard to the sequence of the other sides after this process.

If we find nodes at which the external forces are applied – all of which are on the perimeter of the framework – then one must take these forces in the order in which they are encountered when one runs through the aforementioned perimeter. If one does not follow all of these rules then one can nevertheless still solve the problem of the graphic determination of the internal forces, but one will then no longer have reciprocal figures, but probably complicated or disconnected figures in which one and the same member, when it is not found at that associated plane, must be repeatedly shifted or carried back in order to give enough space for the further constructions, which happened very often with the older methods, where one constructed a separate funicular polygon for each node. If the polygon of external forces is constructed in this way then one will complete the diagram by gradually constructing the polygons that correspond to the nodes of the framework. The problem of constructing a polygon whose sides should have well-defined directions is always determinate then when only two (consecutive) sides are unknown. On this basis, one must begin with a node at which only three lines intersect, such as the line of resistance of two members of the framework and the line of action of an external force. The member that is equipollent to that force will be a triangle side that corresponds to that node and the triangle can thus be constructed. Furthermore, no ambiguity will exist in this construction when one always considers that a member of the framework that belongs to the perimeter of a polygon of diagram *a*, along with the lines of action of two external forces, will always correspond to a line in *b* that goes through the vertex that is common to the sides that are equipollent to those two forces.

In this way, one will gradually go on to the other nodes in such a way that only two sides of each new polygon will ever seem to be unknown. (In the figures, the lines in each diagram are denoted by numbers, in order to specify the sequence.)

A consideration that relates to the outer surface can make one believe that the solution is also determinate in each case for which no node is the junction of only three lines. One imagines, e.g., the sketch of the framework that consists of the sides **5**, **6**, **7**, **8** of a four-vertex and the lines **9**, **10**, **11**, **12**, which join the vertices of the four-vertex with a fifth point. Let the external forces be **1**, **2**, **3**, **4** act upon the vertices (**8**, **5**, **9**), (**5**, **6**, **10**), (**6**, **7**, **11**), and (**7**, **8**, **12**) of the four-vertex, respectively (*).

One now constructs the polygon of external forces 1, 2, 3, 4, and draws the lines 5, 6, 7, 8, resp., which run parallel to the sides in the other figure that have the same names, through the points (1, 2), (2, 3), (3, 4), and (4, 1), resp. One now poses the problem of finding a four-vertex whose sides 9, 10, 11, 12 run parallel to the sides of the given figure with the same name, and whose vertices (9, 10), (10, 11), (11, 12), (12, 9), resp., fall upon the lines 5, 6, 7, 8, respectively. Since problem of constructing a polygon whose sides have given directions and whose vertices fall upon a fixed line generally admits only a single solution, one can believe that the force diagram will be determined completely. One then convinces oneself that this is not the case when one considers that this

(*) These considerations remain unchanged when the framework is defined by whatever sort of polygon and the lines that connect its vertices with a point.

geometric problem will also have cases in which the solution is impossible or undetermined. In fact, of one removes one of the latter conditions – i.e., one subjects the four-vertex to merely the conditions that its sides have to be in given directions and the first three vertices have to be on the given lines 5, 6, 7 – then the fourth vertex will describe a line *r* (**), and the point that is common to this and the given line 8, when considered to be the locus of the fourth vertex, would give the desired solution. However, if *r* lies parallel to line 8 then we will find ourselves in one of the impossible cases. Therefore, the problem can also seem to be undetermined when the line 8 coincides with 8 (which is an even more specialized case); finitely many four-vertices would then satisfy the given conditions.

Now, in order to convince oneself that in the construction of the given reciprocal diagram one confronts this case of indeterminacy, it will suffice to ponder the fact that when one considers the given diagram to be a polygon of forces whose magnitudes are expressed by the members **5**, **6**, **7**, **8** and whose pole is the point (**9**, **10**, **11**, **12**), and the desired Figures 9, 10, 11, 12 will exhibit the corresponding funicular polygon. Now, should the construction of the funicular polygon be possible, then the forces would have to be in equilibrium, which is why when one chooses their magnitudes **5**, **6**, **7**, **8** and the lines of action **5**, **6**, **7** of three of them as given, the line of action of the fourth force must coincide with the line 8. The problem will then be indeterminate, since one can construct infinitely many funicular polygons for a given system of forces and a given pole.

Any rectilinear piece or member of a framework is the line of action of two equal and opposite forces that act on the two nodes that are linked by that member. The magnitude of one of these forces or the measure of the effect that the member in question seems to be subjected to will be given by the corresponding line in diagram *b*. One can consider these two forces to be actions or reactions; in order to go from one case to the other, one only has to invert the sense of both forces. If one considers the forces as actions then when they act from the point of application into the interior of the member, it will be compressed, while in the opposite case, it will be stretched. (In the Figures, the tie rods (*Zugstangen*) are denoted by thin lines, while the struts (*Streben*) are denoted by thick ones.)

Every node of this framework is the point of application of a system of at least three forces that are in equilibrium, and one of the forces can be external; all of the other ones are then reactions to the members that are joined at the node. It suffices to know the sense of one of these forces of the system in order to then derive the senses of all others, for which purpose one only has to traverse the perimeter of the polygon that corresponds to this node. If an external force acts at a node and one traverses the equipollent sides of the polygon in its sense then each of the other sides will also be traversed in the sense that corresponds to the respective internal force when one considers it as a reaction that acts upon that node. By contrast, if one thinks of the internal forces as acting in the sense that

(**) This is a special case of the theorem that was mentioned on page 8. (The fixed points lie at infinity, here.)

they acquire as actions then one must invert the sense of the external forces under traversal.

If only internal forces act upon a node then it will likewise suffice to know the sense of only those forces in order to derive the senses of all of the other ones from it. One calls the type and manner of traversing the perimeter of a polygon in diagram *b* in the sense that corresponds to considering the internal forces as actions the *cyclic ordering* of that perimeter.

If one considers an internal force to be an action that acts upon one of the two nodes, between which the member in question lies, then one will recognize at once whether the member is stretched or compressed.

Any line in diagram *b* is common to two polygons. If one traverses the peripheries of then in the cyclic ordering that they acquire then that side will be described once in the one sense and the other time in the opposite sense (*).

This also corresponds to the fact that the line is the measure of two equal and opposite forces that act along the corresponding member of the framework.

It is known that the algebraic sum of the projections of the faces of a polyhedron is equal to zero. If one applies this theorem to the aforementioned polyhedron *P* and considers that the projections of the outer surface of *S* is composed of the polygons in diagram *b* that correspond to the nodes of the framework, while the projection of the remaining outer surface of *P* represents nothing but the polygon of external forces, then one will obtain the following result:

If one consider a face of a polygon as being positive or negative according to whether it lies on the left or right side when one traverses it perimeter in the cyclic ordering then the sum of the faces of the polygons of diagram *b* that correspond to the nodes of the framework will be equal and opposite to the faces of the polygon of external forces. A theorem that *Maxwell* had posed in another way for an arbitrary planar net will also be valid when it is not possible to construct the diagram of forces.

The so-called *method of sections* will give the draftsman a means of control. If one considers one of the two parts into which the framework is divided by an ideal section then the external forces that act upon that part will be in equilibrium with the reactions of the sectioned member.

If only three of these reactions are unknown then they can be determined from that equilibrium condition, since the problem of decomposing a force *P* into three components whose lines of action **1**, **2**, **3** are given and should define a complete tetragon with **0**, which is the line of action of *P*, is determined completely and admits only *one* solution. In fact, it suffices (Fig. 6) to draw one of the diagonals of the tetrangle – e.g., the line **4**, which links the points **01**, **23**, and furthermore to decompose the given force **0** into two

(*) Since the two polygons that have that side in common will correspond to the nodes that are linked by the member in question, and the sense of the internal force will be an opposite one when one consider it to be an action relative to the one node, while one previously chose it to be an action relative to the other node. This property is also in harmony with the so-called edge law in the polyhedra that are regarded as having inner and outer surfaces. (See *Möbius*: "Ueber die Bestimmung des Inhaltes eines Polyeders," in the Berichte der Gesellschaft der Wissenschaft, 1865, page 33.)

components along **1** and **4**, and finally, to decompose the force **4** into two components along the lines **2** and **3** (*).

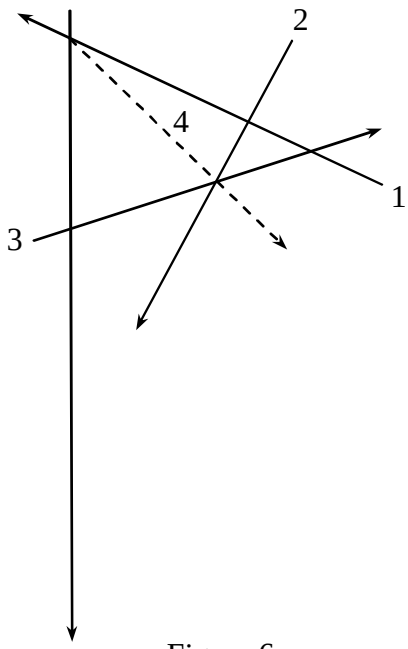

Figure 6.a

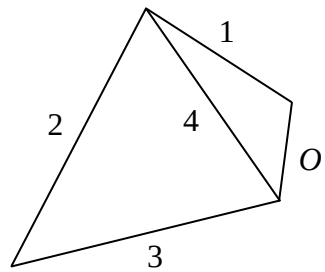

Figure 6.b

This method – which one can call "static" – will suffice for the graphic determination of the internal forces in their own right, as will the geometric method that was just derived, which is derived from the theory of reciprocal figures, and consists of the successive construction of the polygons that correspond to the various nodes of the framework. The static method, however, seems to be less simple, but it can prove to be expedient, in particular, for the verification of the graphic solution that was obtained already. It can also be considered from an entirely different viewpoint. Namely, if one lets **0** denote the resultant of all known forces and lets **1**¸**2**, **3** denote the unknown reactions then the sum of the moments of these four forces must be equal to zero. If follows from this that when one chooses the moment point to be the point of intersection of two lines of resistance, the moment of the third reaction will be equal and opposite to the moment of the force **0**. In this way, one will get a proportion between four quantities (viz., the two forces and their lever arms), of which, the single unknown will be the magnitude of the one reaction. The method of static moments consists of that, which one applies to the numerical determination of the internal forces of a framework (**).

(*) The former decomposition will be accomplished when one constructs the force triangle 0 4 1, whose side *O* is given in magnitude and direction, and the latter, by an analogous construction of the force triangle 4 3 2.

(**) See A. Ritter's, *Elementare Theorie und Berechnung eiserner Dach- und Brücken-Constructionen*, 2nd. ed. (Hannover, 1870).

What now follow are some examples that are suitable for illustrating the simplicity and elegance of the graphical method. In these examples, no appeal will be made to regularity or symmetry in form, although one can almost never avoid them in practice. Thus, the regular forms of practice are only special cases of the irregular forms in abstract geometry, and therefore contain the treatment of all of these practical cases.

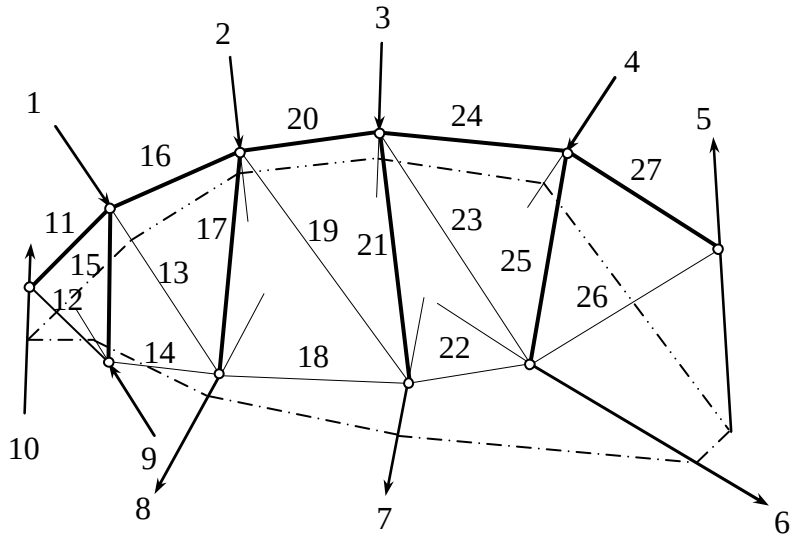

Figure 7.a

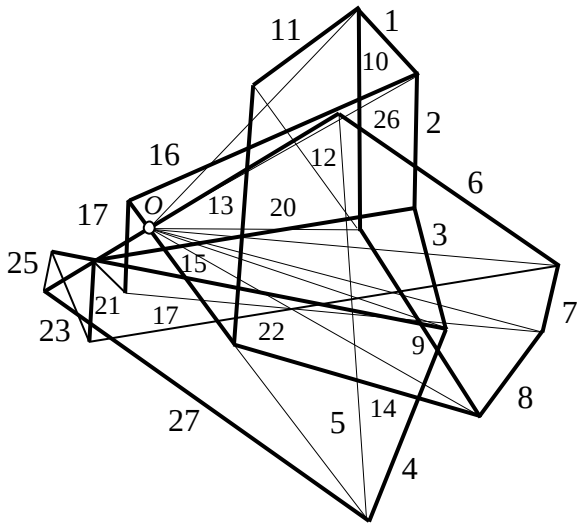

Figure 7.b

1. As a first (theoretical, general) example, let **1**, **2**, …, **10**, sheet *Q* (Fig. 7.a) be a system of ten external forces in equilibrium, such that after the polygon of these forces has been specified and its vertices are linked to a pole *O* (Fig. 7.b, in which the force polygon is drawn with double lines), one can construct a funicular polygon whose vertices lie upon the lines of action **1**, **2**, …, resp., and whose sides (which are dash-dotted in Fig. a) are parallel to the rays that emanate from *O* in sequence. Let the

aforementioned forces be thought of as acting upon the nodes of a framework whose rectilinear members are denoted with the numerals **11**, **12**, **13**, …, **27**.

One begins with the construction of the triangles that correspond to the nodes (**10**, **11**, **12**), when one leads with the two lines 11, 12 – which are parallel to **11**, **12**, respectively – at the ends of 10; to that end, one remarks that 11 must go through the point (1, 10), since in diagram *a* the lines **1**, **10**, **11** belong to the perimeter of a polygon (*). On the same grounds, **12** must go through the point (**9**, **10**). If one traverses the perimeter of the triangle thus-obtained in the sense that is opposite to that of force **10** then one will obtain the sense of the actions that are exerted along the lines **11**, **12** at the nodes under consideration. One then sees that the member **11** is compressed, while **12** is stretched.

One now constructs the tetrangle that corresponds to the node at which the force **9** acts, when one draws **13** through the point (11, 12) and **14** through the point (**8**, **9**). The term **13** will be compressed and **14**, stretched.

One further constructs the pentangle that corresponds to the node at which the external force **1** acts when one draws **15** through the point (13, 14) and **16** through the point (1, 2). The perimeter of the pentagle thus-obtained will intersect itself. The member **15** is stretched, while **16** is compressed.

One further constructs the pentangle that corresponds to the node at which the external force **8** acts. To that end, one draws **17** through the point (15, 16) and **18** through the point (7, 8). One sees that the member **17** is compressed, while **18** is stretched.

If one proceeds in that way then one will find all of the other internal forces. The latter construction will give the triangle that corresponds to the point of application of the force **5**.

20, **21**, **24**, **25**, **27** are compressed, while **19**, **22**, **23**, **26** are stretched.

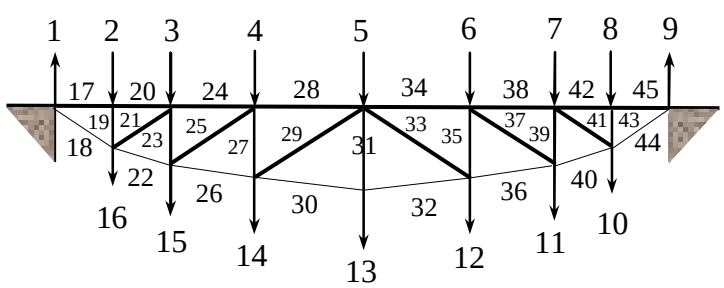

Figure 8.a

2. Fig. 8.a represents a bridge girder, at whose nodes, the forces **1**, **2**, …, **16**, which are all vertical forces, are applied. **1** and **9** are directed from below to above and represent the reactions of the abutment. All of these forces are taken in the order in which they will be encountered when one traverses the perimeter of the framework. The sides of the polygon of external forces are arranged similarly in diagram b, which is a polygon whose sides all point vertically, in which the sum of the sections 1, 9, is equal

(*) Which is a tetrangle whose fourth side is the side of the funicular polygon that lies between the forces 1, 10. As was remarked already, the funicular polygon can also lie completely at infinity.

and opposite to the sum of the sections 2, 3, …, 8, 10, 11, …, 16, since the system of external force must be in equilibrium.

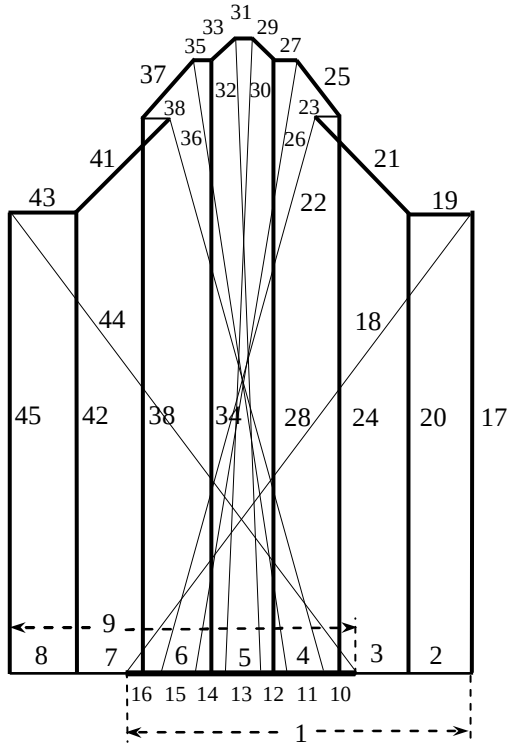

Figure 8.b

Diagram *b* is now easy to construct by the given rules. Beginning from nodes **1**, **17**, **18**, one draws 17 through the point (1, 2); i.e., through the point where section 1, which is directed from below to above and section 2, which is directed from above to below, end. One then does likewise for **18** through the point (16, 1).

Going on to the nodes (**2**, **17**, **19**, **20**), one draws **19** through the point (17, 18) and **20** through the point (2, 3). One then obtains the polygon (2, 17, 19, 20), which is rectangle.

Arriving at the nodes (**16**, **18**, **19**, **21**, **22**), one draws **21** through the point (19, 20) and **22** through the point (15, 16). One thus obtains a pentangle whose perimeter intersects itself.

If one proceeds in the same way then one will gradually go on to the nodes at which the forces **3**, **15**, **4**, **14**, **13**, **5**, **12**, **6**, **11**, **7**, **10**, **9** act. Diagram b, like diagram a, will appear to be symmetric.

All of the members of the top chord are compressed, while those of the bottom chord are stretched. The skew linking members are all compressed. Two of the vertical beams, viz., **23**, **39**, are stretched, while all of the other ones are compressed.

3. In Fig. 9.a, one considers half of a locomotive shed (*).

The external forces are the ones **1**, **2**, **3**, **4**, **5** that act upon the upper nodes of the roof and the reactions of the wall and pillar are **6**, **7**, resp.

All external forces are also parallel, here, so the force polygon will reduce to a line.

The lines 8, 13 will coincide in diagram b; the first one is a subset of the second line. Thus, the polygon that corresponds to the nodes (**8**, **10**, **12**, **13**) will be of that abnormal form that was mentioned previously; namely, one will have a tetrangle 8, 10, 12, 13, three vertices of which, (13, 8), (8, 10), (12, 13), will lie along a line.

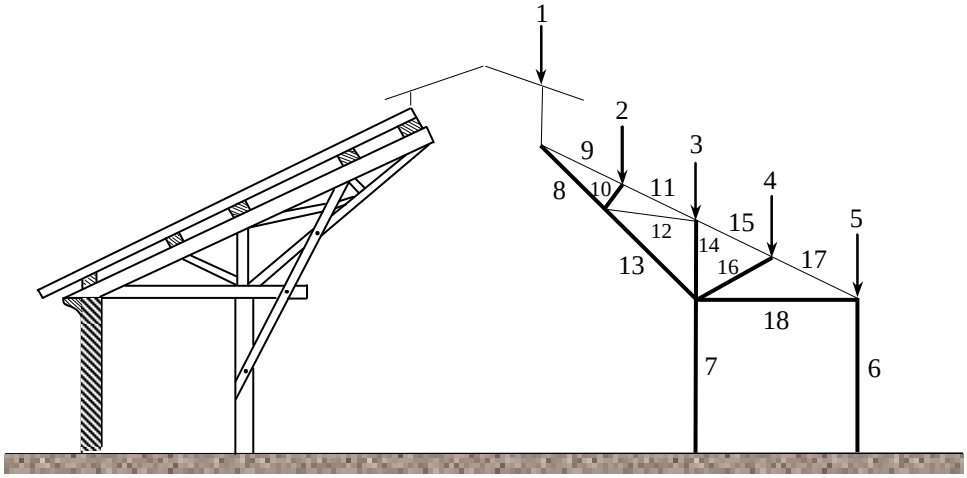

Figure 9.a

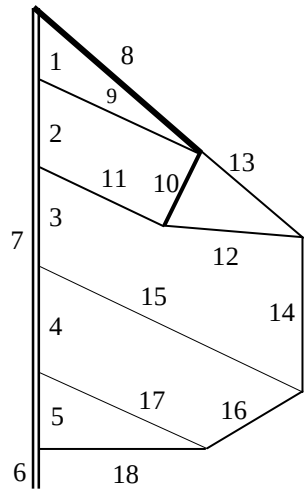

Figure 9.b

An analogous abnormal form is that of the tetrangle 5, 17, 18, 6 that corresponds to the points at which the roof meets the wall.

(*) This example is drawn from Table 19 in Culmann's atlas of graphical statics. However, the two diagrams are not strictly reciprocal there.

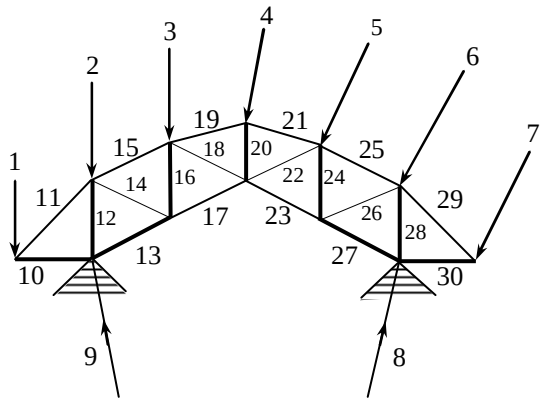

Figure 10.a

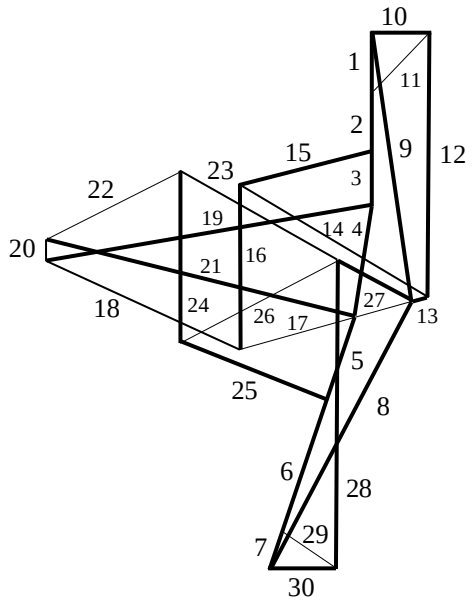

Figure 10.b

4. Diagram *a* in Figure 10 represents a bridge girder, to whose nodes the skew forces **1**, **2**, …, **7** are applied, which one can consider to be resultants of the actions of the weight and the wind; the forces **8**, **9** represent the reactions of the abutment.

The polygon of external forces in diagram *b* is drawn with double lines. One constructs, in turn, the triangle 1, 10, 11, the tetrangle 9, 10, 12, 13, the pentangle 2, 11, 12, 14, 15, the tetrangle 13, 14, 16, 17, the pentangle 3, 5, 16, 18, 19 (whose perimeter intersects itself), the tetrangle 4, 19, 20, 21 (which is likewise intertwined), the pentangle 17, 18, 20, 22, 23, etc.

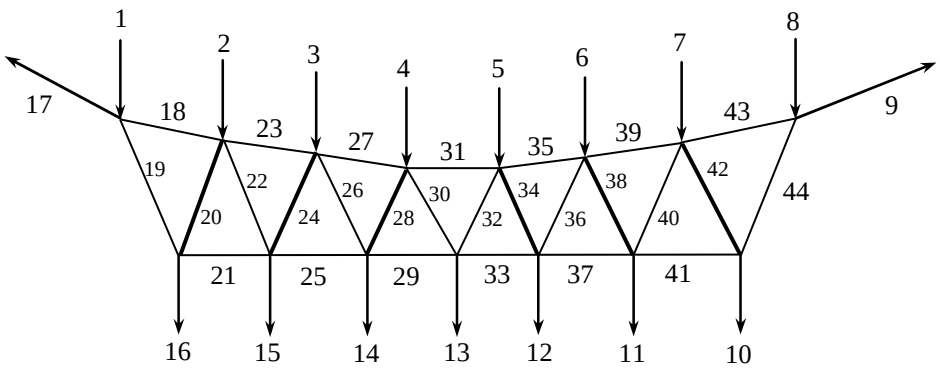

Figure 11a.

5. Diagram *a* in Figure 11 represents a suspension bridge that is loaded in the upper nodes by the weights **1**, **2**, …, **8**, and in the lower ones by the weights **10**, **11**, …, **16**. The skew reactions at the extreme points of the framework **9**, **17** represent equilibrium.

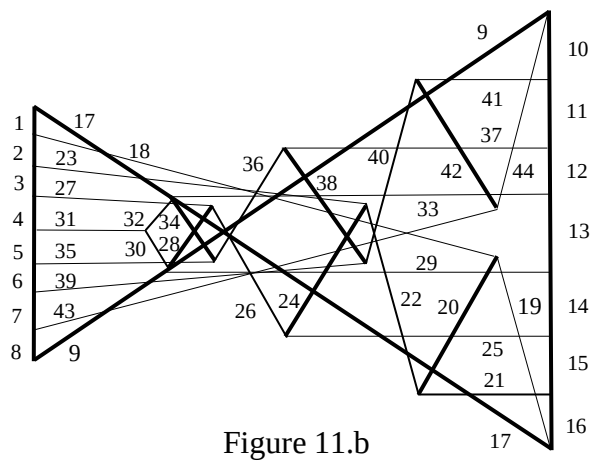

Figure 11.b

The polygon of external forces has its first eight sides pointing vertically, and likewise the lines 10, 11, …, 16 point along other verticals. The skew sides 7 and 19 intersect, such that the perimeter of the entire polygon will also intersect itself.

One constructs, in turn, the polygons:

1, 17, 19, 18; 16, 19, 20, 21; 2, 18, 20, 22, 23; 15, 21, 22, 24, 25; 3, 23, 24, 26, 27; …,

whose perimeters intersect themselves, for the most part.

The two diagrams are also symmetric in this example. Diagram *b* shows that all members of the upper part are stretched, and that this tension will diminish at the ends compared to the middle, and likewise, the fact that all members of the bottom chord are likewise stretched, but that there the tension diminishes at the middle compared to the ends. The connecting members are alternately stretched and compressed, except in the middle, where two successive rods are stretched. If one considers only the stretched or only the compressed connecting members then one will see that their responses will diminish at the ends compared to the middle.

6. Figure 12 represents a reticulated crane (*Netzkrahn*) in diagram *a*. Its dead weight is distributed over various forces **1**, **2**, **3**, …, **9** that are applied to the nodes. The force **5** also includes the extra load that the crane must withstand. All of these weights are equilibrated by the reactions of the support, whose magnitude one will obtain when one decomposes the resultant of all weights into three forces along the lines **10**, **11**, **12**. When taken in the opposite sense, they will already give the pressures of the strut **10**, the column **11**, and the stress in the connecting rod **12**.

In the figure, one also finds the expression of the stated determination of the external forces. Namely, once the sections, which represent the weights 1, 2, …, 9, are carried by a vertical, a pole will be fixed, and the corresponding funicular polygon will have been constructed when one has projected the points (0, 1), (1, 2), (2, 3), …, (8, 9), (9, 0) (*) from that pole. The vertical that goes through the point of intersection of the extreme sides (**0**, **1**), (**9**, **0**) is the line of action of the resultant weight whose magnitude is determined already by the sum of the given forces. If one decomposes this resultant, which is now a known force in all of its members, along the three lines of action **10**, **11**, **12** into three components – which one does when one applies the previously-mentioned construction that was illustrated in the figure – then one will obtain the three forces 10, 11, 12, which, when taken in the opposite sense, will give the complete system of external forces, along with the given weights.

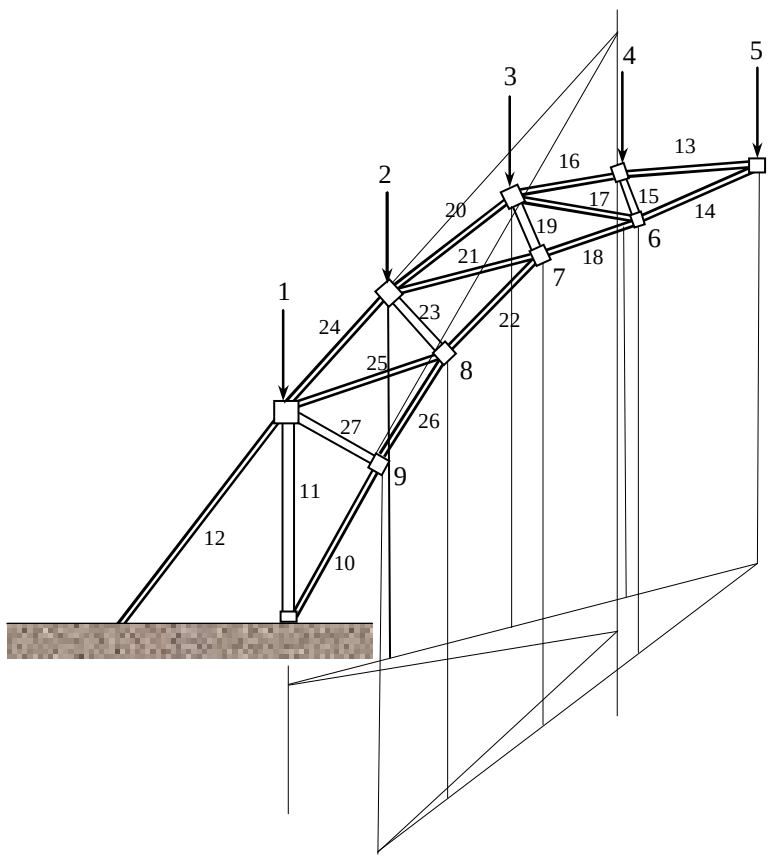

Figure 12.a

(*) Here, (0, 1) refers to the start of section 1 and (9, 0) refers to the end of section 9.

In order to obtain diagram *b*, one begins with the construction of the polygon of external forces, takes it in the order in which they are encounters when one traverse the perimeter of the framework. One then constructs, in turn, the polygons that correspond to the nodes (**5**, **13**, **14**), (**4**, **13**, **15**, **16**), (**6**, **14**, **15**, **17**, **18**), …, in the known way.

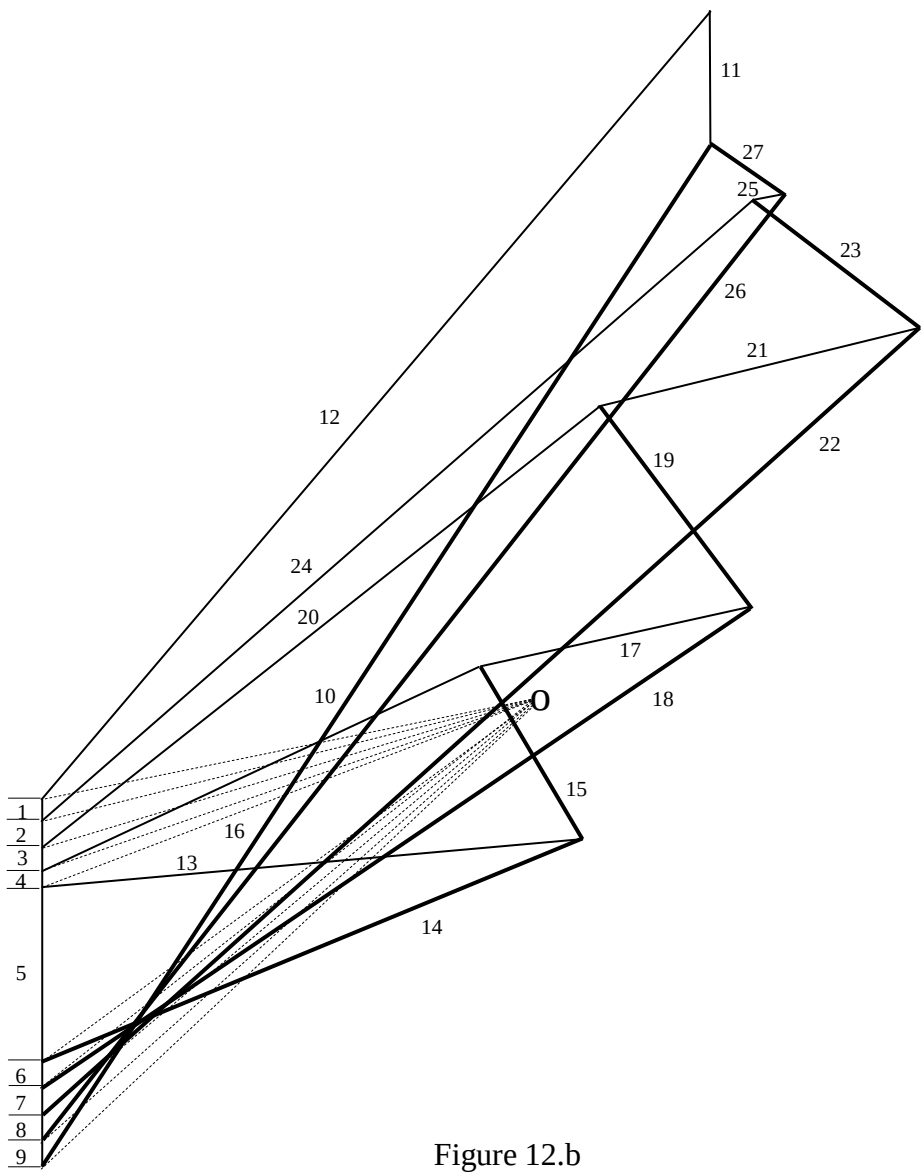

Figure 12.b

The resulting diagram will show that all of the upper members are stretched, while the lower ones are compressed; as for the connecting rods, they will be alternately stretched and compressed.