“Ueber die Biegung gewisser Flächen,” J. f. d. reine u. angew. Math. **18** (1838), 297-302.

On the bending of certain surfaces

(By Hrn. Dr. Ferd. Minding in Berlin)

Translated by D. H. Delphenich

If the rectangular coordinates of a curved surface are expressed in terms of two variable quantities p and q then it is known that one will have the formula:

$$ds^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$$

for the element (ds) of any curve that is found upon it, in which E, F, G are functions of p and q, namely:

$$E = \left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2 + \left(\frac{dz}{dp}\right)^2,$$

$$F = \frac{dx}{dp}\cdot\frac{dx}{dq} + \frac{dy}{dp}\cdot\frac{dy}{dq} + \frac{dz}{dp}\cdot\frac{dz}{dq},$$

$$G = \left(\frac{dx}{dq}\right)^2 + \left(\frac{dy}{dq}\right)^2 + \left(\frac{dz}{dq}\right)^2.$$

Any other surface whose coordinates can be represented by expressions in p and q that once again yield the values above for the line element will obviously be a mere bending of it, at least, as long as the same values of p and q give real points on both surfaces. Due to the equality of corresponding line elements, any triangle of three infinitely-close points on the first surface will be congruent to the triangle of the three corresponding points on the second one. Both surfaces will then be composed of the same elementary triangles in the same order, as the concept of bending would demand.

Up to now the capacity to bend has been proved only for developable surfaces. Here, it shall be show that this property is found for all surfaces that *arise by the motion of a straight line;* one can convince oneself of this in the following way: Let a, a', a'', … be infinitely-close successive lines that define a piece (s) of such a surface. One can then leave the first strip between a and a' unmoved, while rotating the remaining part of s infinitely-little around a as a fixed axis. Thereupon, the strips aa' and $a'a''$ are left unmoved, while the still-remaining pieces of s are rotated around a'', etc., and in that way, the surface will be bent without the lengths of any of the lines that are found on its changing. As one sees, that bending comes about in exactly the same way as it does for developable surfaces. The restriction to them that has been made up to now is therefore entirely inessential.

One chooses an arbitrary curve on the bent surface that cuts all of the lines that generate the surface; it might be called the *guiding curve*. Let A be its intersection with any line, and let B be an arbitrary point on that curve. Let q denote the distance to the point B from A, while u', υ', w' are the cosines of the inclination of the line AB with respect to the axes, u, υ, w are the coordinates of A, and x, y, z are those of B. One will then have:

$$x = u + q\,u', \quad y = \upsilon + q\,\upsilon', \quad z = w + q\,w'. \tag{1}$$

The quantities u, υ, w, u', υ', w' are independent of q and are functions of another variable p. One initially gets:

$$u'^2 + \upsilon'^2 + w'^2 = 1. \tag{2}$$

If one sets:

$$du^2 + d\upsilon^2 + dw^2 = \alpha\, dp^2, \tag{3}$$

$$du\, du' + d\upsilon\, d\upsilon' + dw\, dw' = \beta\, dp^2, \tag{4}$$

$$du^2 + d\upsilon^2 + dw^2 = \gamma\, dp^2, \tag{5}$$

$$u'\, du + \upsilon'\, d\upsilon + w'\, dw = \varepsilon\, dp \tag{6}$$

then α, β, γ, ε will be functions of p, and one will get:

$$dx^2 + dy^2 + dz^2 = ds^2 = (\alpha + 2\beta\, q + \gamma q^2)\, dp^2 + 2\varepsilon\, dp\, dq + dq^2 \tag{7}$$

for the line element. As a result of (2), in order to find other surfaces that give the same line element, as required, one sets:

$$u = \cos\varphi \cos\psi, \qquad \upsilon = \cos\varphi \sin\psi, \qquad w = \sin\varphi\,. \tag{8}$$

φ and ψ are two new variables. If one develops the differentials of u', υ', w' in this and sums their squares then [from (5)] one will get:

$$d\varphi^2 + \cos\varphi^2 \cdot d\psi^2 = \gamma\, dp^2. \tag{9}$$

If one now makes φ into an arbitrary function of p then the foregoing equations will yield an expression for ψ in terms of p. A system of values for u', υ', w' will then follow from (8) that contain one undetermined function. In order to find the associated values of u, υ, w, one then appeals to equations (3), (4), (6). Let:

$$\left\{\begin{aligned} A\, dp &= \upsilon' dw' - w' d\upsilon', \\ B\, dp &= w' du' - u' dw', \\ C\, dp &= u' d\upsilon' - \upsilon' du', \end{aligned}\right. \tag{10}$$

so [due to (8)]:

$$
(11)\qquad \left\{\begin{aligned} A\,dp &= \quad\sin\psi\,d\varphi - \sin\varphi\cos\varphi\cos\psi\,d\psi, \\ B\,dp &= -\cos\psi\,d\varphi - \sin\varphi\cos\varphi\sin\psi\,d\psi, \\ C\,dp &= \quad\cos^2\varphi\cdot d\psi, \end{aligned}\right.
$$

and

$$
(12)\qquad A^2 + B^2 + C^2 = \gamma.
$$

If one eliminates dw, $d\upsilon$, du from (4) and (6) then one will get:

$$
(13)\qquad \left\{\begin{aligned} A\,d\upsilon - B\,du &= \varepsilon' dw' - \beta\,w'd\upsilon', \\ C\,du - A\,dw &= \varepsilon\,d\upsilon - \beta\,\upsilon' dp, \\ B\,dw - C\,d\upsilon &= \varepsilon\,du' - \beta\,u'\,dp, \end{aligned}\right.
$$

and when one adds the squares of these, while recalling (3) and (12), one will get:

$$
\alpha\,\gamma\,dp^2 - (A\,du + B\,d\upsilon + C\,dw)^2 = (\varepsilon^2\gamma + \beta^2)\,dp^2,
$$

or, if one sets $(\alpha - \varepsilon^2)\,\gamma - \beta^2 = g^2$:

$$
(14)\qquad A\,du + B\,d\upsilon + C\,dw = g\,dp.
$$

The quantity that is denoted by g^2 is either positive or zero (which will be true when the surface is developable). Ultimately, $\dfrac{-g^2}{(\alpha - \varepsilon^2 + 2\beta q + \gamma q^2)^2}$ is the measure of curvature of the surface that is considered here, which is implied by the most general expressions that GAUSS developed in his treatment of curved surfaces. One finds from (13) and (14) that one can determine u, υ, w from:

$$
(15)\qquad \left\{\begin{aligned} du &= \frac{1}{\gamma}(A\,g\,dp + \beta\,du' + \varepsilon\,\gamma\,u'dp), \\ d\upsilon &= \frac{1}{\gamma}(B\,g\,dp + \beta\,d\upsilon' + \varepsilon\,\gamma\,\upsilon'dp), \\ dw &= \frac{1}{\gamma}(C\,g\,dp + \beta\,dw' + \varepsilon\,\gamma\,w'dp). \end{aligned}\right.
$$

If one sets u, υ, w in (1) equal to the values that one gets by integrating the foregoing ones and sets u', υ', w' equal to the values in (8) then one will get an expression for the surfaces that can arise from given ones by the bending that is assumed here.

One can simplify the formula above (7) for the line element by a suitable choice of guiding curve. If one chooses another guiding curve instead of the previous one then q must be set to an expression of the form $P + q$ in which P is a function of only p, and q

now means the same thing with respect to the new guiding curve that it did for the first one. Let

$$(\alpha + 2\beta q + \gamma q^2)\, dp^2 + 2\varepsilon\, dp\, dq + dq^2$$

be the value of ds^2 that emerges in that way. One will then have:

$$\alpha' = \alpha + 2\beta P + \gamma P^2 + 2\varepsilon \frac{dP}{dp} + \left(\frac{dP}{dp}\right)^2, \qquad \beta' = \beta + \gamma P, \qquad \gamma' = \gamma, \qquad \varepsilon' = \varepsilon + \frac{dP}{dp}.$$

One then set $P = -\beta / \gamma$, in particular, which will make $\beta' = 0$.

This raises the question of whether amongst the uncountable bends of a surface that were represented above and that define an associated type of the surfaces that are considered here, there is a simplest one of the type that was just pointed out, such as the plane amongst the developable surfaces. By analogy, it seems that those (always possible) bends of the given surface can be chosen conveniently to make the generating lines all be *parallel to a plane.* One will get their expressions directly from the formulas above when one sets $\varphi = 0$; any line will then be assumed to be parallel to the xy-plane. If one then imagines, according to the foregoing remark, that the guiding curve has been chosen such that $\beta = 0$ then for $\varphi = 0$, one will have:

$$d\psi = \sqrt{\gamma} \cdot dp, \qquad u' = \cos \psi, \qquad v' = \sin \psi, \qquad w' = 0, \qquad A = 0, \qquad B = 0, \qquad C = \sqrt{\gamma},$$

$$du = \varepsilon u'\, dp, \qquad du = e\, v' dp, \qquad dw = \frac{g\, dp}{\sqrt{\gamma}} = \sqrt{\alpha - \varepsilon^2}\, dp,$$

and therefore the equations for the desired surface will be the following ones:

$$x = \int \varepsilon \cos\psi\, dp + q \cos\psi,$$

$$y = \int \varepsilon \sin\psi\, dp + q \sin\psi,$$

$$z = \int \sqrt{\alpha - \varepsilon^2} \cdot dp.$$

Here, the values of x and y represent a system of straight lines in the plane that all contact a curve whose coordinates can be found when one sets $q = 0$. For developable surfaces (so when $\alpha - \varepsilon^2 = 0$), that curve will correspond to the edge of regression. However, when $\alpha - \varepsilon^2$ is not zero, one will find that the curve is the baseline of a right cylinder. One can then exhibit the desired surface as having been generated by a succession of straight lines that contact that cylinder perpendicularly to its side in a certain curve.

In order to develop an example of bending, let us consider the helicoid (cochlea). If one assumes that its axis is the guiding curve, as well as the z-axis, then that will imply that:

$$x = q \cos p, \quad y = q \sin p, \quad z = n\,p.$$

It follows from this that $\alpha = n^2$, $\beta = 0$, $\gamma = 1$, $\varepsilon = 0$; hence:

$$ds^2 = (n^2 + q^2)\,dp^2 + dq^2.$$

Furthermore, one will have $g = n$, from which one ultimately gets the following expressions for the bending of that surface:

$$x = n\int A\,dp + q\cos\varphi\cos\psi,$$

$$y = n\int B\,dp + q\cos\varphi\sin\psi,$$

$$z = n\int C\,dp + q\sin\varphi.$$

A, B, C are determined as functions of φ and ψ by equations (11). φ is an arbitrary function of p, and ψ follows from the equation:

$$d\varphi^2 + \cos^2\varphi\,d\psi^2 = dp^2.$$

If one sets, – e.g., φ = const. = μ – then one will have $dp = \cos\mu \cdot d\psi$, and one will get:

$$x = -\,n \sin\mu\cos\mu\sin\psi + q\cos\mu\cos\psi,$$

$$y = \quad n \sin\mu\cos\mu\cos\psi + q\cos\mu\sin\psi,$$

$$z = \qquad\qquad n\cos^2\mu + q\sin\mu.$$

ψ is equal to $p / \cos\mu + h$, while h is a new constant. If one sets $q = 0$ here then that will imply that the axis of the helicoid will take on the form of a certain helix under this bend that will be found on a cylinder of radius $n \sin\mu\cos\mu$. The bend of the helicoid that is represented here arises from the fact that one has erected an altitude at each point of the helix to the plane of its curvature circle.

In order to find the bends of a simple hyperboloid, one multiplies the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$$

by

$$1 = \cos^2 p + \sin^2 p$$

and gets:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\cos p\pm\frac{z}{c}\sin p\right)^2+\left(\sin p\mp\frac{z}{c}\cos p\right)^2;$$

one can then set:

$$\frac{x}{a}=\cos p\pm\frac{z}{c}\sin p,\qquad \frac{y}{a}=\sin p\mp\frac{z}{c}\cos p.$$

Depending upon whether one chooses the upper or the lower sign, these equations will represent one or the other system of lines that are known to generate the simple hyperboloid. Since there will be only one of them here, one takes the upper sign and sets:

$$\frac{z}{c}=\frac{q}{m}\qquad\text{and}\qquad m^2=a^2\sin^2 p+b^2\cos^2 p+c^2$$

in order to introduce q with its previous meaning. The coordinates of the hyperboloid can then be expressed in terms of p and q as follows:

$$x=a\cos p+\frac{a\sin p}{m}\cdot q,$$

$$y=b\sin p-\frac{b\cos p}{m}\cdot q,$$

$$z=\frac{c}{m}\cdot q,$$

and from that, the line element on that surface will be:

$$ds^2=\left(m^2-c^2-\frac{2c^2(a^2-b^2)\sin p\cos p}{m}q+\frac{a^2b^2+a^2c^2\cos^2 p+b^2c^2\sin^2 p}{m^4}q^2\right)dq^2$$

$$-2\,\frac{m^2-c^2}{m}\,dp\,dq+dq^2.$$

One can derive uncountably more surfaces with the help of that surface that are bends of the simple hyperboloid. However, I regard the exhibition of such examples as unnecessary, since the path of further calculation is sufficiently well mapped out.

One is requested to confer an extension of this article that is submitted later in the conclusion of this volume, since its orientation did not permit it to be inserted here, according to the wishes of the author.