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On the bending of curved surfaces

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The ways of bending surfaces that are generated by straight lines that have been considered up to now are not the only ones possible; only the bending of the *plane* can always be generated by straight lines. By contrast, the following equations will represent the bending of the *helicoid*, which contains no straight lines (as is easy to prove), namely:

$$x = \frac{1}{a}\sqrt{n^2+q^2}\cos ap, \qquad y = \frac{1}{a}\sqrt{n^2+q^2}\sin ap,$$

$$z = \frac{1}{a}\int_0^q \sqrt{\frac{a^2n^2+(a^2-1)\,q^2}{n^2+q^2}}\,dq\,.$$

a is an arbitrary constant. These formulas give:

$$ds^2 = (n^2 + q^2)\,dp^2 + dq^2,$$

as one has for the helicoid above. It suffices to prove that they express a bending of it, which is a surface of revolution, moreover, which often follows when one sets q equal to a constant. For $q = 0$, one will get $\sqrt{x^2+y^2} = r = n / a$. The axis of the helicoid (which corresponds to the value zero for q) has therefore been bent into a circle of radius n / a here, which is wound around an uncountable number of times. The axis of rotation (z) goes through the center of the circle and is perpendicular to its plane. The generating curves correspond to the straight lines of the helicoid radiate the circumference of that circle, at which it has its vertex, in a manner that is symmetric on both sides. Its basic property will be given by the equations $a^2 r^2 = n^2 + q^2$ when one remarks that q means the arc length of the curve from the vertex to the point whose abscissa is r $(=\sqrt{x^2+y^2})$. For $a = 1$, the curve is the catenary, but the surface itself is the well-known one that possesses the property that it has the smallest area for a given boundary, just like helicoid. If one takes $a < 1$ then the surface will be imaginary when $q > \frac{a\,n}{\sqrt{1-a^2}}$. The bending will take place for only for a part of the helicoid then. However, when $a = 1$ or > 1, the entire helicoid will be bent.

Above all, one has a very attractive, but at the same time, extremely complicated, problem: Represent all possible ways of bending a given surface. Some of them can be found in the following way for certain cases:

One can always assume that $F = 0$ in the expression for the linear element and then set: $ds^2 = E\,dp^2 + G\,dq^2$. That will demand only that all curves on the surface for which only p changes should intersect the ones for which only q varies at right angles. That assumes that one takes z to be an undetermined function of p or q; e.g., let $dz = Q\,dq$, where Q depends upon just q. One will then get $dx^2 + dy^2 = E\,dp^2 + (G - Q^2)\,dq^2$. If one multiplies this by the equation $1 = \cos^2\psi + \sin^2\psi$ and sets $\sqrt{E} = u$, $\sqrt{G - Q^2} = \upsilon$, for brevity, then it will follow that:

$$dx^2 + dy^2 = (u \cos\psi\,dp + \upsilon \sin\psi\,dq)^2 + (u \sin\psi\,dp - \upsilon \cos\psi\,dq)^2.$$

If one then sets:

$$dx = \quad u \cos\psi\,dp + \upsilon \sin\psi\,dq,$$
$$dy = -u \sin\psi\,dp + \upsilon \cos\psi\,dq$$

then one can occasionally determine the functions Q and ψ in such a way that the values of dx and dy will become integrable. To that end, one must have:

$$\frac{du}{dq}\cdot\cos\psi - u\sin\psi\cdot\frac{d\psi}{dq} = \frac{d\upsilon}{dp}\cdot\sin\psi + \upsilon\cos\psi\cdot\frac{d\psi}{dp},$$

$$\frac{du}{dq}\cdot\sin\psi + u\cos\psi\cdot\frac{d\psi}{dq} = -\frac{d\upsilon}{dp}\cdot\cos\psi + \upsilon\sin\psi\cdot\frac{d\psi}{dp},$$

or, more simply:

$$(A) \qquad \frac{du}{dq} = \upsilon\,\frac{d\psi}{dp} \quad \text{and} \quad \frac{d\upsilon}{dp} = -u\,\frac{d\psi}{dq}.$$

Now, should the determination of ψ be possible then the condition:

$$\frac{d\left(\frac{1}{\upsilon}\cdot\frac{du}{dq}\right)}{dq} = -\frac{d\left(\frac{1}{u}\cdot\frac{d\upsilon}{dq}\right)}{dp}$$

would have to be satisfied by the choice of the function Q (for which $u = \sqrt{G - Q^2}$). This equation gives:

$$\frac{1}{\upsilon}\cdot\frac{d^2u}{dq^2} + \frac{1}{u}\cdot\frac{d^2\upsilon}{dp^2} = \frac{1}{\upsilon^2}\cdot\frac{d\upsilon}{dq}\cdot\frac{du}{dq} + \frac{1}{u^2}\cdot\frac{du}{dp}\cdot\frac{d\upsilon}{dp},$$

or since:

$$\upsilon \frac{d\upsilon}{dp} = \frac{1}{2}\frac{dG}{dp},$$

$$\upsilon \frac{d\upsilon}{dq} = \frac{1}{2}\frac{dG}{dq} - Q\frac{dQ}{dq},$$

$$\upsilon \frac{d^2\upsilon}{dp^2} = \frac{1}{2}\frac{d^2G}{dp^2} - \frac{1}{4\upsilon^2}\cdot\left(\frac{dG}{dp}\right)^2,$$

one will have:

$$\frac{d^2u}{dq^2} + \frac{1}{u}\left[\frac{1}{2}\frac{d^2G}{dp^2} - \frac{1}{4\upsilon^2}\left(\frac{dG}{dp}\right)^2\right] = \frac{1}{\upsilon^2}\cdot\frac{du}{dq}\left(\frac{1}{2}\frac{dG}{dq} - Q\frac{dQ}{dq}\right) + \frac{1}{u^2}\cdot\frac{du}{dp}\cdot\frac{1}{2}\frac{dG}{dp},$$

or when one sets υ^2 equal to its value $G - Q^2$ and develops it:

$$\frac{du}{dq}\cdot Q\frac{dQ}{dq} = MQ^2 + N,$$

in which:

$$M = \frac{d^2u}{dq^2} + \frac{1}{2u}\cdot\frac{d^2G}{dp^2} - \frac{1}{2u^2}\frac{du}{dp}\cdot\frac{dG}{dp},$$

$$N = \frac{1}{4u}\left(\frac{dG}{dp}\right)^2 + \frac{1}{2}\frac{du}{dq}\cdot\frac{dG}{dq} - GM.$$

Now, if the expressions $M / (du / dq)$ and $N / (du / dq)$ contain only q, but not p, then Q, and therefore ψ, can be found from them.

Among the applications that admit the foregoing remarks, I would like to point out only that of the *surface of revolution.* For one of them, one can set:

$$x = \varphi(q)\cdot\cos p, \qquad y = \varphi(q)\cdot\sin p, \qquad z = f(q);$$

hence:

$$ds^2 = \varphi(q)^2\, dp^2 + [\varphi'(q)^2 + f'(q)^2]\, dq^2.$$

Here, the quantities $u = \varphi(q)$, $G = \varphi'(q)^2 + f'(q)^2$ are independent of p, so the condition above will be satisfied. In fact, since $du / dp = 0$, one will get from equations (A) that:

$$\frac{d\psi}{dq} = 0, \qquad \frac{d\psi}{dp}\cdot\sqrt{G - Q^2} = \varphi' q,$$

from which, it will follow that:

$$\frac{d\psi}{dp} = \frac{1}{a}, \qquad G - Q^2 = a^2\,\varphi'(q)^2.$$

(a is an arbitrary constant.) One gets the equations:

$$x = a\,\varphi(q)\cdot\sin\left(\frac{p}{a}\right), \qquad x = a\,\varphi(q)\cdot\cos\left(\frac{p}{a}\right),$$

$$z = \int\sqrt{[\,f'(q)^2 + (1-a^2)\varphi'(q)^2\,]}\cdot dq$$

from this, which again represent a surface of revolution that will coincide with the previous one for $a = 1$, but will represent a bent version of it for other values of a. For example, one can set $\varphi(q) = \cos q$, $f(q) = \sin q$ for the *sphere*, from which one will obtain:

$$x = a\cos q\cdot\sin\left(\frac{p}{a}\right), \qquad x = a\cos q\cdot\cos\left(\frac{p}{a}\right), \qquad z = \int_0^q\sqrt{1-a^2\sin^2 q}\cdot dq\,.$$

For $a = 1$, these equations give a sphere of radius 1; however, for other values of a, they will give a bending of it.

In order to avoid misunderstanding, I shall remark that it is known that a closed convex surface cannot be bent as a whole without damaging it. Therefore, when one speaks of bending them, on must then think of their connectivity as having been interrupted in a certain extent. For example, in the initial formula here, a can mean an improper fraction, which will show that some parts of the original spherical surface overlap each other in the bent surface. In order to obtain the entire bent surface, one must then assume that p takes on only values from to 2π; after bending, the remaining spherical strips that range from $p = 2a\pi$ to $p = 2\pi$ will then cover the strips of the bent surface that are found between $p = 0$ and $p = 2\pi(1 - a)$. As one sees, one can get that result from the formulas themselves only when one establishes that the same values of p and q belong to the same points of the surface before and after the bending.

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