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On a special kind of dual relationship between figures in space.

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Translated by D. H. Delphenich

I was first induced to carry out the present geometric investigations by some results that are concerned with statics and are very simple in themselves. Namely, I found that of the two forces to which a system of forces in space is always reducible, and which do not lie in one and the same plane, the direction of one of them can be taken arbitrarily, and that furthermore when the direction of the one force goes through a given point, the direction must lie in a plane that is determined by that point and contains it, and that conversely, when the one direction is contained in a given plane, the other one must meet a point that is given with the point and is attached to it. In this way, every point in space then corresponds to a certain plane relative to a system of forces, and any plane to a point, and any line, as the direction of one of the two forces that have the same effect as the system, to another line, as the direction of the other force. There thus arise dual relationships between all spatial elements that generally have the same nature as the dual relationships between figures that have been often treated in recent times, except that here the restricting conditions appear that the plane that corresponds to a point must go through that point, and the point that corresponds to any plane must lie in it.

I have sought to treat these relationships purely geometrically, ignoring their origins in statics, and to communicate what I have found now in the hope that some of the relations that emerge from that special assumption will not be without interest. In particular, the construction that is given here of polyhedra that are likewise described in and around each other, as well as the system of lines, each of which corresponds to itself, and which are the axes for a system of forces for which the instantaneous sum of the forces is zero, can be worthy of some notice. In conclusion, I will discuss the connection that prevails between these duality relationships and statics theorems.

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1. Let x, y, z and x', y', z' be the rectangular or oblique coordinates of two points P and P, and let these quantities be coupled to each other by a single equation:

$$V = 0.$$

If one gives the coordinates x, y, z or x', y', z' definite values then $V = 0$ will become an equation between just three indeterminates, namely, x', y', z' or x, y, z. The two points will then be dependent upon each other in such a way that when the position of P or P' is well-defined, the other P' or P will lie in a plane that is given by that.

2. One will now find that the surface in which P' lies for a given position of P is always a plane, in which P can also be chosen. To that end, V must be of the form:

$$Lx' + My' + Mz' = 0,$$

where L, M, N are arbitrary functions of x, y, z, and the nature of these functions dictates the nature of the surface in which the point P is found for an arbitrarily-given position of P'. Therefore, should the latter surface always be a plane, L, M, N, O must be linear functions of x, y, z, and thus the equation $V = 0$ must have the form:

$$(A) \qquad \begin{aligned} &(ax + by + cz + d)\, x' + (a'x + b'y + c'y + d')\, y' \\ &+ (a''x + b''y + c''z + d'')\, z' + a'''x + b'''y + c'''z + d''' = 0. \end{aligned}$$

If the constants a, b, c, d, a', …, d''' are assumed to be given then any point P will correspond to a plane p', as the geometric locus of the points P', and every P', to a plane p, as the locus of P. Furthermore, if a linear equation between x', y', z' is given as the equation of the plane p', and one sets the coefficients of that equation proportional to the coefficients of the same equation in (A) then one will obtain three linear equations between x, y, z from which the latter coordinates, and thus the point P, can be determined. Hence, conversely, every point P' in any plane P' will belong to one and the same point P, just as every point P in a plane p will belong to one and the same point P'.

One thus has two systems of points and planes – the one, whose coordinate will be denoted by x, y, z, and which will be called S, while the other one S' will have the coordinates x', y', z' – and these two systems have a reciprocal relationship to each other such that every point P and every plane p of the one will correspond to a plane p' and a point P' of the other, respectively.

For the sake of brevity, we would like to call the plane that corresponds to a point the *opposite plane* of the point and the point that corresponds to a plane the *opposite point* of the plane.

3. Therefore, the relationship always exists between a point and its opposite plane or a plane and its opposite point that the coordinates of that point and the coordinates of any point in the plane must satisfy equation (A). Conversely: If one has two points whose coordinates satisfy the equation then each of them will lie in the opposite plane of the other.

As a result, if P is a point in the opposite plane to P' then the coordinates of P and P' will fulfill equation (A), and P' will also be a point in the opposite plane of P. In other words:

> *If one has a plane p and a point P that lies in it then the opposite point P′ of the former will also lie in the opposite plane p′ of the latter.*

Thus, if four or more points lie in a plane then the opposite planes of the points must contain the opposite point of the plane, and thus intersect that point together. Conversely: If four or more planes intersect as a point then the opposite points of the planes must lie in a plane, namely, the opposite plane of the point.

We further conclude: If several point R, S, T, … are common to two planes p, q then the opposite plane of each point must contain the opposite point P' of the plane p, as well as the opposite point Q' to the plane q, and therefore the line $P'Q'$; i.e., if three or more points lie in a line then the opposite planes of the point will intersect in a line, in any case. The converse theorem is proved in a similar way, namely, that when three or more planes intersect in a line, the opposite points will, at the same time, be contained in a line.

Therefore, just as any point has its opposite plane, and any plane has its opposite point, every straight line will also correspond to an *opposite line*, such that every opposite plane to a point that is chosen in the one line will contain the other line, an every opposite point to any plane that goes through the one line will be found on the other one.

4. Without stopping to pursue the further development of this reciprocal relationship, which has already been treated many times before, we would now like to assume the special relationship between the two systems that *every point P' in the system S' is contained its opposite plane p in the system S*, so equation (A), since one can regard it as the equation of the plane p, will also be true when one sets $x = x'$, $y = y'$, $z = z'$. However, this will give:

$$ax'^2 + b'y'^2 + c''z'^2 + (b'' + c')\,y'\,z' + (c + a'')\,z'x' + (a' + b')\,x'y' + (d + a''')\,x' \\ + (d' + b''')\,y' + (d'' + c''')\,z' + d''' = 0,$$

and since this equation should be valid for any arbitrary choice of point P' or (x', y', z'), one must have:

$$a = 0, \quad b' = 0, \quad c'' = 0, \quad c' = -\,b'', \quad a'' = -\,c, \quad b = -\,a', \\ a'' = -\,d, \quad b''' = -\,d', \quad c''' = -\,d'', \quad d''' = 0.$$

With that, equation (A) will consolidate to:

$$(-\,a'y + cz + d)\,x' + (a'x - b''z + d')\,y' + (-\,cx + b''y + d'')\,z' - dx - d'y - d''z = 0,$$

or, if, for the sake of greater simplicity, we write a, b, c, f, g, h, instead of b'', c, a', d, d', d'', from now on:

$$(B) \qquad (bz - cy + f)\,x' + (cx - az + g)\,y' + (ay - bx + h)\,z' - fx - gy - hz = 0.$$

Since, as one easily confirms, this equation remains unchanged when x, y, z are permuted with x', y', z', not only does every point P' of the system S' lie in its opposite plane p in the system S, but also every point P of the latter system will lie in its opposite plane p' in the former, moreover. On the same basis, we can shed light on the fact that the opposite planes of two coincident points P and P' of the one system and the other must also coincide, while previously the point (p, q, r) took on an opposite plane whose equation was:

$$(ax + by + \ldots)\, p + (a'x + b'y + \ldots)\, q + \ldots = 0,$$

when it was considered to belong to the system S', and the same point, when considered to be a point in the system S, corresponded to an opposite plane whose equation was:

$$(ax' + a'y' + \ldots)\, p + (bx' + b'y' + \ldots)\, q + \ldots = 0.$$

Thus, all points and planes in space are put into a reciprocal relationship such that a point and a plane are associated, and the former is contained in the latter. Any differences between the two systems S and S' are to be regarded as having been eliminated completely now.

5. Nevertheless, the theorems that were found by the general considerations in no. **3** will also remain valid now. Therefore, if p is a plane, and Q is any point in it then the opposite point P' of p will also be contained in the opposite plane q' of Q. Since P' now lies in p, and Q now lies in q', P' and Q will lie in the mutual intersection of p and q', and we can also state the theorem in the following form:

I. *If the opposite point of one of two intersecting planes lies along their line of intersection then the opposite point of the other plane will also be contained in it, and if the opposite plane of one of two points meets that of the other point then the opposite plane of the latter will also contain the former point.*

Just as before, it will follow further from this that:

II. *The opposite planes of several points that lie in a plane will intersect in a point that lies in the former plane and is its opposite point.*

III. *The opposite points of several planes that intersect at a point lie in a plane that contains the former point and is its opposite plane.*

IV. *All straight lines in space can be grouped pair-wise as lines and opposite lines, and each of these pairs possesses the property that the opposite planes of all points that are taken from one line will contain the other line, such that every point of the one line will have the plane that goes through it and the other line for its opposite plane, and such that conversely, the opposite point of every plane that goes through the one line will be the intersection of the plane with the other line.*

V. *A line that is drawn through two points then has the intersection of the opposite planes of the points for its opposite line, and the line that connects the opposite points of two intersecting planes will be the opposite line to that line of intersection.*

An immediate consequence of these properties of opposite lines is then:

VI. *The opposite lines of several lines that lie in a plane intersect in a point of that plane – namely, at the opposite point to the latter – and the opposite lines to several lines that meet at a point will be contained in a plane that goes through the point, namely, the opposite plane to the point.*

6. In order to make these theorems clearer by an example, we would like to seek to determine the opposite points of the three coordinate planes.

The equation for the *yz*-plane in terms of x', y', z' is $x' = 0$, which might also be true for y' and z'. Therefore, the coefficients of y' and z' in (*B*) must be zero, along with the sum of the terms that are not endowed with x', y', z'; hence:

$$\begin{aligned} cx - az + g &= 0, \\ ay - bx + h &= 0, \\ fx + gy + hz &= 0. \end{aligned}$$

If one multiplies these three equations by h, $- g$, a, respectively, and adds them then that will give $(af + bg + ch)\, x = 0$, so:

$$x = 0, \quad \text{and therefore} \quad y = -\frac{h}{a}, \; z = \frac{g}{a}.$$

These are then the three coordinates of the opposite point of the *yz*-plane. We would like to call it *A*; since $x = 0$, it will lie in the plane itself, as would be proper.

One gets the coordinates of the opposite point *B* to the *zx*-plane in the same way:

$$x = \frac{h}{b}, \quad y = 0, \quad z = -\frac{f}{b},$$

and the opposite point *C* of the *xy*-plane:

$$x = -\frac{g}{c}, \quad y = \frac{f}{c}, \quad z = 0,$$

so *B* is contained in the *zx*-plane, and *C* is contained in the *xy*-plane. All three points, however, lie in the plane that has the equation:

$$fx + gy + hz = 0.$$

From III, that plane will be the opposite plane of the point at which the three coordinate planes intersect, which will then be the coordinate origin *M*. In fact, equation (*B*) will also reduce to $fx + gy + hz = 0$ for $x' = 0$, $y' = 0$, $z' = 0$.

Finally, the lines *BC*, *CA*, *AB* will be the opposite lines of the *x*, *y*, *z* axes, resp.

7. The few results that we have obtained up to now suffice to enable us to construct a system of corresponding points and plane in the stated way without the aid of calculation. Namely, since the angles that coordinate planes make with each other is completely arbitrary, one can draw three planes α, β, γ at will through a point *M* (see Fig. 1) for coordinate planes. One takes two points *A* and *B* in α and β to be the opposite points of these planes arbitrarily. The ratios $h : a$, $g : a$ are determined by the position of *A*, and the ratios $h : b$, $f : b$ are determined by the position of *B*, so the ratios between *f*, *g*¸ *h*, *a*, *b* are determined by the two points.

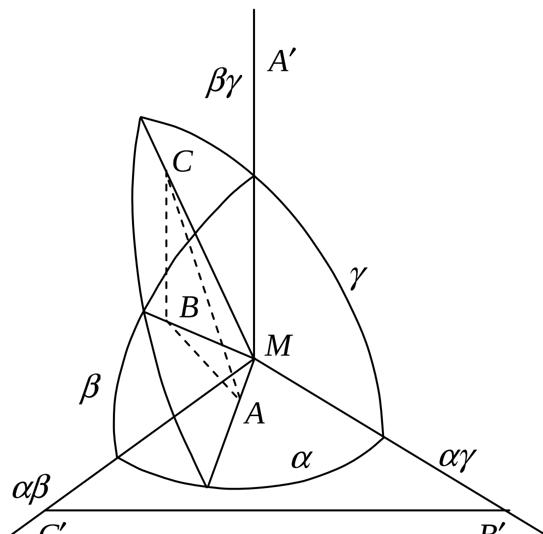

Figure 1.

One lays a new plane μ through *A*, *B*, *M*; it is the opposite plane of *M* and contains the opposite point to the plane γ in its intersection with that plane. One then takes a point *C* arbitrarily from that intersection of μ and γ to be the opposite point to γ. The ratio of *c* to *f* or *g* is then given by it, and since the ratios of all six constants that enter into equation (*B*) are then determined, it must also be possible for one to determine the opposite plane δ to any other point *D* and the opposite point *D* to any plane δ.

8. *a*. First of all, a given point *D*, whose opposite plane we desire to seek lies in the intersection of the planes β and γ – or in $\beta\gamma$, if we express the intersection of two planes by the juxtaposition of the symbols that denote the planes. Since the opposite points of β and γ are *B* and *C*, resp., from **V**, *BC* will be the opposite line of $\beta\gamma$, and as a result of **IV**, *BCD* will be opposite plane of *D*. In the same way, *CAD* or *ABD* will be the opposite plane of *D* when *D* is found in $\gamma\alpha$ or $\alpha\beta$.

b. One similarly illuminates the fact that when the point D lies along one of three lines BC, CA, AB, its opposite plane must be laid through it and $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, resp.

c. Now, let the point D be taken arbitrarily. One lays the planes BCD, CAD, ABD, which intersect the lines $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, resp., at A', B', C', resp., through it and through the three lines BC, CA, AB, so those points will be the opposite points of the three planes that intersect at D (**IV**), and as a result (**III**), A', B', C', along with D, will be in a plane that is the opposite plane to D, therefore the desired δ.

Since D itself lies in δ, two of the three points A', B', C' will already suffice for the determination of δ. The fact that the third point is also contained in δ leads us to a remarkable property of our figure. It relates to two tetrahedra $A'B'C'M$ and $ABCD$, if we consider them somewhat more carefully. The former is bounded by the four faces α, β, γ, δ, and circumscribes the other one, since α, β, γ, δ, resp., contain the points A, B, C, D as their opposite points, resp. However, it is, at the same time, also inscribed in the other one, since A', B'¸ C', M are the opposite points of the faces BCD, CAD, ABD, ABC of the other one. Therefore, we have two tetrahedra, each of which is, at the same time, circumscribed by and inscribed in the other one, and we can now put the previously-imagined property into words of the following form:

> *If the four vertices A, B, C, M of one of two tetrahedra $A'B'C'M$ and $ABCD$ lie in the four faces BCD, CDA, DAB, ABC of the other one and three vertices A, B, C of the other one lie in the faces $B'C'M$, $C'MA'$, $MA'B'$ of the former then the fourth vertex D of the other one will also lie in the fourth face $A'B'C'$ of the first one, and the one tetrahedron is, at the same time, circumscribed by and inscribed in the other one.*

d. The opposite plane of D can also be found when one lays three planes $D\beta\gamma$, $D\gamma\alpha$, $D\alpha\beta$ through D and the lines $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$. If these lines BC, CA, AB intersect in A'', B'', C'', resp., then (**IV**) they will be the opposite points to the three planes, and the plane that is laid through A'', B'', C'', D will likewise be the opposite plane of D (**III**).

Meanwhile, one remarks that since the three planes α, β, γ have the point M in common, and since the three planes $D\beta\gamma$, $D\gamma\alpha$, $D\alpha\beta$ then intersect in the line DM, the opposite points A'', B'', C'' of the latter planes will then lie simultaneously along a line, namely, the opposite line to DM. Thus, the opposite plane of D will be determined by not just these three points, but also by any two of them already and the point D itself. Moreover, since a point can have only one opposite plane, A'', B'', C'' must be contained in a plane with A', B', C', and since the former three points lie along the lines BC, CA, AB, resp., they will be nothing but the intersection of the sides of the triangle ABC with the plane $A'B'C'$, from which, it will, in turn, emerge that A'', B'', C'' lie on a line, namely, the mutual intersection of the planes ABC and $A'B'C'$.

9. We now go on to the inverse problem, which is solved in an entirely analogous way: Find the opposite point D to a given pane δ.

a. If the plane δ is drawn through one of the six lines *BC*, *CA*, *AB*, $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ then its opposite point will lie where it is cut by $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, *BC*, *CA*, *AB*, $\beta\gamma$, resp.

b. If the plane δ has any position then the lines $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ will intersect at the points A', B', C'. $A'BC$, $B'CA$, $C'AB$ will be the opposite planes of these three points that lie in δ, and their mutual point of intersection will then likewise lie in δ (**II**), and *D* will be the opposite point of δ that we seek.

Since δ and the three planes $A'BC$, … intersect in *D*, two of these planes, along with δ, will already suffice in order to find *D*. One easily verifies that this construction also leads to the theorem above on the circumscribed and inscribed tetrahedra.

c. A process for finding the opposite point to an arbitrary plane *s* consists in the following: Let δ cut the lines *BC*, *CA*, *AB* in the points A'', B'', C'', resp. If one lays the planes $A''BG$, $B''GA$, $C''AB$ through $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$ then they will be the opposite plane of A'', B'', C'', and will therefore intersect in a straight line, since A'', B'', C'' lie along a line in the intersection of δ with *ABC*. Any of these two lines will then be the opposite line of the other one, and since the latter line lies in the plane δ, the intersection of the former line with δ must be the opposite point of δ.

10. I have already shown that, and how, one can construct a second tetrahedron to a given one that will be simultaneously inscribed in and circumscribed by the first one in a small paper that is found in Band III of this Journal, page 273, *et seq*. However, with the help of the theory that was just developed, one easily sees how this construction can be generalized, and how:

> *It is possible to construct a second polyhedron from a given one that has just as many vertices and faces as the first one and whose vertices lie in the faces of the first one, but whose faces contain the vertices of the first one.*

In order to solve this problem, let *S* be a vertex of the given polyhedron, and let α, β, γ, … be the faces of *S*, in the order in which they meet each other, such that α has a common edge with β, β has a common edge with γ, etc. One seeks the opposite points to α, β, γ, … (which will be called *A*, *B*, *C*, …, resp.), and thus to construct the polygon *ABC*, … It will be planar, and contain the point *S* in its plane, since α, β, γ, … come together at the point *S*. In the same way, one constructs another polygon from every other vertex of the polygon. Now, every two polygons will simultaneously belong to the sides of all these polygons: The side *AB* of the polygon *ABC*, … at *S*, e.g., will also be a side of the polygon at the vertex *T* when the other endpoint of the edge of the polyhedron in which the faces α and β meet each other, and which has *S* for one endpoint, is *T*. All of the polygonal faces thus obtained will then be connected to a new polyhedron as faces, and that second polyhedron will be said to be simultaneously inscribed in and circumscribed by the first one. It is inscribed by it, since its vertices *A*, *B*, … lie in the faces a, b, … of the first one; it is circumscribed by it, since its faces ABC, …, etc. encounter the vertices S, etc., of the first one.

Above all, one sees that a reciprocal relationship exists between the two polyhedra such that the vertices, edges, and faces of the one are the opposite points, opposite lines, and opposite planes of the faces, edges, and vertices of the other one, resp. A tetrahedron will, in turn, correspond to another tetrahedron under this relationship, a hexahedron, to an octahedron, a dodecahedron to an icosahedron, and conversely, an octahedron to hexahedron, etc.

11. For the further pursuit of our investigations into reciprocal relationships, we would now like to seek to determine the opposite point to a plane that is parallel to the yz-plane. The equation of such a plane in the coordinates x', y', z' is x' = a constant length l, which might also be true for y' and z'. The coefficients of y' and z' must then be zero in (B); thus:

$$cx - az + g = 0, \qquad ay - bx + h = 0,$$

and in addition:

$$l\,(bz - cy + f) - fx - gy - hz = 0.$$

The coordinates of the desired opposite point can be determined from these three equations; they are then the equations of three planes intersect at the opposite point. Since only the third equation contains the distance l from the given plane to the yz-plane, the opposite point of any other plane that is parallel to the yz-plane will also belong to the first two equations. The opposite points of all planes that are parallel to the yz-plane thus lie along a straight line whose equations are:

$$cx - ax + g = 0 \qquad \text{and} \qquad ay - bx + h = 0.$$

However, the latter line runs parallel to the previous one, since each of the two is parallel to the line that goes through the origin of the coordinates and has the equations:

$$cx - az = 0,\ ay - bx = 0, \qquad \text{so } bz - cy = 0, \text{ as well,}$$

or – what amounts to the same thing – the proportions:

$$x : y : z = a : b : c.$$

Now, since the choice of coordinates is entirely open, we conclude:

VII. *The opposite point of three or more mutually parallel planes lies on a straight line, and the straight lines in which the opposite points of two, and therefore also more, systems of parallel planes lie are all parallel to each other.*

Thus, e.g., the line that connects the planes that are parallel to the xy-plane must be likewise parallel to the former parallel lines, which can also be recognized by calculations that are similar to the previous ones. Namely, the equations of those lines will be:

$$bz - cy + f = 0 \qquad \text{and} \qquad cx - az + g = 0.$$

The direction that all of these parallel lines possess is then the only one of its kind for any system of planes and points that one has endowed with the previous relationship. For that reason, we would like to call it the *principal direction* of the system. In equation (B), it is given by the ratios of the coefficients a, b, c, which are coefficients that will be proportional to the cosines of the angles between the principal direction and the x, y, z axes when the coordinate system is a rectangular one.

12. If we assume that this principal direction is parallel to the z-axis then a and b will be zero, and equation (B) will take on the simpler form:

$$(f - cy)\, x' + (g + cx)\, y' + hz' - fx - gy - hz = 0.$$

The equations of the line that goes through the opposite point of the planes that are parallel to the xy-plane will then be:

$$f - cy = 0 \qquad \text{and} \qquad g + cx = 0.$$

If we now take this line that is parallel to the z-axis to be the z-axis itself, then a and b, as well as f and g will be equal to zero, and equation (B) will then take on the following, simplest-possible, form:

$$(C) \qquad xy' - yx' = k\,(z' - z),$$

where k is set to $-h / c$, instead of the previous value.

For a plane that is parallel to the xy-plane whose equation is $z' = n$, one will then have $xy' - yx' = k\,(n - z)$, and as a result, $x = 0$, $y = 0$, $z = n$, i.e., the opposite point of the plane is, as it should be, its intersection with the z-axis. However, if the plane whose opposite point is to be determined is parallel to the yz-plane, and therefore its equation will be $x' = l$, then one will have $xy' - l\,y = k\,(z' - z)$, or – what amounts to the same thing:

$$\frac{x}{y}\,y' - \frac{k}{y}\,z' = l - l\,\frac{x}{y}, \qquad \text{so} \qquad \frac{x}{y} = 0,\ \frac{k}{y} = 0,\ l - k\frac{z}{y} = 0,$$

i.e., ultimately, y and z must be large and behave with respect to each other as k does to l. The opposite point of the plane will then lie in it at infinity along a direction that is determined by the ratio $k : l$. If the given plane is the yz-plane itself – so $l = 0$ – then one merely has to take y to be infinitely large, and as a result, the opposite point will lie at infinity in the direction of the y-axis. The latter result will also follow from the fact that the opposite point of the xy-plane lies at the origin of the coordinates along the y-axis, in which the yz and xy-planes intersect. In the same way, one finds that the opposite point of the zx-plane is at infinity along the x-axis.

Now, since only the z-axis has a well-defined direction in the present coordinate system, the directions of the other two axes can, however, be arbitrary, and it then follows that:

VIII. *Each of the planes that is parallel to the principal direction has an opposite point at infinity, and conversely, the opposite plane of a point at infinity is parallel to the principal direction.*

If one then takes x, y, z to be infinitely large and have given ratios a: b : c then equation (C) will become $ay - bx + kc = 0$, and will belong to one of the planes that are parallel to the z-axis.

We now direct our attention to a plane that is parallel to the x-axis, and therefore has the equation $z' = ay' + b$. If we substitute that value of z' into (C) then we will get:

$$xy' - yx' = k\,(ay' + b - z),$$

and thus $x = ak$, $y = 0$, $z = b$, as the coordinates of the opposite point of the plane. Since $y = 0$, this point will always lie in the zx-plane (which might also be laid in plane that is parallel to the x-axis). Due to the indeterminacy of the x-axis, we draw the conclusion from this that:

IX. *The opposite points of two or more planes that are parallel to one and the same line will lie in one and the same plane that is parallel to these lines and the principal direction of the system.*

13. The theorems **VII**, **VIII**, and **IX**, which were obtained by analysis, can also be derived from the earlier theorems **I**, …, **IV** by simple geometric considerations.

a. Since, from **III**, the opposite point to several planes that intersect in a point lies in a plane along with that point of intersection, and that point of intersection is the opposite point of the latter plane, and since several planes that intersect in parallel can also be regarded as ones that intersect at a point at infinity, the opposite point to several planes that intersect in parallels must be contained in a plane that is likewise parallel to the parallel lines of intersection, and whose opposite point lies at infinity in the direction that is determined by the parallels.

b. Since, furthermore, the opposite points to planes that intersect in one and the same line likewise lie on a line (**IV**), when the first line lies at infinity, and therefore the planes are parallel to each other, their opposite points must also lie along a line.

c. One now imagines two systems of planes; let the planes of either system be parallel to each other, but not with those of the other system. Call the line in which the opposite points of the one system lie a and the line for opposite points of the other system b. Now, since two planes from one and the other system also intersect in parallel lines, from (a), the opposite point of all planes – and thus the two lines a and b, as well – must lie in a plane, and as a result, must intersect or be parallel to each other. However, is a and b intersect in a point A then it would be the common opposite point to two planes of one and the other system. Thus, these two intersecting planes would also be the opposite planes of one and the same point A, which is not possible. Therefore, a and b are parallel to each other, and it also follows that every other line that contains the opposite points of any parallel planes of a third system will be parallel to them, as well.

d. Conversely: The opposite planes α and β of two points *A* and *B* that lie along a parallel to the principal direction are parallel to each other. If this were not the case then one could lay a plane β' that is parallel to α through *B*. The opposite point of β' would then be the point at which β' is met by a parallel to the principal direction that goes through *A*, and as a result, *B* itself. Therefore, *B* would have two different opposite planes β and β', which is not possible.

e. Any plane γ that is parallel to the principal direction has an opposite point at infinity. Then, let *A* and *B* be two points in *g* that lie in a line that is parallel to the principal direction. The opposite plane α and β to *A* and *B*, are, as a result, parallel to each other, and therefore the lines *a* and *b* in which γ is intersected by α and β are also two parallel lines. Now, from **I**, the opposite point of γ must lie along *a*, as well as along *b*, and therefore at infinity along a direction that is determined by those parallels.

One can then remark that all planes whose opposite points lie in a plane γ that is parallel to the principal direction intersect that plane in parallel lines; each of these planes must go through the opposite point of γ, which lies at infinity.

f. If the direction along which a point at infinity lies is given, and should its opposite plane be found then one would draw three planes α, β, γ parallel to that direction, determine their opposite points *A*, *B*, *C*, and *ABC* would then be the desired opposite plane. If one takes, as is therefore possible, α and β to be parallel to each other then *AB* will be parallel to the principal direction, and *ABC* will therefore be a plane that is parallel to the principal direction. Just as any plane that is parallel to the principal direction will then have an opposite point at infinity, conversely, so will the opposite plane of a point at infinity be parallel to the principal direction.

We add that:

X. *The opposite plane to a point U that lies at infinity along the principal direction is likewise at infinity, but it does not have a well-defined position, and conversely, the opposite point to any plane u at infinity lies at infinity along the principal direction.*

The first part of this theorem sheds light upon the fact that when α is any plane that is not parallel to the principal direction, and *A* is its opposite point, the line *AU* can be considered to be parallel to the principal direction, and as a result, the plane that is drawn through α parallel to *U* will be the opposite plane of *u*. In order to convince oneself of the validity of the converse theorem, one remarks that when α is a plane that is parallel to *u*, but not the plane at infinity, and *A* is its opposite point, the opposite point to the plane *u* must be its intersection with a line that is drawn through *A* and parallel to the principal direction.

14. Since the principal direction is the only one of its kind, and therefore a certain symmetry in the positions of the planes and their opposite points can arise in relation to it, but the symmetric properties of a figure can be developed most conveniently by applying a rectangular coordinate system, we would now like to take any plane that meets the principal plane at a right angle to be the *xy*-plane, and, as before, choose the *z*-axis to be the parallel to the principal direction that meets the *xy*-plane at its opposite point. The equation in the coordinates x', y', z' for the plane p' that has the point (x, y, z) or P for its opposite point will then be the equation (C) that was already obtained in no. **12**.

Now, if the point P lies in the *xy*-plane to begin with, then $z = 0$, and the equation for p' will then become:

$$xy' - yx' = k\, z'.$$

It will, as is proper, be fulfilled by $x' = x$, $y' = y$, $z' = 0$, and for $x' = 0$, $y' = 0$, $z' = 0$; i.e., the plane p' will go through its opposite point P and through the origin of the coordinates, and the latter will follow from the fact that P lies in the *xy*-plane and has the point M for its opposite point. (Cf., **I**) Since the opposite plane to every point P that is contained in the *xy*-plane must be drawn through the line MP that links it to M, if one wishes to determine that opposite plane completely then one will need to know only the angle that it makes with the *xy*-plane.

For that reason, one sets $x = r \cos \varphi$, $y = r \sin \varphi$, $x' = r' \cos \varphi'$, $y' = r' \sin \varphi'$, where r is then the distance of the point P from M, φ is the angle that MP makes with the *x*-axis, and r', φ' denote the same things for the projection of any point in the plane p' onto the *xy*-plane. The equation will then be:

$$rr' \sin(\varphi' - \varphi) = kz', \qquad \text{and as a result,} \qquad \frac{r}{k} = \frac{z'}{r' \sin(\varphi' - \varphi)}.$$

Now, $r' \sin(\varphi' - \varphi)$ is nothing but the perpendicular that is dropped in the *xy*-plane from the projection (x', y') of a point (x', y', z') in the plane p' onto the line MP or the intersection of p' with the *xy*-plane, and therefore $\dfrac{z'}{r' \sin(\varphi' - \varphi)}$ equals the tangent of the angle that p' makes with the *xy*-plane. From a previous equation, that tangent is, however, equal to r / k, and thus the necessary angle that we have yet to know will be determined as simply as possible.

If we then call the present *z*-axis, or the line that can be drawn through the opposite points of the planes that meet the principal direction at right angles, the *principal line* of the system and ponder the fact that any plane that is normal to it can be taken to be the *xy*-plane, then we can put the result that was obtained into words in the following way:

XI. *The opposite plane of a point contains the perpendicular that is dropped from that point to the principal line, and makes an angle with the principal line whose cotangent is proportional to the perpendicular.*

All points that lie at the same distance from the principal line will then lie on the surface of a cylinder that is described with the principal line as its *axis*, so their opposite planes will make equal angles with the principal line along the same side. Of the opposite planes to points at unequal distances from the principal line, the closest one will make the largest angle, and when the distance increases from zero to infinity, the angle will decrease from a right angle to zero. (Cf., **VIII**)

Theorem **XI** next teaches us how the opposite plane to a given point can be determined with the help of a principal line. Conversely, should the opposite plane to a given plane *p* be found, one would then lay a plane through the point of intersection of *p* with the principal line that is perpendicular to the latter. The opposite point to *p* would then lie along the line of intersection of that plane with *p* and be at a distance from the principal line in it that is proportional to the cotangent of the angle between *p* and the principal line.

15. It still remains for us to consider the dual relationships that prevail between lines and their opposite lines more closely. In no. **5**, **VI**, we saw that the opposite lines to several lines in a plane all intersect at the opposite point of the plane. Now, let *a*, *b*, *c*, … be several lines that are parallel to just one and the same plane. One lays the planes α, β, γ, …, which are parallel to that plane, through *a*, *b*, *c*, …, resp., and lets *A*, *B*, *C*, …, resp. be their opposite points. Now, since the opposite lines to *a*, *b*, *c*, … go through *A*, *B*, *C*, …, resp. (**IV**), and *A*, *B*, *C*, … will lie in lines that are parallel to the principal direction (**VII**), we conclude that:

XII. *If several lines are parallel to one and the same plane then their opposite lines will meet one and the same line. That line will be parallel to the principal direction, and will meet the plane of its opposite points.*

If a' is the opposite line of *a*, and one lays a plane through *a* and a point *A* at infinity in a' (so it is a plane that is parallel to a') then that plane will be the opposite plane to *A* (**IV**) and will be parallel to the principal direction (**VIII**); as a result:

XIII. *Every plane that is simultaneously parallel to a line and its opposite line is also parallel to the principal direction.*

XIV. *The opposite line to any line that is parallel to the principal direction lies at infinity.*

Every plane that goes through such a line will have an opposite point at infinity.

16. Those lines that are identical with their opposite lines merit special attention. The opposite line to a given line *a* will be found by **V** when one determines the opposite points *A* and *B* to two planes α and β, resp., that go through *a*; the line *AB* will then be the desired opposite line. If one now assumes that the opposite point *A* of one of two planes α lies in *a* itself then, from **I**, the opposite point *B* of β – as well as the opposite point of any other planes that goes through *a* – will also lie in *a*. With that, the line *a* will then coincide with its opposite line.

A line that coincides with its opposite line is called a *double line*. We can present the following theorems about them by means of **I**, **II**, and **III**:

XV. *The opposite point to any plane that goes through a double line lies along the double line, and a double line is contained in the opposite plane of every point that lies along it.*

XVI. *Every line that is drawn in a plane through its opposite point, and every line that goes through a point and simultaneously contains the opposite plane of the point is a double line.*

XVII. *If two double lines lie in a plane then their mutual intersection (which can also be at infinity) will be the opposite point of the plane.*

From **XV**, this opposite point must lie along the one double line, as well as the other one. It follows further from this that:

XVIII. *All double lines that lie in a plane intersect in a point, and conversely, all double lines that go through a point are contained in a plane,*

since, otherwise, due to **XVII**, it would have one more point than an opposite plane.

17. The system of double lines thus fills up all of space. Uncountably many double lines then intersect at any point in space, although they will all lie in a plane, and there will be uncountably many double lines in any plane, which will, however, all meet at a point.

If a line is given by its equations, and one desires to know whether it belongs to the system of double lines then one must examine whether the coordinates of two of its points will satisfy equation (C) when they are substituted for x, y, z and x', y', z' in it. In that equation, (x', y', z') will then be a point of the plane that has (x, y, z) for its opposite point, but the line that connects two such points will be a double line (**XVI**). Therefore, let:

(a) $\quad \dfrac{x}{a}+\dfrac{z}{c} = 1, \qquad$ (b) $\quad \dfrac{y}{b}+\dfrac{z}{c} = 1$

be the two equations of a line. A point that lies on it is (a, b, 0). If one then sets $x' = a$, $y' = b$, $z' = 0$ in (C) then one will get:

(c) $$ay - bx = kz$$

for the equation of the opposite plane of the point (a, b, 0), and a line must lie in this plane if it is to be a double line. Equation (c) and one of the two (a) and (b) are therefore the two general equations of a double line. One can also propose the equation:

$$\frac{1}{c}-\frac{1}{c'} = \frac{k}{ab},$$

which emerges by eliminating x, y, z from (a), (b), (c), as the condition under which the line that is expressed by (a) and (b) will be a double line.

If one assumes that the two points (x, y, z) and (x', y', z') in (C) are infinitely close to each other, so one sets $x' = x + dx$, etc., then (C) will go to:

$$(D) \qquad x\,dy - y\,dx = k\,dz.$$

This is then the general differential equation of a double line, and in fact, one will come back to this differential equation when one differentiates (a) or (b), and (c), and then eliminates the arbitrary constants a, b, c.

18. Let a be a line that has a distinct line a' for its opposite line, and let A, A' be two arbitrarily-chosen points in a, a'. A will then be the opposite plane of Aa' (**IV**) and therefore AA' will be a double line (**XVI**); i.e.:

XIX. *Any line that simultaneously cuts a simple line and its opposite line is a double line.*

Furthermore, if l is a double line and a is a simple line that is cut by l then the opposite point to the plane la will be contained in l (**XV**) since that plane goes through l. Likewise, however, the opposite point of the plane la, as one that is drawn through a, must lie in the opposite line to a (**IV**); as a result, l must cut this opposite line; i.e.:

XX. *A double line that cuts a simple line also meets the opposite line of the simple one.*

A remarkable third theorem can be derived from this theorem, in conjunction with the foregoing one. Namely, if a, b are two simple lines, a', b' are their opposite lines, and l is a line that cuts a, b, a' at the same time then due to the fact that a and a' encounter each other, l will be a double line, and as such, since it meets b, it must also cut b'; thus:

XXI. *If one has two lines and their opposite lines then any line that encounters three of these four lines will also meet the fourth one. Four such lines can then always be regarded as just as many different positions of a line that generates a hyperbolic hyperboloid.*

Connection between the reciprocal relationships that were explained up to now and some theorems in statics.

19. As in no. **7**, let α, β, γ (Fig. 1) be three planes that intersect at M, let A, B two arbitrarily points in α, β, resp., and let C be an arbitrary point in the line of intersection of the planes γ and MAB. One can let two forces act in the directions $\alpha\beta$ and AB, which one denotes by [$\alpha\beta$] and [AB]. One decomposes the force [$\alpha\beta$]into two other ones [MA] and [$\alpha\gamma$] along the direction MA and $\alpha\gamma$, which is always possible, since the latter directions and $\alpha\beta$ lie in a plane and intersect at a point M. Moreover, one can also understand the force [MA] to mean one that is negative or directed along AM, and the same thing will also be true for the remaining forces that are denoted in the same way.

Thus, the initial two forces [$\alpha\beta$] and [AB] have now been converted into three: [$\alpha\gamma$], [MA], [AB]. The last two of these have a resultant that goes through A and is contained in the plane MAB. However, which of all the lines that go through A and lie in MAB is the direction of that resultant will depend upon the ratio of the intensities of the initial two forces [$\alpha\beta$] and [AB]. We would therefore like to assume that this ratio has been determined in such a way that AC itself is the direction of the resultant, and we have then reduced the initial forces [$\alpha\beta$] and [AB] to [$\alpha\gamma$] and [AC], which gives us the following two theorems:

a) *If one has two forces whose directions $\alpha\beta$, AB do not lie in a plane and a direction $\alpha\gamma$ that lies in a plane α with one $\alpha\beta$ of the first two, and therefore has a point M in common with $\alpha\beta$, then it will always be possible to convert the two forces into two other forces that have the same effect as them, and one of which has the direction $\alpha\gamma$. The direction of the other one will then go through the point A in which the plane α is met by the direction AB, and will be contained in the plane that is drawn through M and AB.*

b) *If A, B, C the opposite points of the plane α, β, γ, resp., and therefore AB is the opposite line of $\alpha\beta$, and AC is the opposite line of $\alpha\gamma$, then it will always be possible to let two forces act along the direction $\alpha\beta$ and AB that are related to each other in such a way that they can be converted into two other ones along the direction $\alpha\gamma$ and AC.*

Now, if the forces [$\alpha\beta$] and [AB] are related to each other in that way then one can further show that, above all:

c) *If one of any two forces R and R′ that have the same effect as the latter lie in a given plane δ then the other one will meet the opposite point of that plane.*

Let C' and B' be two arbitrary points in $\alpha\beta$ and $\alpha\gamma$ then. From *a*), [ab] and [AB] can then be converted into two other forces Q and Q', and we would like to assume that when one of them Q has the direction $C'B'$, and therefore cuts $\alpha\beta$ at C', the other one Q' will lie in the plane $C'AB$, and thus, in the opposite plane to C'. However, since [$\alpha\beta$] and [AB] have the same effect as [$\alpha\gamma$] and [AC], and $\alpha\gamma$ is cut by $C'B'$ at B', Q' will be, on the same basis, contained in the plane $B'AC$ or the opposite plane of B'. Therefore, the force Q' will have the intersection of the planes $C'AB$ and $B'AC$ for its direction; i.e., the opposite line of the line $C'B'$ or the direction of Q.

Now, let one (say, R) of two forces R and R', which should have the same effect as [$\alpha\beta$] and [AB], and as a result Q and Q', as well, be contained in an arbitrarily-given plane. That plane cuts the lines $\alpha\beta$ and $\alpha\gamma$ at C' and B', resp., and thus R will meet the direction $C'B'$ of Q. From a), the direction of R will then go through the point at which the plane δ, which contains both R and Q, is cut by Q'. However, since Q' is the opposite line to Q, this point will be the opposite point of the plane δ, which was to be proved. We draw the following conclusions from this:

d) *The direction of one of the two forces R and R', which should have the same effect as* $[\alpha\beta]$ *and* $[AB]$ (*or P and P', as* $[\alpha\beta]$ *and* $[AB]$ *might be called, from now on*), *can be chosen arbitrarily, in general.*

e) *One of the directions of two forces R and R' that have the same effect as P and P' will be the opposite line of the other one.*

Let D and E be given, so, from c), the force R' must go through D, as well as E; however, DE is the opposite line to R. This sheds light on the facts that:

f) *Conversely, when a' is the opposite line of a, two forces that lie in the directions a and a' can always be given that have the same effect as P and P'.*

g) *Finally, just as when R lies in a given plane, R' meets the opposite point of that plane, it is also true that conversely, when R goes through a given point D, R' will be contained in the opposite plane of that point.*

Then, since R' is the opposite line of R, the plane that goes through D and R' will be the opposite plane of D.

20. These theorems now sufficiently illuminate the intrinsic connection that exists between the dual geometric relationships that were treated before and some entirely elementary static theorems. One also easily grasps how, conversely, that purely geometric theory can be derived from the elements of statics.

Namely, every line corresponds to another line relative to two forces P, P' whose directions do not lie in a plane, in such a way that two forces can always be given that have their directions and the same effect. Moreover, every point corresponds to a certain plane that contains it, and every plane corresponds to a point that lies in it, such that when one of two forces that have the same effect as P and P' meet the point, the other one will be contained in the plane, and conversely.

The concepts of opposite line, opposite plane, and opposite point are established with these purely-statically provable theorems, and as we saw already in nos. **13**, **15**, **16**, and **18**, the remaining properties of these dual relationships can be derived from them without the assistance of any new principle. In what follows, some especially remarkable relations between both kinds of relationships – viz., geometric and static – shall be discussed in more detail.

21. The fact that only one of the directions of two forces R, R' that have the same effect as P, P' can be chosen arbitrarily is only true in general. One first excludes all of directions are parallel to the principal direction, since if R is parallel to it then the force R' would lie at infinity (**XIV**), and would therefore not be constructible.

In order to ascertain the static interpretation of the principal direction, one considers that, from **XIII**, the two planes, one of which is parallel to P, and the other of which is parallel to R and R' (and, as a result, also to the common intersection of the two planes), will be parallel to the principal direction. Therefore, if the four forces P, P', R, R' are carried parallel to their directions to one and the same point then the intersection of the planes PP' and RR' will likewise be parallel to the principal direction. However, it is then a result of a known theorem of statics that the resultant of P and P' will be the same as the resultant of R and R', and will therefore have the intersection of the planes PP' and TT' for its direction. In the context of statics, the principal direction will then be the one that is parallel to the resultant of the forces P and P' or two forces that have the same effect as them, once these forces are carried parallel to their directions to one and the same point.

It then follows from this, in another way, that it is impossible to take R to be parallel to the principal direction. If R had that direction then once R and R' were carried to the same point they would fall upon the same line that would parallel to the principal direction, and in their original position they must therefore be either reducible to a single force or be equal to each other, as well as parallel and opposite; i.e., they must be a force-couple in the restricted sense. However, neither of these two conditions is possible, since P and P' are not supposed to lie in a plane.

22. One excludes the double lines or the lines that simultaneously intersect P and P' or any other two forces R and R' that reduce to P and P' from the directions that two forces that have the same effect as P, P' can take on (**XIV**). Thus, if S and S' are two forces that have the same effect as P and P' (and thus, R and R', as well) then R, R', and $-S$ will be reducible to a single force (= S'). However, this is obviously not possible, since R and R', which do not lie in a plane, cut S simultaneously.

However, double lines have another remarkable property in statics that consists of the fact that the sum of the moments of P and P' relative to such a line as an *axis* will be zero. s will then be a double line, r will be a simple line that cuts it, and r' will be the opposite line to the latter, so, from **XX**, r' will also be met by s. However, from **19**.*f*, P and P' can be converted into two forces R and R' that have the same directions as r and r', respectively. Thus, since the directions of R and R' will simultaneously meet s, the moment of R with s as its axis, as well as the moment of R', will be equal to zero, so the sum of these moments will also be equal to zero, and as a result, the sum of the moments of the equivalent forces P and P' will also be zero.

Therefore, amongst all of the axes that lie in a plane, the sum of the moments will be zero for the ones that meet the opposite point of the plane, and amongst all of the axes that go through a point, the sum of the moments will be zero for the ones that simultaneously lie in the opposite plane of the point (**XVIII**). For any other axis that goes through the point, and I will mention this here only for its historical interest, the sum of the moments will be proportional to the angle that the axis makes with the opposite plane of the point; therefore, it will be greatest for the axes that are normal to the opposite plane. However, the smallest of all greatest moment sums occurs for the axes that coincide with the principal line of the system (14). For this fact, one should confer the conclusion of Poinsot's statics-related *Mémoire sur la composition des momens et des aires.*

23. If two forces P, P' that are not contained in a plane are laid through a point parallel to their directions, and then combined with a force T then a force-couple U, $-U$ will arise from this construction that will provoke the same effect as P and P' in conjugation with T. If one now lays an arbitrary line in the plane of U, $-U$ that goes through the point D at which the plane is cut by T then it will always meet the forces T, U, $-U$ simultaneously. Therefore, the sum of the moments of T, U, $-U$ relative to such a line will be equal to zero, so the moment sum of P and P' will also be equal to zero, and as a result, the line will be a double line (22) and the point D will be the opposite point of the plane of U, $-U$. Therefore, just as the forces P, P' might be converted into to simple force and a couple, the former will always meet the plane of the couple at its opposite point. The direction of the former force is also parallel to the principal direction, since when T, U, $-U$ are carried to a point, U and $-U$ are mutually eliminated, and only T will remain as the resultant. (Cf., no. **21**)

One can not only shift a force-couple arbitrarily in its plane without changing its effect, but also move it to a plane that is parallel to that plane. If one then shifts the couple U, $-U$ from its initial plane into one that is parallel to it then the latter plane must also be cut by T at its opposite point, in agreement with the theorem (**VII**) that the opposite points of parallel planes lie along a line that is parallel to the principal direction.

24. I fear that I would fatigue the reader by drawing his attention to all of the theorems of statics that would correspond to all of the remaining theorems that were dealt with in the first part of this paper. Thus, I shall content myself by pointing out Theorem **XXI**, which then reads, when it is expressed statically: If two forces have the same effect as two other ones, or – what amounts to the same thing, here:

If four forces are in equilibrium with each other then every line that encounters the directions of three of them will also encounter the direction of the fourth one.

This also follows most simply from the theory of moments. The moment of each of three forces with respect to an axis that meets the directions of those forces will be equal to zero. Now, since the four forces should be in equilibrium, and thus, the sum of their moments must be zero for any axis, the moment of the fourth force with respect to that axis must also be zero. However, that will not be possible, except when that axis simultaneously cuts the direction of the fourth one.

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