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On the quantum theory of rotating electrons

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In connection with a paper of Dirac, those transformation matrices will be determined that take the place of Lorentz transformations for electron spin and the eigenfunctions. Furthermore, an invariant will be given, from which, by variation of the Dirac eigenfunctions and the electromagnetic potential, the equations for the matter waves and the electromagnetic field can be obtained. In connection with that, one gets an expression for the four-vector of the particle current.

In an amazing, recently-appearing paper, P. A. M. Dirac (*) made relativistic quantum mechanics more powerful. The following work is connected directly with the aforementioned Dirac paper, and shall, moreover, represent nothing but some formal extensions of the Dirac argument. Above all, those transformation shall be considered here that allow one and the same evolution for the eigenfunctions, as observed from different reference systems, when they are converted into each other. With their help, the vector properties of certain bilinear forms that are defined by the wave functions can also be verified. One further finds a "Lagrange function" L, from which one can not only obtain the Dirac equations by variation, but also the Maxwell equations. With the help of the transformations of the wave functions, the proof of the invariance of L is very simple.

On the contrary, a series of questions that are connected with the energy-impulse tensor for matter waves will not be discussed. It may be computed very easily from the aforementioned function L, and is, in contrast to the tensor that was derived by Schrödinger (**), no longer symmetric. We hope to return to this very singular behavior shortly.

I. The spatial rotation and its associated unitary transformation.

According to Dirac, in the absence of an external electromagnetic field, the differential equations for the rotating electron read:

$$\eta_0\left(-\frac{h}{2\pi i}\frac{\partial}{\partial x_0}+\frac{e}{c}\Phi_0\right)\psi+\sum_{j=1}^{3}\eta_j\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_j}+\frac{e}{c}\Phi_j\right)\psi - imc\ \psi = 0. \tag{1}$$

(*) P. A. M. Dirac, Proc. Roy. Soc. (A) **117** (1928), 610.
(**) E. Schrödinger, Ann. d. Phys. (4) **82** (1927), 265.

Here:

$$x_0 = c \cdot t, \qquad \Phi_0 = V, \qquad \Phi_1 = \mathfrak{A}_1\,, \qquad \Phi_2 = \mathfrak{A}_2\,, \qquad \Phi_3 = \mathfrak{A}_3\,.$$

The quantities η_α are the spin matrices; we (following Dirac) would like to given them the following appearance (*):

$$\left.\begin{array}{l} \eta_0 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & +i & 0 \\ 0 & 0 & 0 & +i \end{pmatrix} \quad \eta_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \\ \eta_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \quad \eta_3 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}. \end{array}\right\} \tag{2}$$

The "product" $(\eta_\alpha\,\psi)$ has the following sense:

$$(\eta_\alpha\,\psi)_j = \sum_{k=1}^{4} (\eta_\alpha)_{jk}\,\psi_k\,.$$

If we now subject equation (1) to a transformation that corresponds to a spatial rotation then $\partial / \partial x_0$ and Φ_0 remain unchanged, while the operators $\partial / \partial x_k$ transform in exactly the same way as the spatial components of Φ_k ; i.e., like three-dimensional vector components. Let α_{ik} be the coefficients of the transformation matrix, so equation (1) (under the assumption that the $\Phi_k = 0$, which is inessential) goes to:

$$\frac{h}{2\pi i}\left\{-\eta_0' \frac{\partial \psi}{\partial x_0'} + \sum_{j=1}^{3} \eta_j' \frac{\partial \psi}{\partial x_j'}\right\} - mc\;\psi = 0, \tag{3}$$

where:

$$\eta_\alpha' = \sum_{j=1}^{3} \alpha_{\alpha j}\eta_j \qquad (\alpha = 1, \ldots, 3)$$

and the x_k' are the coordinates in the new system (so one observes that $x_0' = x_0$ and $\eta_0' = \eta_0$).

Dirac has proved that the quantities η_α' emerge from the η_α by a transformation of the form $S^{-1}\eta_\alpha\,S$, without, however, exhibiting that transformation. In order to do this, we

(*) The η_i are not integrals of the equations of motion (i.e., $\dot{\eta}_i \neq 0$). According to (2), the special choice of these quantities can only be valid for a certain time point. However, this will suffice in what follows. Cf., P. A. M. Dirac, Proc. Roy. Soc. (A) **118** (1928), 351.

observe that the coefficients of a matrix that admits a representation of the form $S^{-1}X\,S$ must be bilinear forms of the coefficients of S and S^{-1}. However, from the equation:

$$\eta'_\alpha = \sum_{j=1}^{3} \alpha_{\alpha j}\eta_j \qquad (\alpha = 1, \ldots, 3) \tag{4}$$

it follows immediately that the coefficients of η'_α are linear forms of the rotation transformation coefficients a_{ik} . From this, it further follows that coefficients of the matrices S and S^{-1} are so arranged that the a_{ik} can be represented as bilinear forms in them. From now on, one will then attempt to write out the a_{ik} as bilinear forms of quadratic forms. One achieves this with the help of the Cayley-Klein parameters.

The three-dimensional rotation group contains three parameters; e.g., the Euler angles ϑ, φ, ω. How one expresses the a_{ik} in terms of them is well-known (*). One easily sees that this representation still does not give us the a_{ik} as forms of second degree, as desired. However, if we set:

$$\left.\begin{aligned} &\alpha = \cos\frac{\vartheta}{2}e^{i(\varphi+\omega)/2}, \quad \beta = i\sin\frac{\vartheta}{2}e^{-i(\varphi-\omega)/2}, \\ &\gamma = i\sin\frac{\vartheta}{2}e^{i(\varphi-\omega)/2}, \quad \gamma = \cos\frac{\vartheta}{2}e^{-i(\varphi+\omega)/2} \end{aligned}\right\} \tag{5}$$

then this effortlessly yields the well-known representation of the Euler angles by the Cayley-Klein parameters:

$$\left.\begin{array}{c|c|c|c} & x_1 & x_2 & x_3 \\ \hline x'_1 & \frac{1}{2}(\alpha^2-\beta^2-\gamma^2+\delta^2) & -\frac{i}{2}(\alpha^2-\beta^2+\gamma^2-\delta^2) & -(\alpha\gamma-\beta\delta) \\ \hline x'_2 & \frac{i}{2}(\alpha^2+\beta^2-\gamma^2-\delta^2) & \frac{1}{2}(\alpha^2+\beta^2+\gamma^2+\delta^2) & -i(\alpha\gamma+\beta\delta) \\ \hline x'_3 & \alpha\beta-\gamma\delta & i(\alpha\beta+\gamma\delta) & \alpha\delta+\beta\gamma \end{array}\right\} \tag{6}$$

If one observes (4) then one finds for the matrices η'_α:

$$\eta'_1 = \begin{pmatrix} 0 & 0 & i(\alpha\gamma-\beta\delta) & i(\gamma^2-\delta^2) \\ 0 & 0 & -i(\alpha^2-\beta^2) & -i(\alpha\gamma-\beta\delta) \\ -i(\alpha\gamma-\beta\delta) & -i(\gamma^2-\delta^2) & 0 & 0 \\ i(\alpha^2-\beta^2) & i(\alpha\gamma-\beta\delta) & 0 & 0 \end{pmatrix},$$

(*) One finds the schema in, e.g., Klein-Sommerfeld, *Über die Theorie des Kreisels*, pp. 19. Our x'_1, x'_2, x'_3 is identical with X, Y, Z, resp., in it, and our x_1, x_2, x_3 are identical with x, y, z, resp. The angle that is denoted by ω here is called ψ there.

$$\eta_2' = \begin{pmatrix} 0 & 0 & -(\alpha\gamma+\beta\delta) & -(\gamma^2+\delta^2) \\ 0 & 0 & (\alpha^2+\beta^2) & \alpha\gamma+\beta\delta \\ \alpha\gamma+\beta\delta & \gamma^2+\delta^2 & 0 & 0 \\ -(\alpha^2+\beta^2) & -(\alpha\gamma+\beta\delta) & 0 & 0 \end{pmatrix}, \tag{7}$$

$$\eta_3' = \begin{pmatrix} 0 & 0 & -i(\alpha\delta+\beta\gamma) & -2i\,\gamma\,\delta \\ 0 & 0 & 2i\,\alpha\,\beta & i(\alpha\delta+\beta\gamma) \\ i(\alpha\delta+\beta\gamma) & 2i\,\gamma\,\delta & 0 & 0 \\ -2i\,\alpha\,\beta & -i(\alpha\delta+\beta\gamma) & 0 & 0 \end{pmatrix}.$$

These matrices obviously decompose into two two-rowed matrices that differ from each other only by their signs. If we let ζ_1', ζ_2', ζ_3' denote the two-rowed matrices:

$$\zeta_1' = \begin{pmatrix} i(\alpha\gamma-\beta\delta) & i(\gamma^2-\delta^2) \\ -i(\alpha^2-\beta^2) & -i(\alpha\gamma-\beta\delta) \end{pmatrix}, \qquad \zeta_2' = \begin{pmatrix} -(\alpha\gamma+\beta\delta) & -(\gamma^2+\delta^2) \\ \alpha^2+\beta^2 & \alpha\gamma+\beta\delta \end{pmatrix},$$

$$\zeta_3' = \begin{pmatrix} -i(\alpha\delta+\beta\gamma) & -2i\,\gamma\,\delta \\ 2i\,\alpha\,\beta & i(\alpha\delta+\beta\gamma) \end{pmatrix},$$

and let ζ_1, ζ_2, ζ_3, resp., denote the ones that result from the matrix equations (2):

$$\zeta_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \zeta_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \zeta_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

then we pose the problem of giving two-rowed transformation matrices P and P^{-1} for which the equations:

$$\zeta_1' = P^{-1}\zeta_1\,P, \qquad \zeta_2' = P^{-1}\zeta_2\,P, \qquad \zeta_3' = P^{-1}\zeta_3\,P \tag{8}$$

are fulfilled.

However, this problem was already solved by Pauli (*). One easily verifies that the matrices:

$$P = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \tag{9}$$

satisfy equations (8).

If one observes the relation that follows from equations (5):

$$\alpha\delta - \beta\gamma = 1$$

then one can easily show that one also actually has $P\,P^{-1} = 1$.

(*) W. Pauli, Zeit. f. Phys. **43** (1927), 601.

With the help of the matrices P and P^{-1}, we now define four-rowed matrices that take the η_α [equation (2)] to the η'_α [equation (7)]. Namely, one immediately shows that under the assumption of the validity of equations (8) the following equations are also true:

$$R^{-1}\eta_0 R = \eta_0, \qquad R^{-1}\eta_1 R = \eta'_1, \qquad R^{-1}\eta_2 R = \eta'_2, \qquad R^{-1}\eta_3 R = \eta'_3, \tag{10}$$

if we understand R and R^{-1} to mean the following matrices:

$$R = \begin{pmatrix} \beta & \delta & 0 & 0 \\ \alpha & \gamma & 0 & 0 \\ 0 & 0 & \beta & \delta \\ 0 & 0 & \alpha & \gamma \end{pmatrix}, \qquad R^{-1} = \begin{pmatrix} -\gamma & \delta & 0 & 0 \\ \alpha & -\beta & 0 & 0 \\ 0 & 0 & -\gamma & \delta \\ 0 & 0 & \alpha & -\beta \end{pmatrix}. \tag{11}$$

If we now substitute (10) into equation (3) then because R and R^{-1} commute with imc we can write R^{-1} on the left of the entire equation. Since R^{-1} is not zero, the product can be zero only when the second factor vanishes, and this means:

$$\frac{h}{2\pi i}\left\{-\eta_0 \frac{\partial}{\partial x_0} + \sum_{j=1}^{3} \eta_j \frac{\partial}{\partial x'_j} + \frac{2\pi mc}{h}\right\} R\,\psi = 0. \tag{3a}$$

However, this is equation (1) (*) with the following difference: The coordinates x_1, x_2, x_3 are replaced with the primed coordinates by a spatial rotation of the former coordinate system, and then the eigenfunctions are no longer the ψ, but they have the form $R\,\psi$. If we have established that the functions ψ_1, ψ_2, ψ_3, ψ_4 are the eigenfunctions for some physical process in any coordinate system then the eigenfunctions in another coordinate system that arises from that one by rotation [transformation equations (6)] are:

$$\psi'_1 = \beta\,\psi_1 + \delta\,\psi_2, \quad \psi'_2 = \alpha\,\psi_1 + \gamma\,\psi_2, \quad \psi'_3 = \beta\,\psi_3 - \delta\,\psi_4, \quad \psi'_4 = \alpha\,\psi_3 + \gamma\,\psi_4. \tag{12}$$

From the matrix R [equation (11)], one can then obviously obtain the matrix R^{-1} by first going over to the matrix with complex-conjugate elements and the switching the rows with the columns (**). One calls a matrix that arises from another one in this way the *adjoint* matrix to it; in general, one denotes the adjoint matrix by a †. We then see that in our case the matrix $R^\dagger$ that is adjoint to R and the matrix R^{-1} that is reciprocal to R are identical to each other. One calls a transformation for which the matrix equation:

$$R\,R^\dagger = 1 \tag{13}$$

(*) The fact that the potentials are zero here is inessential, since the result is true precisely when the Φ_i are non-vanishing, because the R and Φ_i commute.

(**) One then has $\alpha^* = \delta$, $\beta^* = -\gamma$ [cf., equation (5)].

is true *unitary*. These unitary transformations are the Hermitian analogues of real-orthogonal transformations. Whereas, e.g., for the real-orthogonal transformations the norm of a vector remains constant, for unitary transformations, this is the case for the sum of the absolute values of the components. In our case, one easily sees that, e.g.:

$$J = \sum_{k=1}^{4} \psi_k \psi_k^* \tag{14}$$

is invariant under the transformation (12), a fact whose proof essentially stems from equation (13), when one observes that, along with (12), the equations:

$$\psi_1'^* = -\gamma \psi_1^* + \alpha\, \psi_2^*, \quad \psi_2'^* = \delta\, \psi_1^* - \beta\, \psi_2^*, \qquad \psi_3'^* = -\gamma \psi_3^* + \alpha\, \psi_4^*, \quad \psi_4'^* = \delta\, \psi_3^* - \beta\, \psi_4^*$$

are true.

The invariance of the expression (14) under spatial rotations of the coordinate system allows one to interpret the invariant (14) as the density of the particle, which is also very closely related as an analogy to the Born (*) and Pauli (**) Anzsatz. The possibility of such an interpretation will then depend very crucially upon the behavior of the expression (14) under Lorentz transformations. Obviously, the particle density must transform like the time component of a vector under a Lorentz transformation, and furthermore, the three spatial vector components must be found that will be extended to a four-current with the particle density that is then divergence-free.

It might be the case that the expression (14) is no longer invariant under a Lorentz transformation. From that, it would follow that the canonical transformation $S^{-1}\ \eta_\alpha\, S$ of the η_α cannot be unitary if it corresponds to a Lorentz transformation. In the next section, we will see that this condition is fulfilled, and we will recognize that this condition is fulfilled only because of the fact that the Lorentz transformation is no longer real, but represents an imaginary rotation.

II. The transformations *S*, *T*, *U*, and their three special Lorentz transformations.

The Lorentz group contains six independent parameters, corresponding to the displacements of the six two-dimensional coordinate planes in four dimensions. Among the six parameters, one can choose three of them such that their variation leaves the time axis unchanged. This is the group of spatial rotations that we have already spoken of. A three-parameter subgroup then remains, under which the time-axis is always transformed. We can compose this subgroup from three special Lorentz transformations, namely, the one in the x_1-direction, the one in the x_2-direction, and the one in the x_3-direction. Each of these transformations has the property that two of the three spatial axes remain unchanged. For the Lorentz transformation in the x_1-direction, one has the coefficient matrix (***):

(*) M. Born, Zeit. f. Phys. **38** (1926), 803.
(**) W. Pauli, *loc. cit.*, pp. 606, equations (1a), (1b).
(***) Cf., M. von Laue, *Die Relatvitätstheorie*, vol. I, sec. 3, pp. 80, Braunschweig, 1919.

$$\begin{array}{l|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x'_1 \ldots & \dfrac{1}{\sqrt{1-\beta^2}} & 0 & 0 & \dfrac{i\beta}{\sqrt{1-\beta^2}} \\ x'_2 \ldots & 0 & 1 & 0 & 0 \\ x'_3 \ldots & 0 & 0 & 1 & 0 \\ x'_4 \ldots & \dfrac{-i\beta}{\sqrt{1-\beta^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-\beta^2}} \end{array} \qquad \begin{array}{l} x_4 = ix_0 \\ \beta = \dfrac{v}{c} \end{array} \tag{15}$$

If we introduce an imaginary angle Θ_1 by way of the equations:

$$\cos \Theta_1 = \frac{1}{\sqrt{1-\beta^2}}, \qquad \sin \Theta_1 = \frac{i\beta}{\sqrt{1-\beta^2}}$$

then we get:

$$\begin{array}{l|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x'_1 \ldots & \cos\Theta_1 & 0 & 0 & \sin\Theta_1 \\ x'_2 \ldots & 0 & 1 & 0 & 0 \\ x'_3 \ldots & 0 & 0 & 1 & 0 \\ x'_4 \ldots & -\sin\Theta_1 & 0 & 0 & \cos\Theta_1 \end{array} \tag{15a}$$

When one applies the transformation (15a) to equation (1) (assuming vanishing potentials Φ_i), it goes to the following equation:

$$\frac{h}{2\pi i}\left\{\eta'_1 \frac{\partial\psi}{\partial x'_1} + \eta_2 \frac{\partial\psi}{\partial x'_2} + \eta_3 \frac{\partial\psi}{\partial x'_3} + \eta'_4 \frac{\partial\psi}{\partial x'_4}\right\} - i\, mc\, \psi = 0. \tag{16}$$

Here:

$$x'_4 = i\, x'_0, \qquad -\eta_4 = i\,\eta_0, \qquad -\eta'_4 = i\,\eta'_0,$$

and (*):

$$\eta'_j = \sum_{k=1}^{4} a_{jk}\eta_k . \tag{17}$$

The matrices η'_1 and η'_4 then have the form:

(*) The a_{jk} are now the coefficients of (15a). One then has:

$$\eta'_2 = \eta_2, \qquad \eta'_3 = \eta_3 .$$

$$\eta_1' = \begin{pmatrix} \sin\Theta_1 & 0 & 0 & -i\cos\Theta_1 \\ 0 & \sin\Theta_1 & -i\cos\Theta_1 & 0 \\ 0 & i\cos\Theta_1 & -i\sin\Theta_1 & 0 \\ i\cos\Theta_1 & 0 & 0 & -\sin\Theta_1 \end{pmatrix},$$

$$\eta_4' = \begin{pmatrix} \cos\Theta_1 & 0 & 0 & -i\sin\Theta_1 \\ 0 & \cos\Theta_1 & i\sin\Theta_1 & 0 \\ 0 & -i\sin\Theta_1 & -\cos\Theta_1 & 0 \\ -i\sin\Theta_1 & 0 & 0 & -\cos\Theta_1 \end{pmatrix}.$$

If one further observes that the coefficients of the primed matrices must be bilinear forms of the coefficients of the transformation matrices S and S^{-1}, then one comes to the representation of S and S^{-1} in terms of the half-angle $\Theta_1 / 2$. One then easily finds that the matrices have the following form:

$$S = \begin{pmatrix} \cos\frac{\Theta_1}{2} & 0 & 0 & i\sin\frac{\Theta_1}{2} \\ 0 & \cos\frac{\Theta_1}{2} & i\sin\frac{\Theta_1}{2} & 0 \\ 0 & i\sin\frac{\Theta_1}{2} & \cos\frac{\Theta_1}{2} & 0 \\ i\sin\frac{\Theta_1}{2} & 0 & 0 & \cos\frac{\Theta_1}{2} \end{pmatrix},$$

$$S^{-1} = \begin{pmatrix} \cos\frac{\Theta_1}{2} & 0 & 0 & -i\sin\frac{\Theta_1}{2} \\ 0 & \cos\frac{\Theta_1}{2} & -i\sin\frac{\Theta_1}{2} & 0 \\ 0 & -i\sin\frac{\Theta_1}{2} & \cos\frac{\Theta_1}{2} & 0 \\ -i\sin\frac{\Theta_1}{2} & 0 & 0 & \cos\frac{\Theta_1}{2} \end{pmatrix}. \tag{18}$$

One verifies that the equations:

$$\eta_1' = S^{-1}\,\eta_1\,S, \qquad \eta_4' = S^{-1}\,\eta_4\,S$$

are fulfilled, but above all, also the fact that the transformation S leaves η_2 and η_3 unchanged, so one has:

$$S^{-1}\eta_2\,S = \eta_2\,, \qquad S^{-1}\eta_3\,S = \eta_3\,.$$

If the angle Θ_1 were real then S^{-1} would be identical with $S^\dagger$, as one sees immediately. However, since the angle Θ_1 is pure imaginary then S is a matrix with real coefficients. The transition to conjugate-complex changes nothing, and the switching of rows and columns yields:

$$S^\dagger = \begin{pmatrix} \cos\frac{\Theta_1}{2} & 0 & 0 & i\sin\frac{\Theta_1}{2} \\ 0 & \cos\frac{\Theta_1}{2} & i\sin\frac{\Theta_1}{2} & 0 \\ 0 & i\sin\frac{\Theta_1}{2} & \cos\frac{\Theta_1}{2} & 0 \\ i\sin\frac{\Theta_1}{2} & 0 & 0 & \cos\frac{\Theta_1}{2} \end{pmatrix} = S,$$

and this matrix is completely different from S^{-1}. We thus find it confirmed that the matrix transformation that corresponds to a Lorentz transformation is no longer unitary. The expression (14) is therefore not invariant under the special Lorentz transformation (15). However, before we examine the behavior of the expression (14) closer, we would first like to consider the other two special Lorentz transformations.

If we subject equation (1) to the transformation:

$$\begin{array}{c|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1'' \ldots & 1 & 0 & 0 & 0 \\ x_2'' \ldots & 0 & \cos\Theta_2 & 0 & \sin\Theta_2 \\ x_3'' \ldots & 0 & 0 & 1 & 0 \\ x_4'' \ldots & 0 & -\sin\Theta_2 & 0 & \cos\Theta_2 \end{array} \tag{19}$$

then (assuming $\Phi_i = 0$) it goes to:

$$\frac{h}{2\pi i}\left\{\eta_1\frac{\partial\psi}{\partial x_1''} + \eta_2''\frac{\partial\psi}{\partial x_2''} + \eta_3\frac{\partial\psi}{\partial x_3''} + \eta_4''\frac{\partial\psi}{\partial x_4''}\right\} - i\,mc\,\psi = 0.$$

Only the matrices η_2 and η_4 will be changed; they go to:

$$\eta_2'' = \begin{pmatrix} \sin\Theta_2 & 0 & 0 & -\cos\Theta_2 \\ 0 & \sin\Theta_2 & \cos\Theta_2 & 0 \\ 0 & \cos\Theta_2 & -\sin\Theta_2 & 0 \\ -\cos\Theta_2 & 0 & 0 & -\sin\Theta_2 \end{pmatrix},$$

$$\eta_4'' = \begin{pmatrix} \cos\Theta_2 & 0 & 0 & \sin\Theta_2 \\ 0 & \cos\Theta_2 & -\sin\Theta_2 & 0 \\ 0 & -\sin\Theta_2 & -\cos\Theta_2 & 0 \\ \sin\Theta_2 & 0 & 0 & -\cos\Theta_2 \end{pmatrix}.$$

One immediately verifies that the matrices:

$$T = \begin{pmatrix} \cos\frac{\Theta_2}{2} & 0 & 0 & \sin\frac{\Theta_2}{2} \\ 0 & \cos\frac{\Theta_2}{2} & -\sin\frac{\Theta_2}{2} & 0 \\ 0 & \sin\frac{\Theta_2}{2} & \cos\frac{\Theta_2}{2} & 0 \\ -\sin\frac{\Theta_2}{2} & 0 & 0 & \cos\frac{\Theta_2}{2} \end{pmatrix},$$

$$T^{-1} = \begin{pmatrix} \cos\frac{\Theta_2}{2} & 0 & 0 & -\sin\frac{\Theta_2}{2} \\ 0 & \cos\frac{\Theta_2}{2} & \sin\frac{\Theta_2}{2} & 0 \\ 0 & -\sin\frac{\Theta_2}{2} & \cos\frac{\Theta_2}{2} & 0 \\ \sin\frac{\Theta_2}{2} & 0 & 0 & \cos\frac{\Theta_2}{2} \end{pmatrix} \tag{20}$$

satisfy the equations:

$$\eta_2'' = T^{-1}\,\eta_2\,T, \qquad \eta_4'' = T^{-1}\,\eta_4\,T, \qquad \eta_1 = T^{-1}\,\eta_1\,T, \qquad \eta_3 = T^{-1}\,\eta_3\,T$$

and are therefore the desired transformation matrices.

Finally, the Lorentz transformation:

$$\begin{array}{c|c|c|c|c} & x_1 & x_2 & x_3 & x_4 \\ \hline x_1'''\ldots & 1 & 0 & 0 & 0 \\ x_2'''\ldots & 0 & 1 & 0 & 0 \\ x_3'''\ldots & 0 & 0 & \cos\Theta_3 & \sin\Theta_3 \\ x_4'''\ldots & 0 & 0 & -\sin\Theta_3 & \cos\Theta_3 \end{array} \tag{21}$$

corresponds to the matrices:

$$U = \begin{pmatrix} i\cos\frac{\Theta_3}{2} & 0 & -\sin\frac{\Theta_3}{2} & 0 \\ 0 & i\cos\frac{\Theta_3}{2} & 0 & \sin\frac{\Theta_3}{2} \\ -\sin\frac{\Theta_3}{2} & 0 & i\cos\frac{\Theta_3}{2} & 0 \\ 0 & \sin\frac{\Theta_3}{2} & 0 & i\cos\frac{\Theta_3}{2} \end{pmatrix},$$

$$U^{-1} = \begin{pmatrix} -i\cos\frac{\Theta_3}{2} & 0 & -\sin\frac{\Theta_3}{2} & 0 \\ 0 & -i\cos\frac{\Theta_3}{2} & 0 & \sin\frac{\Theta_3}{2} \\ -\sin\frac{\Theta_3}{2} & 0 & -i\cos\frac{\Theta_3}{2} & 0 \\ 0 & \sin\frac{\Theta_3}{2} & 0 & -i\cos\frac{\Theta_3}{2} \end{pmatrix}. \tag{22}$$

They satisfy the equations:

$$\eta_1 = U^{-1}\,\eta_1\,U, \qquad \eta_2 = U^{-1}\,\eta_2\,U, \qquad \eta_3''' = U^{-1}\,\eta_3\,U, \qquad \eta_4''' = U^{-1}\,\eta_4\,U,$$

in which:

$$\eta_3'' = \begin{pmatrix} \sin\Theta_3 & 0 & -i\cos\Theta_3 & 0 \\ 0 & \sin\Theta_3 & 0 & i\cos\Theta_3 \\ i\cos\Theta_3 & 0 & -\sin\Theta_3 & 0 \\ 0 & -i\cos\Theta_3 & 0 & -\sin\Theta_3 \end{pmatrix},$$

$$\eta_4''' = \begin{pmatrix} \cos\Theta_3 & 0 & i\sin\Theta_3 & 0 \\ 0 & \cos\Theta_3 & 0 & -i\sin\Theta_3 \\ -i\sin\Theta_3 & 0 & -\cos\Theta_3 & 0 \\ 0 & i\sin\Theta_3 & 0 & -\cos\Theta_3 \end{pmatrix}.$$

None of the three matrices S, T, U are unitary. They would be so if the three quantities Θ_1, Θ_2, Θ_3 were real angles. The quantities (14) are then unchanged under all Lorentz transformations.

In the same way as we showed for the spatial rotations, it follows that the eigenfunctions ψ transform under the transformations (15), (19), and (21) like $S\psi$, $T\psi$, $U\psi$, resp. Naturally, we can also combine the transformations (6), (15), (19), and (21) into a six-parameter transformation; the eigenfunctions then transform like:

$$\psi' = RSTU\ \psi.$$

Forming the products $RSTU$ and $(RSTU)^{-1} = U^{-1}\ T^{-1}\ S^{-1}\ R^{\dagger}$ is quite complicated, and might therefore be omitted here.

Remark added in proof: From a friendly remark of Wigner, one can arrive at the fact that the matrices S, T, U assume the reduced form by a somewhat different choice of η_i, into which we can bring the matrix R. In order to do this, as one easily confirms, one needs only to replace the quantity η_0 that we chose with:

$$\alpha_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

This is permissible, since α_0 also has the eigenvalues 1 and -1, and satisfies the equations:

$$\alpha_0\ \eta_i + \eta_i\ \alpha_0 = 0, \qquad j = 1, 2, 3,$$

in addition.

III. The Lagrange function and the current vector.

We replace x_0 with x_4 / i and Φ_0 with $-\ i\ \Phi_4$ in equation (1). We further denote $-\ i\ \eta_0$ by η_4, as we have already done. If we then multiply on the right by $i\ \eta_4$ then this finally yields:

$$i\sum_{j=1}^{4}\eta_4\eta_j\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_j}+\frac{e}{c}\Phi_j\right)\psi + mc\ \eta_4\ \psi = 0. \tag{23}$$

For the sake of convenience, we would like to write out the matrices $\eta_4\eta_1$, $\eta_4\eta_2$, …, and η_4 explicitly. They are:

$$\eta_4\eta_1 = -\ i\ \varepsilon_1 = \begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}, \qquad \eta_4\eta_2 = -\ i\ \varepsilon_2 = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\eta_4\eta_3 = -\ i\ \varepsilon_3 = \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & -i \\ +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \qquad \eta_4^2 = 1, \qquad -\ \eta_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We introduce an abbreviation: We shall understand $\underline{\psi^*, \varepsilon_j \psi}$ to mean the "scalar" product of $\psi^* = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$ and the quantities ($\varepsilon_j\ \psi$); e.g.:

$$\underline{\psi^*, \varepsilon_2 \psi} = i(\psi_1^* \psi_4 - \psi_2^* \psi_3 + \psi_3^* \psi_2 - \psi_4^* \psi_1).$$

The quantities $\underline{\psi^*, \varepsilon_j \dfrac{\partial \psi}{\partial x_k}}$ then have a corresponding meaning.

We now define the function:

$$L_m = \frac{h}{2\pi}\left\{\underline{\psi^*, \frac{\partial \psi}{\partial x_4}} - \underline{\psi, \frac{\partial \psi^*}{\partial x_4}}\right\} + \frac{h}{2\pi i}\sum_{j=1}^{3}\left(\underline{\psi^*, \varepsilon_j \frac{\partial \psi}{\partial x_j}} - \underline{\psi, \varepsilon_j^* \frac{\partial \psi^*}{\partial x_j}}\right)$$
$$+ \frac{ie}{c}\Phi_4 \underline{\psi^*, \psi} + \frac{e}{c}\sum_{\alpha=1}^{3}\Phi_\alpha \underline{\psi^*, \varepsilon_\alpha \psi} + mc\,\underline{\psi^*, \eta_4 \psi}. \tag{24}$$

We now demand that the integral:

$$\int\int\int\int L_m \, dx_1 \, dx_2 \, dx_3 \, dx_4$$

must be an extremum when the ψ_i and ψ_i^* are considered to be independently-varied functions, and the Euler-Lagrange equations become identical with equations (23) for the ψ_i and the equation for the ψ_i^* that one obtains from (23) when one replaces i with $-i$ (†). The proof is carried out most simply by calculation.

In the relativistic quantum mechanics that was developed by Gordon (††) and Klein (†††), there is also a Lagrange potential whose variational equations are the equations of matter waves. A further and much more important feature of this theory was the fact that the derivatives of the Lagrange potentials represented the four components of the four-potential of the electric current vector. We must see whether this is also true in the present case.

Performing the differentiation of equation (24) yields (††††):

(†) The replacement of i with $-i$ is to be performed everywhere, so it is also done in the matrices ε_j, but not in x_4 and Φ_4; these quantities prove to be real. However, because the i exists in the fourth term of the sum in (23) in such a way that one goes from x_0 to x_4, this i is not to be replaced with $-i$. It is simplest for one to start with equation (1), replace all i with $-i$, and then go from x_0 to x_4; since $\eta_4^* = \eta_4$, this yields:

$$0 = i\left(\frac{h}{2\pi}\frac{\partial}{\partial x_4} - \frac{e}{c}\Phi_4\right)\psi^* + i\sum_{j=1}^{3}\eta_4\eta_j^*\left(-\frac{h}{2\pi i}\cdot\frac{\partial}{\partial x_j} + \frac{e}{c}\Phi_j\right)\psi^* - \eta_4\, mc\, \psi^*.$$

(††) W. Gordon, Zeit. f. Phys. **40** (1926), 117.
(†††) O. Klein, *ibidem*, **41** (1927), 432.
(††††) Naturally, the expressions on the right are only instantaneous values of the current components. Cf., footnote (†) on pp. 15 and the "Remark added in proof" on pp. 16.

$$\left.\begin{aligned}
\frac{\partial L_m}{\partial \Phi_1} &= \frac{e}{c}\,\underline{\psi^*,\varepsilon_1\psi} = -\frac{e}{c}[\psi_1^*\psi_4 + \psi_2^*\psi_3 + \psi_3^*\psi_2 + \psi_4^*\psi_1],\\
\frac{\partial L_m}{\partial \Phi_2} &= \frac{e}{c}\,\underline{\psi^*,\varepsilon_2\psi} = \frac{ie}{c}[\psi_1^*\psi_4 - \psi_2^*\psi_3 + \psi_3^*\psi_2 - \psi_4^*\psi_1],\\
\frac{\partial L_m}{\partial \Phi_3} &= \frac{e}{c}\,\underline{\psi^*,\varepsilon_3\psi} = -\frac{e}{c}[\psi_1^*\psi_3 - \psi_3^*\psi_4 + \psi_3^*\psi_1 - \psi_4^*\psi_2],\\
\frac{\partial L_m}{\partial \Phi_4} &= \frac{ie}{c}\,\underline{\psi^*,\psi} = \frac{ie}{c}[\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4].
\end{aligned}\right\} \tag{25}$$

If we now subject the wave equations (23) to the transformation (15a) then we now obtain the functions $S\psi$ in place of the wave function ψ, where S means the matrix (18). When written out, the transformation looks like:

$$\left.\begin{aligned}
\psi_1' &= \cos\frac{\Theta_1}{2}\psi_1 + i\sin\frac{\Theta_1}{2}\psi_4, \qquad \psi_2' = \cos\frac{\Theta_1}{2}\psi_2 + i\sin\frac{\Theta_1}{2}\psi_3,\\
\psi_3' &= i\sin\frac{\Theta_1}{2}\psi_2 + \cos\frac{\Theta_1}{2}\psi_3, \qquad \psi_4' = i\sin\frac{\Theta_1}{2}\psi_1 + \cos\frac{\Theta_1}{2}\psi_4.
\end{aligned}\right\} \tag{26}$$

If we now form the expression (25) with the ψ_i' precisely as we did with the ψ_i then we find, after some calculations (†):

$$\left.\begin{aligned}
&\underline{\psi'^*,\varepsilon_1\psi'} = \cos\Theta_1\cdot\underline{\psi^*,\varepsilon_1\psi} + \sin\Theta_1\cdot i\cdot\underline{\psi^*,\psi},\\
&\underline{\psi'^*,\varepsilon_2\psi'} = \underline{\psi^*,\varepsilon_2\psi},\\
&\underline{\psi'^*,\varepsilon_3\psi'} = \underline{\psi^*,\varepsilon_3\psi},\\
&i\cdot\underline{\psi'^*,\psi'} = -\sin\Theta_1\cdot\underline{\psi^*,\varepsilon_1\psi} + \cos\Theta_1\cdot i\cdot\underline{\psi^*,\psi},
\end{aligned}\right\} \tag{27}$$

i.e., the four quantities (25) transform according to the Lorentz matrix (15a), and are therefore vector components.

It still remains for us to show that that vector (25) is divergence-free. We differentiate each component of (25) with respect to its corresponding coordinate, add them, after we have multiplied them by $h / 2\pi i$, and finally add $\frac{e}{c}\sum\Phi_j\,\underline{\psi^*,\varepsilon_j\psi}$ + $\frac{ie}{c}\Phi_4\,\underline{\psi^*,\psi} - i\,mc\,\underline{\psi^*,\eta_4\psi}$, and subtract the same expressions. We then obtain:

$$i\frac{h}{2\pi i}\left(\underline{\psi^*,\frac{\partial\psi}{\partial x_4}} + \underline{\frac{\partial\psi^*}{\partial x_4},\psi}\right) + \frac{h}{2\pi i}\sum_{j=1}^{3}\underline{\psi^*,\varepsilon_j\frac{\partial\psi}{\partial x_j}} + \frac{h}{2\pi i}\sum_{j=1}^{3}\underline{\frac{\partial\psi^*}{\partial x_j},\varepsilon_j\psi}$$

(†) One observes the reality property that $i\sin\Theta_1/2$ is real.

$$+\frac{ie}{c}\Phi_4\underline{\psi^*,\psi}-\frac{ie}{c}\Phi_4\underline{\psi^*,\psi}+\frac{e}{c}\sum_{j=1}^{3}\Phi_j\underline{\psi^*,\varepsilon_j\psi}$$

$$-\,mc\underline{\psi^*,\eta_4\psi}-\frac{e}{c}\sum_{j=1}^{3}\Phi_j\,\underline{\psi^*,\varepsilon_j\psi}+mc\underline{\psi^*,\eta_4\psi}\,.$$

If we now combine the differential quotients $\partial\psi / \partial x_j$ – thus, the ones that are multiplied by ψ^* – with the positive potential terms and the quantities $mc\;\underline{\psi^*,\eta_4\psi}$ then we get, when we set $\varepsilon_4 = i$:

$$\underline{\psi^*,\sum_{j=1}^{4}\varepsilon_j\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_j}+\frac{e}{c}\Phi_j\right)\psi+mc\,\eta_4\psi}\,.$$

However, this quantity vanishes because the right-hand factor agrees with (23). We convert what remains slightly by observing that for j = 1, 2, 3 the following equations are valid (†):

$$\underline{\frac{\partial\psi^*}{\partial x_j},\varepsilon_j\psi}=\underline{\psi,\varepsilon_j^*\frac{\partial\psi^*}{\partial x_j}},\qquad \underline{\psi^*,\varepsilon_j\psi}=\underline{\psi,\varepsilon_j\psi^*}\,. \tag{28}$$

We then effortlessly get:

$$\underline{\psi,i\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_4}-\frac{e}{c}\Phi_4\right)\psi^*+\sum_{k=1}^{3}\varepsilon_k^*\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_k}-\frac{e}{c}\Phi_k\right)\psi^*-mc\,\eta_4\psi^*}\,.$$

This expression likewise vanishes because the right-hand factor agrees precisely with the differential equations for the complex-conjugate wave functions [footnote (†) on pp. 13]. The vanishing of the divergence of (25) is thus proved (††).

The invariance of the quantity L_m can now be easily shown. Since the operator $\partial / \partial x_j$ transforms like a vector, what one finds in the curly bracket in (24) is essentially the scalar product of it and the vector (25). The second expression is the scalar product of the four-potential and the current vector, while the invariance of $\underline{\psi^*,\eta_4\psi}$ can be easily shown by calculation with the help of the transformations (18), (20), and (22). Naturally, this quantity is also a rotational invariant, which comes from the fact that two invariants under rotations actually exist, namely, $|\psi_1|^2 + |\psi_2|^2$ and $|\psi_3|^2 + |\psi_4|^2$. One then easily recognizes from equations (12) that ψ_1 and ψ_2 will transform amongst themselves, as will ψ_3 and ψ_4, under a spatial rotation. Whereas the sum of these two rotational invariants is then just rotationally-invariant, the difference of the two is also Lorentz invariant.

(†) These equations are true for arbitrary Hermitian matrices. If the ε_j were no longer Hermitian then equations would be true in which one substitutes the "transposes" $\tilde{\varepsilon}_j$ for the ε_j^*. The function Φ, which is the solution of the transpose of equation (23), would then appear in place of ψ^*. The function L_m can also be generalized in this way. Cf., Dirac, *loc. cit.*, pp. 118.

(††) One finds another proof in Dirac, *loc. cit.*

Remark added in proof: Since equations (28) are independent of any special choice of the ε_j , we have proved the vanishing of the divergence of the current and the vector character of the quantities:

$$\underline{\psi^*, \varepsilon_j \psi} \qquad (j = 1, 2, 3) \qquad \text{and} \qquad \underline{\psi^*, \psi}$$

for an arbitrary time point. The invariance of L_m for all time then follows from this. This result can be concluded from just the Lorentz invariance of the Dirac equations. On the other hand, the proof of the Lorentz invariance of the Dirac equation proceeds only under the assumption that p_0 and the matrices R, S, T, U commute with each other; i.e., they are constant in time. It first follows from this fact that the matrices R, S, T, U that we obtained experience rotations and Lorentz transformations for all time.

The fact that the derivatives of the Lagrange potential L_m with respect to the components of the four-potential give the current components allows one to give a Lagrange function L under which not just the Dirac equations, but also Maxwell's equations can be preserved under variation of the ψ_i, the ψ_i^*, and the Φ_i . One easily concludes that this function L has the form:

$$L = L_m + \frac{1}{8\pi c}(\mathfrak{H}^2 - \mathfrak{E}^2). \tag{29}$$

One arrives at the energy-impulse tensor of matter waves in the same way that Schrödinger did. One can hardly confirm that this, together with the tensor of the electromagnetic field, yields a complete field theory, but one can also not contradict it, either. Certainly, one will have to regard the ψ_i-functions as q-number functions, in the same way that Klein and Jordan (*) did. Naturally, whether this alone will suffice to evade all of the difficulties with the many-body problem (e.g., the back-reaction of the proper field) cannot be decided without further information. The author hopes to return to these questions in the near future.

(*) P. Jordan and O. Klein, Zeit. f. Phys. **45** (1927), 751.