"Sur la dynamique des systèmes ayant un moment angulaire interne," Ann. de la Inst. Henri Poincaré **2** (1949), 251-278.

On the dynamics of systems that have internal angular momentum

by

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One can show that the well-known difficulties that one encounters in the theory of the electron and in the theories of other elementary particles are due to two types of singularities:

1. Singularities that are coupled with a point-like image of the electron and which exist already in the classical theory.

2. Singularities in the fluctuations that essentially have their origin in quantum theories ([1]).

Numerous attempts have been made to remedy the singularities of the first kind. Abraham and Lorentz ([2]) have represented the electron as a small, but finite, system that possesses an electric charge that is distributed uniformly, and Lorentz has even proposed that the hypothesis that all of the mass of the electron is of electromagnetic origin. Meanwhile, one rapidly perceives that the principle of relativity demands the existence of non-electromagnetic forces and energies inside the electron, at least if one supposes that Maxwell's equations are valid in a general fashion. Those considerations led Poincaré ([3]) to his well-known model of the electron in which the repulsive Coulomb forces are counterbalanced by cohesive forces of an unknown nature. The Poincaré model provided a coherent classical image of the electron that satisfied all of the conditions of the theory of relativity. Poincaré's theory is essentially a dualistic theory. Mie ([4]) was the first to introduce the idea of a unitary theory of the field in which all electromagnetic phenomena were described by one field. Inside of the electron, it must be very different from the electromagnetic field that one infers from Maxwell equations. Mie's theory must be

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([1]) W. PAULI, "Difficulties of Field Theories and of Field Quantization," *Report of the International Conference in Cambridge*, 1946, vol. I, Fundamental Particles.

([2]) Cf., for example, H. A. LORENTZ, *On the Theory of Electrons*.

([3]) H. POINCARÉ, "Sur la dynamique de l'électron," Rend. Pal. **21** (1906), pp. 129.

([4]) G. MIE, "Grundlagen einer Theorie der Materie," Ann. Phys. (Leipzig) **37** (1912), pp. 511; *ibid.*, **39** (1942), pp. 1.

abandoned, because it contradicts some well-established physical facts. As a result, Born ([5]) once more took up the idea of a unitary theory of electrodynamical phenomena and succeeded in developing a coherent, nonlinear electrodynamical theory.

Although it is not at all certain (or perhaps even true) that a quantum theory of electrons and electrodynamical fields can be obtained by a simple quantization of the equations that were considered by Poincaré and Born, it is nonetheless not without interest to investigate the degree to which it is possible to attribute the properties of a particle to a small classical system of finite dimension. Here, we intend that the term *classical spinning particle* to mean a system that is defined by its space-time coordinates, it impulse-energy quadri-vector, and its internal kinetic moment.

From the theory of relativity, a finite-dimensional system has an infinite number of degrees of freedom, and it seems difficult to describe such a system by a particle that has a finite number of degrees of freedom. That amounts to knowing if it is, in general, possible to define a point of the system whose position can be considered to be that of a representative particle of the system.

In Newtonian mechanics, and for an arbitrary system, such a point is provided by the center of gravity. The motion of the latter is indeed identical to that of a particle whose mass, quantity of motion, and energy are equal to the corresponding quantities of the system, respectively. One can, moreover, attribute an internal kinetic moment to the center of gravity that is equal to the kinetic moment that the system possesses with respect to its center of gravity. In this paper, we shall discuss the following problem: To what degree can a representative point be defined for an arbitrary relativistic system? In the first chapter, we shall consider a free system with no external forces, and in the second one, we shall consider the general case of a classical system that is subject to given external forces.

In a series of articles that appeared in *Acta Physica Polonica,* Mathisson ([6]) and Weyssenhoff ([7]) have developed a classical theory of spinning particles. That theory was based, on the one hand, upon the principles of relativity, and on the other hand, upon some new hypotheses. The equations of motion of a spinning particle that were obtained by those authors have some strange consequences. If one is given that a particle can be considered to be the limit of a system that we shall envision here then our study will give us a better comprehension of the meaning and physical interpretation of Mathisson's equations.

I. – DEFINITION OF THE CENTER OF GRAVITY OF A FREE SYSTEM IN THE THEORY OF RELATIVITY.

Consider an arbitrary finite, isolated system in special relativity; i.e., a system that is not subject to external forces, but whose constituents have arbitrary interactions. The definition of the center of gravity of such a system was discussed in a series of conferences that took place in Dublin in 1947 and were published in the *Communications*

([5]) M. BORN, Proc. Roy. Soc. A **143** (1934), pp. 410.
([6]) M. MATHISSON, Acta Phys. Pol. **6** (1937), pp. 163 and 218.
([7]) J. WEYSSENHOFF, Nature (1938), pp. 328; Acta Phys. Pol. **9** (1947), 1-62.

of the Dublin Institute for Advanced Studies ([8]). We shall give an outline here of the principal results of that article without entering into all of the details of the proofs. The system considered is defined by its impulse-energy tensor $T_{ik} = T_{ik}(x_l)$, which is a function of the space-time coordinates:

$$x_l = \{x, y, z, t, \ddot{\imath}ct\} = \{\mathbf{x}, \ddot{\imath}ct\}.$$

x, y, z are the components of the radius vector $\mathbf{x}$ in an arbitrary Lorentz reference system S, t is the time variable, c is the speed of light, and $\ddot{\imath}$ represents a quantity whose square is equal to – 1. (It is convenient to employ the symbol $\ddot{\imath}$, in order to distinguish it from the i that appears in the commutation relations of quantum mechanics.)

For a free system, we have the fundamental equation:

$$\frac{\partial T_{ik}}{\partial x_k} = 0. \tag{1}$$

We make the convention of summing over dummy indices (viz., indices that appear twice) from 1 to 4 for Latin indices, and from 1 to 3 for only Greek indices.

Set:

$$\frac{1}{\ddot{\imath}c}T_{i4} = \frac{1}{\ddot{\imath}c}T_{4i} = g_i = \left\{\mathbf{g}, \frac{\ddot{\imath}}{c}h\right\}, \tag{2}$$

in which $\mathbf{g}$ and h are the density of the quantity of motion and the energy density, respectively. For $i = 4$, equation (1) can be written:

$$\frac{\partial g_k}{\partial x_k} = 0. \tag{3}$$

It results from (1) that the four quantities:

$$P_i = \int g_i(\mathbf{x}, t)\, dV = \left\{\mathbf{P}, \frac{\ddot{\imath}}{c}H\right\} \tag{4}$$

that are obtained by integration over all of ordinary space at given instants are independent of t and transform like the components of a quadri-vector under a Lorentz transformation.

The constants of motion $\mathbf{P}$, H represent the total impulse and total energy of the system, respectively.

The invariant quantity M_0 that is defined by the equation:

([8]) C. MØLLER, "On the definition of the center of gravity of an arbitrary closed system in the theory of relativity," Communications of the Dublin Institute for Advanced Studies, Series A, No. 5. Most of the results that are contained in that article have been obtained independently by M. H. L. Pryce in a paper in Proc. Roy. Soc. London A **195** (1948), pp. 62.

(5) $$P_l\, P_l = -M_0^2\, c^2$$

is also independent of time and determines the proper mass of the global system. In the Lorentz reference system S^0, in which the impulse $\mathbf{P}^0$ is zero, we will obviously have:

(6) $$P_i^0 = \{0, 0, 0, \ddot{\imath} M_0\, c\}.$$

The velocity of S^0 with respect to a Lorentz reference frame S is given by:

(7) $$\mathbf{U} = \frac{c^2\mathbf{P}}{H}.$$

In an analogous fashion, it results from (1) and the symmetry of the impulse-energy tensor that the quantities:

(8) $$M_{ik} = -M_{ki} = \int (x_i\, g_k - x_k\, g_i)\, dV$$

are the components of an antisymmetric tensor that is independent of time. That four-dimensional tensor is the tensor of kinetic moment with respect to the arbitrary origin of our space-time reference.

In Newtonian mechanics, the coordinates of the center of gravity of a system whose mass density is μ ($\mathbf{x}$, t) are given by the vector:

(9) $$\mathbf{X} = \frac{1}{M}\int \mu(\mathbf{x},t)\,\mathbf{x}\, dV ,$$

in which:

$$M = \int \mu\, dV$$

is the total mass of the system. The center of gravity is then the center of mass, and its position is, in a sense, the mean position of the mass of the system.

However, from the theory of relativity, any quantity of energy will correspond to a quantity of inertial mass that is given by Einstein's well-known relation. Let h ($\mathbf{x}$, t) be the energy density of the system, so the corresponding mass density μ ($\mathbf{x}$, t) will be given by:

(10) $$h = \mu\, c^2.$$

In a given Lorentz reference system S, the position of the center of mass is determined by the equation:

(11) $$\mathbf{X} = \frac{1}{H}\int h(\mathbf{x},t)\,\mathbf{x}\, dV .$$

With the aid of (3) and (4), one easily sees that the point that is defined by (11) is animated with a constant velocity:

(12) $$\frac{d\mathbf{X}(S)}{dt} = \frac{c^2\mathbf{P}}{H} = \mathbf{U};$$

i.e., with the same velocity as the system S^0 in which the total quantity of motion is zero.

A deeper study will show us that the point that is defined by (11) depends upon the Lorentz reference system S in which the integral of the right-hand side of (11) is calculated. That amounts to saying that the centers of mass in the two Lorentz reference systems are two different points, in general. In fact, in an arbitrary physical system, there is an infinitude of centers of mass that corresponds to the various Lorentz reference systems S. Fokker ([9]) pointed that out already, for the special system that is composed of a certain number of particles with no mutual interaction.

From (12), all of the centers of mass are at rest in the reference system S^0 that is their proper reference system. That system plays a special role: The center of mass belongs to the reference system S^0 itself. We call that point whose radius vector is $\mathbf{X} = \mathbf{X}(S^0)$ the *center of gravity* of the system. In the proper reference system S^0, that radius vector will have the constant value:

(13) $$\mathbf{X}^0 = \frac{c^2}{H}\int\frac{h^0(\mathbf{x}^0, t^0)}{c^2}\mathbf{x}^0\,dv^0 = \frac{1}{M_0}\int h^0(\mathbf{x}^0, t^0)\,\mathbf{x}^0\,dv^0 .$$

Let X_i be the space-time coordinates of the center of gravity, thus-defined, in an arbitrary Lorentz reference system. If τ is proper time, and $X_i = X_i(\tau)$ is a linear function of τ and the velocity quadri-vector that corresponds to the constant value then:

(14) $$U_i = \frac{dX_i}{d\tau} = \frac{P_i}{M_0} .$$

We now define the quadri-dimensional tensor m_{ik} that represents the internal kinetic moment with respect to the center of gravity by the equations:

(15) $$m_{ik} = \int[(x_i - X_i)g_k - (x_k - X_k)g_i]\,dv = M_{ik} - (X_i P_k - X_k P_i).$$

With the aid of (14), we get:

(16) $$\frac{dm_{ik}}{d\tau} = -(U_i P_k - U_k P_i) = 0.$$

Hence, $m_{ik} = - m_{ki}$ is a constant antisymmetric tensor.

Introduce two spatial vectors $\mathbf{m}$ and $\mathbf{n}$ that are defined by:

(17) $$\begin{cases} \mathbf{m} = \{m_x, m_y, m_z\} = \{m_{23}, m_{31}, m_{12}\}, \\ ï\mathbf{n} = ï\{n_x, n_y, n_z\} = \{m_{14}, m_{24}, m_{24}\}. \end{cases}$$

([9]) A. D. FOKKER, *Relativiteitstheorie*, pp. 170, Nordhoff, Gronnigen, 1929.

One deduces from (15) that:

$$\mathbf{m} = \int (\mathbf{x} - \mathbf{X}) \times \mathbf{g}\, dv \tag{18}$$

is the vector of kinetic moment with respect to the center of gravity, which we call the *internal kinetic moment*. Moreover, we deduce from (17) and (15) that if we take $x_4 = X_4$ (τ) then we will have:

$$\mathbf{n} = \int (\mathbf{x} - \mathbf{X}) \frac{h(\mathbf{x}, t)}{c} dv \bigg|_{x_4 = X_4} = \frac{1}{c} \int h\, \mathbf{x}\, dv - \mathbf{X} \frac{H}{c}. \tag{19}$$

That means that the time variable $x_4 = ï\, ct$ in the integral must be taken to be equal to the time X_4 of the center of gravity.

The first expression for $\mathbf{n}$ in (19) shows that $\mathbf{n} / c$ is equal to the moment of the mass of the system with respect to the center of gravity.

If one takes (11) into account then the last expression for $\mathbf{n}$ in (19) will give us:

$$\mathbf{X} = \frac{1}{c} \int h\, \mathbf{x}\, dv \bigg|_{x_4 = X_4} - \frac{c\mathbf{n}}{H} = \mathbf{X}\,(S) - \frac{c\mathbf{n}}{H}, \tag{20}$$

in which $\mathbf{X}$ and $\mathbf{X}$ (S) correspond to two simultaneous positions of the center of gravity and center of mass, respectively. Therefore, we see that the center of gravity is a center of mass in any Lorentz reference system if and only if the internal kinetic moment tensor m_{ik} is equal to zero – i.e., in the case of a system without spin.

In the proper reference frame S^0, the center of gravity is, by definition, identical to the center of mass. Consequently, the vector $\mathbf{n}$ must be zero S^0; i.e.:

$$\mathbf{n}^0 = 0, \qquad m_{i4}^0 = 0. \tag{21}$$

That relation is identical to the invariant relation:

$$m_{ik}\, P_k = 0, \tag{22}$$

as one will deduce from (6) and (21) when (22) is written in the reference system S^0.

By reason of (14), equation (22) must also be written:

$$m_{ik}\, U_k = 0, \tag{23}$$

which, in turn, expresses the condition that the center of gravity is the center of mass in its proper reference system in an invariant form.

If we choose the same orientation for the spatial axes in S and S^0 then the Lorentz reference system S is defined uniquely by S^0 and the vector:

$$\mathbf{v} = \mathbf{U} = \frac{c^2 \mathbf{P}}{H}, \tag{24}$$

which represents the relative velocity of S^0 with respect to S.

From the properties of the transformation of an antisymmetric tensor m_{ik} , (21) will give us:

(25) $$\mathbf{n} = \frac{\mathbf{v}\times\mathbf{m}_0}{c\sqrt{1-\frac{v^2}{c^2}}},$$

in which $\mathbf{m}^0$ is the internal kinetic moment in S^0.

From (20), (24), and (25), the difference between the simultaneous positions of the center of gravity and the center of mass in the system S is given by the spatial vector:

(26) $$\mathbf{a}\,(S) = \mathbf{X}\,(S) - \mathbf{X} = \frac{c\,\mathbf{n}}{H} = \frac{\mathbf{v}\times\mathbf{m}^0}{M_0c^2},$$

which is independent of time.

Since the passage from S to S^0 is effected by a Lorentz transformation without rotation of the spatial axes, and since $\mathbf{a}$ is perpendicular to the relative velocity $\mathbf{v}$, the distance measured in S^0 is given by the spatial vector $a^0(S)$, which is equal to $\mathbf{a}\,(S)$ in (26); i.e.:

(27) $$\mathbf{a}^0\,(S) = \mathbf{a}\,(S) = \frac{\mathbf{v}\times\mathbf{m}^0}{M_0c^2}.$$

In the proper reference system, S^0 all of the centers of mass that are obtained by varying S or $\mathbf{v}$ in (27) will have a locus that consists of the circle that is perpendicular to the internal kinetic moment $\mathbf{m}^0$ and a radius of:

(28) $$\rho = \frac{|\,\mathbf{m}^0\,|}{M_0c^2}.$$

As a result, we call the circle the *center of mass disc*, or simply the *disc*. The center of that disc is the point $C = C\,(S^0)$, which has been called the center of gravity of the system. If $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\|$ is decomposed into the sum of two vectors $\mathbf{v}_\perp$, $\mathbf{v}_\|$ which are perpendicular to $\mathbf{m}^0$ and parallel to it, respectively, then we will see that $\mathbf{a}^0\,(S)$ in (27) depends upon only the perpendicular component $\mathbf{v}_\perp$. Each point of the disc is then a center of mass in an infinitude of systems S that correspond to the various values of $\mathbf{v}_\|$ that are found in the interval $-\sqrt{c^2-\mathbf{v}_\perp^2} \le \mathbf{v}_\| \le \sqrt{c^2-\mathbf{v}_\perp^2}$. The disc of the centers of mass is at rest in S^0, and consequently, it will displace in an arbitrary system S like a rigid body with a constant velocity.

Consider a system that is completely contained in a sphere of center C of radius r in S^0; i.e., a system in which all of the components of the impulse-energy tensor zero outside of that sphere. Moreover, if we suppose that the energy density h is positive in any reference frame then it will be clear that the center of mass disc must be completely inside of the sphere. Indeed, if we consider an arbitrary point $C\,(S)$ on the disc then that point will the center of mass in the Lorentz reference system S, and since h is positive, it must be inside of the system.

We then get the inequality:

$$r \geq \frac{|\mathbf{m}^0|}{M_0 c}. \tag{29}$$

In other words:

A classical system that has a positive energy density, a given internal kinetic moment $\left|\mathbf{m}^0\right|$, *and a given proper mass* M^0 *always has finite dimensions that are given by* (29) *in the center of gravity system.*

If the system is smaller then the energy density h cannot be everywhere positive in every reference system.

As we pointed out in the introduction, Mathisson ([6]) and Weyssenhoff ([7]) have developed a theory of the motion of classical spinning particle in which the motion of such a particle will not be determined uniquely by the initial position and velocity of the particle. Indeed, the equations of motion have an infinitude of solutions for given initial values of those quantities. In the case of a free particle, those solutions will correspond to circular motion around a center that itself displaces with a constant velocity. By reason of the resemblance between that motion and Schrödinger's *zitterbewegung* of a Dirac electron, Mathisson and Weyssenhoff have considered the Mathisson particle to be the classical image of the Dirac electron.

Since a particle can be considered to be a limiting case of the general system that is considered here, and since the center of gravity of any free system displaces with a constant velocity, the coordinates of the Mathisson particle obviously cannot be identified with the coordinates of the center of gravity as they were defined above.

Meanwhile, in an arbitrary physical system, as we see, there exist a certain number of points that properties that are very similar to those of the center of gravity, and a deeper study will show that the Mathisson equations are, in fact, the equations of motion of those pseudo-centers of gravity. The existence of an infinitude of solutions of those equations indicates simply that there are an infinitude of pseudo-centers of gravity for any system that possesses an internal kinetic moment. Among all of the centers of mass of the center of gravity disc, only one of them has the property that is it a center of mass in its proper reference system S^0. Any other point C (S) whose radius vector a^0 (S) = $\dfrac{\mathbf{v}\times\mathbf{m}^0}{M_0 c^2}$ is a center of mass in a system S that displaces with the velocity – $\mathbf{v}$ with respect to its proper reference system S^0. Now imagine that the rigid disc that was considered above is put into rotation around the center of gravity C with a constant angular velocity:

$$\boldsymbol{\omega}^0 = -\frac{M_0 c^2}{|\mathbf{m}^0|^2}\ \mathbf{m}^0. \tag{30}$$

$\boldsymbol{\omega}^0$ is then a vector in the same direction as $\mathbf{m}^0$, with the opposite sense, and a length of:

$$\omega^0 = \frac{M_0 c^2}{|\mathbf{m}^0|^2}. \tag{31}$$

Any fixed point p on the rotation disc will then be a center of mass in its instantaneous proper reference system at any instant, because if $\mathbf{r}^0$ (p) is the radius vector at the instant considered then the velocity of that point in the system S^0 will be:

$$\mathbf{u}^0 = (\boldsymbol{\omega}^0 \times \mathbf{r}^0) = \frac{M_0 c^2}{|\mathbf{m}^0|^2} (\mathbf{r}^0 \times \mathbf{m}^0),$$

or, since $\mathbf{r}^0$ is perpendicular to $\mathbf{m}^0$ and $\mathbf{u}^0$:

(32) $$\mathbf{r}^0 = \frac{(-\mathbf{u}^0 \times \mathbf{m}^0)}{M_0 c^2}.$$

The comparison of (32) and (27) shows that p is the center of mass in the Lorentz reference system S^* that is animated relative to S^0 with a velocity $\mathbf{u}^0$; i.e., S^* is the instantaneous proper reference system at the point p. Hence, any point of the rotating disc is a pseudo-center of gravity that is, at each instant, the center of mass in its instantaneous proper reference system. The number of those pseudo-centers of gravity is equal to the number of points of the rotating disc. The distance r^0 (p) can take all values that are found between 0 and ρ, which is defined by (28):

(33) $$0 \leq r^0\,(p) \leq \frac{|\mathbf{m}^0|}{M_0 c}.$$

The speed of p tends to the speed of light c when r^0 (p) tends to the upper limit r; indeed:

(34) $$u^0\,(p) = r^0\,\omega^0 = r^0\,\frac{M_0 c^2}{|\mathbf{m}^0|} \to c,$$

when:

$$r^0 \to \frac{|\mathbf{m}^0|}{M_0 c}.$$

We shall now see that the equations of motion of those pseudo-centers of gravity are identical with the Mathisson equations. Let $x_i^{(p)}$ be the space-time coordinates of a pseudo-center of gravity in an arbitrary Lorentz reference system, and let τ be the corresponding proper time.

The quadri-dimensional Ω_{ik} that represents the kinetic moment with respect to the point $x_i^{(p)}$ is given by:

(35) $$\Omega_{ik} = \int [(x_i - x_i^{(p)}) g_k - (x_k - x_k^{(p)}) g_i]\, dV = M_{ik} - (x_i^{(p)} P_k - x_i^{(p)} P_i).$$

Upon differentiating (35) with respect to proper time τ, we will get:

$$\dot{\Omega}_{ik} = \frac{d\Omega_{ik}}{d\tau} = -\,(u_i\,P_k - u_k\,P_i), \tag{36}$$

in which:

$$u_i = \frac{dx_i^{(p)}}{d\tau}$$

is the velocity quadri-vector of the point p, which satisfies the equations:

$$u_i\,u_i = -\,c^2, \qquad u_i = \frac{dx_i^{(p)}}{d\tau}\,. \tag{37}$$

If $S^*(t)$ is the proper reference system of p that corresponds to the instant τ then p will be, by definition, the center of mass in S^* at that instant. By an argument that is analogous to the one that we made in the case of the center of gravity, we can conclude that the mixed spatial-temporal components of Ω_{ik} must be zero in S^*; i.e.:

$$\Omega_{ik}^* = 0,$$

which is an equation that can be written in an invariant fashion as:

$$\Omega_{ik}\,u_k = 0, \tag{38}$$

by analogy with (23). Upon substituting the expression (35) for Ω_{ik} in that, one will have:

$$\Omega_{ik}\,u_k - x_i^{(p)}\,(P_k\,u_k) + P_i\,(x_k^{(p)}\,u_k) = 0, \tag{39}$$

and by differentiating that with respect to τ, and with the aid of (37):

$$M_{ik}\dot{u}_k - u_i(P_k\,u_k) - x_i^{(p)}(P_k\,\dot{u}_k) + P_i\,(-c^2 + x_k^{(p)}\dot{u}_k) = 0. \tag{40}$$

Upon multiplying that equation by $\dot{u}_k$, we will obtain:

$$P_i\,\dot{u}_i = \frac{d}{d\tau}(P_i\,u_i) = 0, \tag{41}$$

since M_{ik} is antisymmetric, and $u_i\,\dot{u}_i = 0$.

The invariant $P_i\,u_i$ is then a constant of motion. In the instantaneous proper reference system S^*, we will have:

$$u_i^* = \{0, 0, 0, \ddot{\imath}\,c\},$$

and in turn:

$$P_i\,u_i = \ddot{\imath}\,c\;P_4^* = -\,E^* = -\,M^*c^2, \tag{42}$$

in which M^* is the total mass of the system in S^*. That mass is then independent of τ.

Upon multiplying (35) by $\dot{u}_k$, we will get, with the aid of (41):

$$\Omega_{ik}\,\dot{u}_k = M_{ik}\dot{u}_k + P_i(x_k^{(p)}\dot{u}_k)\,. \tag{43}$$

Upon taking (42) and (43) into account, equations (40) can be written:

$$P_i = M^* u_i + \frac{\Omega_{ik}\,\dot{u}_k}{c^2} = M^* u_i + \pi_i\,, \tag{44}$$

with

$$\pi_i = \frac{\Omega_{ik}\,\dot{u}_k}{c^2}\,, \tag{45}$$

and $\pi_i\,u_i = 0$, due to (38); as a result:

$$P_i\,P_i = -\,M_0^2\,c^2 = -\,M^{*2}\,c^2 + \pi_i\,\pi_i\,. \tag{46}$$

If one multiplies (36) by then one will have, upon taking (37) and (41) into account:

$$\dot{\Omega}_{ik}\,\dot{u}_k = 0. \tag{47}$$

By differentiating equation (44), and recalling that M^* is constant, one will then get the following equations for the motion of the pseudo-centers of gravity:

$$M^*\dot{u}_i + \frac{\Omega_{ik}\,\ddot{u}_k}{c^2} = 0. \tag{48a}$$

Furthermore, introduce (44) into (36); one will get:

$$\dot{\Omega}_{ik} + \frac{1}{c^2}(u_i\,\Omega_{kl}\,\dot{u}_l - \Omega_{il}\,\dot{u}_l\,u_k) = 0. \tag{48b}$$

Equations (48*a*) and (48*b*), when combined with equation (38), are formally identical to the equations that were given by Mathisson for the motion of a spinning particle. Equations (42) and (44) represent first integrals of equations (48). One will see immediately that:

$$u_i = U_i = \text{const.}, \qquad \Omega_{ik} = m_{ik} = \text{const.} \tag{49}$$

is a solution of equations (48).

That solution corresponds to the motion of the center of gravity. In that case, $\dot{u}_i = 0$, and from (45), (46), and (44), we will have:

$$\pi_i = 0,\ M^* = M_0\,, \qquad P_i = M^*\,u_i = M_0\,U_i\,. \tag{50}$$

As one will easily see, all of the other solutions of (48) and (38) for the given values of M_0 and $\mathbf{m}^0$ will correspond to motions of the points p on the rotating disc considered. For all of those solutions, the quadri-vector π_i is non-zero, and $M^* > M_0$; i.e., the quadri-vector P_i is not proportional to u_i , but to a component π_i that is perpendicular to u_i . Only the center of gravity both equations (38) and (50) at once.

II. –DYNAMICS OF SYSTEMS SUBJECT TO EXTERNAL FORCES.

Now consider an arbitrary system that is subject to given external forces. We first treat the case in which the external forces are not gravitational forces. In a well-defined Lorentz reference system, those forces will then be described by a quadri-vector:

(51) $$f_i = \{\mathbf{f}, \frac{\ddot{i}}{c}\mathbf{q}\},$$

in which **f** represents the force density, and **q** represents the energy that is expended per unit time and volume. Instead of (1), we will now have:

(52) $$\frac{\partial T_{ik}}{\partial x_k} = f_i ,$$

in which T_{ik} is, once more, the impulse-energy tensor of our system. The four-dimensional spatial domain in which $T_{ik} \neq 0$ is a tube whose direction is timelike.

We shall now try to determine the world-line of the center of gravity in our system. Let L be a curve that has a tangent at each of its points whose direction is timelike. L can then be considered to be the world-line of a moving point that we call the *representative point*. If $x_i^{(r)}$ are the space-time coordinates of that representative point, and τ is the corresponding proper time then L will be well-defined if the $x_i^{(r)}$ are given as functions of the parameters τ:

(53) $$x_i^{(r)} = x_i^{(r)}(\tau).$$

The velocity quadri-vector $u_i = dx_i / d\tau$ of the representative point verifies the equation:

(54) $$\begin{cases} u_i\, u_i = -c^2, \\ u_i\, \dot{u}_i = \quad 0, \end{cases}$$

in which the dot indicates differentiation with respect to τ.

Consider an arbitrary point p (τ) of L that corresponds to a well-defined value of τ, and let V (τ) be the three-dimensional hyperplane that is perpendicular to the tangent at p; i.e., it is orthogonal to the vector u_i at p. The consecutive hyperplanes V (τ) are determined uniquely when L is given. Consider a volume element dV in the hyperplane V

(t) that is composed of three independent infinitesimal dx_i , δx_i , Δx_i . That volume element is represented by the antisymmetric tensor:

$$dV_{ikl} = \begin{vmatrix} dx_i & \delta x_i & \Delta x_i \\ dx_k & \delta x_k & \Delta x_k \\ dx_l & \delta x_l & \Delta x_l \end{vmatrix}, \tag{55}$$

or by the corresponding pseudo-vector dV_i that is dual to it, which is defined by:

$$dV_1 = -\,ï\, dV_{234}\,, \qquad dV_2 = ï\, dV_{241}\,, \qquad dV_3 = -\,ï\, dV_{412}\,, \qquad dV_4 = ï\, dV_{123}\,. \tag{56}$$

The pseudo-vector dV_i is orthogonal to the hyperplane $V(\tau)$ and is then proportional to $u_i(\tau)$. Since:

$$dV_i\, dV_i = -\,(dV)^2, \tag{57}$$

in which dV is the invariant volume of that element, we will obviously have:

$$dV_i = \frac{dV}{c}\, u_i\,, \qquad dV = -\frac{1}{c}(dV_i\, u_i). \tag{58}$$

In a Lorentz reference system $S^*(\tau)$ whose time axis is parallel to $u_i(\tau)$, we have:

$$\begin{cases} u_i^* = \{0, 0, 0, \quad ïc\}, \\ dV_i^* = \{0, 0, 0, ï dV^*\}. \end{cases} \tag{59}$$

$S^*(\tau)$ is the proper differential of the representative point at the instant considered.

Now consider two consecutive hyperplanes $V(\tau)$ and $V(\tau + d\tau)$ and the four-dimensional domain Ω that is bounded by those surfaces and a cylindrical surface s that encloses the tube in which $T_{ik} \neq 0$. Consider an infinitesimal element $d\Sigma$ in Ω that has a cylindrical form whose axis dl_i of length dl is perpendicular to $V(\tau)$, and whose cross-section is dV. If L is a straight line then the two hyperplanes $V(\tau)$ and $V(\tau + d\tau)$ will be parallel, and dl will simply be equal to the distance $ïc\ d\tau$ between the points of intersection $p(\tau)$ and $p(\tau + d\tau)$ with the curve L. If one takes the curvature of L into account then one will easily see that one has:

$$dl = ïc\, d\tau \left(1 + \frac{\xi_i\, \dot{u}_i}{c^2}\right), \tag{60}$$

in which

$$\xi_i = x_i - x_i^{(r)}, \quad \xi_i\, u_i = 0 \tag{61}$$

is a vector that links the point $p(\tau)$ to the element dV in $V(\tau)$. By virtue of (60) and (58), the volume $d\Sigma$ of the infinitesimal four-dimensional cylindrical element will be equal to:

(62) $$d\Sigma = dV\,dl = \frac{dV_i\,u_i}{\ddot{i}}\left(1+\frac{\xi_k\dot{u}_k}{c^2}\right)d\tau\,.$$

Upon integrating equation (52) in the domain Ω, we will get from (62):

(63) $$\int_\Omega \frac{\partial T_{ik}}{\partial x_k}\,d\Sigma = d\tau\int_{d\tau} f_i\,\frac{dV_i\,u_i}{\ddot{i}}\left(1+\frac{\xi_k\dot{u}_k}{c^2}\right).$$

Upon applying Gauss's theorem in four-dimensional space, one can transform the left-hand side of that equation into an integral over the surface that bounds Ω. Since $T_{ik} = 0$ on the cylindrical surface s, only the hyperplanes $V(\tau)$ and $V(\tau + d\tau)$ give a contribution to the integral, and we will obtain for the left-hand side of (63):

$$\int_{V(\tau+d\tau)} T_{ik}\,\frac{dV_k}{\ddot{i}} - \int_{V(\tau)} T_{ik}\,\frac{dV_k}{\ddot{i}}\,.$$

Hence, after dividing by $\ddot{i}c\;dt$ and passing to the limit $d\tau \to 0$, equation (63) will become:

(64) $$\frac{d}{d\tau}\int_{V(\tau)} \frac{T_{ik}}{\ddot{i}c}\,\frac{dV_k}{\ddot{i}} = \int_{V(\tau)} f_i\left(1+\frac{\xi_k\dot{u}_k}{c^2}\right)\frac{dV_l u_l}{-c}\,.$$

Upon defining two quadri-vectors $P_i(\tau)$ and $F_i(\tau)$ by:

(65) $$P_i(\tau) = \int_{V(\tau)} \frac{T_{ik}}{\ddot{i}c}\,\frac{dV_k}{\ddot{i}}\,,$$

(66) $$F_i(\tau) = \int_{V(\tau)} f_i\left(1+\frac{\xi_k\dot{u}_k}{c^2}\right)\frac{(dV_l u_l)}{-c}\,,$$

equation (64) can be written:

(67) $$\frac{dP_i}{d\tau} = F_i\,.$$

In the instantaneous proper reference system $S^*(\tau)$ of our representative point, the expression (65) for P_i will reduce to:

(68) $$P_i^* = \int \frac{T_{i4}^*}{\ddot{i}c}\,dV^* = \{\mathbf{P}^*,\ \frac{\ddot{i}}{c}H^*\},$$

in which $\mathbf{P}^*$ and H^* are the total impulse and energy in the reference system S^*, respectively.

In the case of a free system, we have $f_i = 0$ and $F_i = 0$. In that case, P_i will then be independent of τ, as well as the choice of representative curve L. Meanwhile, in general, the quadri-vector P_i (τ) will depend upon the point p (τ) considered, as well as on the direction of the tangent at that point.

If we take into account the symmetry of the tensor T_{ik} then we will deduce the following relation from (52):

$$\frac{\partial}{\partial x_l}(x_i\, T_{kl} - x_k\, T_{il}) = x_i\, f_k - x_k\, f_i\,.$$

Upon integrating that equation over Ω and applying Gauss's theorem, as we did in order to establish (67), we will obtain:

(69) $$\frac{d}{d\tau} M_{ik} = D_{ik}\,,$$

in which M_{ik} and D_{ik} are defined by the relations:

(70) $$M_{ik}\,(\tau) = \int_{V(\tau)} \frac{x_i T_{kl} - x_k T_{il}}{\ddot{i}c} \frac{dV_l}{\ddot{i}}\,,$$

(71) $$D_{ik}\,(\tau) = \int_{V(\tau)} (x_i\, f_k - x_k\, f_i)\left(1 + \frac{\xi_k\, \dot{u}_k}{c^2}\right)\frac{(dV_l\, u_l)}{-c}\,.$$

M_{ik} is the four-dimensional tensor in our arbitrary Lorentz reference system that represents the total kinetic moment with respect to the origin and corresponds to the value τ of proper time for the representative point. Similarly, the tensor that represents the kinetic moment with respect to the representative point p (τ) will be given by:

(72) $$\Omega_{ik}\,(\tau) = \int_{V(\tau)} \frac{\xi_i T_{kl} - \xi_k T_{il}}{\ddot{i}c} \frac{dV_l}{\ddot{i}} = \int_{V(\tau)} \frac{(x_i - x_i^{(r)})T_{kl} - (x_k - x_k^{(r)})T_{il}}{\ddot{i}c} \frac{dV_l}{\ddot{i}}\,,$$

which can also be written:

(73) $$\Omega_{ik}\,(\tau) = M_{ik}\,(\tau) - [\,x_i^{(r)}(\tau)\, P_k\,(\tau) - x_k^{(r)}(\tau)\, P_i\,(\tau)].$$

Upon taking (69) and (67) into account, if we differentiate with respect to τ then we will obtain:

$$\begin{aligned}\frac{d\Omega_{ik}}{d\tau} &= \dot{M}_{ik} - (u_i P_k - u_k P_i) - (x_i^{(r)} \dot{P}_k - x_k^{(r)} \dot{P}_i) \\ &= D_{ik} - (u_i\, P_k - u_k\, P_i) - (x_i^{(r)} F_k - x_k^{(r)} F_i)\,.\end{aligned}$$

That equation can also be written:

(74) $$\frac{d\Omega_{ik}}{d\tau} = d_{ik} - (u_i\, P_k - u_k\, P_i)\,,$$

with

(75) $$d_{ik} = D_{ik} - (x_i^{(r)} F_k - x_k^{(r)} F_i) = \int (\xi_i f_k - \xi_k f_i)\left(1 + \frac{\xi_i \dot{u}_k}{c^2}\right)\frac{dV_l\, u_l}{-c},$$

in which have used equations (71) and (66).

Equations (67) and (74) – i.e.:

(76*a*) $$\dot{P}_i = F_i ,$$

(76*b*) $$\dot{\Omega}_{ik} = d_{ik} - (u_i P_k - u_k P_i) ,$$

are valid for any curve L; i.e., for any choice of representative point. We would now wish that L should represent the motion of the center of gravity and then try to find the other conditions that define that point.

In the case of a free system, the center of gravity is determined uniquely by the two equations (50) and (38); i.e., by:

(77) $$P_i = M_0\, u_i ,$$

(78) $$\Omega_{ik}\, u_k = 0.$$

The first equation expresses the proportionality of the impulse-energy vector and the velocity quadri-vector of the center of gravity, where M_0 is the total proper mass of the system. The second equation expresses the condition that the center of gravity is the center of mass in the reference system in which it is at rest. In order to define the center of gravity in the presence of external forces, it seems natural to likewise utilize equations (77) and (78). As we see, it is nonetheless not generally possible to demand that equations (77) and (78) should both be satisfied, because they are not compatible with the *equations of motion* (76), in general.

If we suppose that the relation (77) remains verified in the general case then we can write the equations of motion (76):

(79) $$\begin{cases} M_0 \dot{u}_i + \dot{M}_0 u_i = F_i , \\ \qquad\qquad \dot{\Omega}_{ik} = d_{ik} . \end{cases}$$

Upon multiplying it by u_i , we will deduce from the first of these equations that:

(80) $$\dot{M}_0 = -\frac{F_i\, u_i}{c^2},$$

in which:

(81) $$\dot{u}_i = \frac{F_i + \dfrac{F_k u_k}{c^2} u_i}{M_0}.$$

If we define a quadri-vector a_i by:

(82) $$a_i = \frac{\Omega_{ik}\, u_k}{M_0 c^2}, \quad a_i\, u_i = 0$$

then equation (78) will signify that the vector a_i is constant and equal to zero. Now, it is easy to evaluate the derivative of a_i by means of (79), (80), and (81); one will get:

(83) $$\dot{a}_k = 2a_i \, \frac{F_k\, u_k}{M_0 c^2} + \frac{\Omega_{ik}\, F_k}{M_0^2 c^2} + \frac{d_{ik}\, u_k}{M_0 c^2}\,.$$

In general, a_i will not be constant then. It is only if:

(84) $$\frac{\Omega_{ik} F_k}{M_0} + d_{ik}\, u_k = 0$$

that

(85) $$a_i = 0$$

will be a solution to (83). It is only in the special case that is defined by (84) then that our representative will be the center of mass in its proper reference system.

In the general case, a_i will not remain zero, even if it is zero at the initial instant. That signifies that the representative point will become the position of the center of mass in the instantaneous reference system $S^*(\tau)$. The space-time coordinates of the center of mass in $S^*(\tau)$ are given at any instant by:

(86) $$X_i\,(\tau) = x_i^{(r)}(\tau) - a_i\,(\tau),$$

as one will see immediately upon considering equation (86) in the reference frame $S^*(\tau)$, and upon using the definition of a_i and Ω_{ik} . Since $a_i\, u_i = 0$, the components a_i in $S^*(\tau)$ have the form = $\{\mathbf{a}^*, 0\}$ and $a_i\, a_i = |\, \mathbf{a}^*\,|^2$ is equal to the square of the distance between the representative point and the center of mass in $S^*(\tau)$.

Consider a system that is initially free, in which $a_i = 0$ for the center of gravity, and make external forces act during a certain time interval. After that time interval, the values of a_i, and in turn, those of $a_i\, a_i$, can be non-zero by a certain quantity, at least as long as the forces that enter into equation (83) do not satisfy special conditions. After that, the system will become free again, so our representative point can then be very different from the center of gravity, although it will displace with the same constant velocity. The representative point that is defined by (77) and (76) can even be found far outside the system of the latter is subject to external forces during a certain time interval. It might then be reasonable to assume that the relation (78) is always valid. In that case, our representative point will in fact always be a center of mass in the instantaneous proper system, which signifies that it will always be situated inside of the system, at least if the energy density is everywhere positive.

We then define our representative point by equation (78), combined with the equations of motion (76). We deduce from (78) by differentiating:

(89) $$\dot{\Omega}_{ik} u_k + \Omega_{ik} \dot{u}_k = 0.$$

Upon multiplying (76*b*) by u_k , we will get, with the aid of (89):

(90) $$\dot{\Omega}_{ik} u_k = - \Omega_{ik} \dot{u}_k = d_{ik} u_k - u_i (P_k u_k) - P_i c^2.$$

If we set:

(91) $$P_k u_k = - M^* c^2$$

then M^* is an invariant that represents the total mass of the system in the reference system $S^*(\tau)$ in which our representative point is at rest at the instant considered. Upon substituting (91) into (90) and solving that equation with respect to P_i, we will get:

(92) $$P_i = M^* u_i + \frac{\Omega_{ik} \dot{u}_k}{c^2} + \frac{d_{ik} u_k}{c^2}.$$

The impulse-energy P_i is then the sum of two terms:

(93) $$P_i = M^* u_i + \pi_i ,$$

the first of which is proportional to u_i, while the second one:

(94) $$\pi_i = \frac{\Omega_{ik} \dot{u}_k}{c^2} + \frac{d_{ik} u_k}{c^2}$$

is orthogonal to u_i ; i.e., by virtue of (78):

$$\pi_i u_i = 0.$$

Upon substituting (93) into (76), we will get the following equations of motion:

(95*a*) $$\frac{d}{d\tau}(M^* u_i) + \dot{\pi}_i = F_i ,$$

(95*b*) $$\dot{\Omega}_{ik} + u_i \pi_k - u_k \pi_i = d_{ik} .$$

In the particular case where:

(96) $$d_{ik} = 0,$$

i.e., the one in which the forces produce no precession of the vector that represents the internal kinetic moment, we will have $\pi_i \dot{u}_i = 0$, $\dot{\Omega}_i \dot{u}_i = 0$, and equations (95) will take the form:

(96*a*) $$\frac{d}{d\tau}(M^* u_i) + \Omega_i \ddot{u}_i = F_i ,$$

(96*b*) $$\dot{\Omega}_{ik} + u_i \pi_k - u_k \pi_i = 0.$$

Those equations have the same form as the ones that were given by Mathisson for the *equation of motion of a spinning particle in the case of an external force that has a zero moment with respect to the particle.* Equation (96*b*) shows that $\dot{\Omega}_{ik} \neq 0$, but the precession that is described by that equation is simply the precession that is called

Thomas precession. It constitutes a purely kinematical effect and is due to the fact that the succession of a large number of infinitesimal Lorentz transformations with no spatial rotation can possibly product a finite Lorentz transformation with a well-defined rotation of the spatial axes.

For a free system (i.e., for $f_i = 0$) we have $d_{ik} = 0$, $F_i = 0$, and equations (95), which reduce to equations (48), describe the motion of the false centers of gravity that are situated on the rotating disc that was mentioned in the first chapter. Equations (95), combined with equation (78), will then determine the world-lines of the pseudo-centers of gravity in the presence of external forces. In the case of a free system, the center of gravity can be distinguished from pseudo-centers by the condition $\pi_i = 0$. However, that condition is generally incompatible with the equations of motion (95). Equation (95*a*) can be written:

$$M^*\dot{u}_i + \dot{M}^* u_i + \dot{\pi}_i = F_i \,.$$

Upon multiplying this by u_i, we will deduce that:

$$\dot{M}^* = -\frac{1}{c^2}(F_i\,u_i - \dot{\pi}_i\,u_i).$$

The preceding equation then gives:

$$\dot{u}_i = \frac{F_i - \dot{\pi}_i + \frac{1}{c^2}(F_k\,u_k - \dot{\pi}_k u_k)\,u_i}{M^*}\,.$$

Upon substituting this expression into (94) and remembering that $\Omega_{ik}\,u_k = 0$, we will get:

(98) $$\pi_i = \frac{\Omega_{ik}F_k}{M^*c^2} + \frac{d_{ik}u_k}{c^2} - \frac{\Omega_{ik}\dot{\pi}_k}{M^*c^2}\,.$$

We then see that it is only in the case where:

(99) $$\frac{\Omega_{ik}F_k}{M^*c^2} + d_{ik}\,u_k = 0$$

that $\pi_i = 0$ will be a solution of (98). As in the case $M^* = M_0$, the condition (99) is identical with the condition (84) that was found before.

In the general case in which the forces do not satisfy the condition (99), the relation (98) will show that if $\pi_i = 0$ at a certain instant then the derivatives $\dot{\pi}_i$ will be non-zero, which also signifies that the π_i will be non-zero soon after that. Now consider a free system before and after a certain time interval during which that system is subject to arbitrary external forces. Before and after, the center of gravity will then be defined in an unambiguous manner. If our representative point is chosen in such a way that $\pi_i = 0$ at the onset, which signifies that it coincides with the center of gravity, then the action of the external forces will produce a change in the value of π_i that will be non-zero when the system becomes free again. The representative point will not coincide with the center of gravity much later, but it will be one of the pseudo-centers of gravity of the rotating disc. In the center of gravity system, the distance between the representative point and the center of gravity can, in turn, have any value between zero and the radius $\rho = |\mathbf{m}^0| / M_0 c$ that is given by (28) or (33).

The world-lines of the various pseudo-centers of gravity completely fill up a tube whose thickness has order $|\mathbf{m}^0| / M_0 c$. In the case of a free system, it is possible to choose one of those world-lines uniquely and to define it to be the center of gravity. However, as we just saw, any external force, no matter how weak, will generally provoke a mixture of world-lines of the pseudo-centers of gravity, which will make it impossible to distinguish a particular line as the center of gravity. That signifies that making an exact, unambiguous definition of the center of gravity is generally impossible for a system that is subject to external forces, since the center of gravity or its world-line are defined only with the uncertainty that is given by the tube of world-lines that was in question above. Those general results are valid for any system, and as a result, also in the limit of a very small system. In our opinion, the Mathisson equations must not be considered to be the equations of motion of a spinning particle, but rather as equations that describe the tube of world-lines of the false centers of gravity, which is a tube that will determine the uncertainty in the definition of the center of gravity of the system in the presence of external forces.

It is true that for most macroscopic systems, the dimensions of the cross-section of the uncertainty tube that was mentioned above will be very small, since $|\mathbf{m}^0| / M_0 c$ will then be very small. However, for a classical system that has a mass that is equal to the mass m_0 of the electron and a kinetic moment of order a quantum of action h, the uncertainty in the definition of the center of gravity will have order the Compton wave length $h / m_0 c$. In the *non-relativistic* limiting case (i.e., for $c \to \infty$) the uncertainty tube will become infinitely thin, and in that domain the notion of center of gravity will likewise acquire a precise significance for systems that are subject to arbitrary external forces.

Up to now, we have considered only external forces that are not gravitational forces. We shall see that the situation is different in the case of pure gravitational forces. We will show that one can then give an unambiguous significance to the notion of center of gravity, at least when the system is sufficiently small. That amounts to saying that one can make the forces of gravitation disappear in a small region of space-time by a convenient space-time coordinate transformation.

Let x^i be the space-time coordinates in an arbitrary reference system in general relativity, and let $g_{ik} = g_{ik}(x^l)$ be the covariant components of the metric tensor that describes the external gravitational field. (We must now make a distinction between the covariant and contravariant components of tensors.) We must study the effect of that gravitational field on a physical system that is arbitrary, but small, and is described by an impulse-energy tensor whose contravariant components are $T^{ik} = T^{ki}$. The theorem of conservation of energy and impulse is expressed in relativity by the relation:

(100) $$\frac{\partial T^{ik}}{\partial x_k} + \frac{T^{ik}}{\sqrt{|g|}} \frac{\partial \sqrt{|g|}}{\partial x_k} + \Gamma^i_{kl} T^{ki} = 0,$$

which replaces equation (1) of special relativity. g is the determinant of the metric tensor, and:

(101) $$\Gamma^i_{kl} = g^{im} \frac{1}{2}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

are the geodesic three-index Christoffel components. In full rigor, the function g_{ik} in (100) must also contain the gravitational field that is produced by the system itself, but we shall assume that the field is weak enough to be neglected with respect to the external field.

In four-dimensional space, the domain in which T^{ik} is non-zero is, for a small system, a thin tube whose thickness is given by the dimensions of the system. Let L be the world-line of the center of gravity, which we shall now determine. Since X^i are the space-time coordinates of the center of gravity, and t is the corresponding proper time, the world-line L will be well-defined if the X^i are given as functions of τ:

(102) $$X^i = X^i(\tau).$$

Let $p(\tau)$ be an arbitrary point of L. We can introduce a coordinate system $\overset{0}{x}{}^i$ that is geodesic at the point p – i.e., a system in which the first derivatives of the metric tensor are zero at the point p :

(103) $$\left| \frac{\partial \overset{\circ}{g}_{ik}}{\partial \overset{\circ}{x}{}^i} \right|_p = 0,$$

and in an infinitude of ways. As we see, in that *local inertial system,* it will be possible to treat our physical system as a free system with no gravitational forces, on the condition that the system should be small enough that we can neglect the tidal effects (*effets de marée*); i.e., on the condition that the curvature of space-time should be sufficiently small in relation to the given dimensions of the system. We can, in turn, define the coordinates of the center of gravity in that system by proceeding in the same manner as in special relativity for a free system.

Now choose a system of normal Riemann coordinates, in particular, for a geodesic system. By a convenient linear transformation, we can, moreover, arrange that the coordinate lines are orthogonal at the point p. If the origin of the coordinates $\overset{0}{x}{}^i = 0$ is taken at p then the components of the metric tensor in the neighborhood of p will have the form:

(104) $$\overset{\circ}{g}_{ik} = G_{ik} + \tfrac{1}{3} \overset{\circ}{R}_{ijkl}(p)\, \overset{\circ}{x}{}^l \overset{\circ}{x}{}^m ,$$

in which $\overset{\circ}{R}_{ijkl}$ is the Riemann-Christoffel curvature tensor, and:

$$G_{ik} = \begin{cases} 0 & \text{for} \quad i \neq k, \\ 1 & \text{for} \quad i = k = 1, 2, 3, \\ -1 & \text{for} \quad i = k = 4 \end{cases} \tag{105}$$

is the constant metric tensor in a Lorentz reference system of special relativity.

With the aid of (104), we will get from a direct calculation, upon neglecting terms of order higher than the second in $\overset{\circ}{x}{}^i$:

$$\overset{\circ}{\Gamma}{}^i_{kl} = \tfrac{1}{3}[\overset{\circ}{R}{}^i_{klm}(p) + \overset{\circ}{R}{}^i_{lkm}(p)]\,\overset{\circ}{x}{}^m \tag{106}$$

and

$$\frac{1}{\sqrt{|\overset{\circ}{g}|}}\frac{\partial\sqrt{|\overset{\circ}{g}|}}{\partial\overset{\circ}{x}{}^k} = -\tfrac{2}{3}\overset{\circ}{R}_{kl}(p)\,\overset{\circ}{x}{}^l, \tag{107}$$

in which $\overset{\circ}{R}_{ik}$ is the contracted curvature.

In the coordinate system $\overset{\circ}{x}{}^i$, the fundamental equation (100) can be written:

$$\frac{\partial\overset{\circ}{\Gamma}{}^{ik}}{\partial\overset{\circ}{x}{}^k} = -\tfrac{1}{3}[\overset{\circ}{R}{}^i_{klm}(p) + \overset{\circ}{R}{}^i_{lmm}(p)]\,\overset{\circ}{x}{}^m\,\overset{\circ}{T}{}^{kl} + \tfrac{2}{3}\overset{\circ}{T}{}^{kl}\,\overset{\circ}{R}_{kl}(p)\,\overset{\circ}{x}{}^l. \tag{108}$$

If one divides that equation by c and integrates over all values of $\overset{\circ}{x}{}^1$, $\overset{\circ}{x}{}^2$, $\overset{\circ}{x}{}^3$ (where $\overset{\circ}{x}{}^i$ has a small constant value) then the left-hand side will be equal to $\partial\overset{\circ}{P}{}^i/\partial\overset{\circ}{x}{}^4$, in which:

$$\overset{\circ}{P}{}^i = \int\frac{\overset{\circ}{T}{}^{i4}}{c}\,d\overset{\circ}{x}{}^1 d\overset{\circ}{x}{}^2 d\overset{\circ}{x}{}^3 \tag{109}$$

is the quadri-vector that represents the total energy and impulse in our local inertial system.

Now suppose that the curvature of space-time is small in comparison to the spatial dimensions $\overset{\circ}{d}$ of the system. We can then neglect the right-hand side of (108) after integrating, due to its terms of the form $\overset{\circ}{R}{}^i_{klm}\,\overset{\circ}{x}{}^m$. The derivatives of $\overset{\circ}{P}{}^i$ with respect to

$\overset{\circ}{x}{}^i$ or with respect to τ will, in turn, be small compared to the $\overset{\circ}{P}{}^i$ themselves, and in the limiting case of a very small system, one can set:

$$\frac{d\overset{\circ}{P}{}^i}{d\tau} = 0. \tag{110}$$

In the same way, we will find that the derivative of the quantity $\overset{\circ}{m}_{ik}$ that is defined by:

$$\overset{\circ}{m}{}^{ik} = \frac{1}{c}\int(\overset{\circ}{x}{}^i\overset{\circ}{T}{}^{k4} - \overset{\circ}{x}{}^k\overset{\circ}{T}{}^{i4})\,d\overset{\circ}{x}{}^1 d\overset{\circ}{x}{}^2 d\overset{\circ}{x}{}^3 \tag{111}$$

is small with respect to the $\overset{\circ}{m}_{ik}$ themselves under the conditions that were mentioned. If the system is small enough then we can set:

$$\frac{d\overset{\circ}{m}{}^{ik}}{d\tau} = 0. \tag{112}$$

The quantities $\overset{\circ}{P}{}^i$ and $\overset{\circ}{m}{}^{ik}$ transform as a vector and a tensor, respectively, under any linear, orthogonal transformation of the coordinates $\overset{\circ}{x}{}^i$. $\overset{\circ}{m}{}^{ik}$ is the tensor that represents the internal kinetic moment in the geodesic system $\overset{\circ}{x}{}^i$.

We now define the center of gravity by the equations:

$$\overset{\circ}{m}_{ik}\overset{\circ}{U}{}^k = 0, \tag{113}$$

$$\overset{\circ}{P}_i = \overset{\circ}{M}_0\overset{\circ}{U}_i, \tag{114}$$

in which:

$$\overset{\circ}{U}{}^i = \frac{d\overset{\circ}{X}{}^i}{d\tau} \tag{115}$$

is the velocity vector of the center of gravity in the geodesic system.

Equation (113) expresses the idea that the center of gravity is the center of mass in the local inertial system, while equation (114) establishes the proportionality between $\overset{\circ}{U}{}^i$ and $\overset{\circ}{P}{}^i$. Since the derivative of $M_0 = -\overset{\circ}{P}_i\overset{\circ}{P}_i / c^2$ is zero, equation (110) can be written:

$$\frac{d\overset{\circ}{U}{}^i}{d\tau} = 0, \tag{116}$$

which is the equation of motion of the center of gravity in the system $\overset{\circ}{x}{}^i$. If we return to the original coordinate system x^i then equation (116) will become:

$$\frac{dU^i}{d\tau} = -\Gamma^i_{rs}\, U^r U^s \,, \tag{117}$$

or

$$\frac{d^2 X^i}{d\tau^2} = -\Gamma^i_{rs}\, \frac{dX^r}{d\tau}\frac{dX^s}{d\tau}. \tag{118}$$

Those equations show that the world-line of the center of gravity is a geodesic – i.e., the motion of that point is identical to the motion of a particle that falls freely in the given external gravitational field.

$\overset{\circ}{m}_{ik}$ behaves like a tensor with respect to linear transformations of coordinates $\overset{\circ}{x}{}^i$. We can now consider m_{ik} to be an antisymmetric tensor that is attached to the center of gravity and *define* its components m_{ik} (τ) in the system x^i by the usual transformation laws of a tensor that is attached to the point X^i. The transformation to the system x^i will make equations (112) take the form:

$$\frac{dm^{ik}}{d\tau} = -\ \Gamma^i_{rs}\, U^r m^{sk} - \Gamma^k_{rs}\, U^r m^{is}\,. \tag{119}$$

We have similar equations for the derivatives of the covariant components m_{ik}. Equation (113) can then be written in the invariant form:

$$m_{ik}\, U^k = 0, \tag{120}$$

and we see that it is compatible with the equation of motion (117) and (119). Those equations show that the vector U^i and the tensor m^{ik} propagate by parallel-displacing along the world-line of the center of gravity. The direction of the internal kinetic moment will not generally be identical to the original direction then when the center of gravity traverses a closed circuit. That *geodesic precession* of the direction of a vector that represents the kinetic moment conforms to the rule that was given by Fokker ([10]).

In the present article, we have considered exclusively classical physical systems for which all quantum effects can be neglected. As we have seen, the notion of center of gravity generally has an unambiguous significance only for systems that are not subject to external forces in the case where they are free. Now, it is possible to immediately extend the theory that was developed here to arbitrary quantum systems. The existence of the quantum of action introduces a supplementary limitation in the definition of the center of gravity even in the case of a free system. That limitation takes the form of a quantum uncertainty relation that is due to the fact that the coordinates of the center of gravity are represented by operators that do not commutate with each other in that case.

([10]) A. D. FOKKER, *loc. cit.* ([9]), pp. 249.

For more details, the reader is requested to refer to the article in the *Communications of the Dublin Institute for Advanced Studies* that was cited above ([8]).

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