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ADDITION

ON THE INTEGRATION OF FIRST-ORDER PARTIAL DIFFERENCE EQUATIONS IN THREE VARIABLES

The integration of an arbitrary first-order partial difference equation in three variables depends upon only the integration of just one ordinary difference equation in two variables in which the differential of one of the variables is regarded as constant. The procedure consists of first seeking the ordinary difference equations of the characteristic of the surface and then integrating those equations. That will naturally divide the subject that we shall treat into two distinct parts.

PART ONE

SEARCH FOR THE ORDINARY DIFFERENCE EQUATIONS OF THE CHARACTERISTIC

In order to give more clarity to the search for the equations of the characteristic, I would like to undertake it by only geometric considerations. However, what I said about the characteristic up to now (§ **VIII**, pp. 53) is not complete, and I would like to recall what was said above.

Imagine an arbitrarily-given curved surface whose construction depends upon two parameters α, β, and whose equation is represented by:

$$f(x, y, z, \alpha, \beta) = 0 .$$

Suppose, moreover, that the two parameters are not mutually independent, but that there is a well-defined relation between them that is expressed by $\beta = \varphi(\alpha)$, in which φ is a certain given function, in such a way that α is the principal parameter: The equation of the surface will be:

$$f(x, y, z, \alpha, \varphi(\alpha)) = 0 .$$

Having done that, one then gives different values to the parameter α, so that equation will belong to the various surfaces, in each of which the quantities α, $\varphi(\alpha)$ will both be constants with various values. Therefore, if one supposes that the parameter α takes all possible values from $-\infty$ up to $+\infty$ in succession then the surface will move while changing form. It will successively pass through all forms and all positions that it is susceptible to, and it will sweep out a certain space.

Finally, if one supposes that all of those surfaces exist together and that another surface envelopes everything then that latter surface, up to which each of the former extend and outside of which none of them reach, and which is consequently their limit, is also the limit of the space that is traversed by the former, when it is regarded as mobile and variable in form by virtue of the variation of the parameter α.

It is the latter surface to which I give the name of *envelope*, in comparison to the moving surface, whereas I have given the name of the *enveloped surface* to the moving surface.

II.

It is obvious that the envelope touches each of the enveloped surfaces along a curve that is found, at the same time, on the envelope and the enveloped surface. It is that contact curve to which I have given the name of the *characteristic* of the envelope. Now, there is no point on the envelope at which that surface does not touch a certain enveloped surface for which the constant parameter α has a certain well-defined value. Therefore, there is no point on the envelope through which there does not pass a certain characteristic for which the parameter α has the same value that it has on the enveloped surface on which that curve is found, i.e., for which the quantities α, $\varphi(\alpha)$ are both constants.

The envelope can then be regarded as the locus of all characteristics that correspond to the different values of α, and consequently, it can be regarded as being generated by the motion of the characteristic, which is considered to be mobile and variable in form by virtue of the variation of the parameter α. Later on (art. XI), we shall see why I have found it necessary to distinguish all possible generators of the envelope by a special name.

III.

If one considers two consecutive enveloped surfaces, such that the parameter α has a certain well-defined value for one of them, while it is $\alpha + d\alpha$ for the other, then those two surfaces will intersect along a certain curve that will also be a line of contact between the two, and which will not differ from the one along which the first of the two enveloped surfaces touches the envelope: That curve will then be the characteristic that corresponds to the first of the two envelopes. Now, the equations of those two consecutive enveloped are:

$$f(x, y, z, \alpha, \varphi(\alpha)) = 0 ,$$
$$f(x, y, z, \alpha + d\alpha, \varphi(\alpha) + d\varphi(\alpha)) = 0 ,$$

or, if one represents the first one by:

$$f = 0$$

then the second one will become:

$$f + \left(\frac{df}{d\alpha}\right) = 0 .$$

Therefore, those two equations are those of the characteristic, and because they must be valid at the same time for that curve, the first one will reduce the second one, and they will become:

$$f = 0 ,$$

$$\left(\frac{df}{d\alpha}\right) = 0 ,$$

in which the value of α determines the form and position of each of the characteristics. Finally, the envelope, which is the general locus of all characteristics, will then have an integral equation that is result of eliminating the indeterminate α from the two preceding equations.

IV.

One sees that if the form of the function φ is given, and consequently that of its derivative φ', then the elimination of α will always be possible, and that the equation of the envelope will be delivered entirely by the variable α. However, it will always contain some traces of the function φ and its derivative φ': Therefore, one will have a different envelope for each different form that one can give to the function φ. Therefore, if one would like for the equation to the belong to all possible envelopes that can be produced by the various motions that one can give to the moving enveloped surface then one would have to regard the form of the function φ as arbitrary. However, the elimination of the parameter α would no longer be practicable, or at least it could be performed only in certain particular cases, and, in general, the equation of the envelope could no longer be expressed by the system of two equations:

$$f = 0 ,$$

$$\left(\frac{df}{d\alpha}\right) = 0 ,$$

between which one must eliminate the indeterminate α, and in which the function φ is arbitrary.

V.

Nevertheless, the envelope can be expressed by a single partial difference equation. Indeed, of the two previous equations, the second one says that the differential of the first one, when taken while regarding α alone as variable, is equal to zero: One can then differentiate the first one while regarding α as constant. However, that equation $f = 0$ belongs to a curved surface for which two variables – for example, x, y – are independent, as well as their differentials dx, dy. It will then be necessary that the two differentials of that equation, taken while first regarding x as the only variable, and then y, must each be true by themselves: The first one will be in terms of:

$$p, x, y, z, \alpha, \varphi(\alpha),$$

while the second one will be in terms of:

$$q, x, y, z, \alpha, \varphi(\alpha).$$

Therefore, one can eliminate the two quantities α, $\varphi(\alpha)$ from those two equations and $f = 0$, and one will get a first-order partial difference equation:

$$F(p, q, x, y, z) = 0.$$

Now:

1. Upon eliminating α, one will transfer all of the things that one first said about the enveloped surface over to the envelope.

2. Upon eliminating $\varphi(\alpha)$, one will transfer all of the things that one first said about the enveloped surface that are determined by the supposed form of the function φ to all possible envelopes.

Therefore, the first-order partial difference equation:

$$F(p, q, x, y, z) = 0$$

will belong to all possible envelopes that can be produced in an arbitrary manner by the motion of the same moving enveloped surface.

VI.

One knows that if two arbitrary curves are given in space then if one of them is approached by a plane that touches both of them, which will not determine its position, and if one supposes that the plane rolls in such a manner that it does not cease to touch the two curves during its motion, which will be determined by that, then it will sweep out a space whose envelope is a developable surface that passes through the two curves.

Having said that, imagine two consecutive characteristics, the first of which corresponds to a well-defined value of α, and suppose that a plane rolls without ceasing to touch both of them. The motion of that plane will determine a developable surface that passes through two consecutive characteristics and will be tangent to the envelope as a consequence. In the equation of that developable surface, α will have the same well-defined value that it has for the characteristic at which it touches the envelope, and the quantities α, $\varphi(\alpha)$ will both be constants.

Now, if we perform the same operation on the second and third characteristic that we just performed on the first and second one then we will get a second developable surface that similarly passes through the second and third characteristics, which will likewise touch the envelope, and whose equation will differ from that of the first one only because the quantity α will become $\alpha +$

$d\alpha$, and it will be obvious that those two consecutive developable surfaces will cut along the second characteristic, which is common to them.

Therefore, if one continues to make the developable surfaces pass through all characteristics that are taken pairwise consecutively then one will have a sequence of developable surfaces that are tangent to the envelope and any two consecutive ones will intersect in one of the characteristics. Hence, if one supposes that the developable surface moves in space then by virtue of the variation of the parameter α that is contained in its equation, it will sweep out a space whose envelope will be the same as the one that we have considered up to now. That developable surface can then be regarded as a new envelope that moves by virtue of the variation of the same parameter α, and which will belong to the same envelope.

VII.

What we just said about the developable surface whose general partial difference equation $rt - s^2 = 0$ is of second order can also be said of any other surface whose partial difference equation is of second order: We shall discuss only one example.

Imagine that a sphere of constant radius rolls while always touching the first and second characteristic. It will sweep out a space whose envelope will be the surface of a tube with a circular section of constant radius, and that surface, in whose equation the parameter α will have a well-defined value that is appropriate to the first characteristic, will pass through the two consecutive characteristics and will be tangent to the original envelope.

If one performs the same operation on the second and third characteristic then one will have the surface of a second tube of circular section with the same radius that passes through the second and third characteristic, which will likewise touch the original envelope, and whose equations will differ from that of the first only because the parameter a will have taken the values $\alpha + d\alpha$. The surfaces of those two consecutive tubes will intersect in the second characteristic, which is common to them.

Therefore, if one continues to make the surfaces of circular tubes pass through all of the characteristics considered pairwise in succession then one will have a sequence of surfaces that are tangent to the original envelope, and which will successively intersect in all of the characteristics when they are themselves considered pairwise in succession. Hence, if one imagines that the surface of the tube, into whose equation the parameter α will enter, moves in space by virtue of the variation of the that parameter, it will sweep out a space that will have the same envelope.

VIII.

It follows from this that a surface, when considered to be an envelope, will not necessarily have enveloped surfaces. That is, it is the common envelope of an infinite number of different spaces that are swept out by different enveloped surfaces that each move by virtue of the variation of the same parameter α, which is found in the equation of the enveloped surface, along with $\varphi(\alpha)$. Now, we saw (art. IV) that the integral equation of a surface that is considered to be an envelope is, in general, the result of eliminating α from the equation of the moving enveloped surface:

$$f(x, y, z, \alpha, \varphi(\alpha)) = 0$$

and its differential, when taken while regarding α as the only variable. Moreover, we just saw that the function f can take an infinitude of different forms depending upon the nature of the moving enveloped surface. Therefore, when the integral equation of an envelope is presented as the result of eliminating α from two equations such as:

$$f = 0 ,$$

$$\left(\frac{df}{d\alpha}\right) = 0 ,$$

that system of two equations will hardly be necessary, and it can be replaced with an infinitude of other systems of two equations that are different from the first two and will produce the same result upon eliminating α.

Nonetheless, it is advantageous to choose from among all moving enveloped surfaces that produce the same envelope by their motion, the one whose equation is the simplest or the one whose construction is easiest or whose consideration one is most accustomed to.

IX.

Consider an arbitrary envelope that is the locus of characteristics and begin with that one of the curves that corresponds to a well-defined valued of α. Take an arbitrary point on it and imagine its tangent at a point. Next, draw an arbitrary plane through that tangent that is tangent to the curve, but whose position is undetermined, and which is not tangent to the envelope. Having done that, imagine that the plane turns around the tangent until it touches the following characteristic a certain point that will be determinate. The position of the plane will then be determinate, it will be tangent to the envelope, and it will contain the tangent to the second characteristic.

Next, imagine that the plane turns around the tangent to the second characteristic until it touches the third one at another point that will likewise be determinate. In that position, it will again be tangent to the envelope, and it will contain the tangent to the third characteristic.

Finally, imagine that the plane continues to roll in that way while always turning around the tangent to the characteristic that it touches until it touches the following characteristic. It is obvious that while it moves, it will not cease to be tangent to the envelope, since it will always pass through the tangent to two consecutive characteristics. There will then be as many curves on the envelope that are determined in that manner as one can imagine points that are taken arbitrarily on the first characteristic. Each of those curves will cut all of the characteristics, and conversely, and because the consideration of those curves will become necessary for us, I shall give them the name of *trajectories* of the characteristic.

The angles at which the trajectories cut the characteristics will generally depend upon the nature of the envelope. For example, in the surfaces of revolution, the parallels are the characteristics, while the meridians are the trajectories, and in that case the trajectories will be orthogonal.

The plane that moves in such a manner as to always touch the envelope along the same trajectory will determine a developable surface that will itself touch the envelope along the entire extent of the trajectory.

X.

We have made the tangent plane roll along the envelope in two different ways.

In the first of them, the point of contact of the moving plane does not leave the same characteristic, and two consecutive planes will intersect along the tangent to the trajectory that passes through the point of contact.

In the second one, the contact plane of the moving plane will not leave the same trajectory, and two consecutive planes will intersect along the tangent to the characteristic that passes through the point of contact.

Therefore, the developable surface that touches the envelope along the characteristic and the one that touches the envelope along the trajectory are reciprocal, in the sense that the first one is the locus of tangents to the different trajectories whose contact points are taken along the same characteristic, while the second one is the locus of tangents to the different characteristics whose contact points are taken along the same trajectory.

That property deserves a great deal of attention since its expression will produce the two ordinary difference equations for the characteristic for us.

XI.

Since the envelope is the locus of all trajectories whose various integral equations differ by only the value of a certain parameter β that varies from one to the other, the trajectory can also be regarded as a generator of the envelope that it generates by virtue of the motion that gives it the variation of β. However, there is a great difference between that generator and the characteristic.

For the characteristic, the quantities α, $\varphi(\alpha)$ that enter into its integral equations are both constants and must be regarded as such when one differentiates the ordinary difference equations of those equations. One can then eliminate them like two true arbitrary constants, and the ordinary difference equations that one obtains will no longer include the function φ, which belongs to all of the characteristics that are found on all possible envelopes. On the contrary, the two equations of the trajectory can, in truth, even be regarded as independent of α, but they do not have the form of the function φ. That curve is necessarily derived from the one that directs the motion of the enveloped surface, and it will depend upon the nature of the envelope that one considers.

Hence, the characteristic is the generator that is common to all of the different envelopes, and there are an infinite number of them that one can produce from the same enveloped surface, whereas the trajectory of only the generator of one of those envelopes. The characteristic, whose nature absolutely depends upon only that of the enveloped surface, is therefore the only one that carries the general character of the generation that is expressed by the partial difference equation. That is why I have given them a special name that will distinguish them from all other generators.

For example, for the surface of revolution, the meridian, which is the trajectory, is indeed a generator. It is even the only one that one is accustomed to considering. However, that generator is specific to an individual surface of revolution. It will vary without any dependency upon one individual surface or the other, and that generator is not what gives the surface its character of being in revolution. That distinction belongs to only the parallel, i.e., the circumference of the circle whose plane is always normal to the axis, which is constant in position and passes through its center. I say that the parallel is the common generator to all of the surfaces of revolution around the same axis. It is the curve that gives the surface its character of being in revolution. That is the characteristic of that method of generation.

Now that we have posed those preliminaries, we shall now move on to a study of the ordinary difference equations of the characteristic.

XII.

Let the general first-order partial difference equation be given:

$$F(p, q, x, y, z) = 0 , \tag{A}$$

into which the given quantities enter in a manner that is arbitrary but given. If one differentiates it by ordinary differences then one will have an ordinary difference equation of the form:

$$P\, dp + Q\, dq + X\, dx + Y\, dy + Z\, dz = 0 ,$$

in which the five coefficients P, Q, X, Y, Z that are obtained by differentiation are known functions of p, q, x, y, z, and since one has $dz = p\, dx + q\, dy$, moreover, that differential equation will be:

$$P\, dp + Q\, dq + (X + p\, Z)\, dx + (Y + q\, Z)\, dy = 0 . \tag{B}$$

Having said that, consider the tangent plane that is supported by two arbitrary consecutive characteristics that each touch it at one point. If one would like to pass from the first of those two points on the envelope to the second one, i.e., to sweep out the element of the trajectory, then since those points are on the same tangent plane, the quantities p, q will not change during that passage, and it will then be necessary to have $dp = 0$, $dq = 0$ in (B), which will give:

$$(X + p\, Z)\, dx + (Y + q\, Z)\, dy = 0 , \tag{C}$$

in which the value of dy / dx indicates the direction in the x, y-plane along which that passage must take place, i.e., the direction of the projection of the element of the trajectory. That differential equation is therefore nothing but the projection of the trajectory itself onto the x, y-plane.

The tangent plane is not the only surface that passes through the trajectory. If one makes:

$$(D) \qquad P\,dp + Q\,dq = 0$$

then one will likewise have (C). Therefore, the surface to which equation (D) belongs will also pass through the trajectory. Now, that surface is developable, since its ordinary difference equation will involve dp, dq. Moreover, it will touch the envelope, since the quantities p, q have the same values for that surface and the envelope: Therefore, equation (D) is that of the developable surface that touches the envelope along the trajectory.

Up to now, we have regarded the tangent plane whose equation, when limited to the extent of the element of the envelope and when referred to the point of contact are its origin, is:

$$(E) \qquad dz = p\,dx + q\,dy$$

as immobile. However, if we wish to begin rolling that plane on the envelope in an arbitrary manner in such a manner as to give rise to the formation of an arbitrary developable surface then the second plane will cut the first one along a line whose equation will be obtained by differentiating (E) without varying the coordinates dx, dy, dz, which will give:

$$(F) \qquad dp\,dx + dq\,dy = 0\,.$$

In the latter equation, it one replaces dy / dx with the value pertains to the straight line around which the plane turns then one will have the ordinary difference equation of the developable surface that is produced by the motion of the plane in terms of dq, dq, and if one replaces dq / dp with the value that pertains to the developable surface that is produced by the motion of the plane then one will have the equation of the element of the straight line around which the plane turns in terms of dx, dy.

From that, if the plane rolls in such a manner that the point of contact does not leave the characteristic then it will turn around the tangent to the trajectory, and the value of dy / dx must be the one that provides the equation (C) of the trajectory. Therefore, if one substitutes that value, which will produce:

$$(G) \qquad (X + p\,Z)\,dq - (Y + q\,Z) = 0\,,$$

then one will have the differential equation of the enveloped developable, which is an equation that also belongs to the characteristic that is found on that enveloped surface. On the contrary, if the plane rolls in such a manner that the point of contact does not leave the trajectory then it will turn around the tangent to the characteristic and give rise to the formation of the developable surface that touches the envelope along the trajectory, and the value dq / dp must be the one that gives equation (D) for that surface. Hence, if one substitutes that value, which will produce:

$$(H) \qquad P\,dy - Q\,dx = 0\,,$$

then one will have an equation that belongs to the element of the characteristic around which the plane turns, and consequently, the differential equation of the projection of the characteristics onto the x, y-plane. Thus, the two ordinary difference equations (G), (H) will belong to the characteristic.

In summary, one will see that the equation:

$$(B) \qquad P\,dp + Q\,dq + (X + p\,Z)\,dx + (Y + q\,Z)\,dy = 0$$

splits into the following two:

$$(C) \qquad (X + p\,Z)\,dx + (Y + q\,Z)\,dy = 0\,,$$

$$(D) \qquad P\,dp + Q\,dq = 0\,,$$

and if one has posed the equation:

$$(F) \qquad dp\,dx + dq\,dy = 0$$

then when one replaces dy / dx or dq / dp in (C), (D) with the values that give the latter, one will have the two equations:

$$(G) \qquad (X + p\,Z)\,dq - (Y + q\,Z)\,dp = 0\,,$$

$$(H) \qquad P\,dy - Q\,dx = 0\,,$$

which both belong to the characteristic.

Equations (B), (E), which both belong to the envelope on which the characteristic is found, will also belong to that curve: Therefore, one will have the four ordinary difference equations (B), (E), (G), (H) for the characteristic.

XIII.

The four ordinary difference equations that we just found for the characteristic involve the five differentials dp, dq, dx, dy, dz. One can then eliminate any three of those differentials from them, and one will have an equation that will contain only the other two, which will produce the following ten results:

1. $P\,dy - Q\,dx = 0\,,$

2. $P\,dz - (P\,p + Q\,q)\,dx = 0\,,$

3. $Q\,dz - (P\,p + Q\,q)\,dx = 0\,,$

4. $P\,dp + (X \quad + p\,Z)\,dx = 0\,,$

5. $$Q\,dp + (X + p\,Z)\,dy = 0\,,$$

6. $$P\,dq + (Y + q\,Z)\,dx = 0\,,$$

7. $$Q\,dq + (Y + q\,Z)\,dy = 0\,,$$

8. $$(P\,p + Q\,q)\,dp + (X + p\,Z)\,dz = 0\,,$$

9. $$(P\,p + Q\,q)\,dq + (Y + p\,Z)\,dz = 0\,,$$

10. $$(X + p\,Z)\,dq - (Y + q\,Z)\,dp = 0\,.$$

Those ten equations, which all belong to the characteristic, will be useful to us in what follows. However, we must indeed recall that they are not exactly independent, and that they say nothing more than the four equations (*B*), (*E*), (*G*), (*H*), which they are a necessary consequence of.

One can form the preceding ten equations in a convenient manner, because since the five quantities:

$$\boxed{\begin{array}{c|c} \dfrac{dx}{P},\quad \dfrac{dy}{Q},\quad \dfrac{dz}{P\,p+Q\,q} & \dfrac{dp}{X+p\,Z},\quad \dfrac{dq}{Y+q\,Z} \end{array}}$$

are arranged into two classes, as one sees, when the sum of any two of those quantities is equated to, with the – sign if they are taken from the same class and the + sign if they are taken from different ones, will produce one of the ten equations of the characteristic.

PART TWO

INTEGRATING THE EQUATIONS OF THE CHARACTERISTIC

XIV.

Two cases can present themselves: Either the partial difference equation is linear in p, q or it is not. The first of those two cases is the simplest to treat, so we shall address it first.

Therefore, let the general linear equation be given:

$$P\,p + Q\,q = L\,,$$

in which the three coefficients L, P, Q are given arbitrarily as functions of x, y, z. The first three equations of the characteristic, which are the only ones necessary in this case, will become:

$$P\,dy - Q\,dx = 0\,,$$
$$P\,dz - L\,dx = 0\,,$$
$$Q\,dz - L\,dy = 0\,.$$

They belong to the projections of characteristic onto the three coordinates planes. Therefore, any two of them will imply the third. It will then suffice to consider two of them, for example, the first two.

First suppose that those equations are both integrable and that their integrals are represented by:

$$M = \alpha\,, \qquad N = \beta\,,$$

in which α and β are the two arbitrary constants that are introduced by the integrations. In that state, since those two integrals will belong to all possible characteristics that can be found on all possible envelopes to which the partial difference equation belongs (i.e., all possible characteristics, whose number is infinite of order two), there will not be one for which the two integrals cannot become proper equations if one gives a conveniently-determined value to each of the two arbitrary constants α, β.

However, if one does intend to consider all of those characteristics at once then if one proposes to consider only a certain series of them, and if one would like for all of the ones that comprise that series to be coupled to each other by a law in such a way that one can pass from any one of them to the following one only in a well-defined manner, then the constants α, β will no longer be both arbitrary. When any one of them is taken arbitrarily, the other one will necessarily follow: The second one will then be a certain function of the first one, and the form of that function will depend upon the law that couples all of the characteristics of the series. Upon letting φ represent the form of the function in question, the integral equations, which will then become:

$$M = \alpha\,, \qquad N = \varphi(\alpha)\,,$$

will no longer both belong to all of the characteristics, but only to those of the curves that are included in the series that is determined by the form of the function φ, and each individual characteristic in that series will be determined by the particular value of the arbitrary constant α.

Therefore, if one supposes that any one of the characteristics of the series is made mobile and variable in form by virtue of the variation of α then that curve will successively coincide with all other ones in the same series in the course of its motion. It will generate their general locus, which will be nothing but one of the envelopes to which the partial difference equation belongs, and one will have the unique equation of the envelope, or of the general locus, upon eliminating α from the preceding two equations, which will give:

$$N = \varphi(M)\,.$$

In that equation, it is only the form of the function φ that determines the particular series of characteristics whose envelope is the general locus, in such a way that if one changes the series, and consequently the envelope, then nothing will change in the equation besides the form of the function φ. Therefore, if one regards the function φ as arbitrary then the preceding equation will belong to the general locus of each of the possible series, i.e., to each of the possible envelopes. Consequently, it will be the complete integral of the partial difference equation.

XV.

In the preceding article, we supposed that the ordinary difference equations of the characteristic:

$$P\,dz - L\,dx = 0\,,$$
$$Q\,dz - L\,dy = 0$$

were both integrable. That might follow immediately only in some very special cases because each of them will contain one of the three variables, but not its differential. Nevertheless, their integrations will depend upon only the integration of just one ordinary difference equation in two variables.

Indeed, since those two equations belong to one curve, one can consider only one of the three variables x, y, z, to be the principal variable. Let it be z and regard the differential dz as constant. Having done that, of the two equations, the first one:

(*A*) is in terms of $x, y, z, \dfrac{dx}{dz}$,

while the second one:

(*B*) is in terms of $x, y, z, \dfrac{dy}{dz}$.

If one differentiates either one of them – (*A*), for example – using ordinary differences then one will get a third equation:

(*C*) in terms of $x, y, z, \dfrac{dx}{dz}, \dfrac{dy}{dz}, \dfrac{ddx}{dz^2}$.

If one eliminates the two quantities y and dy / dz from those two equations then one will have an equation:

(*D*) in terms of $x, z, \dfrac{dx}{dz}, \dfrac{ddx}{dz^2}$,

which will belong to all possible characteristics and will be an ordinary second difference equation in two variables, and its integration will depend upon only the integration of the partial difference equation, because if one integrates that equation twice for the first differences and one completes each of its integrals with a particular arbitrary constant then one will have two equations: One of them:

(E) is in terms of $x, z, \dfrac{dx}{dz}, \alpha,$

while the other:

(F) is in terms of $x, z, \dfrac{dx}{dz}, \beta,$

in which when one eliminates dx / dz by means of (A), and one will have two integral equations for the characteristic: One of them:

(G) is in terms of $x, y, z, \alpha,$

while the other:

(H) is in terms of $x, y, z, \beta,$

and when each of them is solved for the arbitrary constant that it contains, they will become:

$$M = \alpha, \qquad N = \varphi(\alpha).$$

If we argue with those two equations in the same way that we did for the ones in the previous article then the complete integral of the partial difference equation will be:

$$N = \varphi(M).$$

Therefore, the integration of a linear first-order partial difference equation will depend upon only the integration of just one second-order ordinary difference equation in two variables, in which the differential of one of the two variables is regarded as constant.

XVI.

One must observe that it is not necessary for a partial difference equation to have the form $P\,p + Q\,q = L$ in order for it to be regarded as linear. It will suffice that it can be reduced to one when one supposes that the analysis is perfect. Hence, in order to give the generality that is appropriate to what was just said in the preceding article, one must regard any equation that is composed in any way from the four quantities $P\,p + Q\,q$, x, $y\,z$, i.e., any one of the form:

$$F(P\,p + Q\,q, x, y, z) = 0$$

in which P and Q contain neither p nor q, to be linear.

XVII.

If the linear equation $P\,p + Q\,q = L$ lacks one term (i.e., if one of the three quantities L, P, Q is equal to zero) then the integration will no longer depend upon that of a first-order ordinary difference equation in two variables. For example, if one has $L = 0$, i.e., if the proposed equation is:

$$P\,p + Q\,q = 0$$

then the two equations of the characteristic will become:

$$P\,dy - Q\,dx = 0\,,$$
$$dz = 0\,.$$

The integral of the second one $z = \alpha$ expresses the idea that z is constant for the same characteristic, and that the curve is in the plane perpendicular to z: Thus, replacing z with its constant value α in the first one will make that equation involve:

$$x, y, z, \varphi(\alpha),$$

in which $\varphi(\alpha)$ is the arbitrary constant that is introduced by integration, and one will need only to replace α with its value z in order to have the equation of the surface, which is an equation that be, consequently, in terms of:

$$x, y, z, \varphi(\alpha)\,.$$

The same thing will be true for the case in which one has:

$$P = 0 \quad \text{or} \qquad Q = 0\,,$$

and one will find that the integral involves:

$$x, y, z, \varphi(x)$$

in the first case, and:

$$x, y, z, \varphi(a)$$

in the second.

XVIII.

It follows from the foregoing that no matter how the quantities P, Q are composed from the three coordinates x, y, z, the curved surface that is associated with the linear first-order partial difference equation:

$$F(P\,p + Q\,q, x, y, z) = 0$$

can always be regarded as being generated by the motion of a well-defined curve that is mobile and variable in form by virtue of the variation of the two parameters, one of which is an arbitrary function of the other, and into which the equations of which the derivatives of that arbitrary function do not enter. In that case, the integral will be expressed by just one equation.

It is easy to show that, conversely, when a surface is generated by the motion of a well-defined curve that is mobile and variable in form by virtue of the variation of the two parameters, one of which is an arbitrary function of the other, provided that the derivatives of that function do not enter into the equations of the generator, the partial difference equation of that surface will always be linear in p, q, i.e., it will always have the form:

$$F(P\,p + Q\,q, x, y, z) = 0\,.$$

Hence, the surfaces that belong to first-order linear partial difference equations have a special character. Although they can be considered to be envelopes, they are nonetheless capable of being generated in a simpler way, and the integral equation is unique for each of them, whereas for the surfaces that are expressed by equations in which p and q are raised to powers, the integrals are always expressed by the system of two equations, between one must eliminate a parameter α that is found in the arbitrary function and its derivatives.

We shall now move on to the general case.

XIX.

In the general case, i.e., when the partial difference equation does not have the form:

$$F(P\,p + Q\,q, x, y, z) = 0\,,$$

it will not suffice to consider only the first three equations of art. XIII:

1. $$P\,dy - Q\,dx = 0\,,$$

2. $$P\,dz - (P\,p + Q\,q)\,dx = 0\,,$$

3. $$Q\,dz - (P\,p + Q\,q)\,dy = 0\,,$$

which are the equations of the projections of the characteristic onto the three coordinate planes, because the surface can no longer be considered to be generated by a curve whose integral equations involve x, y, z, α, $\varphi(\alpha)$, without derivatives, and which are mobile by virtue of the variation of α. The surface must be regarded as an envelope. It will then be necessary to employ the equation of an envelope. However, the seven other equations in art. XIII will be those of just as many different envelopes by means of which, one can, if desired, form an infinitude of other ones, and of all those envelopes, the one that belongs to the equation:

10. $$(X + p\,Z)\,dq - (Y + q\,Z)\,dp = 0$$

will be that of an enveloped developable, whose method of generation is simple and whose considered is easy. That is the one that we shall then employ.

Otherwise, when the quantities P, Q that enter into the three equations of the projections of the characteristic contain the five quantities p, q, x, y, z, those three equations cannot suffice, because if we operate on any two of them as we did in art. XVII, i.e., if we differentiate them with ordinary differences, which will give two new equations, then we will also introduce the two new differentials dp, dq that we can eliminate only by taking their values in the other seven equations that contain them.

One knows that the general equation of a developable surface involves only the two quantities p, q, of which only one of them can be regarded as the principal variable as a consequence. Let that one be p and let its differential dp be regarded as constant. Having done that, let:

$$(A) \qquad F(p, q, x, y, z) = 0$$

be the general first-order partial difference equation that must be integrated. Then consider those four of the ten equations of the characteristic that contain dp, namely:

The fourth one (B), which involves $\frac{dx}{dp}, p, q, x, y, z,$

The fifth one (C), which involves $\frac{dy}{dp}, p, q, x, y, z,$

The eighth one (D), which involves $\frac{dz}{dp}, p, q, x, y, z,$

The tenth one (E), which involves $\frac{dq}{dp}, p, q, x, y, z$.

If one differentiates (E) using ordinary differences then the equation to which one arrives directly will involve:

$$\frac{ddq}{dp^2}, \frac{dq}{dp}, \frac{dx}{dp}, \frac{dy}{dp}, \frac{dz}{dp}, p, q, x, y, z,$$

and upon replacing $\frac{dx}{dp}, \frac{dy}{dp}, \frac{dz}{dp}$ with their values that one infers from the three equations (B), (C), (D), one will have a second-order ordinary difference equation:

(F) that involves $\frac{ddq}{dp^2}, \frac{dq}{dp}, p, q, x, y, z.$

Differentiating the latter once more, while replacing $\frac{dx}{dp}, \frac{dy}{dp}, \frac{dz}{dp}$ with their values, one will have a third-order difference equation:

(*G*) that involves $\frac{d^3q}{dp^3}, \frac{ddq}{dp^2}, \frac{dq}{dp}, p, q, x, y, z.$

One will then have four equations (*A*), (*E*), (*F*), (*G*) that will be delivered by the differentials of the three coordinates x, y, z. Therefore, upon eliminating x, y, z from those four equations, the resultant:

(*H*) will involve $\frac{d^3q}{dp^3}, \frac{ddq}{dp^2}, \frac{dq}{dp}, p, q,$

and consequently, the third-order ordinary differences that involve the two variables p, q. That equation (*H*), in which the differential of the variable p is regarded as constant, belongs to the enveloped developable, and its integration will depend upon only the integration of the proposed equation.

Indeed, let that equation be integrated three times for the second differences, and suppose that those integrals, each of which is completed with one of the three arbitrary constants α, β, γ, are:

(*J*) which involves $\frac{ddq}{dp^2}, \frac{dq}{dp}, p, q, \alpha,$

(*K*) which involves $\frac{ddq}{dp^2}, \frac{dq}{dp}, p, q, \beta,$

(*L*) which involves $\frac{ddq}{dp^2}, \frac{dq}{dp}, p, q, \gamma.$

We first observe that in those equations, all three of which belong to the enveloped developable, the quantities α, β, γ are constants for the same enveloped surface, which is considered to be fixed, and all three will vary when the enveloped surface moves. Moreover, that enveloped surface can move in just one way. Therefore, any two of those constants will be arbitrary functions of the third one: Hence, one will have:

$$\beta = \varphi(\alpha), \qquad \gamma = \psi(\alpha).$$

If one replaces $\frac{ddq}{dp^2}, \frac{dq}{dp}$ with their values in terms of p, q, x, y, z that are inferred from equations (*E*), (*F*) in the three equations (*J*), (*K*), (*L*) then they will become:

(J') which involves $p, q, x, y, z, \alpha,$

(K') which involves $p, q, x, y, z, \varphi(\alpha),$

(L') which involves $p, q, x, y, z, \psi(\alpha).$

Having done that, if one employs any two of those equations – for example, (J'), (K') – then one can use those two equations to eliminate p and q from the given one (A), which will then involve:

$$x, y, z, \alpha, \varphi(\alpha),$$

which I represent by $M = 0$, and which will be that of the enveloped developable.

When that enveloped surface is made mobile by virtue of the variation of a, it will successively touch the envelope in all of the characteristics: The second integral equation of the characteristic will then be $dM / d\alpha = 0$. Therefore, the locus of all the characteristics, or the required envelope, will have an equation that is the result of eliminating a from the two equations:

$$M = 0 ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0 .$$

What we just said is not complete, and it is necessary to add what will be said in the following article.

XX.

Of the three equations (J'), (K'), (L'), we have employed only two of them that were taken at random, namely, the first two. Nonetheless, all three of them are generally necessary.

Indeed, those three equations give values for α, $\varphi(\alpha)$, $\psi(\alpha)$ that involve p, q, x, y, z, all three of which are constant for each enveloped developable and will vary from one envelope to the other. Now, it can happen that the given equation:

$$(A) \qquad F(p, q, x, y, z) = 0$$

expresses only one relation between the values of the two quantities α, $\varphi(\alpha)$. In that case, upon eliminating p, q from the three equations (A), (J'), (K'), the three coordinates x, y, z will also disappear, and the resulting equation, which will involve α, $\varphi(\alpha)$, will express only the manner by which the values of those two quantities enter into the given equation, and consequently, they will determine the form of the function $\varphi(\alpha)$, which will no longer be arbitrary.

In general, the given equation (A) is nothing but the expression of a relation that exists between the value of the three quantities α, $\varphi(\alpha)$, $\psi(\alpha)$. Therefore, if one eliminates three of the five quantities p, q, x, y, z from the four equations (A), (J'), (K'), (L') then the other two will disappear, and what will remain is an equation in α, $\varphi(\alpha)$, $\psi(\alpha)$ that expresses that relation and serves to determine the form of one of the two functions, so only one of them will be arbitrary. Suppose that it is $\psi(\alpha)$ that one must determine and that one substitutes its value in terms of α, $\varphi(\alpha)$ in (L'): If one eliminates p, q from the three equations then the resultant equation, which will be in terms of x, y, z, α, $\varphi(\alpha)$, will be that of the moving enveloped developable by virtue of the variation of α and which will touch the envelope at the characteristic in all of its positions. Therefore, if that equation is represented by $M = 0$ then the result of the elimination of α from the two equations:

$$M = 0\,,$$

$$\frac{dM}{d\alpha} = 0$$

will be the integral of the given equation, when completed by the arbitrary function φ.

One sees that the integration of any first-order partial difference equation in three variables will depend upon only the integration of a third-order ordinary difference equation in two variables, and in which the differential of one of the variables is regarded as constant. However, methods that are as general as the ones that we just discussed are rarely useful, due to their lengthiness and the analytical difficulties that they entail, and one cannot dispense with the applicable methods in the less-general cases. One can find a great number of them. We shall discuss only the ones that apply to some examples.

XXI.

Surfaces whose enveloped developables are cylinders parallel to a given plane.

Let $A\,x + B\,y + z = 0$ be the equation of a fixed plane that is drawn through the origin and to which the cylindrical surfaces must be parallel. Let $y + a\,x = 0$ be the equation of the projection onto the x, y-plane of the straight line to which the generator of each cylindrical surface must be parallel. It is easy to see that the integral equation of the cylindrical surface will be:

$$A\,x + B\,y + z = \varphi(y + a\,x)$$

and that its partial difference equation will be:

$$p + A = \alpha\,(q + B)\,,$$

in which A, B are the invariable constants of the fixed plane, and α, which is a constant for each of the cylindrical surfaces, changes value from one of those surfaces to the other. If one would then like to have an equation that is appropriate to all cylindrical surfaces that are parallel to the fixed plane then one must make that arbitrary constant disappear by differentiation using ordinary differences, which will give:

$$(p + A)\, dq - (q + B)\, dp = 0.$$

Now, we have seen (art. XII) that the equation of the enveloped developable is:

$$(X + p\, Z)\, dq - (Y + q\, Z)\, dp = 0\ .$$

Therefore, in order for that enveloped surface to be cylindrical and parallel to the fixed plane, it is necessary that the values of dq / dp that are provided by the last two equation must be equal to each other, or that one must have:

$$(X + p\, Z)\, dq - (Y + q\, Z)\, dp + Z\,(B\, p - A\, q) = 0\ .$$

However, upon representing the partial difference equation of the envelope by $U = 0$, it will be clear that the three quantities X, Y, Z will be $\left(\frac{dU}{dx}\right)$, $\left(\frac{dU}{dy}\right)$, $\left(\frac{dU}{dz}\right)$, respectively, in which one regards p and q as constants. The last equation is therefore a first-order partial difference equation in the four variables, U, x, y, z, the last three of which are principal. Since that equation is linear and has constant coefficients, it is easy to treat, and it expresses the idea that the quantity U must be composed in an arbitrary manner from the two quantities $A\, x + B\, y + z$ and $z - p\, x - q\, y$, and the quantities p, q, which are constants in that equation. Therefore, the general equation of the surfaces whose enveloped developable is cylindrical and parallel to a fixed plane will be:

$$(A) \qquad F\,(A\, x + B\, y + z,\ z - p\, x - q\, y,\ p,\ q) = 0\ .$$

It is easy to verify that result, because if one differentiates it using ordinary differences then one will find the following values for X, Y, Z :

$$\begin{aligned} X &= A\,F' - p\,F''\ , \\ Y &= B\,F' - q\,F''\ , \\ Z &= F' + F''\ , \end{aligned}$$

which will give:

$$\begin{aligned} X + p\, Z &= (p + A)\, F'\ , \\ Y + q\, Z &= (q + B)\, F'\ , \end{aligned}$$

and consequently, the equation of the enveloped developable:

$$(X + p\, Z)\, dq - (Y + q\, Z)\, dp = 0$$

will become:

$$(p + A)\, dq - (q + B)\, dp = 0\ ,$$

which is that of a cylindrical surface that is parallel to the fixed plane.

We shall now see that the integration of equation (*A*), no matter how it is composed from the four quantities, will depend upon only the integration of just one ordinary difference equation in two variables. Indeed, the equation of the enveloped developable:

$$(p + A)\, dq - (q + B)\, dp = 0$$

is integrated and will give:

$$q + B = \alpha\, (p + A)\ ,$$

in which α is the arbitrary constant. If one infers the values of p, q from that equation and from $dz = p\,dx + q\,dy$, while setting:

$$A\,x + B\,y + z = u\ ,$$
$$x + \alpha\,y = v\ ,$$

to abbreviate, then one will find that:

$$p = \frac{du}{dv} - A\,,$$

$$q = \frac{du}{dv} - B\,,$$

$$z - p\,x - q\,y = \frac{u\,dv - v\,du}{dv}\ .$$

When those values are substituted in the given equation (*A*), it will become:

$$F\left(u, \frac{u\,dv - v\,du}{dv}, \frac{du}{dv} - A, \alpha\,\frac{du}{dv} - B\right) = 0\ ,$$

which is a first-order ordinary difference equation between just the two variables u, v in which α is constant, and which belongs to the enveloped developable. The integral of that equation, when completed by an arbitrary function $\varphi\,(\alpha)$, will be in terms of:

$$u,\ v,\ \alpha,\ \varphi\,(\alpha)\ .$$

Hence, if one represents that equation by $M = 0$ then the integral of the given equation (*A*) will be the result of eliminating a from the two equations:

$$M = 0\ ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\ .$$

XXII.

The equation $F(A\,x + B\,y + z, z - p\,x\,q\,y, p, q) = 0$ that we just treated is a little more general than we said it was, because it is not only appropriate to the surfaces whose enveloped developable is cylindrical and parallel to the fixed plane, but also to the ones that have no other enveloped developable than themselves, i.e., to developable surfaces whose equation is:

$$F(z - p\,x - q\,y, p, q) = 0 .$$

The same general equation refers to a large part of the ones that Lagrange treated in his beautiful work on particular integrals that was printed in the *Mémoires de l'Académie de Berlin* for the year 1774.

For example, if the quantity $z - p\,x - q\,y$ does not enter into the equation, and if one has $A = 0$, $B = 0$, moreover, which will locate the fixed plane in the x, y-plane, then the equation will become:

$$F(z, p, q) = 0$$

that Lagrange integrated by means of the hypothesis $q = \alpha\,p$. In that case, the enveloped developable will have the equation:

$$p\,dq - q\,dp = 0 ,$$

whose integral $q = \alpha\,p$ belongs to a cylindrical surface that is parallel to the x, y-plane and is provided by the method.

If the general equation lacks only the quantity $z - p\,x - q\,y$ then it will become:

$$F(A\,x + B\,y + z, p, q) = 0 ,$$

which includes the previous example as a special cases, and which (as we saw in **§ IX**, pp. 6) belongs to the surface whose enveloped surface, which is invariable in form and size, moves without turning in such a manner that the curves that are swept out by all of its points are similar to each other, equal, and in planes parallel to the fixed plane: In that case, the integral that is given by that method is not the most elegant, and one will arrive at a simpler result that is easier to represent in space by employing the equation of the invariable enveloped surface.

One can treat the general equation of surfaces whose enveloped developable is cylindrical and directed in an arbitrary manner, but that equation is a second-order partial difference equation that we shall not go into any further.

XXIII.

Surfaces whose enveloped developables are conical with all of their summits along the same given line.

One knows that if the coordinates of a point in the directions of x, y, z are α, β, γ, respectively, then the partial difference equation of the conical surface, which has its summit at that point will be:

$$z - \alpha = p\,(x - \beta) + q\,(y - \gamma)\,. \tag{A}$$

Suppose that the given straight line on which the summit must be determined has the equation:

$$x = A\,z + a\,, \qquad y = B\,z + b\,,$$

in which A, B, a, b are given constants. The same equations will be valid between the coordinates of the summit, and one will have:

$$\beta = A\,\alpha + a\,, \quad \gamma = B\,\alpha + b\,.$$

If one substitutes those values then the conical surface will be:

$$z - p\,(x - a) - q\,(y - b) = \alpha\,(1 - A\,p - B\,q)\,,$$

in which α is an arbitrary constant whose magnitude determines the position of the summit along the given line.

One will infer the following values for α, β, γ from the last three equations:

$$\alpha = \frac{z - p\,(x-a) - q\,(y-b)}{1 - A\,p - B\,q}\,,$$

$$\beta = A\,\frac{z - p\,(x-a) - q\,(y-b)}{1 - A\,p - B\,q} + a\,,$$

$$\gamma = B\,\frac{z - p\,(x-a) - q\,(y-b)}{1 - A\,p - B\,q} + b\,,$$

which are values that are constant for the same conical envelope.

If one now differentiates equation (A) while regarding α, β, γ as constants then one will have:

$$(y - \gamma)\,dq + (x + \beta)\,dp = 0\,,$$

which is the ordinary difference equation of all the conical surfaces that have their summit along the straight line, and into which one will only need to replace β, γ with their values. Now, the general equation of the enveloped developable is:

$$(x + p\,Z)\,dq - (Y + q\,Z)\,dp = 0\ .$$

Hence, in order for that enveloped developable to be conic and have its summit along the give line, it is necessary that the last two equations must coincide, i.e., that one must have:

$$X\,(x - \beta) + Y\,(y - \gamma) + Z\,(z - \alpha) = 0\ .$$

However, upon representing the partial difference equation of the envelope by $U = 0$, it will be obvious that the latter equation is a first-order partial difference equation in the four variables U, x, y, z, and that one treats the quantities p, q as constants. It is also clear that the quantities α, β, γ are constants, because all three of the differentials of those quantities, when taken while regarding p and q as constants, are zero. Therefore, if one takes the value of Z in that equation and substitutes it in:

$$dU = X\,dx + Y\,dy + Z\,dz\ ,$$

which will give:

$$\frac{dU}{z-\alpha} = X\,d\left(\frac{x-\beta}{z-\alpha}\right) + Y\,d\left(\frac{y-\gamma}{z-\alpha}\right)\ ,$$

then one will see that U must be composed in an arbitrary manner from the two quantities $\frac{x-\beta}{z-\alpha}$, $\frac{y-\gamma}{z-\alpha}$, and the two quantities p and q, which have been regarded as constants. Hence, the general equation of the surfaces whose enveloped developable is conical and has its summit along the given line will have the form:

$$(B) \qquad F\left(\frac{x-a-A\alpha}{z-\alpha}, \frac{y-b-B\alpha}{z-\alpha}, p, q\right) = 0\ ,$$

in which one must replace a with its value:

$$(C) \qquad \alpha = \frac{x - p\,(x-a) - q\,(y-b)}{z - A\,p - B\,q}\ .$$

Having done that, it will be easy to show that the integration of equation (B) depends upon only that of a first-order ordinary difference equation between two variables. Indeed, equation (C), which is nothing but the equation of the conical envelope, in which α is an arbitrary constant, can be put into the following form:

$$p\,(x - a - A\,\alpha) + q\,(y - b - B\,\alpha) = z - \alpha\ ,$$

or upon setting:

$$\frac{x-a-A\alpha}{z-\alpha} = u\,, \qquad \frac{y-b-B\alpha}{z-\alpha} = v\,,$$

to abbreviate, the equation of the conical envelope will be:

$$p\,u + q\,v = 1\,.$$

If one infers the value of p and q from that equation and from:

$$p\,dx + q\,dy = dz$$

then one will have:

$$p\,(u\,dy - v\,dx) = \quad dy - v\,dz\,,$$
$$q\,(u\,dy - v\,dx) = -\,dx - u\,dz\,,$$

or rather:

$$p = \frac{dv}{u\,dv - v\,du}\,,$$

$$q = \frac{-\,du}{u\,dv - v\,du}\,.$$

Therefore, upon substituting all of those values in the given equation (B), it will take the following form:

$$F\left(u, v, \frac{dv}{u\,dv - v\,du}, \frac{-\,du}{u\,dv - v\,du}\right) = 0\,,$$

which is a first-order ordinary difference equation between just the two variables u, v, and in which α is constant.

The integral of that equation, when completed by an arbitrary function of α, will be expressed in terms of:

$$u, \quad v, \quad \varphi(\alpha),$$

or in terms of:

$$\frac{x-a-A\alpha}{z-\alpha}, \quad \frac{y-b-B\alpha}{z-\alpha}, \quad \varphi(\alpha),$$

and will belong to the conical envelope. Therefore, if one represents that equation by $M = 0$ then the complete integral of the given equation (B) will be the result of eliminating α from the two equations:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,.$$

One can treat the general equation of the surfaces whose enveloped developable is conical in an analogous manner. However, the summits of the cones will no longer be required to lie along a straight line; they will be along an arbitrary curve: The two coordinates β, γ will be arbitrary functions of α, and the equation of the envelope will be a third-order partial difference equation. In that case, the envelope will be comprised of the space that is swept out by a given arbitrary surface that moves without turning, but which changes form while always remaining similar to itself.

XXIV.

Upon considering neither the eighth nor the ninth of the ten equations of the characteristic that were stated in art. XIII, *while any one of the other eight are integrable, either immediately or by means of what was proposed, the integration of the latter will depend upon only that of just one first-order ordinary difference equation in two variables.*

We shall prove that for the case of the tenth equation by analogy with the preceding articles.

Suppose that an arbitrary first-order partial difference equation $U = 0$ is such that the equation:

$$(A) \qquad (X + p\,Z)\,dq - (Y + q\,Z)\,dp = 0$$

is integrable either directly or by means of $U = 0$, and that the integral is represented by:

$$(B) \qquad f(p, q, \alpha) = 0 ,$$

in which α is the arbitrary constant that is introduced by the first integration, so we will have, consequently:

$$(C) \qquad f'\,dp + f''\,dq = 0 .$$

From that alone, the quantity U can take on a general form that must be found.

It is obvious that the values of dq / dp that provide the two equations (A), (C) must be equal to each other, which will give:

$$X\,f' + Y\,f'' + Z\,(p\,f' + q\,f'') = 0 .$$

Now the latter equation, in which p, q, α are regarded as constants, is a partial difference equation for the four variables U, x, y, z. Hence, if one then takes the value of Z and substitutes it in:

$$dU = X\,dx + Y\,dy + Z\,dz ,$$

then one will have:

$$(p\,f' + q\,f'')\,dU = X\,[(p\,f' + q\,f'')\,dx - f'\,dz] + Y\,[(p\,f' + q\,f'')\,dy - f''\,dz] ,$$

in which p, q, α are constants and which indicates that the quantity U must be composed in an arbitrary manner from the following two: $(p\,f' + q\,f'')\,x - z\,f'$, $(p\,f' + q\,f'')\,y - z\,f''$, along with p, q, which are regarded as constants.

One will then have:

$$U = F[x(p\,f' + q\,f'') - z\,f', y(p\,f' + q\,f'') - z\,f'', p, q] \ ,$$

or what amounts to the same thing:

$$U = F(z - p\,x - q\,y, x\,f'' - y\,f', p, q) \ ,$$

and consequently, the given equation $U = 0$ can be put into the form:

$$(D) \qquad F(z - p\,x - q\,y, x\,f'' - y\,f', p, q) = 0 \ ,$$

in which the derivative functions f', f'' are given at p, q, and contain the constant α, moreover, which can be eliminated by means of (B). It is pointless to form that equation (D); it will suffice to prove that it must be true.

Having done that, equation (B), which belongs to the enveloped developable, is itself a partial difference equation. It is easy to integrate, and one knows that in order to get its integral, it will be necessary to:

1. Set:

$$(E) \qquad z - p\,x - q\,y = \psi(p) \ .$$

2. Eliminate q by means of its value that is taken from (B).

3. Eliminate p by means of the differential of (E), which is taken while regarding x, y, z as constants, i.e., by means of:

$$-x\,dp - y\,dq = \psi'(p)\,dp \ ,$$

and when one gives dq its value from (C), it will become:

$$(F) \qquad x\,f'' - y\,f' = f''\psi' \ .$$

Hence, the integral equation of the enveloped developable is the result of eliminating the two quantities p, q from the three equations (B), (E), (F), and if the function ψ, which enters into the equation, along with its derivative, is regarded as arbitrary then the integral will belong to the general enveloped developable.

However, if one would like for that envelope to be the one that pertains to the surface that is expressed by $U = 0$, in particular, then one must determine the form of the function ψ in such a manner that the given one $U = 0$, or what amounts to the same thing, equation (D), is satisfied.

Now, if one replaces:

$z - p\,x - q\,y$	with its value $\psi(p)$	that one infers from (E),
$x\,f'' - y\,f'$	with its value $f''\psi'(p)$	that one infers from (F), and
q	with its value in terms of p that one infers from (B)	

in (D) then all that will remain are $\alpha, p, \psi(p), \psi'(p)$. Hence, without forming equation (D), it will always be possible to eliminate the four quantities q, x, y, z from the four equations $U = 0$, (B), (E), (F), and one will have a result:

(G) in terms of $\alpha, p, \psi(p), \psi'(p)$,

which will be a first-order ordinary difference equation between the two variables p, $\psi(p)$, in which α is constant, and which will serve to determine the form of the function ψ.

Suppose that equation (G) is integrated and that its integral, when completed with an arbitrary function of α, is:

(H) in terms of $\alpha, p, \psi(p), \varphi(\alpha)$.

Having done that, if one eliminates the four quantities $p, q, \psi(p), \psi'(p)$ from the five equations (B), (E), (F), (G), (H) then one will have the integral equation of the enveloped developable in terms of:

$$x, y, z, \alpha, \varphi(\alpha) .$$

Hence, upon representing that integral by $M = 0$, the integral of the given equation $U = 0$ will be the result of eliminating α from the two equations:

$$M = 0 ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0 .$$

XXV.

Upon arguing analogously with the other seven equations of the characteristic, one will show that the proposition that was stated in the preceding article will be true for each of them. We shall not go into the details of that, but will be content to state the results:

1. If the equation $P\,dy - Q\,dx = 0$ is integrable, directly, or by means of the given equation $U = 0$, and if the known integral is represented by:

$$(B) \qquad f(x, y, \alpha) = 0\,,$$

in which α is the arbitrary constant, then the given equation can take on the form:

$$(D) \qquad F(p\,f'' - q\,f', x, y, z) = 0\,.$$

One sets:

$$(E) \qquad z = \psi(x)\,,$$

$$(F) \qquad p\,f'' - q\,f' = f''\psi'\,.$$

It will always be possible to eliminate the four quantities p, q, y, z from the given equation $U = 0$ and the three equations (B), (E), (F), and the result will be:

(G) in terms of $\alpha, x, \psi(x), \psi'(x)$,

whose integral will be

(H) in terms of the $\alpha, x, \psi(x), \varphi(\alpha)$,

and will determine the form of the function ψ. If one puts $\psi(x)$ equal to its value z then the latter equation will become one:

(K) in terms of $\alpha, x, z, \varphi(\alpha)$.

Having done that, one can infer the values of α and φ from (B), (K), and if one represents those values by:

$$\alpha = M\,, \qquad \varphi(\alpha) = N$$

then the complete integral of the given equation $U = 0$ will be:

$$N = \varphi(M)\,.$$

2. If the equation $(P\,p + Q\,q)\,dx - p\,dz = 0$ is integrable, and if its known integral is represented by:

$$(B) \qquad f(x, y, \alpha) = 0$$

then the given equation $U = 0$ can be put into the form:

$$(D) \qquad F\left(\frac{p\,f'' + f'}{q}, x, y, z\right) = 0 \,.$$

If one sets:

$$(E) \qquad y = \psi(z)\,,$$

$$(F) \qquad p\,f'' + f' = q\psi'$$

then it will be possible to eliminate the four quantities p, q, x, y from the four equations $U = 0$, (B), (E), (F). The result of that elimination will be:

(G) in terms of $\alpha, z, \psi(z), \psi'(z)$,

whose integral will be

(H) in terms of $\alpha, z, \psi(z), \varphi(\alpha)$.

If one puts $\psi(z)$ equal to its value y that is inferred from (E) in that integral then one will have:

(K) in terms of $\alpha, y, z, \varphi(\alpha)$.

Having done that, if one infers the values of α, $\varphi(\alpha)$ from (B) and (K), and if one represents those values by:

$$\alpha = M, \qquad \varphi(\alpha) = N$$

then the complete integral of the given equation $U = 0$ will be:

$$N = \varphi(M)\,.$$

3. If it is the equation $(P\,p + Q\,q)\,dy - Q\,dy = 0$ that is integrable, and if its known integral is represented by:

$$(B) \qquad F(y, z, \alpha) = 0$$

then the given equation $U = 0$ can be put into the form:

$$F\left(\frac{q\,f'' + f'}{p}, x, y, z\right) = 0\,,$$

and one operates as in the preceding case.

4. The two equations of the characteristic:

$$P\,dp + (X + p\,Z)\,dx = 0\,,$$
$$Q\,dq + (Y + p\,Z)\,dy = 0$$

belong to the same envelope whose integral equation has the general form:

$$z = \psi(x) + \pi(y)\,,$$

and whose two partial difference equations are:

$$p = \psi'(x)\,, \qquad q = \pi'(y)\,.$$

Therefore, if the first of those two equations of the characteristic can be reduced to two variables p, x by means of the given equation $U = 0$ then the second one will be reducible to the two variables q, y, and conversely.

Let their integrals be:

$$p = f(x,\,\alpha)\,, \quad q = f(y,\,\varphi(\alpha))\,,$$

in which α and $\varphi(\alpha)$ are two arbitrary constants. The integral equation of the enveloped surface will be:

$$z = \int f(x,\alpha)\,dx + \int f(y,\varphi(\alpha))\,dy\,,$$

and if one represents that equation by $M = 0$ then the complete integral of the given equation $U = 0$ will be the result of eliminating α from the two equations $M = 0$ and $dM / d\alpha = 0$.

5. If the equation of the characteristic:

$$Q\,dp + (X + p\,Z)\,dy = 0$$

is made integrable by means of the given equation $U = 0$, and if that known integral is:

(B) $$f(p,\,y,\,\alpha) = 0\,,$$

in which α is the arbitrary constant, then the given equation can be put into the form:

(D) $$F(q\,f' + x\,f'',\,p\,q\,f' + z\,f'',\,p,\,y) = 0\,.$$

One sets:

$$(E) \qquad p\,q\,f' + z\,f'' = f''\psi(y)\,,$$

$$(F) \qquad q\,f' + x\,f'' = f'\psi'\,.$$

It will always be possible to eliminate the four quantities p, q, x, z from the four equations $U = 0$, (B), (E), (F), and the resultant will be:

(G) in terms of $\alpha, y, \psi(y), \psi'(y)$,

whose integral will be:

(H) in terms of $\alpha, y, \psi(y), \varphi(\alpha)$.

Having done that, if one eliminates the four quantities p, q, $\psi(y)$, $\psi'(y)$ from the five equations (B), (E), (F), (G), (H) then one will have the integral equation of the enveloped surface in terms of $x, y, z, \alpha, \varphi(\alpha)$. Thus, if one represents that equation by $M = 0$ then the complete integral of the given equation $U = 0$ will be the result of eliminating α from the two equations:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,.$$

6. It is obvious that if the equation of the characteristic:

$$P\,dq + (Y + q\,Z)\,dx = 0$$

is integrable directly, or it can be made integrable by means of the given equation $U = 0$, then everything will be as it was in the previous case when one changes p and y into q and x, respectively.

The geometric considerations upon which we just based the search for equations of the characteristic are familiar to students at l'École Polytechnique. However, they can be troublesome for other readers, and we believe that we can arrive at the same equations by a purely-analytical process. We shall first do that for the case of linear equations and then proceed to the general case.

XXVI.

One knows that an arbitrary first-order partial difference equation belongs to the envelope of the space that is swept out by a given surface that moves by virtue of the variation of a parameter α, and whose integral equation includes an arbitrary function of α that will disappear upon differentiation, moreover. However, that same equation will also belong to each of the enveloped surfaces that are included in the envelope, because each enveloped surface is a particular case of the envelope: It is what the envelope will become when the parameter α is constant.

With that, let the general linear equation be given:

$$(A) \qquad P\,p + Q\,q = L\,,$$

in which the three coefficients P, Q, L are given arbitrarily in terms of x, y, z: If that equation, which does not contain α explicitly, can belong to each of the individual enveloped surfaces then it will necessarily contain the quantity α, at least implicitly, which is constant for each enveloped surface, and whose different values determine each of them in particular, and it is only the quantities p, q that can contain it. However, those two quantities are not exactly independent. They have a relation between them that is expressed by the equation:

$$dz = p\,dx + q\,dy\,,$$

which results from their definition. One can then use the latter equation to eliminate one of the two quantities from the given equation, and one will have one of the following two equations:

$$(B) \qquad \begin{aligned} p\,(P\,dy - Q\,dx) &= \quad L\,dy - Q\,dz\,, \\ q\,(P\,dy - Q\,dx) &= -L\,dy + Q\,dz\,. \end{aligned}$$

It will suffice for one to consider just one of them, the first, for example.

Equation (B), in which p includes α explicitly, is an ordinary difference equation. It will then belong to all of the enveloped surfaces, for each of which the arbitrary parameter α will have a constant value that is different for each of them. Therefore, if one considers a certain enveloped surface for which the arbitrary constant has a certain value α and passes to the infinitely-close enveloped surface, for which the arbitrary parameter will have the value $\alpha + d\alpha$, then it will be clear that the equation of that second enveloped surface will be:

$$(D) \qquad \left[p + \left(\frac{dp}{d\alpha}\right)d\alpha\right](P\,dy - Q dx) = L\,dx - Q\,dy\,.$$

Now, the characteristic of the envelope is nothing but the intersection of two consecutive enveloped surfaces. Therefore, equations (B), (D), which belong to two consecutive enveloped surfaces, will be those of the characteristic.

If one subtracts (B) from (D) then one will have:

$$P\,dy - Q\,dx = 0\,,$$

by virtue of which (D) will give:

$$L\,dy - Q\,dz = 0\,.$$

Finally, upon eliminating *dy* from the last two, one will once more have:

$$L\,dx - P\,dz = 0\,.$$

Those three ordinary difference equations, any one of which is a necessary consequence of the other two, are those of the projections of the characteristics onto the three rectangular planes that we have treated in art. XIV and the following ones.

It is obvious that the operation that we just carried out will reduce to differentiating equation (*B*) while regarding *p* alone as variable, which will immediately give:

$$P\,dy - Q\,dx = 0\,,$$
$$L\,dy - Q\,dz = 0\,.$$

If one differentiates (*C*), while regarding *q* alone as variable, then one will likewise have:

$$P\,dy - Q\,dx = 0\,,$$
$$L\,dx - P\,dz = 0\,,$$

which is the same result. We shall now proceed to the general case.

XXVII.

When the first-order partial difference equation is not linear, the operation that we just performed, which consists of replacing one of the quantities *p*, *q* with its value that we take for it in $dz = p\,dx + q\,dy$ and then differentiating it, while regarding only the remaining one of the two quantities as variable, will produce only one equation for the characteristic, and that is not sufficient. In order to have two equations for that curve, we must partially differentiate the given equation once.

Let the given general equation be:

$$F\,(p, q, x, y, z) = 0\,,$$

and suppose that its ordinary differential is:

$$P\,dp + Q\,dq + X\,dx + Y\,dy + Z\,dz = 0\,, \tag{A}$$

in which *P*, *Q*, *X*, *Y*, *Z* are given in terms of *p*, *q*, *x*, *y*, *z* by differentiation. Its two partial differentials, which are taken by first regarding *x* as the only variable, and then *y*, will be:

$$P\,r + Q\,s + X + p\,Z = 0\,, \tag{B}$$

$$P\,s + Q\,t + Y + q\,Z = 0\,. \tag{C}$$

It will suffice to consider just one of those two equations, the first one, for example.

Equation (B), which is a linear second-order partial difference equation, belongs to a more general surface than the one that was given, and which has two characteristics: However, one of those two characteristics is common to the surface of the given equation, and the other one is known, because one knows that the new characteristic will have $dy = 0$ or $y = \beta$ for its equation. Hence, the equations of the first characteristic of the surface to which equation (B) belongs will also be those of the characteristic of the surface to which the given equation belongs.

Upon considering equation (B) to be something that belongs to the moving enveloped surface by virtue of the variation of a parameter α, the two quantities r, s will be the ones that are affected by the variation of α when one passes from any enveloped surface to the following one, because if the characteristic is the intersection of two consecutive enveloped surfaces then it will also be a line of contact for those two surfaces, and the quantities p, q will not change upon passing from one to the other, but the two quantities will not be independent. They are coupled by the equation:

$$dp = r\,dx + s\,dy\,,$$

which is the expression of their definition, and by means of which, it is easy to eliminate any one of them from (B), which will produce one of the following two equations:

$$(D) \qquad s\,(P\,dy - Q\,dx) - [P\,dp + (X + p\,Z)\,dx] = 0\,,$$

$$(E) \qquad r\,(P\,dy - Q\,dx) + [Q\,dp + (X + p\,Z)\,dy] = 0\,,$$

each of which belongs to a moving enveloped surface and contains only one quantity s or r, which is variable by virtue of the variation of the parameter α. It will suffice to consider one of those two equations, the first one, for example.

Upon arguing as we did in the preceding article, it will be obvious that in order to have the equation of the characteristic, one must differentiate equation (D) of the enveloped surface while regarding α, and consequently s, as the only variable, and since s is linear, that operation will produce the two equations:

$$P\,dy - Q\,dx = 0\,,$$
$$P\,dp + (X + p\,Z)\,dx = 0\,,$$

which will belong to the characteristic, and by means of which, along with the following two:

$$(A) \qquad \begin{cases} P\,dp + Q\,dq + (X + p\,Z)\,dx + (Y + q\,Z)\,dy = 0, \\ dz = p\,dx + q\,dy, \end{cases}$$

one can form the ten equations of the art. XIII.

XXVIII.

If one differentiates (E), while regarding α, and consequently r, as the only variable, then one will have two equations:

$$P\,dy - Q\,dx = 0\,,$$
$$Q\,dp + (X + p\,z)\,dy = 0\,,$$

which will likewise produce the ten equations of art. XIII.

Instead of operating on equation (B) as we did, we can likewise treat equation (C). Indeed, if we eliminate one of the two quantities s, t from that equation by means of:

$$dq = s\,dx + t\,dy$$

then we will get one of the following two equations:

$$(F) \qquad t\,(P\,dy - Q\,dx) - [P\,dq + (Y + q\,Z)\,dx] = 0\,,$$
$$(G) \qquad s\,(P\,dy - Q\,dx) + Q\,dp + (Y + q\,Z)\,dx = 0\,,$$

the first of which, when differentiated while regarding α (i.e., t) as the only variable, will produce the two equations:

$$P\,dy - Q\,dx = 0\,,$$
$$P\,dq + (Y + q\,Z)\,dx = 0\,,$$

which provide the ten equations of art. XIII, and the second of which, when differentiated while regarding α (i.e., s) as the only variable, will produce the two equations:

$$P\,dy - Q\,dx = 0\,,$$
$$Q\,dq + (Y + q\,Z)\,dy = 0\,,$$

which will achieve the same goal.
