"Où l'on fait voir que les Équations aux différences ordinaires, pour lequelles the conditions d'intégrabilité ne sont pas satisfaites, sont susceptibles d'une véritable intégration, et que c'est de cette intégration que dépend celle des équations aux différences partielles élevées," Hist. Acad. Roy. Sci. (1784), 502-576.

In which one is made to see that the ordinary difference equations for which the integrability conditions are not satisfied are susceptible to a true integration, and that integration will depend upon the integration of higher-degree partial difference equations.

By MONGE

Translated by D. H. Delphenich

I.

One knows that all ordinary difference equation in two variables will belong to real curves. Indeed, for first order, they can all (except one) be put into the form $M\,dx + N\,dy = 0$. Consequently, if one takes a point at random, i.e., one is given the values of x and y, then one can always find the inclination of the tangent to the curve at that point by that equation, or what amounts to the same thing, one can find the direction of the describing point, because that direction is determined by the ratio of dy to dx, which will give that equation. For the second order, all of the equations in two variables (except one) can be put into the form $L\,ddy + M\,ddx + N = 0$, L, M, N, which includes the variables with their first differences. Hence, if one is given a point at random and the direction of the tangent at that point then that will determine x, y, dy / dx, and one will know the value of ddx, which is an arbitrary hypothesis that we are the master of. It is always possible to find the value of ddy from the differential equation, and consequently the change in direction that the describing point exhibits at that location on the curve, or what amounts to the same thing, one can find the radius of curvature of the curve. The same thing is true in higher orders.

When the first-order differential equations involve more than two variables, one can divide them into two classes. The one of them, like the following ones:

$$dz^2 = a^2(dx^2 + dy^2) ,$$

$$z^2(dx^2 + dy^2 + dz^2) = a^2(dx^2 + dy^2) ,$$

are raised with respect to the differences, and they are all regarded as absurd. The other can be reduced to the linear form:

$$L\,dx + M\,dy + N\,dz \ldots = 0 ,$$

in which the coefficients L, M, N, … can include the variables under the radical, and among the latter, the ones that satisfy certain conditions are once more regarded as absurd. The number of conditions that one deals with here is always equal to the number of variables minus two. For example, for the case of three variables, if the differential equation is put into the form $dz = p\,dx + q\,dy$ then the equation of condition is $\left(\frac{dp}{dy}\right) = \left(\frac{dq}{dz}\right)$, which one can develop in the following manner:

$$L\left[\left(\frac{dM}{dz}\right)-\left(\frac{dN}{dy}\right)\right]+M\left[\left(\frac{dN}{dx}\right)-\left(\frac{dL}{dz}\right)\right]+N\left[\left(\frac{dL}{dy}\right)-\left(\frac{dM}{dz}\right)\right] = 0\,.$$

For the case of four variables, if the differential equation is put into the form $dz = p\,du + q\,dx + r\,dy$ then the conditions will be:

$$\left(\frac{ddp}{dx\,dy}\right) = \left(\frac{ddq}{du\,dy}\right),$$

$$\left(\frac{ddp}{dx\,dy}\right) = \left(\frac{ddr}{du\,dx}\right),$$

and one can develop them as in the preceding case.

If there are five variables and the differential equation takes the form $dz = p\,dv + q\,du + r\,dx + s\,dy$ then the three conditions will be:

$$\left(\frac{d^3p}{du\,dx\,dy}\right) = \left(\frac{d^3q}{dv\,dx\,dy}\right),$$

$$\left(\frac{d^3p}{du\,dx\,dy}\right) = \left(\frac{d^3r}{dv\,du\,dy}\right),$$

$$\left(\frac{d^3p}{du\,dx\,dy}\right) = \left(\frac{d^3s}{dv\,du\,dx}\right),$$

and so on, for a much larger number of variables.

I propose to show that there is no differential equation that is absurd if one nonetheless intends that word to mean an impossible or imaginary property, etc. I will show that all of the differential equations express real properties whether they do or do not satisfy the conditions that I just referred to. I will show that they are all susceptible to a true integration, and in order to shed a bright light upon that matter, I shall discuss what the ones that include three variables signify in space.

II.

Of all the first-order ordinary difference equations in two variables, there is only one of them that is not linear, and that equation is:

$$M^2\, dx^{2m} + N^2\, dy^{2m} = 0\ ,$$

in which M, N are functions of x, y. Now, that equation cannot express anything in reality, unless one has, at the same time, $M = 0$, $N = 0$ or $dx = 0$, $dy = 0$. The first of those two results cannot be regarded as an integral, because it does not include an arbitrary constant. Therefore, the true integral of that equations is the system of two simultaneous equation $x = a$, $y = b$, i. e, the equation in question does not belong to a curved line, but to an arbitrary unique point that is taken on the *xy*-plane. One will then have the difference between the first-order linear equation in two variables and the only equation of that order that is higher-degree such that the former all belong to curves, and that the integral of each of them is a unique equation, when completed by just one arbitrary constant, while the latter belong a point, and its integral is the system of two simultaneous finite equations, when completed by two arbitrary constants.

The property of the higher-degree first-order ordinary difference equation in two variables has already been observed. However, since that equation is unique, one must regard it as an exception to the general rule, and one does not need to remark that it was the beginning of an immense chain that is attached to the greatest difficulties in in integral calculus. Indeed, among the ordinary difference equation in three variables, the ones that satisfy the condition that I referred to in the *preceding article* and which is known by the name of the *integrability condition* will all belong to curved surfaces, and the integral of each of them is a unique equation, when completed by just one arbitrary constant. However, all of the equations that do not satisfy that condition are infinite in number and do no belong to surfaces. The loci are the curves of double curvature that are traced out in space, and the integral of each of them is the system of two simultaneous equations. Finally, among those equations, there is only one of them, namely:

$$M^2\, dx^{2m} + N^2\, dy^{2m} + P^2 dz^{2m} = 0\ ,$$

whose integral is the system of three simultaneous finite equations $x = a$, $y = b$, $z = c$. It belongs to neither a curved surface nor a curve of double curvature. Its locus is a unique point that is taken arbitrarily in space.

For the first-order ordinary difference equations in four variables, the integral of which satisfies both of the two conditions that I just referred to, there is just one equation that is completed by just one constant. The integral of the ones that satisfy only one of the two condition is the system of two simultaneous equations. The integral of the ones that satisfy neither of the two conditions is the system of three simultaneous finite equations. Finally, there is only one of them:

$$M^2\, du^{2m} + N^2\, dx^{2m} + P^2 dy^{2m} + Q^2 dz^{2m} = 0\ ,$$

whose integral is the system of four simultaneous equations $u = a$, $x = b$, $y = c$, $z = e$. The same thing is true for a greater number of variables.

Hence, the goal of the equations that are known by the name of *integrability conditions* is not, as has been believed up to now, to indicate those of the differential equations for which integrals are possible, but to show the number of simultaneous finite equations that can be composed of integrals that are always possible. Before moving on, I shall clarify what I just said by way of some simple examples.

III.

Example I. **–** Let the proposed equation be:

$$(A) \qquad dz^2 = a^2 (dx^2 + dy^2) \, ,$$

in which a is a given constant. It is even obvious that this equation belongs to the curve of double curvature whose elements form a constant angle with *xy*-plane. Hence, the equations of all lines that make the same angle with the *xy*-plane must satisfy the given one, no matter what the directions of those lines, moreover. Now, those equations are:

$$(B) \qquad x = \alpha z + \beta ,$$

$$(C) \qquad y = z \sqrt{\frac{1}{\alpha^2} - \alpha^2} + \gamma ,$$

in which α, β, γ are three arbitrary constants. Thus, the system of those two equations, taken together, will be a solution of the given equation. Indeed, if one differentiates those two equations then the two arbitrary constants β, γ will vanish, and one will have:

$$dx = \alpha \, dz ,$$

$$dy = dz \sqrt{\frac{1}{\alpha^2} - \alpha^2} ,$$

and if one eliminates α from the last two equations then one will have:

$$dz^2 = a^2 (dx^2 + dy^2) .$$

Although the system of two equations (A), (B) is completed by three arbitrary constants, α, β, γ, one just saw that it is not the complete integral of equation (A) and that the complete integral is even more general.

If one eliminates the constant between (A) and (B) then the resulting equation:

$$(x-\beta)^2+(y-\gamma)^2=\frac{z^2}{a^2}$$

will be that of all conical surfaces whose summits are in the xy-plane and whose sides make a constant angle with that plane. If one sets $\gamma = \varphi(\beta)$, where φ is an arbitrary function, then the equation:

$$(x-\beta)^2+(y-\varphi(\beta))^2=\frac{z^2}{a^2}$$

will belong to only those of the conical surfaces whose summits are placed along a certain curve that is traced in the xy-plane, the equation of that curve being $y = \varphi(x)$, and if one considers two of those consecutive curved surfaces then they will intersect along a line, and one will have the second equation upon differentiating the equation of the cones with respect to the variable parameter β. Thus, the equations of that line will be:

$$(D)\qquad (x-\beta)^2+(y-\varphi(\beta))^2=\frac{z^2}{a^2},$$

$$(E)\qquad x-\beta+(y-\varphi(\beta))\,\varphi'(\beta)=0.$$

That line will also form a constant angle with the xy-plane, and it will be one of the ones that satisfy equation (A). However, if one considers the sequence of conical surfaces then one will have a sequence of lines, as before, that differ in position only by virtue of the variable parameter β, and all of those lines will be found to be pairwise consecutive on the same conical surface, so they will necessarily intersect pairwise consecutively, and they will consequently define the tangents to the same curve of double curvature: Therefore, the tangents to that curve of double curvature are equally inclined with respect to the xy-plane, so the elements of that curve will make constant angles with that plane. Hence, the equations of that curve will ultimately define the complete integral of the given equation.

Now, it is obvious that one will have the equations of the curve of double curvature upon differentiating the two equations (D), (E) with respect to the variable parameter β. Moreover, equation (E) is already the differential of (D) that was taken in that manner: Therefore, the complete integral of equation (A) is the system of three simultaneous equations:

$$(D)\qquad (x-\beta)^2+(y-\varphi(\beta))^2=\frac{z^2}{a^2},$$

$$(E)\qquad x-\beta+(y-\varphi(\beta))\,\varphi'(\beta)=0,$$

$$(F)\qquad -1-(\varphi'(\beta))^2+(y-\varphi(\beta))\,\varphi''(\beta)=0,$$

the last two of which are the first differentials, and second of which (D) is taken while varying only the indeterminate β, and in which φ is an arbitrary function, i.e., that integral is the result of eliminating the indeterminate β from the three equations (D), (E), (F).

It is easy to verify that integral by differentiation: Indeed, if the differentials of the two equations (D), (E) with respect to β having been taken then it will follow that one can differentiate those two equations while regarding β as constant, which will give:

$$(d) \qquad (x - \beta)\, dx + (y - \varphi(\beta))\, dy = \frac{z\, dz}{a^2} ,$$

$$(e) \qquad dx + \varphi'(\beta)\, dy = 0 ,$$

and eliminating the three indeterminates β, $\varphi(\beta)$, $\varphi'(\beta)$ from the four equations (D), (E), (d) (e) will give:

$$(A) \qquad dz^2 = a^2 (dx^2 + dy^2) .$$

The thread of a screw whose axis is perpendicular to the *xy*-plane is a particular case of that example, and the thread of a screw that is traced on a cylindrical surface with an arbitrary base that is perpendicular to the *xy*-plane is the general case.

IV.

EXAMPLE II: Let the given equation be:

$$(A) \qquad z^2 (dx^2 + dy^2 + dz^2) = a^2 (dx^2 + dy^2) ,$$

in which a is a given constant. Due to the proportion:

$$a : z :: \sqrt{dx^2 + dy^2 + dz^2} : \sqrt{dx^2 + dy^2} ,$$

it is obvious that if one imagines a circle whose radius is a, whose center is in the *xy*-plane, and whose plane is perpendicular to the latter one, then the given equation will belong to all of the curves whose elements make the same angle with the *xy*-plane as the element of the circle that is taken at the same height, or what amounts to the same thing, taken with a z equal to that of the curve. Therefore, all of the circles whose radii are a, whose planes are parallel to the z-plane, and whose centers are placed in the *xy*-plane must satisfy the given equation. Now, the equations of those circles are:

$$(B) \qquad (x - \alpha)^2 + (y - \beta)^2 + z^2 = a^2 ,$$

$$(C) \qquad x - \alpha = \gamma\,(y - \beta)\,,$$

in which α, β, γ are three arbitrary constants. Therefore, the system of those two equations taken simultaneously is a particular solution to equation (A). Indeed, if one differentiates the two equations (A), (B) then one will have:

$$(x - \alpha)\,dx + (y - \beta)\,dy + z\,dz = 0\,,$$
$$dx = \gamma\,dy\,,$$

and if one eliminates the three arbitrary parameters α, β, γ from those four equations then the resulting equation will be the given one. Although the system of two equations (A), (B) is completed by three arbitrary parameters, one will nonetheless see that the complete integral of the given equation is even more general.

Equation (B) belongs to a sphere whose radius is a and whose center is placed on the xy-plane at a point whose coordinates are α, β. If one sets $\beta = \varphi(\alpha)$ then the equation:

$$(D) \qquad (x - \alpha)^2 + (y - \varphi(\alpha))^2 + z^2 = a^2$$

will belong to all spheres of the same radius whose centers are placed in the xy-plane along a certain curve, where the equation of that curve is $y = \varphi(x)$. If one considers two consecutive spheres among them then they will cut along a circle, and one will get the second equation by differentiating the equation of the sphere with respect to the variable parameter α. Therefore, the equations of that circle will be:

$$(D) \qquad (x - \alpha)^2 + (y - \varphi(\alpha))^2 + z^2 = a^2\,,$$

$$(E) \qquad x - \alpha + (y - \varphi(\alpha))\,\varphi'(\alpha) = 0\,,$$

and those equations will once more satisfy the given one. However, if one considers the sequence of spheres whose centers are placed along the same curve then one will have a sequence of circles like the previous one that will differ from it only by virtue of the variable parameter α. All of those circles are found pairwise consecutively on the same sphere, so they can intersect pairwise consecutively, and the sequence of their points of intersection will define a curve of double curvature that is touched by all of the circles: Therefore, each element of that curve of double curvature will be common to one of those circles, so that element will make the angle with the xy-plane that is implied by the given equation. Hence, the equations of that curve of double curvature will be the complete integral of equation (A).

Now, it is obvious that in order to have the equations of that curve of double curvature, one must differentiate the equations (D), (E) with respect to the variable parameter α. Moreover, equation (E) is already the differential of (D) when taken in that manner. Thus, the complete integral of the given equation will be the system of three simultaneous equations:

(*D*) $$(x-\alpha)^2 + (y-\varphi(\alpha))^2 + z^2 = a^2 ,$$

(*E*) $$x-\alpha + (y-\varphi(\alpha))\,\varphi'(\alpha) = 0 ,$$

(*F*) $$-1-(\varphi'(\alpha))^2 + (y-\varphi(\alpha))\,\varphi''(\alpha) = 0 ,$$

the last two of which are the first and second differentials of (*D*), when taken while regarding α alone as variable, and in which φ is an arbitrary function, i.e., that the complete integral is the result of eliminating the indeterminate α from the three equations (*D*), (*E*), (*F*), and that in each particular case, that integral can be expressed only by the system of two simultaneous equations.

In order to verify that result by differentiation, one must remark that the differentials of the two equations (*D*), (*E*) with respect to α exist, and that one can then differentiate the two equations (*D*), (*E*) while regarding α as constant. Now, if one can perform that differentiation then one will have the following two equations:

(*d*) $$(x-\alpha)\,dx + (y-\varphi(\alpha))\,dy + z\,dz = 0 ,$$

(*e*) $$dx + \varphi'(\alpha)\,dy = 0 ,$$

and if one eliminates the three indeterminates α, $\varphi(\alpha)$, $\varphi'(\alpha)$ from the four equations (*D*), (*E*), (*d*), (*e*) then one will find that:

(*A*) $$z^2(dx^2 + dy^2 + dz^2) = a^2(dx^2 + dy^2) ,$$

so the integral that was just found is exact.

V.

EXAMPLE III. – Let the given equation be:

(*A*) $$(x\,dy - y\,dx)^2 + (y\,dz - z\,dy)^2 + (z\,dx - x\,dy)^2 = a^2(dx^2 + dy^2 + dz^2) ,$$

in which a is a given constant. If one puts that equation into the form:

$$d\cdot\sqrt{x^2 + y^2 + z^2 - a^2} = \sqrt{dx^2 + dy^2 + dz^2}$$

then it will be easy to recognize that it belongs to all of the curves of double curvature whose tangents are, at the same time, tangent to a sphere of radius a whose center is at the origin. Therefore, the equations of all of the tangents to the sphere will define a particular solution to the given equation. Now, the equations of those tangents are:

(*B*) $$\alpha x + \beta y + z\sqrt{a^2 - \alpha^2 - \beta^2} = a^2,$$

(*C*) $$x - \alpha = \gamma(y - \beta),$$

in which α, β are the coordinates of the contact point, and γ determines the direction of the tangent. Therefore, if one regards the three quantities α, β, γ in those equations as arbitrary then one will have a particular solution of the given one, and it is easy to assure oneself of that fact by differentiation.

Although the two equations (*B*), (*C*) are completed by the three arbitrary parameters, one nonetheless sees that they are not the complete integral of equation (*A*).

Equation (*B*) is that of the tangent plane to the sphere at a point of contact whose coordinates, in the sense of x and y, are α, β, respectively. If one sets $\beta = \varphi(\alpha)$ then one will find that the contact point has been placed along a certain curve whose projection into the xy-plane will have the equation $y = \varphi(x)$, and the equation of the tangent plane will become:

(*D*) $$\alpha x + y\,\varphi(\alpha) + z\sqrt{a^2 - \alpha^2 - \varphi(\alpha)^2} = a^2.$$

If one considers two consecutive tangent planes then those planes will intersect along a tangent line to the sphere, and one will have the second equation for that line upon differentiating the equation of the plane with respect to the variable parameter α. Hence, the two equations of that line will be:

(*D*) $$\alpha x + y\,\varphi(\alpha) + z\sqrt{a^2 - \alpha^2 - \varphi(\alpha)^2} = a^2,$$

(*E*) $$x + y\,\varphi'(\alpha) - z\frac{\alpha + \varphi(\alpha)\,\varphi'(\alpha)}{\sqrt{a^2 - \alpha^2 - \varphi(\alpha)^2}} = 0,$$

and since the line to which those equations belong is tangent to the sphere, it will then follow that they satisfy the given equation. However, if one considers the sequence of all planes that touch the sphere at the point along the curve, one will have a sequence of lines like the preceding one, and when those lines are taken pairwise in succession, they will intersect since they will be pairwise in the same tangent plane. Hence, they will be the tangents to the same curve of double curvature, and the equations of that curve of double curvature will be the complete integral of the given equation. Now, it is obvious that in order to get the equations of that curve of double curvature, one must differentiate equations (*D*), (*E*) of the line with respect to the parameter α. Moreover, equation (*E*) is already the differential of (*D*), when taken in the same manner, so it will suffice to differentiate (*E*): Thus, upon setting:

$$a^2 - \alpha^2 - \varphi(\alpha)^2 = \psi(\alpha)^2,$$

to abbreviate, the complete integral of the given equation will be the system of four simultaneous equations:

$$(D) \qquad \alpha x + y\,\varphi(\alpha) + z\,\psi(\alpha) = a^2,$$

$$(E) \qquad x + y\,\varphi'(\alpha) + z\,\psi'(\alpha) \quad = 0,$$

$$(F) \qquad y\,\varphi''(\alpha) + z\,\psi''(\alpha) = 0,$$

$$\alpha^2 + \varphi(\alpha)^2 + \psi(\alpha)^2 = a^2,$$

in which φ and ψ are arbitrary functions. Of those four equations, the last one is intended to eliminate the extra function, and (*E*), (*F*) are the first and second difference of (*D*), when taken while regarding α as the only variable. Once one of the functions has been eliminated, the integral will be the result of eliminating α from the three equations (*D*), (*E*), (*F*).

It is easy to verify that result by differentiation, as in the two preceding examples.

In my article on the developables of curves of double curvature (tome X of the *Savants étrangers*), I gave the name of *edge of regression* to the curve that is touched by all of the lines that constitute a developable surface. From that, the equation in question here will belong to the edge of regression of an arbitrary developable surface that is circumscribed by the sphere.

VI.

The three examples that I just referred to will suffice to show that:

1. The ordinary higher-degree difference equations in three variables that do not satisfy the integrability condition are not absurd, but they express real properties.

2. Those equations can be truly integrated, and their loci are curves of double curvature that can only be expressed by the system of two simultaneous equations, while the other equations, when their number is greater than two, are intended to eliminate the indeterminates or the extra functions.

3. The integrals of those differential equations must be completed by an arbitrary function that the geometers are allowed to have only as the integral of a partial differential equation.

Those considerations open up a new field of analysis and geometry, and they give rise to an integral calculus that deserves the attention of geometers, because we will see in what follows that the integration of higher-degree partial difference equations depend upon only that type of calculation.

I would like to present some results of very great generality.

VII.

Theorem I:

The complete integral of the ordinary difference equation in three variables:

$$(A) \qquad F\left(\frac{dx}{dz}, \frac{dy}{dz}\right) = 0 ,$$

into which the variables themselves do not enter, and in which F is an arbitrary function of two quantities, whether algebraic or transcendental and well-defined or arbitrary, is the result of eliminating the indeterminate α from the following three equations:

$$(B) \qquad F\left(\frac{x-\alpha}{z}, \frac{y-\varphi(\alpha)}{z}\right) = 0 ,$$

$$(C) \qquad \left(\frac{dF}{d\alpha}\right) = 0 ,$$

$$(D) \qquad \left(\frac{ddF}{d\alpha^2}\right) = 0 ,$$

in which F is the same function of the two quantities as the one in the given equation, and φ is an arbitrary function.

In order to prove that one must observe that the differentials of the two equations (*B*), (*C*) with respect to the indeterminate α, and that as a consequence one can differentiate those equations while regarding α as a constant, and upon setting:

$$\frac{x-\alpha}{z} = m , \qquad \frac{y-\varphi(\alpha)}{z} = n ,$$

to abbreviate, that will give:

$$\left(\frac{dF}{dm}\right) dm + \left(\frac{dF}{dn}\right) dn = 0 ,$$

$$\left[\left(\frac{ddF}{dm^2}\right) + \left(\frac{ddF}{dm\,dn}\right)\varphi'(\alpha)\right] dm + \left[\left(\frac{ddF}{dm\,dn}\right) + \left(\frac{ddF}{dn^2}\right)\varphi'(\alpha)\right] dn = 0 .$$

Now, the last two equations cannot be independently substituted for the value of the function *F*, unless one has, at the same time, the following two equations $dm = 0$, $dn = 0$, or:

$$\frac{x-\alpha}{z} = \frac{dx}{dz} ,$$

$$\frac{y-\varphi(\alpha)}{z} = \frac{dy}{dz} .$$

Hence, if one eliminates α and $\varphi(\alpha)$ from equation (B) by means of the last two equations then one will have the given equation:

$$F\left(\frac{dx}{dz}, \frac{dy}{dz}\right) = 0 .$$

VIII.

The equation $dz^2 = a^2(dx^2 + dy^2)$ of article III falls withing the purview of the preceding theorem, because it can be put into the form:

$$a^2\left(\frac{dx^2}{dz^2} + \frac{dy^2}{dz^2}\right) - 1 = 0 .$$

We have also seen that its complete integral is the result of eliminating the indeterminate α from the following equation:

$$a^2\left[\left(\frac{x-\alpha}{z}\right)^2 + \left(\frac{y-\varphi(\alpha)}{z}\right)^2\right] - 1 = 0 ,$$

along with its two first and second differentials, when taken while regarding α as the only variable.

IX.

Theorem II:

The three quantities X, Y, Z are each composed from the three variables x, y, z, so the complete integral of the ordinary first-order difference equation:

$$F\left(\frac{dx}{dz}, \frac{dy}{dz}\right) = 0$$

is the result of eliminating the indeterminate α *from the following three equations:*

$$F\left(\frac{X-\alpha}{Z}, \frac{Y-\varphi(\alpha)}{Z}\right) = 0 ,$$

$$\left(\frac{dF}{d\alpha}\right) = 0 ,$$

$$\left(\frac{ddF}{d\alpha^2}\right) = 0 ,$$

in which the function F is the same as the one in the given equation and φ is an arbitrary function.

That theorem is proved in the same way as the previous one.

X.

The equation:

$$z^2\,(dx^2 + dy^2 + dz^2) \;=\; a^2\,(dx^2 + dy^2)$$

in article IV falls within the scope of the last theorem, because one can put it into the form:

$$\frac{z^2\,dz^2}{a^2 - z^2} \;=\; dx^2 + dy^2 ,$$

or into the following one:

$$\left[d\sqrt{a^2 - z^2}\,\right]^2 \;=\; dx^2 + dy^2 .$$

We have also seen that its complete integral is the result of eliminating the indeterminate α from the equation:

$$\frac{(x - \alpha)^2}{a^2 - z^2} + \frac{(y - \varphi(\alpha))^2}{a^2 - z^2} \;=\; 1$$

and its first and second differentials when they are taken while regarding α as the only variable.

XI.

It follows from all of what was just said that if one imagines a curved surface whose equation $M = 0$ includes a variable parameter α and an arbitrary function of that parameter, which is represented by $\varphi(\alpha)$, in addition to the three coordinates x, y, z, and that if one imagines all curved surfaces that are different from the ones that are obtained by giving all possible to α in succession and supposes that the form of the function φ is invariable then any two of those surfaces, taken in succession, will intersect along a curve whose equations will be:

$$(A) \qquad M = 0 ,$$

(*B*) $$\left(\frac{dM}{d\alpha}\right) = 0\,.$$

The sequence of those curves of intersection will define a curved surface that will be the envelope of all the first ones, and one will have the finite equation of that envelope in terms of x, y, z upon eliminating the variable parameter α from the two equations (*A*), (*B*). However, that elimination is not possible, in general, because the function φ is arbitrary.

Moreover, if one considers the curves of intersection whose sequence comprises the envelope then any two of those curves, taken in succession, will intersect at a certain point whose coordinates are determined by the three equations:

(*A*) $$M = 0\,,$$

(*B*) $$\left(\frac{dM}{d\alpha}\right) = 0\,,$$

(*C*) $$\left(\frac{ddM}{d\alpha^2}\right) = 0\,.$$

The sequence of those points will comprise a curve of double curvature whose two finite equations will be obtained by eliminating the parameter α from the three equations (*A*), (*B*), (*C*). Not only does the curve of double curvature that is in question here touch all of the possible surfaces that are included in the equation $M = 0$, but also each of its elements is found on three of those surfaces, taken consecutively. Finally, that curve is the limit of the envelope.

In order to have the differential equation of the envelope that is produced by an arbitrary function φ, one must differentiate equation (*A*) with respect to x and y, while regarding α and $\varphi(\alpha)$ as constant in both cases, which is permissible because of equation (*B*). One will then have the three equations:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,,$$

$$\left(\frac{ddM}{d\alpha^2}\right) = 0\,.$$

When one eliminates the two quantities α and $\varphi(\alpha)$ from them, one will have a partial difference equation $V = 0$ that belongs to the envelope independently of the form of the functions φ, which has disappeared, i.e., which belongs to all of the envelopes that one will get by successively giving all possible forms to the function φ in $M = 0$.

As for the limit, we have already seen that in order to have its equations, we must differentiate the two equations (*A*), (*B*) by ordinary differences while regarding α and $\varphi(\alpha)$ as constants, which is permissible by virtue of equations (*B*), (*C*), and eliminate the three quantities α, $\varphi(\alpha)$, from the four equations:

$$(A) \qquad M = 0 ,$$

$$(B) \qquad \left(\frac{dM}{d\alpha}\right) = 0 ,$$

$$d\,(A) \qquad dM = 0 ,$$

$$d\,(B) \qquad d\cdot\left(\frac{dM}{d\alpha}\right) = 0 ,$$

which will produce a higher-degree first-order equation $U = 0$ in ordinary differences in three variables, and for which the integrability condition is not satisfied.

XII.

Now, the two equations:

$$V = 0 ,$$
$$U = 0 ,$$

the first of which is a partial difference equation and the second of which is an ordinary difference equation, are such that if either one of them is given then it will always be easy to get the other one without knowing their integral equations.

1. If one is given the partial difference equation $V = 0$ and one replaces p or q with the value that one takes for it in $dz = p\,dx + q\,dy$ (suppose that it is the value of p that one substitutes) then one will have an equation $V' = 0$ that is composed of the variables x, y, z, their ordinary differences dx, dy, dz, and the quantity q, and the result of eliminating the quantity q from the two equations:

$$V' = 0 ,$$
$$\left(\frac{dV'}{dq}\right) = 0$$

will give the ordinary difference equation $U = 0$.

2. Conversely, if one is given $U = 0$ then if one replaces dz with the value $p\,dx + q\,dy$, one will have an equation $U' = 0$ that is composed of the variables x, y, z, the partial differences p, q,

and the quantity dy / dx ; I shall represent the latter quantity by ω. Having done that, the result of eliminating ω from the two equations:

$$U' = 0 ,$$
$$\left(\frac{dU'}{dq}\right) = 0$$

will give the partial difference equation $V = 0$.

For example, in article III, the equation $M = 0$ is:

$$(x - \alpha)^2 + (y - \varphi(\alpha))^2 = \frac{z^2}{a^2} ,$$

and the two equations $V = 0$, $U = 0$ are:

$$p^2 + q^2 = a^2 ,$$
$$dz^2 = a^2 (dx^2 + dy^2) .$$

If the first of those two equations has been posed then in order to get the second, one must replace p with the value $\frac{dz - q\,dy}{dx}$, which will give:

$$q^2 (dx^2 + dy^2) - 2q\,dy\,dz + dz^2 - a^2 dx^2 = 0 .$$

If one differentiates the last equation while regarding q as the only variable then that will give:

$$q^2 (dx^2 + dy^2) = dy\,dz ,$$

by virtue of which, the preceding equation will become:

$$q\,dy\,dz = dz^2 - a^2\,dx^2 ,$$

and if one eliminates q from the last two then one will find the ordinary difference equation:

$$dz^2 = a^2 (dx^2 + dy^2) .$$

Conversely, if one is given the equation:

$$dz^2 = a^2 (dx^2 + dy^2)$$

then in order to find the partial difference equation, one must replace dz with the value $p\,dx + q\,dy$, and upon setting $dy / dx = \omega$, that will give:

$$\omega^2 (q^2 - a^2) + 2\omega\,p\,q + p^2 - a^2 = 0 .$$

If one differentiates that equation while regarding ω as the only variable, which will give:

$$\omega(q^2 - a^2) + p\,q = 0 ,$$
$$\omega\, p\,q + p^2 - a^2 = 0 ,$$

and if one eliminates ω then one will find the partial difference equation:

$$p^2 + q^2 = a^2 .$$

XIII.

In order to prove that proposition of the preceding article in general, I shall first point out that the result of eliminating the two quantities α, β from the three equations:

$$\varphi(\alpha, \beta) = 0 ,$$

$$\left(\frac{d \cdot \varphi(\alpha, \beta)}{d\alpha}\right) = 0 ,$$

$$\left(\frac{d \cdot \varphi(\alpha, \beta)}{d\beta}\right) = 0$$

is the same as the one that one obtains upon first eliminating α from the two equations:

$$\varphi(\alpha, \beta) = 0 ,$$

$$\left(\frac{d \cdot \varphi(\alpha, \beta)}{d\alpha}\right) = 0 ,$$

which will give a first result $\psi(\beta) = 0$, and then eliminating β from the two equations:

$$\psi(\beta) = 0 ,$$

$$d\left(\frac{\psi(\beta)}{d\beta}\right) = 0 ,$$

Having said that, the ordinary difference equation $U = 0$ is the result of eliminating the quantities α, $\varphi(\alpha)$, $\varphi'(\alpha)$ from the four equations:

$$M \quad = 0 ,$$

$$dM = 0 ,$$

$$\frac{dM}{d\alpha} = 0 ,$$

$$\frac{ddM}{d\alpha^2} = 0 ,$$

or rather, if one represents the result of eliminating $\varphi(\alpha)$ from the first two of them by $k = 0$ then the equation $U = 0$ will be the result of eliminating α from the two equations:

$$k = 0 ,$$

$$\left(\frac{dk}{d\alpha}\right) = 0 .$$

Analogously, the partial difference equation $V = 0$ is the result of eliminating the quantities α, $\varphi(\alpha)$, dz from the four equations:

$$M = 0 ,$$

$$\left(\frac{dM}{dx}\right) = 0 ,$$

$$\left(\frac{dM}{dy}\right) = 0 ,$$

$$dz = p\ dx + q\ dy ,$$

or rather, if one lets ω represent dy / dx then one will have the equation $V = 0$ upon first eliminating $\varphi(\alpha)$ and dz from the equations:

$$M = 0 ,$$

$$dM = 0 ,$$

$$dz = p\ dx + q\ dy ,$$

and then, after representing the result by $k' = 0$, eliminating α and ω from the following ones:

$$k' = 0 ,$$

$$\left(\frac{dk'}{d\alpha}\right) = 0 ,$$

$$\left(\frac{dk'}{d\omega}\right) = 0 .$$

In order to perform the last operation, by virtue of the lemma, one can first eliminate α from $k' = 0$ and $\left(\frac{dk'}{d\alpha}\right) = 0$, which will give a result $k'' = 0$, and one then eliminates ω from $k'' = 0$ and $\left(\frac{dk''}{d\omega}\right) = 0$.

Now, the equation $k' = 0$ is the result of eliminating dz from $k' = 0$ and $dz = p\,dx + q\,dy$, so the result of eliminating α from $k' = 0$ and $\left(\frac{dk'}{d\alpha}\right) = 0$ will be the same as that of eliminating dz from:

$$U = 0$$

and

$$dz = p\,dx + q\,dy .$$

One will then have the equation $V = 0$ upon first eliminating dz from the last two equations, which will give $k'' = 0$ as a result, and eliminating ω from the following two:

$$k'' = 0 ,$$
$$\left(\frac{dk''}{d\omega}\right) = 0 ,$$

which is first part of the proposition.

As for the second part, one must observe that since the equation $V = 0$ results from the following two:

$$U = 0 ,$$

$$dz = p\,dx + q\,dy ,$$

conversely, the equation $U = 0$ must result from these equations:

$$V = 0 ,$$

$$dz = p\,dx + q\,dy$$

upon eliminating the two quantities p, q. Now, if the aforementioned elimination of one of those two quantities has been performed, which will give a result that I shall represent by $h = 0$, then no equation will remain for one to eliminate the other of those quantities. Hence, if it is q that remains then in order to make it disappear, one must eliminate it from the two equations:

$$h = 0\,,$$

$$\left(\frac{dh}{dq}\right) = 0\,,$$

which is the second part of the proposition.

XIV.

We have seen (XII) that the integral of the partial difference equation $V = 0$ is the result of eliminating the indeterminate α from the two equations:

$$(A) \qquad M = 0\,,$$

$$(B) \qquad \left(\frac{dM}{d\alpha}\right) = 0\,,$$

and that the higher-degree ordinary difference equation $U = 0$ is the result of eliminating the same indeterminate α from the following three:

$$(A) \qquad M = 0\,,$$

$$(B) \qquad \left(\frac{dM}{d\alpha}\right) = 0\,,$$

$$(C) \qquad \left(\frac{ddM}{d\alpha^2}\right) = 0\,.$$

It follows from this that of the two equations $V = 0$, $U = 0$, if either one is given and one knows the integral of the other in the form that I just discussed then one will also know that of the first. That is, if one knows the integral of the equation $U = 0$ and that the integral is in the form of the three equations (A), (B), (C) then one will have the integral of the equation $V = 0$ upon suppressing the equation $V = 0$. Conversely, if one knows the integral of the equation $V = 0$ in the form of two equations (A), (B) then one will have that of the equation $U = 0$ upon combining the two equations (A), (B) with the differential of (B) that one takes while regarding the indeterminate α as the only variable.

Therefore, the integral calculus of the higher-degree ordinary difference equations, as well as that of partial difference equations, are absolutely dependent upon each other, and perfecting one of those types of calculations will necessarily follow from that of the other.

XV.

All of the foregoing will present only a useless sphere of ideas if the forms of the equations one must integrate correspond in one and the other calculus. However, I would like to show by two examples that certain ordinary difference equations that are included within the forms that I treated above correspond to the partial difference equations that one cannot integrate by any other method, and conversely.

Example I:

The ordinary difference equation:

$$(U) \qquad (x\,dy - y\,dx)^2 + (y\,dz - z\,dy)^2 + (z\,dx - x\,dz)^2 = a^2\,(dx^2 + dy^2 + dz^2)\,,$$

whose integral I found by geometric considerations, and which belongs to the edge of regression of all developable surfaces that are circumscribed on the same sphere, is not included in any of the general forms that I have given an integral form. However, if one replaces dz with the value $p\,dx + q\,dy$ and then eliminates dy / dx by means of the differential that is taken while regarding dy / dx as the only variable then the partial difference equation that one will obtain will be:

$$z - p\,x - q\,y = a^2 \sqrt{1 + p^2 + q^2}\,,$$

which will belong to all of the developable surfaces that are circumscribed on the same sphere. Now, that equation is included in the ones that Lagrange has integrated, and upon setting:

$$z - \alpha\,x - \varphi(\alpha) \cdot y - a^2 \sqrt{1 + p^2 + q^2} = M\,,$$

to abbreviate, its integral will be the result of eliminating α from the following two equations:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,.$$

Hence, the integral of the ordinary difference equation (U) will be the result of eliminating the same indeterminate α from the three equations:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,,$$

$$\left(\frac{ddM}{d\alpha^2}\right) = 0 .$$

Example II:

Conversely, the partial difference equation:

$$\text{(V)} \qquad b\,x^2\,(z+p\,x-q\,y)^2 + a\,b\,y^2\,(z-p\,x+q\,y)^2 + a\,z\,(z+p\,x-q\,y)^2 = 0$$

cannot be integrated by any known method. However, if one replaces *p* with its value that one takes in $dz = p\,dx + q\,dy$, and one eliminates *q* after differentiating it while regarding *q* as the only variable then one will have:

$$(x\,dz + z\,dx)^2 + a\,(z\,dy + y\,dz)^2 + b\,(x\,dy + y\,dx)^2 = 0 ,$$

which is a higher-degree ordinary difference equation that is included in the case of Theorem II, and upon setting:

$$\left(\frac{x\,z-\alpha}{x\,y}\right)^2 + a\left(\frac{y\,z-\varphi(\alpha)}{x\,y}\right)^2 + b = M ,$$

to abbreviate, it integral will be the result of eliminating α from the three equations:

$$M = 0 ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0 ,$$

$$\left(\frac{ddM}{d\alpha^2}\right) = 0 .$$

Therefore, the integral of the partial difference equation (*V*) is the result of eliminating the indeterminate α from just the first two of those three equations, i.e., from:

$$M = 0$$

and

$$\left(\frac{dM}{d\alpha}\right) = 0 .$$

XVI.

Up to now in this article, only the higher-degree ordinary difference equations have been in question. However, the linear ordinary difference equations in three variables that do not satisfy the old integrability condition are not absurd, since they likewise belong to all curves of double curvature, and they are susceptible to a true integration. Finally, the search for their integrals depends upon only the integration of an ordinary difference equation in two variables.

Indeed, let:

$$L\,dx + M\,dy + N\,dz = 0 \tag{A}$$

be a linear ordinary difference equations that does not satisfy the integrability condition. From what we just saw in regard to higher-degree equations, one replaces dz with its value $p\,dx + q\,dy$, which will give:

$$(L + N\,p)\,dx + (M + N\,q)\,dy = 0 .$$

Then, after differentiating that equation while varying only dy / dz, one eliminates dy / dz, which will come down to equating the coefficients of dx and dy to zero, and one will have the two simultaneous equations:

$$L + N\,p = 0 , \tag{B}$$

$$M + N\,q = 0 . \tag{C}$$

Having done that, one integrates one or the other of the two equations while regarding the variable that does not vary in the partial difference as constant. For example, one integrates the first one:

$$L\,dx + N\,dz = 0$$

while regarding y as constant and completes the integral with an arbitrary function of y. Finally, one replaces q in (C) with its value that one infers from the integral, which will produce a second equation without any differentials, and those two equations will belong to the curve of double curvature that is the locus of the given equation.

XVII.

Example I:

Let the problem be that of integrating the equation:

$$dz = x\,y\,(x\,dx + y\,dy) . \tag{A}$$

In this case, the two partial difference equations will be:

(*B*) $$p = x^2 y\,,$$

(*C*) $$q = x y^2\,,$$

and the integral of the first one is:

$$z = \tfrac{1}{3} x^3 y + \varphi(y)\,,$$

in which φ is an arbitrary function. I shall differentiate it while regarding y alone as variable, which will give:

$$q = \tfrac{1}{3} x^3 + \varphi'(y)\,,$$

and if I replace q with its value in (*C*) then I will find that:

$$x y^2 = \tfrac{1}{3} x^3 + \varphi'(y)\,.$$

Hence, the integral of the given equation is the system of two simultaneous equations:

$$z = \tfrac{1}{3} x^3 y + \varphi(y)\,,$$
$$x y^2 = \tfrac{1}{3} x^3 + \varphi'(y)\,.$$

That integral can be put into another form, because the two equations (*B*), (*C*) can be replaced by the following:

$$p y - q x = 0\,,$$
$$p = x^2 y\,.$$

Now, the integral of the first one is $z = \varphi(x^2 + y^2)$, and in order for the second one to be satisfied, it is necessary that one must have $x y = z\,\varphi'(x^2 + y^2)$. Therefore, the complete integral of the given equation is once more the system of two simultaneous equations:

$$z = \varphi(x^2 + y^2)\,,$$
$$x y = z\,\varphi'(x^2 + y^2)\,,$$

which is easy to verify by differentiation. Therefore, the given equation belong to a curve of double curvature is traced on an arbitrary surface of revolution whose axis coincides with the *z*-line. However, the projection of that curve onto the plane perpendicular to the axis will depend upon the generating curve in a manner that is asserted by the second of the two integral equations.

XVIII.

Example II:

Let the problem be that of integrating:

$$(A) \qquad dz = x\,y\,(dx - dy)\,.$$

The two partial difference equations become:

$$(B) \qquad p = \quad x\,y\,,$$

$$(C) \qquad q = -\,x\,y\,.$$

The integral of the first one is $z = \frac{1}{2}x^2 y + \varphi(y)$, and in order for equation (C) to be satisfied, it is necessary that one must have:

$$x\,y + \tfrac{1}{2}x^2 + \varphi'(y) = 0\,.$$

Hence, the integral of equation (A) is the system of two simultaneous equations:

$$z = \tfrac{1}{2}x^2 y + \varphi(y)\,,$$

$$x\,y + \tfrac{1}{2}x^2 + \varphi'(y) = 0\,.$$

In other words, the two equations (B), (C) can be replaced by the following two:

$$\begin{aligned} p + q &= 0\,, \\ p &= x\,y\,. \end{aligned}$$

The integral of the first one is:

$$z = \varphi\,(x - y)\,,$$

and in order for the second one to be satisfied, it is necessary for one to have:

$$x\,y = \varphi'\,(x - y)\,.$$

Hence, the complete integral of the given equation will again be the system of two simultaneous equations:

$$\begin{aligned} z &= \varphi\,(x - y)\,, \\ x\,y &= \varphi'\,(x - y)\,, \end{aligned}$$

i.e., the given equation (A) belongs to a curve of double curvature that is traced on a cylindrical surface with an arbitrary base such that the lines of the surface are parallel to the line that divides the angle between the x and y axes into two equal parts. However, the projection of the curve onto the xy-plane will depend upon the base of the cylindrical surface in a manner that is expressed by the second of the two integral equations.

XIX.

If the number of variables is greater than three then one proceeds in an analogous manner, i.e., one first replaces dz with its value:

$$p\,du + q\,dx + r\,dy + \ldots$$

and then differentiates the result while regarding each of those quantities as the only variable in order to eliminate dx / du, dy / du, …, which will produce a number of equations that is sufficient for that elimination. One will then have partial difference equations that one treats as in the two preceding cases.

Example III:

Let the problem be that of integrating:

$$u\,du + y\,dx + z\,dy + x\,dz = 0 .$$

I replace dz with its value and find that:

$$(u + p\,x)\,du + (y + q\,x)\,dx + (z + r\,x)\,dy = 0 .$$

I differentiate that while regarding dx / du, dy / du as the only variable in each particular case, and then eliminate those two quantities. In the case where all of the differences are linear, that will reduce to equating the coefficients of du, dx, dy to zero, and I will obtain the three partial difference equations:

$$u + p\,x = 0 ,$$
$$y + q\,x = 0 ,$$
$$z + r\,x = 0 .$$

The integral of the first one is:

$$x\,z + \tfrac{1}{2}u^2 = \varphi(x, y) ,$$

in which φ is a function of two quantities, and in order for the other two equations to be satisfied, it is necessary that one must have:

$$y + \varphi'(x, y) = 0 ,$$
$$z + \varphi''(x, y) = 0 ,$$

in which φ' and φ'' are the coefficients of dx and dy in the differential of the function φ. Therefore, the complete integral of the given equation will be the system of three simultaneous equations:

$$xz + \tfrac{1}{2}u^2 = \varphi(x, y) ,$$
$$z - y = \varphi'(x, y) ,$$
$$-z = \varphi''(x, y) .$$

XX.

The number of integral equations is not always equal to the number of variables, reduced by one unit, as in the preceding case.

Example IV:

Let the given equation be:

$$(A) \qquad u\,du + x\,dx + x\,dy + z\,dz = 0 .$$

One first sees that one can reduce to an expression in three terms with the following form:

$$u\,du + x\,(dx + dy) + z\,dz = 0 ,$$

whose integral is obviously the system of two simultaneous equations:

$$u^2 + z^2 + \varphi(x + y) = 0 ,$$
$$2x' - \varphi'(x + y) = 0 .$$

Therefore, whenever the given equation is capable of being reduced to three terms, its integral will contain no more than two equations. However, in any case where it is capable of being given that form, it will not always be as easy to reduce it as in that simple example. Hence, upon proceeding as in article XX, the calculations will indicate the reduction of the equations. Indeed, if one sets dz equal to its value $p\,du + q\,dx + r\,dy$ in the given equation then one will have the three partial difference equations:

$$u + p\,z = 0 ,$$
$$x + q\,z = 0 ,$$
$$x + r\,z = 0 .$$

The integral of the first one is:

$$(E) \qquad z^2 + u^2 = \varphi(x, y) ,$$

in which φ is supposed to be a function of two quantities, and in order for the other two to be satisfied, it is necessary that one must have:

$$(F) \qquad -2x = \varphi'(x, y),$$
$$(G) \qquad -2x = \varphi''(x, y),$$

in which φ' and φ'' are coefficients of dx and dy in the differential of $\varphi(x, y)$.

However, since the two equations (F), (G) are in terms of the same two variables x and y, it will follow that the function φ, which was regarded as composed of two quantities, is composed of only one. That is because if one sets:

$$\varphi(x, y) = z' \quad \text{and} \quad dz' = p'\,dx + q'\,dy$$

then equations (F), (G) will become:

$$(F') \qquad -2x = p',$$

$$(G') \qquad -2x = q',$$

which will give $p' - q' = 0$, and consequently:

$$z' = \psi(x + y),$$

in which ψ is an arbitrary function of only the quantity $x + y$. Therefore, equation (E) will become:

$$z^2 + u^2 = \psi(x + y),$$

and one of the two equations (F') or (G') will be employed. In order for the other one to be satisfied, it is necessary that one must have:

$$-2x = \psi'(x + y).$$

Therefore, the integral of the given equation (A) is the system of the last two equations, taken simultaneously, which is what we found originally.

Conclusion.

The ordinary difference equations, whether higher-degree or linear, that do not satisfy the integrability conditions are not entirely absurd if one intends that to means that they express properties that are impossible, imaginary, etc. They all assert real properties, and they are capable of being truly integrated into finite quantities. What is absurd is that their integrals can be expressed by just one equation. For example, in the case of three variables, it is absurd that the equation might belong to a curved surface, as one tacitly supposes. In that case, it will belong to a curve of double curvature, which can be determined by just one differential equation, but which cannot be expressed in terms of finite quantities, as in the case of the system of two simultaneous equations.

On the higher-degree second-order ordinary difference equations, and for a number of variables that is greater than two.

XXI.

In a paper that I presented on the developments of curves of double curvature that was printed among the *Mémoires* of the Savans étrangers (tome X), I showed that:

1. Although every curve has an infinite number of developments, it will nonetheless have just one radius of curvature at each of its points.

2. When the curve has double curvature, the series of centers of curvature will form a curved line that is not one of the developments.

3. If x, y, z are the rectangular coordinates of a point on the curve then the expression for the radius of curvature at that point will be:

$$(dx^2 + dy^2 + dz^2)^3 : (dx\,ddy - dy\,ddx)^2 + (dz\,ddx - dx\,ddz)^2 + (dy\,ddz - dz\,ddy)^2$$

when one knows that no second difference is zero.

If one would then like to have the differential equation for all curves of double curvature whose radius of curvature is constant then one must equate the preceding expression to a constant a^2, which will give:

$$(A) \qquad (dx^2 + dy^2 + dz^2)^3 = a^2[(dx\,ddy - dy\,ddx)^2 + (dz\,ddx - dx\,ddz)^2 + (dy\,ddz - dz\,ddy)^2] \ ,$$

which is a higher-degree equation of the type that one ordinarily regards as absurd, but nonetheless expresses a real property, and it admits a true integration, even in terms of finite quantities.

XIII.

It is initially obvious that all of the circles whose radius is a, no matter what their positions in space might be, must satisfy that equation. Now, the equations of a circle that is placed in space in an arbitrary manner are:

$$(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2 = a^2 ,$$
$$x - \alpha + (y-\beta)\,\varepsilon + (z-\gamma)\,\kappa = 0 ,$$

in which α, β, γ are the coordinates of the center, and ε, κ determine the two directions of the plane of the circle. Hence, if one regards the five quantities α, β, γ, ε, κ in those two equations as arbitrary constants then they will satisfy the second-order ordinary difference equation (*A*), which is easy to verify by differentiation. That is because if one differentiates each of those equations by ordinary first and second differences then one will have six equations in all, and when one eliminates the five arbitrary constants from them, one will find equation (*A*). However, although the equations of the circle contain five arbitrary parameters, they once more define only one particular case of the complete integral, because the angles that the consecutive elements of the circumference of the circle make between them are equal and in the same plane, and one can imagine a curve for which those angles are in planes that are perpetually different without ceasing to be equal: The equations of that curve will satisfy equation (*A*), since its radius of curvature is constant, and they will not be included in the equation of a circle, since the curve has double curvature.

If the three rectangular coordinates of the curve that one considers are x, y, z then let α, β, γ be the respective coordinates of the center of curvature that correspond to that point. Furthermore, let $\beta = \varphi(\alpha)$, $\gamma = \psi(\alpha)$ be the equations of the curve that passes through all centers of curvature, where φ and ψ are two functions of α. Having done that, the length of a radius of curvature must be equal to a constant α, so one must first have:

$$(B) \qquad (x-\alpha)^2 + (y-\varphi(\alpha))^2 + (z-\psi(\alpha))^2 = a .$$

Thus, for any curve of double curvature, the distances from any arbitrary center of curvature to three consecutive points on the curve, which it is the center of, will always be equal. Therefore, the distance from the curve to the corresponding center of curvature will not change when one varies the ordinate α twice in succession. Hence, the first and second differences of equation (*B*), when taken while regarding α as the only variable, must exist. Thus, one will once more have:

$$(C) \qquad x-\alpha-(y-\varphi(\alpha))\,\varphi'(\alpha)+(z-\psi(\alpha))\,\psi'(\alpha) = 0 ,$$

$$(D) \qquad -1-(\varphi'(\alpha))^2-(\psi'(\alpha))^2+(y-\alpha)\,\varphi''(\alpha)+(y-\psi(\alpha))\,\psi''(\alpha) = 0 .$$

The consideration that just provided the last equation is general, and it belongs to all curves of double curvature. However, the curve in question has the peculiarity that its radius of curvature is constant, so the distance from the same center of curvature to four consecutive points of the curve will always be the same. Therefore, once again, the distance from the point of the curve to the center of curvature cannot change when one varies the ordinate a third time. Hence, the third difference of equation (*B*), when taken while regarding α as the only variable, must once more exist. Thus, one will have the fourth equation:

$$(E) \qquad -3\,\varphi'(\alpha)\,\varphi''(\alpha)-3\psi'(\alpha)\,\psi''(\alpha)+(y-\alpha)\,\varphi'''(\alpha)+(y-\psi(\alpha))\,\psi'''(\alpha) = 0 .$$

Therefore, upon representing equation (B) by $M = 0$, the system of four simultaneous equations:

$$(B) \qquad M = 0\,,$$

$$(C) \qquad \left(\frac{dM}{d\alpha}\right) = 0\,,$$

$$(D) \qquad \left(\frac{ddM}{d\alpha^2}\right) = 0\,,$$

$$(E) \qquad \left(\frac{d^3M}{d\alpha^3}\right) = 0$$

will belong to the curve whose radius of curvature is constant and equal to α, and they will express the same thing as the second-order ordinary difference equation (A).

The four equations that I just found will be the finite integral of equation (A) if their number does not exceed the number of variables x, y, z. However, one can eliminate x, y, z from those four equations, and what will remain is one equation between α, $\varphi(\alpha)$, $\psi(\alpha)$, which is an equation of condition that must be satisfied in order for the three equations employed in the elimination to be the required integral.

At the very least, it results from this that the functions φ and ψ must not both be arbitrary, and that if one of the two is taken at random then the form of the other one must follow in order for the curve whose equations are $\beta = \varphi(\alpha)$, $\gamma = \psi(\alpha)$ to pass through the centers of curvature of a curve whose curvature is constant. Moreover, if one eliminates the coordinates x, y, z then the resultant equation in terms of α, $\varphi(\alpha)$, $\psi(\alpha)$ will be precisely the same equation as equation (A) in terms of x, y, z, which amounts to a remarkable property of the curves of constant curvature that we shall explain shortly.

XXII.

Although the four equations (B), (C), (D), (E) represent only a differential result, meanwhile, their consideration will lead to the first integral of equation (A). Indeed, if one differentiates the first two of those four equations while regarding α as constant, which is permissible by virtue of the second and third ones, and then eliminates the functions φ, φ' from the four equations (B), (C), d (B), d (C) then one will have the two equations:

$$(b) \qquad (x-\alpha)^2(dy^2+dx^2)+2(x-\alpha)(z-\psi(\alpha))\,dx\,dz+(z-\psi(\alpha))^2(dy^2+dz^2) = a^2\,dy^2\,,$$

$$(c) \qquad (x-\alpha)(dy^2+dx^2)+[(x-\alpha)\,\psi'(\alpha)+(z-\psi(\alpha))]\,dx\,dz+(z-\psi(\alpha))\,\psi'(\alpha)\,(dy^2+dz^2) = 0\,.$$

Then, after taking the value of $\varphi'(\alpha)$ in the equation $d \cdot (B)$, which is $\varphi'(\alpha) = -\dfrac{\psi'(\alpha)\,dz + dx}{dy}$, if one differentiates it while regarding α as constant, which is again permissible, by virtue of equation (E), then one will find:

$$\varphi''(\alpha) = -\frac{dz}{dy}\,\psi''(\alpha) .$$

Finally, upon substituting that value $\varphi''(\alpha)$ in (D), one will have the third equation:

$$(d) \qquad \begin{aligned} &-(dx^2+dy^2)-(dy^2+dz^2)\,\psi'(\alpha)^2-2\,dx\,dz\,\psi'(\alpha)+(x-\alpha)\,\psi''(\alpha)\,dx\,dz \\ &+(z-\psi(\alpha))\,\psi''(\alpha)\,(dy^2+dz^2)=0. \end{aligned}$$

The three equations (b), (c), (d) are the result of eliminating the function φ and its differences from the four equations (B), (C), (D), (E), and those of their differentials that one takes while regarding α as constant.

Now, if one represents equation (b) by $N = 0$ then equations (c), (d) will be $\left(\dfrac{dN}{d\alpha}\right) = 0$, $\left(\dfrac{ddN}{d\alpha^2}\right) = 0$, which is easy to see by inspection. Therefore, one of the integrals, which is the first and complete integral of the second-order ordinary difference equation (A), is the result of eliminating α from the three equations:

$$(b) \qquad N = 0 ,$$

$$(c) \qquad \left(\frac{dN}{d\alpha}\right) = 0 ,$$

$$(d) \qquad \left(\frac{ddN}{d\alpha^2}\right) = 0 .$$

It is easy to verify that result by the differentiation because if one differentiates the first two of those equations by ordinary differences, while regarding α as constant, which is permissible by virtue of the second and the third, as well as the fact that upon eliminating the three quantities α, $\psi(\alpha)$, $\psi'(\alpha)$ from the four equations (b), (c), d (b), d (c), the result of that elimination will be equation (A) of article XXII.

XXIV.

If, rather than eliminating the function φ and its differentials φ', φ'' from equations (B), (C), d (B), d (C), one eliminates the function ψ and the differentials ψ', ψ'', then upon setting:

$$(x-\alpha)^2(dz^2+dx^2)+2\,(x-\alpha)(y-\varphi(\alpha))\,dx\,dy+(y-\varphi(\alpha))^2(dy^2+dz^2)-a^2dz^2 \;=\; N^2\,,$$

to abbreviate, one will find that the other complete first integral of equation (*A*) is the result of eliminating α from the three equations:

$$(b') \qquad N' = 0\,,$$

$$(c') \qquad \left(\frac{dN'}{d\alpha}\right) = 0\,,$$

$$(d') \qquad \left(\frac{ddN'}{d\alpha^2}\right) = 0\,,$$

which is an integral that is completed by the arbitrary function φ, just as the other was completed by the function ψ, and which one can likewise verify by differentiation.

XXV.

Now that we have the two first and complete integrals of equation (*A*), it is easy to find its finite integral. In order to do that, we must observe that if it is possible to eliminate α from the three equations of the first integral, which will produce two equations without α, and to then eliminate α once more from the three equations of the second integral, which will produce two other equations in α, then we will have four equations in α, which will contain the differential quantities dx / dy, dy / dz, and the two arbitrary functions φ, ψ, and if we eliminate the two quantities dx / dz, dy / dz from those four equations then the two resulting equations will be the finite integral of equation (*A*). However, although the eliminations in question here cannot be performed in general, we can nonetheless indicate what they are. Moreover, it is necessary to point out that α must be first eliminated from the equations (*b*), (*c*), (*d*), when taken in particular, and then from the other three (b'), (c'), (d'), which are likewise taken in particular, and before combining those six equations, we must put a prime on α in one of the systems; the second one, for example. After doing that, if we set:

$$(x-\alpha)\,(dy^2+dx^2)+2\,(x-\alpha)(z-\psi(\alpha))\,dx\,dz+(z-\psi(\alpha))\,(dy^2+dz^2)-a^2dy^2 \;=N\,,$$

$$(x-\alpha')\,(dz^2+dx^2)+2\,(x-\alpha')(y-\varphi(\alpha'))\,dx\,dy+(z-\varphi(\alpha'))\,(dy^2+dz^2)-\alpha'^2dz^2 \;=L\,,$$

to abbreviate, then the finite and complete integral of equation (*A*) will be the result of eliminating the four quantities α, α', dx / dz, dy / dz from the following six equations:

$$L = 0\,, \qquad N = 0\,,$$

$$\left(\frac{dL}{d\alpha'}\right) = 0\,, \quad \left(\frac{dN}{d\alpha}\right) = 0\,,$$

$$\left(\frac{ddL}{d\alpha'^2}\right) = 0\,, \quad \left(\frac{ddN}{d\alpha^2}\right) = 0\,,$$

and because the two quantities dx / dz, dy / dz are not included among the arbitrary functions, the present elimination of those quantities will be possible. Having performed that elimination, what will remain are four equations without differentials, and the finite equation of equation (A) will be the result of eliminating the two indeterminates α, α' from those four equations, which I shall not discuss because it would involve too much development.

One proceeds analogously in order to get the finite integral of an ordinary difference equation of arbitrary degree whenever one has all of its complete first integrals.

One then sees that not only is the ordinary second difference equation (A), which does not satisfy the integrability conditions, not absurd and that it expresses a real property, since it belongs to all curves of double curvature whose radius of curvature is constant and equal to a, but also that the equation is susceptible to two true integrations by first differences and an integration in terms of finite quantities. Finally, one sees that the two first integrals are each completed by an arbitrary particular function and that its finite integral is completed by those two functions.

XXVI.

Before leaving that example behind, I shall mention some properties of the curve that it addresses, not so much because they are very remarkable, as because they will give an idea of the results to which the consideration of higher-degree ordinary difference equations can lead.

1. In article XXIII, we saw that if we know:

$$(x - \alpha)^2 + (y - \varphi(\alpha))^2 + (z - \psi(\alpha))^2 - a^2 = M\,,$$

to abbreviate, then the system of four equations:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,,$$

$$\left(\frac{ddM}{d\alpha^2}\right) = 0\,,$$

$$\left(\frac{d^3M}{d\alpha^3}\right) = 0$$

will belong to the curve whose curvature is constant. However, that system is not an integral, because upon eliminating the three coordinates x, y, z from those four equations, one will arrive at a condition equation in terms of α, $\varphi(\alpha)$, $\psi(\alpha)$ that is the same as the ordinary second difference equation in terms of x, y, z. It then results from this that the curve whose coordinates are α, $\varphi(\alpha)$, $\psi(\alpha)$ will also have constant curvature, and because the four equations express the idea that four consecutive point on the second curve will be at equal distance from the same corresponding point on the first one, it will follow that the radius of curvature of the second curvature will have the same magnitude and position as that of the first one.

Hence, *when the curvature of a curve is constant, that curve and the one that passes through its centers of curvature will be reciprocal, such that one of them will be reciprocal to the line of centers of curvature of the other one.*

One can arrive at that result by another consideration, because I showed in the article on developments (*Savans étranger*, tome X) that if the equations of a curve of double curvature are $y = \varphi(x)$, $z = \psi(x)$, and α, β, γ are the coordinates of the curve that passes through the centers then in order to have the equations of the latter in terms of α, β, γ, if one sets:

$$(\alpha - x)^2 + (\beta - \varphi(x))^2 + (\gamma - y(x))^2 - a^2 = M ,$$

to abbreviate, then one must eliminate x from the three equations:

$$\left(\frac{dM}{dx}\right) = 0 ,$$

$$\left(\frac{ddM}{dx^2}\right) = 0 ,$$

$$\left(\frac{d^3M}{dx^3}\right) = 0 .$$

Hence, if the radius of curvature of the first curve is constant and equal to a then it will be further necessary that one must have $M = 0$. Hence, if a curve of constant curvature is given then the equations of the line of its centers must satisfy the four equations:

$$M = 0 , \quad \left(\frac{dM}{dx}\right) = 0 , \quad \left(\frac{ddM}{dx^2}\right) = 0 , \quad \left(\frac{d^3M}{dx^3}\right) = 0 .$$

However, we have seen in article XXIII that if the line of centers is given then the equations of the curve of constant curvature satisfy the four equations:

$$M = 0 , \quad \left(\frac{dM}{d\alpha}\right) = 0 , \quad \left(\frac{ddM}{d\alpha^2}\right) = 0 , \quad \left(\frac{d^3M}{d\alpha^3}\right) = 0 .$$

Furthermore, the x, y, z will enter into M in the same way as the α, β, γ. Hence, the curve of constant curvature and the one that passes through its centers of curvature are deduced from each other in the same manner, and will consequently define reciprocal curves, as I asserted above.

2. The element of the curve that passes through the centers of curvature of another curve always lies in the plane normal to the second, because two consecutive centers of curvature are always on the same normal plane, but when the curvature of a curve is constant, the element of its line of centers will be perpendicular to the common radius of curvature, moreover. Hence, the corresponding tangents to the two curves will always be at the same distance and lie in the rectangular planes.

Therefore, *when the curvature of a curve is constant, that curve and the one that passes through its centers of curvature are everywhere at the same distance from each other. They perpetually intertwine while always presenting their concavities a bit like the strands of a string with two strands, and their tangents at the extremities of the common radius of curvature are in rectangular planes.*

The thread of a screw with a circular base is obviously a curve of constant curvature, and its equations satisfy equation (*A*). The line of centers of that curve is the thread of another screw with the same pitch and the same axis but is traced on a cylinder whose diameter will be greater or lesser in magnitude than that of the first one according to whether the constant inclination of the tangent to the first one is more or less than 45 degrees. Furthermore, the diameter of the cylinder of the second curve is such that for the same pitch, the inclinations of the tangents to the two curves will be complements to each other.

Finally, the entire world knows about the screw with two equidistant threads. One frequently employs them in the arts, and mainly in the pendulum of currencies. When the tangents to those threads make angles of 45 degrees with the plane of the base of the cylinder, those two curves are mutually the lines of centers of each other.

XXVII.

Theorem I:

If the difference dx of the principal variable is constant and one has an ordinary second difference equation in three variables:

$$(A) \qquad F\left(\frac{ddy}{dx^2}, \frac{ddz}{dx^2}\right) = 0\,,$$

into which only the second differences and the constant first difference dx will enter. The first integral of that equation will be the result of eliminating the indeterminate α *from the following three equations:*

$$(B) \qquad F\left(\frac{dy - \alpha\, dx}{x\,dx}, \frac{dz - \varphi(\alpha)\cdot dx}{x\,dx}\right) = 0\,,$$

$$(C) \qquad \left(\frac{dF}{d\alpha}\right) = 0\,,$$

$$(D) \qquad \left(\frac{ddF}{d\alpha^2}\right) = 0\,,$$

in which the function F is the same as the given one, and in which φ *is an arbitrary function.*

In order to prove that one sets:

$$\frac{dy - \alpha\, dx}{x\,dx} = u\,, \qquad \frac{dz - \varphi(\alpha)\cdot dx}{x\,dx} = v\,,$$

to abbreviate, so the first two integral equations (*B*), (*C*) will become:

$$(B') \qquad F(u, v) = 0\,,$$

$$(C') \qquad \left(\frac{dF}{du}\right) + \left(\frac{dF}{dv}\right)\varphi'(\alpha) = 0\,.$$

If one differentiates the last two equations while regarding α as constant, which is permissible by virtue of the two equations (*C*), (*D*), then the differentials will both have the form:

$$M\,du + N\,dv = 0\,.$$

They cannot persist simultaneously and independently of the form of the function *F* unless one does not have, at the same time, $du = 0$, $dv = 0$, or upon developing the values of *du* and *dv*, unless one does not have:

$$\frac{dy - \alpha\, dx}{x\,dx} = \frac{ddy}{dx^2}\,,$$

$$\frac{dz - \varphi(\alpha)\cdot dx}{x\,dx} = \frac{ddz}{dx^2} .$$

Now, if one eliminates the quantities α, $\varphi(\alpha)$ from equation (B) by means of the last two equations then one will have the given equation (A). (Therefore, etc.)

XXVIII.

Theorem II:

Suppose that the difference dx of the principal variable is always regarded as constant and the quantities Y, Z are each composed in an arbitrary manner in terms of x, y, z, and their first differences. If one has an ordinary second difference equation:

$$F\left(\frac{dY}{dx^2}, \frac{dZ}{dx^2}\right) = 0 ,$$

into which only the differences of the quantities x, Y, Z enter, then the first integral of that equation will be the result of eliminating of α from the three equations:

$$F\left(\frac{Y - \alpha\ dx}{x\,dx}, \frac{Z - \varphi(\alpha)\cdot dx}{x\,dx}\right) = 0 ,$$

$$\left(\frac{dF}{d\alpha}\right) = 0 ,$$

$$\left(\frac{ddF}{d\alpha^2}\right) = 0 ,$$

in which F is the function in the given equation and φ is an arbitrary function.

That theorem is a consequence of the preceding one.

XXIX.

Theorem III:

Let u, x, y, z, … be any number of variables, among which, u is the principal variable whose first difference du is regarded as constant. Furthermore, suppose that the three quantities X, Y, Z are composed of all the variables and their first differences in an arbitrary way. If one has an ordinary second difference equation:

$$F\left(\frac{dY}{dX}, \frac{dZ}{dX}\right) = 0$$

that includes only the differences of the quantities X, Y, Z then one of the first integrals of that equation will be the result of eliminating the indeterminate α from the three equations:

$$F\left(\frac{Y-\alpha\, dx}{X}, \frac{Z-\varphi(\alpha)\cdot dx}{X}\right) = 0 ,$$

$$\left(\frac{dF}{d\alpha}\right) = 0 ,$$

$$\left(\frac{ddF}{d\alpha^2}\right) = 0 .$$

XXX.

Theorem IV:

Suppose that one has any number of variables, among which u is the principal variable, and its first difference du is regarded as constant. Suppose, furthermore, that the three quantities U, V, W are composed of all the variables in an arbitrary manner. If one has the ordinary second difference equation:

$$F\left(\frac{ddU}{ddW}, \frac{ddV}{ddW}\right) = 0$$

then, in addition to the first integral that is deduced from the previous theorem, that equation will have yet another one that will be the result of eliminating the indeterminate α' from the following three equations:

$$F\left[\frac{u\,dU-(U-\alpha')\,du}{u\,dW-W\,du}, \frac{u\,dV-(V-\varphi(\alpha'))\,du}{u\,dW-W\,du}\right] = 0 ,$$

$$\left(\frac{dF}{d\alpha'}\right) = 0 ,$$

$$\left(\frac{ddF}{d\alpha'^2}\right) = 0 .$$

The last two theorems are proved just like the first one.

Conclusion.

One then sees that:

1. The ordinary second difference equations, which do not satisfy the old integrability conditions, as well as ones of higher orders, by an obvious analogy, say nothing absurd. They are then susceptible to a true integration, and their integrals are completed by arbitrary functions.

2. Those of the equations that include only three variables belong to the curves of double curvature, which express their manner of generation, and when that manner of generation depends upon not only some points that are given at will, but on curves that are taken arbitrarily, the integrals of those differential equations will be completed by arbitrary functions.

XXXI.

On first-order partial difference equations.

We have already seen that if we are given a first-order partial difference equation in three variables that is represented by $V = 0$ then if we replace p or q in that equation – p, for example – with the value that we take in the equation $dz = p\,dx + q\,dy$ and eliminate q from the result by means of its differential that is taken while regarding only q as variable then we will have an ordinary difference equation $U = 0$ that will generally have a higher degree. We saw, analogously, that if the complete integral of the ordinary difference equation is the result of eliminating α from the three equations:

$$M = 0 ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0 ,$$

$$\left(\frac{ddM}{d\alpha^2}\right) = 0 .$$

Since M is found by integration and contains an arbitrary function of α, the complete integral of the partial difference equation $V = 0$ will be the result of eliminating α from the first two of those integral equations, in such a way that the perfection of the integral calculus of partial difference equations will follow from that of ordinary difference equations.

What I said in the preceding article about the integration of linear partial difference equations is a particular case of the method that I just proposed because the linear equation always has the form:

$$L\,p + M\,q + N = 0 .$$

If one eliminates p by means of the equation $dz = p\,dx + q\,dy$ then one will have:

$$L\,dz + N\,dx + q\,(M\,dx - L\,dy) = 0\,,$$

and if one differentiates the latter equation while regarding q as the only variable then one will have:

$$M\,dx - L\,dy = 0\,,$$

and consequently:

$$L\,dz + N\,dx = 0\,,$$

which are the two equations that I gave, and which Lagrange had published before.

XXXII.

None of the integrals of partial difference equations, even the ones of first order in three variables, are capable of being put into the preceding form, because that form tacitly supposes that the equation belongs to a curved surface, and there is an infinite number of partial difference equations that belong to its curves of double curvature whose integral can only be expressed by a system of two simultaneous equations, between which there is nothing to eliminate, or by a system of three equations, between which one must eliminate one indeterminate, as I would like to show in the following example:

Let the problem be that of integrating the equation:

$$p - A\,y = \varphi\,(q + Ax)\,,$$

in which A is a constant and φ is an arbitrary function.

What is remarkable about that equation is the fact that if one sets $A = 0$ then it will become $p = \varphi\,(q)$, which belongs to all developable surfaces, and whose integral is known, whereas if one lets A remain then it will no longer belong to a curved surface, but to a curve of double curvature.

Indeed, set:

$$q + A\,x = \alpha\,,$$

to abbreviate, which will give:

$$p - A\,y = \varphi\,(\alpha)\,.$$

it is clear that the quantity α is an indeterminate about whose value nothing can be said, and which is destined to disappear by elimination. If one replaces p and q with the preceding values in $dz = p\,dx + q\,dy$ then one will have:

$$dz = \varphi\,(\alpha)\,dx + \alpha\,dy - A\,(x\,dy - y\,dx)\,.$$

Now, if that ordinary difference equation is integrable, while regarding α as constant, and if its integral, when completed by an arbitrary function of α, is $M = 0$ then it will be obvious from the principles of that type of calculus that the complete integral of the proposed equation will be the result of eliminating α from the two equations $M = 0$, $\left(\frac{dM}{d\alpha}\right) = 0$. However, the ordinary difference equation does not satisfy the integrability conditions, so it will belong to a curve of double curvature, which can be expressed in terms of finite quantities only by the system of two equations, and from article XVI, if the quantity α is an absolute constant then those two equations will be:

$$z = x\,\varphi(\alpha) + \alpha\,y + \psi\left(\frac{y}{x}\right),$$

$$\psi'\left(\frac{y}{x}\right) = -\,A x^2 .$$

Moreover, α is not an absolute constant, but it must only be regarded as constant in the preceding integration, i.e., that integration is taken while regarding the two quantities y / x and α as constants. Thus, the function ψ that completes the integral must be composed of only those quantities. Hence, the complete integral of the given equation will be the result of eliminating the indeterminate α from the three equations:

$$(A) \qquad z = x\,\varphi(\alpha) + \alpha\,y + \psi\left(\frac{y}{x}, \alpha\right),$$

$$(B) \qquad 0 = x\varphi'(\alpha) + y + \psi''\left(\frac{y}{x}, \alpha\right),$$

$$(C) \qquad \psi'\left(\frac{y}{x}, \alpha\right) = -\,A x^2,$$

the second of which is the differential of the first one, which is taken while regarding α as the only variable, and in which ψ is an arbitrary function of two quantities, while ψ' and ψ'' are the coefficients of du and dv in the differential of $\psi(u, v)$. The result of that elimination will consist of two equations that are those of the curve of double curvature that the partial difference equation belongs to.

Among the partial difference equations, there are ones that belong to the curves of double curvature then. Their finite integrals can be expressed only by a system of two equations, between which there is nothing to eliminate, and that integral will be completed by an arbitrary function of two quantities that the geometers have allowed only for the integrals of equations in four variables.

Since the preceding conclusion is extraordinary, I would like to verify it in several ways.

1. If one sets $A = 0$ in the integral then equation (C) will give $\psi' = 0$, which indicates that the quantity y / x is not included in the function ψ. Thus, that integral will reduce to two equations:

$$z = x\,\varphi(\alpha) + \alpha\,y + \varphi(\alpha),$$

$$0 = x\varphi'(\alpha) + y + \psi'(\alpha) \ ,$$

which is the known integral of the equation $p = \varphi(q)$ that one obtains upon likewise setting $A = 0$ in the given equation.

2. If one differentiates equation (A) while regarding α as constant, which is permissible by virtue of (B), then one will find that:

$$p = \varphi(\alpha) - \frac{y}{x^2}\psi'\left(\frac{y}{x}, \alpha\right) ,$$

$$q = \quad \alpha + \frac{1}{x}\psi'\left(\frac{y}{x}, \alpha\right) .$$

If one replaces ψ' with its value that is taken from (C) then one will have:

$$p = \varphi(\alpha) + A\,y ,$$
$$q = \quad \alpha \ - A\,x .$$

Finally, upon eliminating α, one will get:

$$p - A\,y = \varphi(q + A\,x) ,$$

which is the given equation.

3. If one takes two special cases, i.e., of one gives well-defined forms to the functions φ, ψ, and eliminates α from the three equations (A), (B), (C) then one will have two equations that will satisfy the given equation when differentiated.

XXXIII.

Upon operating in an analogous manner, one finds that if the given partial difference equation in four variables u, x, y, z is:

$$\left(\frac{dz}{dy}\right) + A u\,x = \varphi\left[\left(\frac{dz}{du}\right) - A\,x\,y, \left(\frac{dz}{dx}\right) - A u\,y\right] ,$$

in which A is an absolute constant, and φ is an arbitrary function of two quantities, then its complete integral will be the result of eliminating the two indeterminates α, β from the following four equations:

$$z = \alpha u + \beta x + y\, \varphi(\alpha, \beta) + \psi\left(\frac{u\,x}{y}, \alpha, \beta\right),$$

$$u + y\varphi'(\alpha,\beta) + \psi'' = 0\,,$$

$$x + y\varphi''(\alpha,\beta) + \psi''' = 0\,,$$

$$\psi' = A\,y^2\,,$$

the second and third of which are the differentials of the first one, which are taken while regarding α as the only variable for one of them and β, for the other. Moreover, ψ is an arbitrary function of the three quantities and, φ' and φ'' are the two coefficients in the difference of φ, while ψ', ψ'', ψ''' are the coefficients of the difference of ψ.

The integral calculus of ordinary difference equations, as well as that of partial difference equations, mutually depend upon each other, so all of the progress that one makes in the second of those two types of calculi will be useful in the first one. Hence, I shall cite some theorems that are not included among those of Lagrange.

XXXIV.

I shall suppose that the number of variables is arbitrary, and I shall set:

$$dz = p\,du + q\,dx + r\,dy + \ldots,$$

to abbreviate.

Theorem I:

If the quantities U, X, Y are given functions of u, x, y, respectively, then the complete integral of the partial difference equation:

$$F(z, p\,U, q\,X, r\,Y, \ldots) = 0$$

will depend upon only quadratures.

Let:

$$p\,U = \alpha\, r\, Y\,,$$
$$q\,X = \beta\, r\, Y\,,$$
$$\ldots\ldots\ldots\ldots\ldots,$$

in which α, β, … are indeterminates about whose values one can say nothing, and which are destined to disappear under elimination. When they are substituted for $p\,U$, $q\,X$, … in the given equation, that equation will become:

$$f(z, \alpha, \beta, \ldots, r\,Y) = 0\,,$$

from which one can infer the value of $r\,Y$ in terms of z, α, β, … Let that value be:

$$r\,Y = f(z, \alpha, \beta, \ldots)\,,$$

so one can conclude that:

$$p\,U = \alpha f(z, \alpha, \beta, \ldots)\,,$$
$$q\,X = \beta f(z, \alpha, \beta, \ldots)\,.$$

If one replaces p, q, r, … with those values in:

$$dz = p\,du + q\,dx + r\,dy + \ldots$$

then one will have:

$$\frac{dz}{f(z, \alpha, \beta, \ldots)} = \frac{\alpha\,du}{U} + \frac{\beta\,dx}{X} + \frac{dy}{Y} + \cdots,$$

which is an equation whose integral, while regarding α, β, … as constants, depends upon only quadratures and must be completed by an arbitrary function of the hypothetical constants α, β, … Let $M = 0$ be that integral, thus-completed. It cannot be employed by itself, since one must indicate what was regarded as constant in the integration. One will then have:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,,$$

$$\left(\frac{dM}{d\beta}\right) = 0\,,$$

$$\ldots\ldots\ldots\ldots\ldots,$$

simultaneously. All of those equations must be true independently of the values of α, β, …, so if one eliminates those indeterminates then one will have the complete integral of the given equation. The elimination in question cannot be performed, in general, because the quantities α, β, … are included in the arbitrary function and the partial differences.

XXXV.

Theorem II:

The quantities L, M, N, P, …, whose number is the same as the number of variables, are composed of variables and first partial differences in such a manner that if all of the following equations, except one, are given then the remaining one will necessarily follow:

$$\begin{aligned} dL &= 0 , \\ dM &= 0 , \\ dN &= 0 , \\ dP &= 0 , \\ &\ldots\ldots\ldots \end{aligned}$$

If one has an equation that is composed from all of those quantities, which I shall represent by:

$$F(L, M, N, P, \ldots) = 0 ,$$

in which F is an arbitrarily-given function, whether algebraic or transcendental and arbitrary or well-defined. One will have the integral of that equation upon first eliminating all of the partial differences and one of the two arbitrary functions φ or ψ from the following equations:

$$\begin{aligned} L &= \varphi(\alpha, \beta, \ldots) , \\ M &= \psi(\alpha, \beta, \ldots) , \\ N &= \alpha , \\ P &= \beta , \\ &\ldots\ldots \\ F(\varphi, \psi&, \alpha, \beta, \ldots) = 0 , \end{aligned}$$

in which F is the same function as the given one. That will produce a unique equation that I shall represent by $M = 0$, and one then eliminates all of the indeterminates $\alpha, \beta, \ldots$ from the equations:

$$M = 0 ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0 ,$$

$$\left(\frac{dM}{d\beta}\right) = 0 ,$$

$$\ldots\ldots\ldots$$

Example: Let the given equation be:

$$F\,[p, q, r, \ldots, (z - p\,u - q\,x - r\,y - \ldots)] = 0\,,$$

in which F indicates an arbitrarily-given function. Since all of the equations:

$$\begin{gathered} dp = 0\,, \\ dq = 0\,, \\ dr = 0\,, \\ \ldots\ldots\ldots \\ d\,(z - p\,u - q\,x - r\,y - \ldots) = 0 \end{gathered}$$

have the property that any one of them will follow from all of the other ones, the given equation falls within the scope of the theorem, and its integral will be the result of eliminating the indeterminates $\alpha, \beta, \ldots$ from the following equations:

$$F\,\{\varphi\,(\alpha, \beta, \ldots), \alpha, \beta, \ldots, [z - u\,\varphi\,(\alpha, \beta, \ldots) - \alpha\,x - \beta\,y - \ldots]\} = 0\,,$$

$$\left(\frac{dF}{d\alpha}\right) = 0\,,$$

$$\left(\frac{dF}{d\beta}\right) = 0\,,$$

$$\ldots\ldots\ldots\ldots\ldots$$

Lagrange gave an analogous result, but only for the case of two principal variables.

XXXVI.

On higher-order partial difference equations.

All that we said about first-order partial difference equations is analogously true for those of higher order. That is, if one sets:

$$\begin{aligned} dz &= p\,dx + q\,dy\,, \\ dp &= r\,dx + s\,dy\,, \\ dq &= s\,dx + t\,dy\,, \end{aligned}$$

to abbreviate, and one replaces r and t in a second partial difference equation $V = 0$ with their values in terms of dp, dq, dx, dy that one takes in the preceding equations then one will have an equation that no longer includes any other second partial differences besides s, and if one differentiates that equation while regarding s as the only variable and then eliminates s by means of that differential then one will have an ordinary difference equation $U = 0$ in the variables x, y, z, p, q whose integral will provide that of the given one.

The result of the ordinary differences that I represented in general by $U = 0$ can take several very different forms.

1. That result can include ordinary difference equations that will always be true when the given one is linear and in some cases of higher-degree differences. If the integrals of those two ordinary difference equations are $M = \alpha$, $N = \beta$, in which α and β are arbitrary constants that were introduced by the integrations, then the first integral of the given one will be $M = \varphi(N)$.

2. If the result of the ordinary differences includes only one higher-degree equation, and the complete integral of that equation is the result of eliminating the indeterminate α from the three equations of that form:

$$M = 0 ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0 ,$$

$$\left(\frac{ddM}{d\alpha^2}\right) = 0 ,$$

in which the first integral of the given equation will be the result of eliminating α from the first two equations:

$$M = 0 ,$$

$$\left(\frac{dM}{d\alpha}\right) = 0 .$$

3. Finally, if the result of the ordinary differences is a unique linear equation for which the old integrability conditions are not satisfies then one will proceed in a manner that is analogous to what was done in article XXXII.

I would like to provide an example of each of those three cases.

XXXVII.

Example I: Suppose that the problem is to integrate the second partial difference equation:

$$r\,t - s^2 + A = 0 ,$$

in which A is a constant.

I replace r and t with their values that are taken from:

$$dp = r\,dx + s\,dy , \qquad dq = s\,dx + t\,dy ,$$

which gives:

$$s\,(dq\,dy + dp\,dx) = dp\,dq + A\,dx\,dy\,,$$

from which one infers that:

$$dp = -\sqrt{A}\,dy\,,$$
$$dq = \sqrt{A}\,dx\,.$$

Now, those two equations are exact differences, and their complete integrals are:

$$p + \sqrt{A}\,y = \alpha\,,$$
$$q - \sqrt{A}\,x = \beta\,.$$

Thus, if one sets $\alpha = \varphi(\beta)$ then the complete first integral of the given equation will be:

$$p + \sqrt{A}\,y = \varphi\left(q - \sqrt{A}\,x\right)\,.$$

If one sets $A = 0$ in the given equation then it will become $r\,t - s^2 = 0$, which is the equation of a developable surface, and if one makes the same assumption in the integral then one will have $p = \varphi(q)$, which is one of the first integrals of that equation.

Now, the integral that was just found is precisely the equation that I treated in article XXXII. Hence, the complete integral of the given equation is the result eliminating the indeterminate α from the three equations:

$$z = x\,\varphi(\alpha) + \alpha\,y + \psi\left(\frac{y}{x}, \alpha\right)\,,$$

$$0 = x\varphi'(\alpha) + y + \psi''\left(\frac{y}{x}, \alpha\right)\,,$$

$$\psi'\left(\frac{y}{x}, \alpha\right) = x^2\sqrt{A}\,,$$

the second of which is the differential of the first one, when taken while regarding α as the only variable, and in which φ is an arbitrary function of one quantity and ψ is an arbitrary function of two quantities, while ψ' and ψ'' are the coefficients of du and dv in the differential of $\psi(u, v)$.

Therefore, the given equation belongs to a curve of double curvature, except in the case where one has $A = 0$. It will then belong to any developable surface.

XXXVIII.

Example II: Suppose that the problem is to integrate the equation:

$$(r\,t - s^2)^2 + 4\,r\,s = 0\,.$$

I eliminate r and t by means of the equations:

$$dp = r\,dx + s\,dy\,, \qquad dq = s\,dx + t\,dy\,,$$

which will give:

$$[dp\,dq - s\,(dq\,dy + dp\,dx)]^2 + 4s\,dx\,dy^2\,(dp - s\,dy) = 0\,.$$

If I differentiate that equation, while regarding s as the only variable and then eliminate s by means of that differential then I will find the ordinary difference equation:

$$dp\,dq = dy^2\,.$$

Now, that higher-degree equation is included in the ones that I treated in article VII, and upon setting:

$$(p - \alpha)\,(q - \varphi(\alpha)) - y^2 = M\,,$$

to abbreviate, its integral will be the result of eliminating α from the three equations:

$$M = 0\,,$$

$$\left(\frac{dM}{d\alpha}\right) = 0\,,$$

$$\left(\frac{ddM}{d\alpha^2}\right) = 0\,.$$

Hence, one of the first integrals of the given equation is the system of the first two of those three equations, i.e., the result of eliminating the indeterminate α from the following two:

$$(p - \alpha)\,(q - \varphi(\alpha)) - y^2 = 0\,,$$
$$(p - \alpha)\,\varphi'(\alpha) + q - \varphi(\alpha) = 0\,.$$

In order to get the integral, I infer the values of p and q from those two equations, which will give:

$$p = \alpha + \frac{y}{\sqrt{-\varphi'(\alpha)}}\,,$$
$$q = \varphi(\alpha) + y\sqrt{-\varphi'(\alpha)}\,,$$

and if I substitute those values in $dz = p\,dx + q\,dy$ then I will find that:

$$dz = \alpha\, dx + \varphi(\alpha)\, dy + \frac{y}{\sqrt{-\varphi'(\alpha)}}\,(dx - \varphi'(\alpha)) .$$

If the latter equation is integrable then upon regarding α as constant and its integral, when completed by an arbitrary function of α, if one knows $M = 0$ then the required finite integral will be the result of eliminating α from the two equations $M = 0$, $\left(\frac{dM}{d\alpha}\right) = 0$, and that integral will belong to a curved surface. However, the latter ordinary difference equation is not integrable while regarding a as constant, and in order for it to become so, it is further necessary to regard $x - y\varphi'(\alpha)$ as constant. I can then integrate under that double hypothesis, and it will then be necessary:

1. To complete the integral by an arbitrary function of the two quantities that are regarded as constants in the integration.

2. To express the two quantities that do not vary by two equations, which will give:

$$z = \alpha\, x + y\, \varphi(\alpha) + \psi\,[(x - y\varphi'(\alpha)), \alpha] ,$$

$$0 = x + y\varphi'(\alpha) - y\varphi''(\alpha)\psi' + \psi'' ,$$

$$\psi' = \frac{y}{\sqrt{-\varphi'(\alpha)}} .$$

Therefore, the finite integral of the given equation is the result of eliminating α from the preceding three equations, the second of which is the differential of the first, which is taken while regarding α as constant, and in which φ is an arbitrary function of one quantity, ψ is an arbitrary function of two quantities, ψ' and ψ'', are the coefficients of du and dv in the differential of $\psi(u, v)$. Hence, the given equation will belong to a curve of double curvature.

XXXIX.

Example III: Let the given equation be:

$$A\, r + B\, q + z = 0 ,$$

in which A and B are constants.

I substitute the value of r from $dp = r\, dx + d\, dy$ in that and find that:

$$A\, dp + (B\, q + z)\, dx = A\, s\, dy .$$

If I eliminate s by means of the differential, which I take while regarding s as the only variable, while regarding s as the only variable, which reduces to equating the two sides of the equation to zero, then I will get the two simultaneous ordinary difference equations:

$$A\,dp + (B\,q + z)\,dx = 0\,,$$
$$dy = 0\,,$$

in such a way that if the first one satisfies the integrability conditions and its integral is $M = 0$ then the first integral of the given one will be $M = \varphi\,(y)$. However, the first equation does not satisfy the integrability conditions, and its integral, when taken independently of the second equation, will be the system of two simultaneous equations:

$$A\,p + \varphi\,(x) = 0\,,$$
$$\varphi'(x) = B\,q + z\,.$$

Hence, that integral must be taken while supposing that the second equation is true, that the arbitrary function that completes it must also be a function of y, and that one must have the system of simultaneous equations:

$$A\,p + \varphi\,(x, y) = 0\,,$$
$$\varphi'(x, y) = B\,q + z$$

as a first integral to the given equation, in which φ' is the coefficient of dx in the differential of $\varphi(x, y)$.

That first integral can be put into another form, because if one infers the following values of p, q from those equations:

$$p = \frac{-\varphi(x, y)}{A}\,,$$
$$q = \frac{\varphi'(x, y) - z}{B}\,,$$

and substitutes that in $dz = p\,dx + q\,dy$ then one will have:

$$dz = \frac{-\varphi(x, y)\,dx}{A} + \frac{\varphi'(x, y)\,dy - z\,dy}{B}\,,$$

which is a first-order ordinary difference equation that includes no indeterminate and must persist unconditionally. It is itself the true first integral of the second-order partial difference equation, i.e., it expresses the same thing and has the same degree of generality as the given one. Because that equation does not satisfy the integrability conditions, and its integral can be expressed only by a system of two simultaneous equations, it will then follow that the given equation belongs to a curve of double curvature.

Now, in order to further integrate that equation once, $\varphi(x, y)$ must be the coefficient of dx in the differential of another arbitrary function $\psi(x, y)$ in such a manner that one has:

$$\varphi(x, y) = \psi'(x, y),$$
$$\varphi'(x, y) = \psi'''(x, y),$$

in which ψ''' is the coefficient of dx^2 in the second difference of ψ, and the ordinary difference equation will become:

$$dz = -\, d\,\frac{\psi(x, y)}{A} + \left[\frac{\psi'''(x, y) - z}{B} + \frac{\psi''(x, y)}{A}\right] dy \ ,$$

in which ψ'' is the coefficient of dy in the differential of $\psi(x, y)$. Now, from what I said about the integration of linear ordinary difference equations, the integral of that equation will be the system of two simultaneous equations:

$$z = -\,\frac{\psi(x, y)}{A} + \pi(y) \ ,$$
$$\pi'(y) = \frac{\psi'''(x, y) - z}{B} + \frac{\psi''(x, y)}{A},$$

in which π is an arbitrary function of the single quantity y. Therefore, that system of equation will also be the finite and complete integral of the proposed one, which is very easy to verify by differentiation.

XL.

The same procedure applies to equations of higher order. Permit me to cite one such example. Let the problem be to integrate the third-order partial difference equation:

$$\left(\frac{d^3 z}{dx^3}\right)\left(\frac{d^3 z}{dy^3}\right) = \left(\frac{d^3 z}{dx^2\, dy}\right)\left(\frac{d^3 z}{dx\, dy^2}\right) .$$

I eliminate three of those partial differences by means of the following three equations:

$$dr = \left(\frac{d^3 z}{dy^3}\right) dx + \left(\frac{d^3 z}{dx^2\, dy}\right) dy \ ,$$

$$ds = \left(\frac{d^3 z}{dx^2\, dy}\right) dx + \left(\frac{d^3 z}{dx\, dy^2}\right) dy \ ,$$

$$dt = \left(\frac{d^3z}{dx\,dy^2}\right) dx + \left(\frac{d^3z}{dy^3}\right) dy \;,$$

and if one preserves $\left(\frac{d^3z}{dx^3}\right)$ then they will give:

$$\left(\frac{d^3z}{dx^3}\right)(dt\,dy^2 - dr\,dx^2) \;= dr\;(ds\;dy - dr\;dx)\;.$$

I differentiate that equation while regarding $\left(\frac{d^3z}{dx^3}\right)$ as the only variable and eliminate that quantity by means of the differential, and since $\left(\frac{d^3z}{dx^3}\right)$ is linear, that comes down to equating each side of the equation to zero, and I will have the following two ordinary difference equations:

$$dr\;(ds\;dy - dr\;dx) = 0\;,$$
$$dt\,dy^2 - dr\,dx^2 \;= 0\;.$$

The first of those two equations has two roots, one of which is $dr = 0$, by virtue of which the second one will become $dt\,dy^2 = 0$, one of whose roots is $dt = 0$. Hence, one will have the two simultaneous equations:

$$dr = 0,$$
$$dt = 0\;.$$

Now, the complete integral of those equations is $r = \alpha$, $t = \beta$. Hence, one of the first integrals of the given equation will be:

$$t = \varphi(r)\;,$$

which is easy to verify by differentiation.

XLI.

The higher-degree partial difference equations are not the only ones that can belong to curves of double curvature. Most of the linear equations are also included in that category. We shall be content to see that by means of the equation:

$$(A) \qquad A\,r + B\,s + C\,t + D\,p + E\,q + F = 0\;,$$

in which all of the coefficients are constants. We have already treated that equation in the previous paper, article XXX, but the integral that we found there was also too specialized.

If one replaces r and t in that equation with their values that are taken from:

$$dp = r\,dx + s\,dy\,, \qquad dq = s\,dy + t\,dy$$

and then eliminates s by means of the differential that is taken while regarding s as the only variable then one will have two simultaneous ordinary difference equations:

$$(B) \qquad A\,dy^2 - B\,dx\,dy + C\,dx^2 = 0\,,$$

$$(C) \qquad A\,dp\,dy^2 - C\,dq\,dx + (D\,p + E\,q + F)\,dx\,dy = 0\,.$$

The roots of the first one are:

$$dy - k\,dx = 0 \quad \text{and} \quad dy + k'\,dx = 0\,,$$

in which k and k' are the roots of the algebraic equation:

$$A k^2 - B k + C = 0\,.$$

Therefore, upon employing the first root and introducing it into equation (C), the two simultaneous ordinary difference equations will become:

$$(D) \qquad dy - k\,dx = 0\,,$$

$$(E) \qquad A\,(dp + k'\,dq) + (D\,p + E\,q + F)\,dx = 0\,.$$

Those are the two equations that must give one of the first integrals of the given equation.

The integral of equation (D) is $y - k\,x = \alpha$, in which α is the arbitrary constant. If the integral of equation (E) is $M = \beta$ then the first integral will be $M = \varphi(\alpha)$. However:

1. Equation (E) does not belong to a curved surface, in general, and its integral can be a unique equation only in the case where the coefficients of the given equation satisfy the equation:

$$C\,D^2 + A\,E^2 = B\,D\,E\,.$$

In all other cases, the integral of the ordinary differential equation (E), when considered independently of equation (D), can be expressed only by the system of two simultaneous equations:

$$A\,(p + k'\,q) + \varphi(x) = 0\,,$$
$$\varphi'(x) = D\,p + E\,q + F\,.$$

2. Equation (E) must not be considered alone, and its integral must be taken while supposing that equation (D) is true, i.e., that α is constant. Thus, that integral must be completed, not by a

function of x alone, but by a function of x and α. Therefore, the form in which the first integral of the given equation first presents itself is the system of two equations:

$$A(p + k'q) + \varphi(x, y - k\,x) = 0\,, \tag{F}$$

$$\varphi'(x, y - k\,x) = D\,p + E\,q + F\,, \tag{G}$$

in which φ is an arbitrary function of two quantities, and φ' is the coefficient of du in the difference of $\varphi(u, v)$.

That integral is verified by differentiation. Furthermore, if one sets $D = 0$, $E = 0$, $F = 0$ then the given equation will become:

$$A\,r + B\,s + C\,t = 0\,,$$

and equation (G) will give $\varphi'(x, y - k\,x) = 0$, which expresses the idea that the function φ is composed from only the quantity $y - k\,x$. Thus, the first integral will reduce to the single known equation:

$$A(p + k'q) + \varphi(x, y - k\,x) = 0\,.$$

One infers the following values of p and q from the two equations (F), (G):

$$p(Dk' - E) = \frac{E}{A}\varphi(x, y - k\,x) - k'[F - \varphi'(x, y - k\,x)]\,,$$

$$q(Dk' - E) = -\frac{D}{A}\varphi(x, y - k\,x) + F - \varphi'(x, y - k\,x)]\,,$$

and upon substituting them in $dz = p\,dx + q\,dy$, one will find that:

$$(Dk' - E)\,dz = \frac{\varphi}{A}(E\,dx - D\,dy) + (F - \varphi')(dy - k'\,dx)\,, \tag{H}$$

which is an ordinary difference equation that expresses just the same thing as the given equation, when completed by a function of two quantities. If one exchanges the two quantities k, k' with each other in that equation then it will be obvious that one will have the other first integral of the given equation.

The two first integrals of the linear partial difference equation (A) will then be true ordinary difference equations in either case, in which the various ways that the quantity z can vary is no longer in question. Since those two equations do not satisfy the integrability condition, and their common integral can be expressed only by a system of two simultaneous equations, it will then follow that the given equation will not belong to a curved surface, in general, but to a curve of double curvature.

In order to integrate equation (H), one must first observe that the integral of the left-hand side must be a function of the two quantities $E\,x - D\,y$ and $y - k'\,x$, and then that the partial differences of that function must be equal to the two respective terms in the right-hand side. Therefore, the finite integral of equation (A) will be comprised of the system of three equations:

$$z\,(D\,k' - E) = \psi\,(E\,x - D\,y, y - k'\,x) \ ,$$
$$A\,\psi'\,(E\,x - D\,y, y - k'\,x) = \varphi(x, y - k\,x) \ ,$$
$$\psi''\,(E\,x - D\,y, y - k'\,x) = F - \varphi'\,(x, y - k\,x) \ ,$$

in which ψ is an arbitrary function of two quantities and ψ' and ψ'' are the coefficients of du and dv in the differential of $\psi\,(u, v)$. Of those three equations, the second one is destined to give the form of the function φ from that of the function ψ, and the other two equations are those of the curve of double curvature that is the locus of the given equation.

If we replace dx, dy in equations (D), (E) with their values that we infer from the following two equations:

$$dy - k\,dx = d\alpha\,, \qquad dy - k'\,dx = d\alpha'$$

and then operate as we did then we will find that the complete integral of the given equation is the system of two equations:

$$z = \pi\,(y - k\,x, y - k'\,x)\,,$$
$$(k\,D - E)\,\pi' + (k'D - E)\,\pi'' + \frac{B^2 - 4AC}{A}\,\pi''' - F = 0\,,$$

in which π is an arbitrary function of two quantities, π' and π'' are the coefficients of du, dv in the differential of $\pi\,(u, v)$, and π''' is one-half the coefficient of $du\,dv$ in the second differential of the same function.

Since the equation of an arbitrary curved surface can always be put into the form:

$$z = \pi\,(y - k\,x, y - k'\,x)\,,$$

it will follow that there is not curved surface on which one can trace out one of the curves of double curvature that satisfy the given equation. When the function p is such that the second equation is naturally satisfied, the equation of the surface will be itself a case of the complete integral because the surface will then be entirely composed of curves that satisfy the given equation.

What we just said about that example must also apply to the other cases that we treated in the preceding paper. By a similar argument, we will find that the complete first integral for the equation $r - t - 2\pi / x = 0$ in article XV of that paper is the ordinary difference equation:

$$dz + \varphi\,(x, y - x)\,dy - \varphi'\,(x, y - x)\,\frac{dx + dy}{2x} = 0\,,$$

in which φ is an arbitrary function of two quantities, and φ' is the coefficient of du in the differential of $\varphi(u, v)$. Finally, the finite integral is the system of two equations:

$$z = \psi(x + y, x - y),$$
$$\psi' + \psi'' = 2x\psi''',$$

in which ψ is an arbitrary function of two quantities, ψ', ψ'' are the coefficients of du, dv, resp., in the differential of $\varphi(u, v)$, and ψ''' is one-half the coefficient of $du\,dv$ in the second differential of the same function.

General conclusions.

It results from this supplement that:

1. The ordinary difference equations for which the integrability conditions are not satisfied contain nothing absurd or impossible, and they are susceptible to a true integration in finite terms.

2. The integrals of those equations are completed by arbitrary functions of variable quantities, which are functions that one will further need to employ only for the integrals of partial difference equations.

3. The integrability conditions have the sole objective of indicating the number of equations that must define the finite integral after any elimination of the indeterminates is supposed to have been performed.

4. The integral of a greater number of partial difference equations is not susceptible to being expressed by just one equation, even when one supposes that the elimination of all indeterminates has been done. For example, in the case of three variables, the greatest number of partial difference equations will belong to curves of double curvature, and not to curved surfaces, as all of the ordinary methods of integration tacitly suppose, so the number of quantities that enter into the arbitrary function will then be greater than that of the principal variables, when reduced by one unit.

5. There are certain partial difference equations whose intermediate integrals are true ordinary difference equations.

6. Finally, that geometry can once more make very great progress because one now has the means to analyze some new ways of generating curves and because one has the power to understand a great number of properties of size that are expressed by relations that involve equations that have been regarded as intractable up to now.

Addition.

In the foregoing supplement, I constructed several higher-degree ordinary difference equations. However, I did not construct any of them for any the linear equations that do not satisfy the integrability conditions that I had integrated. I would like to show what that type of equations signify in space by way of an example.

If one supposes that an eye, when reduced to a single point, is placed in an arbitrary manner with respect to a curved surface then I will say the *apparent contour line* of that surface to mean the curve that is composed of the extreme points of that surface that the eye can perceive. That will be the line of contact between the curved surface and a conical surface that circumscribes it and whose summit is at the point where the eye is: I then suppose that the problem is to find the apparent contour line of an arbitrary surface of revolution around the z-axis, as seen by an eye that is situated at the point whose coordinates are a, b, c, independently of the generating curve of the surface.

The equation of the surface of revolution is:

$$z = \varphi(x^2 + y^2) ,$$

and its partial difference equation is:

$$p\,y - q\,x = 0 ,$$

The equation of a conical surface with an arbitrary base and its summit at the eye-point is:

$$\frac{z-c}{x-a} = \psi\left(\frac{y-b}{x-a}\right) ,$$

and its partial difference equation is:

$$p\,(x - a) + q\,(y - b) = z - c .$$

Now, it is obvious that for the required curve, not only are the x, y, z of the surface of revolution and the conical surface the same, respectively, but the quantities p, q are also the same for the two surfaces, since those two surfaces have the same tangent plane all along the curve. Hence, the five quantities x, y, z, p, q will have the same values in the two equations:

$$p\,y - q\,x = 0 , \qquad p\,(x - a) + q\,(y - b) = z - c .$$

Thus, if one takes the values of p and q in those two equations and substitutes them in $dz = p\,dx + q\,dy$, which is just as true for both surfaces, then one will have:

$$[x\,(x - a) + y\,(y - b)]\,dz = (z - c)\,(x\,dx + y\,dy) ,$$

which is a linear ordinary difference equation that does not satisfy the integrability condition and which, when considered alone, will express the apparent contour of an any surface of revolution around the z-axis, as seen by an eye that is placed at the point whose coordinates are a, b, c. The integral of that equation will be the system of the following two:

$$z = \varphi(x^2 + y^2) ,$$
$$2\,[x\,(x - a) + y\,(y - b)]\,\varphi' = \varphi - c ,$$

in which φ is an arbitrary function.

One then sees that any ordinary difference equation in three variables of first degree that does not satisfy the integrability condition will belong to the contact curve between two general curved surfaces, i.e., two surfaces that are each given by a linear partial difference equation.
