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Canonical systems of total differential equations in the theory of groups of transformations.

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It is known that the complete integration of an involutory system of first-order partial differential equations that does not contain the unknown function and the complete integration of the *associated canonical system* of first-order total differential equations are equivalent problems analytically, just as in the classical Hamilton-Jacobi theory, the search for a complete solution of just one partial differential equation, when solved for one of the derivatives, and the complete integration of the associated Hamiltonian system (*) are equivalent problems.

The canonical systems enjoy the property that they transform into each other under all of the contact transformations that operate on *the dependent variables*, as well as those transformations that contain the independent variables. The infinite group of all those contact transformations contains a subgroup of transformations that is also infinite, namely, the ones that leave an arbitrarily-chosen canonical system, since they are the infinitude of contact transformations that convert one given canonical system into another one (**).

No matter how one proceeds with the integration of an involutory system of first-order partial differential equations, whether with the Jacobi method, properly speaking, the method that I proposed to the Lombardo Institute in 1883 as an extension of Mayer's method, or Cauchy's method, as one might call it, one will always be dealing with the associated system of total differential equations. With the first procedure, one will find only one integral: When one continues with that procedure, one will construct another analogous involutory system with one more equation, which is associated with a canonical system with two less dependent variables, but one more independent one. However, with the second procedure, the associated canonical system can be integrated completely.

The famous *Lie theorem* also takes on its most-general, and at the same time simplest, interpretation when one considers the canonical system that is associated with an involutory system of first-order partial differential equations.

^(*) Cf., my articles on that, which were published in the Rendiconti del R. Istituto Lombardo: "I. Il metodo di Pfaff per l'integrazione equazioni a derivate parziali del I. O.," (2) **16** (1883), pp. 637 and 691; "II. Intorno ai sistemi d'equazioni a derivate parziali del I. O in involuzione," (2) **36** (1903), pp. 775.

^(**) However, there is also an infinitude of *non-contact transformations* that transform a given canonical system into itself or any other one. Cf., the last § of the second of my cited notes and the present one.

In my opinion, the consideration of canonical systems introduces a uniformity and simplicity of method and an elegance of form into the theory of first-order partial differential equations that is worthy of note. Therefore, permit me to point out to the Academy the results of my research into canonical systems of total differential equations and their relationship to the theory of transformation groups.

§ 1. – Consider the *m* infinitesimal transformations of the 2n + m variables:

$$p_1, q_1; p_2, q_2; \ldots; p_n, q_n; t_1, t_2, \ldots, t_m$$

whose symbols are:

(1)
$$U_r f \equiv \frac{\partial f}{\partial t_r} + (f, f_r) \qquad (r = 1, 2, ..., m),$$

in which the f_r are functions of all the variables that verify the m(m-1)/2 relations identically:

(2)
$$\frac{\partial f_r}{\partial t_s} - \frac{\partial f_s}{\partial t_r} + (f_r, f_s) = 0 \qquad (r, s = 1, 2, ..., m).$$

The symbol (f, g) in that formula denotes the Poisson parentheses, or:

$$(f,g) \equiv \sum_{i} \frac{\partial(f,g)}{\partial(p_{i},q_{i})}$$
.

Since it will result identically that:

$$U_r U_s f = U_s U_r f$$
, or: $(U_r, U_s) f \equiv 0$,

as one verifies immediately, the *m* infinitesimal transformations will generate a group of *m* parameters $z_1, ..., z_m$, whose most-general infinitesimal transformation is:

(3)
$$\delta p_i = \sum_{i=1}^m \frac{\partial f_s}{\partial q_s} \delta z_s, \quad -\delta q_i = \sum_{i=1}^m \frac{\partial f_s}{\partial p_s} \delta z_s, \quad \delta t_r = \delta z_r$$
$$(i = 1, 2, ..., n; r = 1, 2, ..., m).$$

If we assume that the values of the parameters that correspond to the identity transformation then we will have:

$$t_r = t_r^0 + z_r^{},$$

and the remaining formulas of the transformation are obtained by integrating the system of total differential equations:

(4)
$$dp_i = \sum_s \frac{\partial f_s}{\partial q_i} dt_s, \qquad dq_i = -\sum_s \frac{\partial f_s}{\partial q_i} dt_s,$$

in such a way that the p_i , q_i will become p_i^0 , q_i^0 , respectively, for $t_r = t_r^0$. That system of total differential equations, which is *completely integrable*, due to (2), is what one calls *a canonical system*.

Let:

(4')
$$\begin{cases} p_i = p_i(p_1^0, \dots, q_n^0; t_1, \dots, t_m; t_1^0, \dots, t_n^0), \\ q_i = q_i(p_1^0, \dots, q_n^0; t_1, \dots, t_m; t_1^0, \dots, t_n^0) \end{cases}$$

be the general solution of (4). One will then have a group of transformations that are paircommuting and have m essential parameters, and their equations will be:

(5)
$$\begin{cases} p_i = p_i(p_1^0, \dots, q_n^0; t_1^0 + z_1, \dots, t_m^0 + z_m), \\ q_i = q_i(p_1^0, \dots, q_n^0; t_1^0 + z_1, \dots, t_m^0 + z_m), \\ t_r = t_r^0 + z_r. \end{cases}$$

Now, one has:

$$U_{s}\left\{\sum_{i}q_{i}\,dp_{i}-\sum_{r}f_{r}\,dt_{r}\right\}=-\sum_{i}\left\{\frac{\partial f_{s}}{\partial p_{i}}\,dp_{i}+q_{i}\,d\frac{\partial f_{s}}{\partial p_{i}}\right\}-\sum_{r}\left\{\frac{\partial f_{s}}{\partial p_{i}}+(f_{i},f_{s})\right\}dt_{r}.$$

By reason of (2), one will also have:

$$\frac{\partial f_r}{\partial t_s} + (f_r, s_r) = \frac{\partial f_s}{\partial t_r},$$

and therefore:

(6)
$$U_{s}\left\{\sum_{i}q_{i}\,dp_{i}-\sum_{r}f_{r}\,dt_{r}\right\}=d\sum_{i}\left\{q_{i}\frac{\partial f_{s}}{\partial q_{i}}-f_{s}\right\}.$$

Consider the integral:

$$I = \int \left(\sum_{i} q_{i} \, dp_{i} - \sum_{r} f_{r} \, dt_{r} \right),$$

which extends over an arbitrary closed line in the domain of the variables p_i , q_i , t_r . For the infinitesimal transformation (3), if:

$$\delta I = \sum_{s} U_{s} I \cdot \delta z_{s}$$

then one will have:

 $\delta I = 0$.

Therefore, I is an integral invariant of the system of differential equations (4) relative to the closed line.

It is known that: $\delta dt_r = 0$, and therefore: $dt_r = dt_r^0$.

If one integrates the canonical system completely and expresses the final values p_i , q_i of the dependent variables as functions of their initial (arbitrary) values p_i^0 , q_i^0 , the t_r , and their initial values t_r^0 , then one will have:

(7)
$$\sum_{i} q_{i} dp_{i} - \sum_{r} f_{r} dt_{r} = \sum_{i} q_{i}^{0} dp_{i}^{0} - \sum_{r} f_{i}^{0} dt_{r} + d\Omega$$

identically, in which Ω denotes a function of all of the independent variables.

In follows from this, in particular, that when one regards the independent variables t as constant parameters, the integral equations of a canonical system will establish a contact transformation between the dependent variables and their initial values.

§ 2. – The integral equations (4') of (4) can certainly be solved for the q_i and p_i^0 since it is legitimate to consider the variables p_i , q_i^0 ; t_r and t_r^0 to be independent in the identity (7). In addition, it is known that when one sets:

$$V=\Omega+\sum_i p_i^0 q_i^0,$$

(7) can be written:

$$\sum_{i} q_{i} dp_{i} - \sum_{r} f_{r} dt_{r} = -\sum_{i} p_{i}^{0} dq_{i}^{0} - \sum_{r} f_{i}^{0} dt_{r} + dV$$

since in order for this equation to be an identity, V must necessarily be expressible in terms of only p_i , q_i^0 ; t_r , t_r^0 , and one will have:

(8)
$$\begin{cases} q_i = \frac{\partial V}{\partial p_i}, \quad p_i^0 = \frac{\partial V}{\partial q_i^0}, \\ \frac{\partial V}{\partial t_r} + \frac{\partial V}{\partial t_i^0} + f_r = f_r(p_1^0, \dots, q_n^0; t_1^0, \dots, t_m^0) \equiv f_r^0. \end{cases}$$

If we use the symbol δ to indicate the infinitesimal variations along the trajectory of the group, as we have done before, and take (6) into account then:

$$\delta d \Omega = \delta \left\{ \sum_{i} q_{i} dp_{i} - \sum_{r} f_{r} dt_{r} \right\} = d \sum_{s} \left(\sum_{i} q_{i} \frac{\partial f_{s}}{\partial q_{i}} - f_{s} \right) \delta t_{s}.$$

If one integrates along the trajectory in the group that starts from the initial values p_i^0 , q_i^0 , t_r^0 then one will have:

$$d\Omega = d\int \sum_{s} \left(\sum_{i} q_{i} \frac{\partial f_{s}}{\partial q_{i}} - f_{s} \right) \delta t_{s},$$

in which, thanks to (4'), the expression under the integral sign will necessarily become an exact differential in the t_r whose variables are the only ones that are varied in the course of the integration (*).

Since only $d \Omega$ can figure in (7), with no further analysis, one assumes that:

$$\Omega = \int_{(t_s^0)}^{(t_0)} \sum_{s} \left(\sum_{i} q_i \frac{\partial f_s}{\partial q_i} - f_s \right) dt_s,$$

which is a function that is calculated by expressing the function under the integral sign in terms of t_r , p_i^0 , q_i^0 , t_r^0 the using (4').

Calculate the variation $D \Omega$ of the Ω directly under infinitesimal variations of only p_i^0 and q_i^0 . Thanks to (4), one will find that:

$$D \Omega = \int \sum (\delta q_i D p_i + q_i D \delta p_i) = \int \delta \sum_i q_i D p_i = \sum_i (q_i D p_i - q_i^0 D p_i^0) .$$

If one also regards the t_r as variable then one will have:

$$d \Omega = D \Omega + \sum_{r} \frac{\partial \Omega}{\partial t_{r}} dt_{r}$$

$$= D \Omega + \sum_{r} \left(\sum_{i} q_{i} \frac{\partial f_{r}}{\partial q_{i}} - f_{r} \right) dt_{r}$$

$$= D \Omega + \sum_{i} q_{i} \delta p_{i} - \sum_{r} f_{r} dt_{r}$$

$$= \sum_{i} (q_{i} dp_{i} - q_{i}^{0} dp_{i}^{0}) - \sum_{r} f_{r} dt_{r},$$

$$\frac{\partial}{\partial t_r} \left[\sum_i q_i \frac{\partial [p_i]}{\partial t_s} - f_s \right] = \sum_i [q_i] \frac{\partial^2 [p_i]}{\partial t_s \partial t_s} - \left[\frac{\partial f_s}{\partial t_r} \right] + \sum_i \left[\frac{\partial f_s}{\partial q_i}, \frac{\partial f_r}{\partial p_i} \right],$$

which is an expression that will not change when one switches the indices r, s, due to the identity (2). Q. E. D.

^(*) It is easy to verify that directly, moreover. One lets $[p_i]$ and $[q_i]$ denote the functions in the right-hand sides of (4'), and in general [g] will denote the result of the substitution (4') in any function g of the p_i and q_i . If one takes (4) into account then one will immediately find that:

in which one writes dp_i^0 in place of Dp_i^0 , if it will not create any confusion, when the p_i^0 are considered to be independent variables.

Therefore, *if one holds only the* t_r^0 *constant* then one will have:

$$dV = \sum_{i} q_{i} dp_{i} + \sum_{i} p_{i}^{0} dq_{i}^{0} - \sum_{r} f_{r} dt_{r},$$

and it will follow from this that:

$$q_i = \frac{\partial V}{\partial p_i}, \qquad p_i^0 = \frac{\partial V}{\partial q_i^0}, \qquad \frac{\partial V}{\partial t_r} + f_r = 0.$$

In addition, when one compares that with the last of (8), one will get:

$$\frac{\partial V}{\partial t_r^0} = f_r^0,$$

which is a conclusion that agrees with the one that Jacobi reached in the nineteenth lecture of his celebrated *Vorlesungen über Dynamik* concerning the second partial differential equation that the principal function of a dynamical problem must satisfy, according to Hamilton, but our conclusion is more general than Jacobi's.

Calculate Ω and then V in the manner that was described. If one eliminates the p_i^0 from the first group using (4) then one will get a complete solution (up to an additive constant) of the involutory system:

(9)
$$0 = \frac{\partial V}{\partial t_r} + f_r \left(t_1, \dots, t_m; p_1, \dots, p_n; \frac{\partial V}{\partial p_1}, \dots, \frac{\partial V}{\partial p_n} \right)$$

that is the associate of the canonical system (4).

That is the procedure for integrating such a system that I arrived at in 1883 in the first of my cited articles.

§ 3. – In order to integrate the canonical system (4), one fixes the *path of integration* in the domain of the independent variables t_r . However, one replaces the t_r and dt_r with:

$$t_r^0 + (t_r - t_r^0)t$$
, $(t_r - t_r^0)dt$

respectively, and considers the t_r to be constants (like the t_r^0), so one varies only t.

In that way, (4) will be converted into the Hamiltonian system:

(10)
$$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}, \qquad \frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$$

in which:

$$H = \sum_{r} (t_r - t_r^0) f_r (\dots, t_s^0 + (t - t_s^0) t, \dots; p_1, \dots, q_n).$$

That is integrated in such a way that the p_i , q_i will become p_i^0 , q_i^0 , respectively, for t = 0. When the integration is complete, one then sets t = 1, and the expressions that are obtained for p_i , q_i will give the general solution of (4).

Now, integrating (10) is analytically equivalent to finding a complete solution of the partial differential equation:

(11)
$$\frac{\partial V}{\partial t} + H\left(t_1; p_1, \dots, p_n; \frac{\partial V}{\partial p_1}, \dots, \frac{\partial V}{\partial p_n}\right) = 0.$$

If one knows the general solution to (10) then a complete solution to (11) will be given by:

$$V = \sum_{s} p_i^0 q_i^0 + \int_0^t \left(\sum_{i} q_i \frac{\partial H}{\partial q_i} - H \right) dt,$$

in which, as usual, the quadrature is performed by expressing the p_i , q_i as functions of t, p_i^0 , q_i^0 , by means of the general solution of (10) and then eliminating the p_i^0 from the first group of integral equations:

$$p_i = p_i (p_1^0, \dots, q_n^0, t)$$
.

The integral equations in (10) can be put into the form:

$$q_i = rac{\partial V}{\partial p_i}, \qquad p_i^0 = rac{\partial V}{\partial q_i^0},$$

and then the ones in (4) can be obtained immediately by setting t = 1, from what was said before.

One easily concludes from this that V will be converted into a complete solution of the involutory system (9) when one sets t = 1 since from the calculation of Ω in the previous §, when one fixes the path of integration, as above, one will have:

$$\Omega = \int_0^1 \left(\sum_i q_i \frac{\partial H}{\partial q_i} - H \right) dt ,$$

and one will then see that the solution that was obtained in the preceding § will coincide with the one that was just indicated.

Therefore:

The complete integration of the involutory system (9) reduces to the integration of the single equation (11). Thus, when one finds the solution of (11) that reduces to the linear function $\sum q_i^0 p_i$

for t = 0, one will obtain a complete solution to the given involutory system by setting t = 1.

That is one of the many forms in which one can put Lie's theorem (*).

§ 4. – If one establishes a contact transformation that depends upon t_r between two systems of variables:

$$(p_i, q_i); (p_i^*, q_i^*)$$
 $(i = 1, 2, ..., n),$

and therefore a transformation such that when one holds the tr constant, one will have:

$$\sum_i q_i dp_i = d\Phi + \sum_i q_i^* dp_i^* .$$

If one also keeps the *t_r* variable then one will have:

(12)
$$\sum_{i} q_i dp_i - \sum_{r} f_r dt_r = d\Phi - \sum_{r} f_r^* dt_r$$

in which the f_r^* denote functions of p_i^* , q_i^* , and t_r .

If one introduces the p^* , q^* in place of the old p, q in the group of transformations (5) then one will get a new group of transformations that is similar to the old ones. Since the product of contact transformations is always a contact transformation, one can conclude that the new group has the same characteristic property as the old one, and in particular, any of its transformations will define a contact transformation between the p_i^* , q_i^* , and their initial values. Therefore, the most general infinitesimal transformation of the new group will again be of type (3).

That leads one to expect that a canonical system of total differential equations will transform into another system that is also canonical under any contact transformation of the p_i , q_i .

When (4) is multiplied by the arbitrary variations δq_i , $-\delta p_i$ and summed over all values of the index *i*, that will give:

$$\sum_{i} \left(\delta q_i \, dp_i - \delta p_i \, dq_i \right) = \sum_{r} \left(\delta f_r \, dt_r - \delta t_r \, df_r \right)$$

^(*) Cf., Rend. Ist. Lomb. (2) 36, pp. 784.

when one agrees to set $\delta t_r = 0$.

That equation can be written:

$$\delta\left\{\sum_{i} q_{i} dp_{i} - \sum_{r} f_{r} dt_{r}\right\} - d\left\{\sum_{i} q_{i} \delta p_{i} - \sum_{r} f_{r} \delta t_{r}\right\} = 0,$$

and one will get (4) from it once more by setting the coefficients of all quantities δp_i , δq_i , and δt_r equal to zero since the *m* equations that one obtains by annulling the coefficients of the δt_r will be a consequence of (5), by reason of (2) (*). It will then follow that when one performs the contact transformation between the (p_i, q_i) and (p_i^*, q_i^*) , (4) will transform into:

$$dp_i^* = \sum_r \frac{\partial f_r^*}{\partial q_i^*} dt_r, \qquad dq_i^* = -\sum_r \frac{\partial f_r^*}{\partial p_i^*} dt_r,$$

due to (12).

In order to prove that this system is also canonical, one needs to prove, in addition, that one has:

$$\frac{\partial f_r^*}{\partial p_s} - \frac{\partial f_s^*}{\partial p_r} + (f_r^*, f_s^*)^* = 0$$

identically, in which the Poisson parentheses (,)^{*} are meant to be formed with the new variables. It follows from (12) that:

$$\begin{cases} q_i = \frac{\partial \Phi}{\partial p_i} + \sum_j q_j^* \frac{\partial p_j^*}{\partial p_i}, \\ 0 = \frac{\partial \Phi}{\partial q_i} + \sum_j q_j^* \frac{\partial p_j^*}{\partial q_i}, \\ -f_r = \frac{\partial \Phi}{\partial t_r} + \sum_j q_j^* \frac{\partial p_j^*}{\partial t_r} - f_r^*, \end{cases}$$

and when one eliminates Φ from that, one will get:

^(*) Cf., § 2 of recent, previously-cited, article: "Intorno ai sistemo di equaz. a deriv. parz. del I. O. in involuzione," Rend. dell'Istituto Lomb. for the current year.

(A)
$$\begin{cases} \sum_{j} \frac{\partial(p_{j}^{*}, q_{j}^{*})}{\partial(p_{i}, p_{k})} = 0, \quad \sum_{j} \frac{\partial(p_{j}^{*}, q_{j}^{*})}{\partial(q_{i}, q_{k})} = 0, \quad \sum_{j} \frac{\partial(p_{j}^{*}, q_{j}^{*})}{\partial(p_{i}, q_{k})} = 0, \quad \sum_{j} \frac{\partial(p_{j}^{*}, q_{j}^{*})}{\partial(p_{i}, p_{i})} = 1, \\ -\frac{\partial f_{r}}{\partial p_{i}} = -\frac{\partial f_{r}^{*}}{\partial p_{i}} + \sum_{j} \frac{\partial(p_{j}^{*}, q_{j}^{*})}{\partial(t_{r}, p_{i})}, \quad -\frac{\partial f_{r}}{\partial q_{i}} = -\frac{\partial f_{r}^{*}}{\partial q_{i}} + \sum_{j} \frac{\partial(p_{j}^{*}, q_{j}^{*})}{\partial(t_{r}, q_{i})}, \\ \frac{\partial f_{r}}{\partial t_{s}} - \frac{\partial f_{s}}{\partial t_{r}} = \frac{\partial f_{s}^{*}}{\partial t_{s}} - \frac{\partial f_{s}^{*}}{\partial t_{r}} - \sum_{j} \frac{\partial(p_{j}^{*}, q_{j}^{*})}{\partial(t_{r}, t_{s})}. \end{cases}$$

One will obtain the following well-known relations from the first four of those equations by a known procedure:

(B)
$$\begin{cases} (p_i^*, p_k^*) = (q_i^*, q_k^*) = 0 & (i, k = 1, 2, \dots, k), \\ (p_i^*, q_k^*) = 0 & (i \neq k); & (p_i^*, q_i^*) = 1, \end{cases}$$

from which, it will follow that the Poisson parentheses are invariant under contact transformations, i.e.:

(C)
$$(f,g) = (f,g)^* = \sum_j \frac{\partial(f,g)}{\partial(p_j^*,q_j^*)},$$

in which the arbitrary functions f, g in the second Poisson parenthesis are regarded as being expressed in terms of the p^*, q^* .

It follows from the fifth and sixth equations in (A) that:

$$\sum_{i} \frac{\partial f_{r}}{\partial p_{i}} \frac{\partial f_{s}}{\partial q_{i}} = \sum_{i} \frac{\partial f_{r}^{*}}{\partial p_{i}} \frac{\partial f_{s}}{\partial q_{i}} + \sum_{i} \sum_{j} \sum_{k} \frac{\partial (p_{j}^{*}, q_{j}^{*})}{\partial (t_{r}, p_{i})} \frac{\partial (p_{k}^{*}, q_{k}^{*})}{\partial (t_{s}, q_{i})} - \sum_{i} \sum_{j} \left\{ \frac{\partial f_{r}^{*}}{\partial p_{i}} \frac{\partial (p_{j}^{*}, q_{j}^{*})}{\partial (t_{s}, q_{i})} + \frac{\partial f_{s}^{*}}{\partial q_{i}} \frac{\partial (p_{j}^{*}, q_{j}^{*})}{\partial (t_{r}, p_{i})} \right\}.$$

With an easy calculation and the help of the first 5 of (A), that will imply:

$$(f_r, f_s) =$$

$$(f_r^*, f_s^*) - \sum_i \sum_j \frac{\partial p_i^*}{\partial t_r} \frac{\partial (q_j^*, f_s)}{\partial (p_i, q_i)} + \sum_i \sum_j \frac{\partial p_j^*}{\partial t_s} \frac{\partial (q_j^*, f_r)}{\partial (p_i, q_i)} + \sum_i \sum_j \frac{\partial q_i^*}{\partial t_r} \frac{\partial (p_j^*, f_s)}{\partial (p_i, q_i)} - \sum_i \sum_j \frac{\partial q_i^*}{\partial t_s} \frac{\partial (q_j^*, f_r)}{\partial (p_i, q_i)} + \sum_i \sum_j \frac{\partial q_i^*}{\partial (p_i, q_i)} \frac{\partial (p_i^*, f_s)}{\partial (p_i, q_i)} - \sum_i \sum_j \frac{\partial q_i^*}{\partial t_s} \frac{\partial (q_j^*, f_r)}{\partial (p_i, q_i)} + \sum_i \sum_j \frac{\partial q_i^*}{\partial (p_i, q_i)} \frac{\partial (p_i^*, f_s)}{\partial (p_i, q_i)} - \sum_i \sum_j \frac{\partial q_i^*}{\partial (p_i^*, f_r)} \frac{\partial (q_i^*, f_r)}{\partial (p_i, q_i)} + \sum_i \sum_j \frac{\partial$$

However, with the use of (C), and when one the expressions for f_r^* , f_s^* in terms of the p_i^* , q_i^* , and t_r by (f_r^*) , (f_s^*) , for the moment, one will have:

$$(f_r, f_s) = (f_r^*, f_s^*) + \sum_j \frac{\partial(p_j, p_j)}{\partial(t_r, t_s)} + \sum_i \left\{ \frac{\partial(f_s^*)}{\partial p_i^*} \frac{\partial p_i^*}{\partial t_r} + \frac{\partial(f_s^*)}{\partial q_i^*} \frac{\partial q_i^*}{\partial t_r} \right\} - \sum_i \left\{ \frac{\partial(f_r^*)}{\partial p_i^*} \frac{\partial p_i^*}{\partial t_s} + \frac{\partial(f_r^*)}{\partial q_i^*} \frac{\partial q_i^*}{\partial t_s} \right\}$$
$$= (f_r^*, f_s^*) + \sum_j \frac{\partial(p_j, p_j)}{\partial(t_r, t_s)} + \frac{\partial f_s^*}{\partial t_r} - \frac{\partial f_r^*}{\partial t_s} + \frac{\partial(f_r^*)}{\partial t_s} - \frac{\partial(f_s^*)}{\partial t_r} \right\},$$

and from the last of (A) and (C):

$$\frac{\partial f_r}{\partial t_s} - \frac{\partial f_s}{\partial t_t} + (f_r, f_s) = \frac{\partial (f_r^*)}{\partial t_s} - \frac{\partial (f_s^*)}{\partial t_r} + (f_r^*, f_s^*)^*$$

Therefore:

Any contact transformation of the p_i , q_i will convert any canonical system into another system that is likewise canonical, or the transform of the group (5) will be a similar group endowed with the same properties.

§ 5. – Consider the contact transformation:

(14)
$$q_i = \frac{\partial \Omega}{\partial p_i}, \qquad -q_i^* = \frac{\partial \Omega}{\partial p_i^*},$$

in which Ω denotes a function of the p_i , p_i^* , t_r that is chosen in such a way that the functional determinant:

$$\frac{\partial \left(\frac{\partial \Omega}{\partial p_1^*}, \dots, \frac{\partial \Omega}{\partial p_n^*}\right)}{\partial (p_1, \dots, p_n)}$$

will not be identically zero.

For such a transformation, (12) will become:

$$\sum_{i} q_{i} dp_{i} - \sum_{r} f_{r} dt_{r} = d \Omega - \sum_{i} q_{i}^{*} dp_{i}^{*} - \sum_{r} \left(f_{r} + \frac{\partial \Omega}{\partial t_{r}} \right) dt_{r}.$$

Consequently, if Ω is a complete solution of the involutory system (9) with the non-additive p_1^* , ..., p_n^* , or if:

(9')
$$\frac{\partial \Omega}{\partial t_r} + f_r \left(t_1, \dots, t_m; p_1, \dots, p_n; \frac{\partial \Omega}{\partial p_1}, \dots, \frac{\partial \Omega}{\partial p_n} \right) = 0$$

then it will result that:

$$\sum_{i} q_i dp_i - \sum_{r} f_r dt_r = d \Omega + \sum_{i} q_i^* dp_i^*,$$

and one will therefore have:

 $f_r^* = 0$.

Therefore, the transform of the canonical system (4) will have the solved form, i.e., the integrals of (4) will be the p_i^* , q_i^* . We then have the theorem:

With a contact transformation (14), if one knows how to obtain an arbitrary complete solution of the associated involutory system (9') then the canonical system (4) can be integrated, and with that translation, the group (5) will transform into a group of translations that is similar to the given group, as is known.

Since the inverse of a contact transformation is also a contact transformation, one can obviously conclude that:

The property of being reducible to the solved for by a contact transformation is characteristic of canonical systems.

Observe that a canonical system (C) with less than *r* independent variables can always be regarded as a particular canonical system with *r* independent variables: That is because if s < r, and one supposes that $f_1, f_2, ..., f_s$ does not depend upon $t_{s+1}, ..., t_r$, and that $f_{s+1} = f_{s+1} = ... = f_r = 0$ then one will have a system (C) with only *s* independent variables.

Consider two canonical systems (C_1) , (C_2) that contain the same number of dependent variables. They can be reduced to the solved form with two contact transformations T_1 , T_2 . However, a system of the solved form will remain invariant under an *arbitrary transformation S* that acts upon the dependent variables, but not the independent ones. Consequently, the most-general transformation that converts (C_1) into (C_2) is:

$$T_1 S T_2^{-1}$$
.

In particular, (C_1) is transformed into itself by the infinite group of transformations that is similar to the group of S:

$$T_1 S T_1^{-1}$$
.

The theory of transformations of canonical systems, and in particular Hamiltonian systems, will take on the greatest simplicity as a result of the preceding theorems (*).

^(*) Cf., my paper: "Sulla trasformazione delle equazioni differenziali di Hamilton," which was published in three articles that were included in the Rend. dell'Accademia dei Lincei (5.a) v. XII, pt. 1, pps. 114-122, 149-152, 297-300.