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On Hamilton's dynamical equations

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The dynamical equations of Lagrangian type are summarized in the equations of virtual work when they are expressed in terms of arbitrary position parameters (*).

The pair of dynamical equations of Hamiltonian type are summarized in a peculiar form into which one can put the work equation, and which is called the *Hamiltonian form.*

The systematic use of the Hamiltonian form of the work equations presents some noteworthy advantages, and above all in the transformation of the canonical equations, which was brought to light by ÉMILE MATHIEU in his invaluable *Dynamique analytique* (Paris, 1878) (**).

The Hamiltonian form of the equation of virtual work makes it obvious that the transformations of the canonical equations are contact transformations, and in particular, one can clearly deduce a generalization of the *fundamental theorem of the theory of perturbations* from that, as I propose to show in the article.

1. – Let $p_1, p_2, \ldots, p_N$ be an arbitrary system of position parameters, and let $p_1', p_2', \ldots, p_N'$ be their derivatives with respect to time *t*. Let *T* be the *vis viva* when it is expressed in terms of the *p*, p', t. Let $\delta\mathcal{L}$ be the virtual work done by the applied forces, when expressed in the form:

$$\delta\mathcal{L} = \sum_{\iota=1} \mathcal{P}_\iota \, \delta p_\iota ,$$

so the Lagrangian equation of virtual work will be the following one:

$$\text{[I]} \qquad \frac{d}{dt} \sum_{\iota=1} \frac{\partial T}{\partial p_\iota'} \delta p_\iota = \delta T + \delta\mathcal{L} .$$

(*) Cf., **§ 2** of my note: "Sulle equazioni di Lagrange," that I presented the Academy at the session on 11 January 1903 (Atti…, vol. XXXVIII).

(**) Cf., also the article by Prof. SIACCI: "Teorema fondamentale nella teoria delle equazioni canoniche del moto," Mem. della R. Acc. dei Lincei (3), vol. XII.

The δp in that equation are coupled by a certain number of homogeneous linear equations:

$$\varphi_\delta^{(\nu)} \equiv \varphi_1^{(\nu)}\,\delta p_1 + \varphi_2^{(\nu)}\,\delta p_2 + \cdots + \varphi_N^{(\nu)}\,\delta p_N = 0 \qquad (\nu = 1, 2, \ldots, N - n),$$

while the p and the time are coupled by the differential equations:

$$\varphi^{(\nu)} \equiv \varphi_0^{(\nu)} + \varphi_1^{(\nu)}\,p_1' + \varphi_2^{(\nu)}\,p_2' + \cdots + \varphi_N^{(\nu)}\,p_N' = 0\,,$$

in which the $\varphi_\iota^{(\nu)}$ are functions of the p and t.

Observe that the $\delta p_\iota' = d\,\delta p_\iota / dt$ vanish identically from [I]. Therefore, the latter quantities *can be considered to be arbitrary*.

Set:

[II] $$\frac{\partial T}{\partial p_1'} = q_1\,,\quad \frac{\partial T}{\partial p_2'} = q_2\,,\ \ldots,\ \frac{\partial T}{\partial p_N'} = q_N\,.$$

As is easy to see, those equations are always soluble for the p_ι'. Indeed, let T_2, T_1, T_0 be the parts of T that are homogeneous of degree 2, 1, 0, resp., in the p_ι'. One will observe that T_2 is obtained from the original expression of the *vis viva:*

$$T = \tfrac{1}{2}\sum m(x'^2 + y'^2 + z'^2)\,,$$

in which one sets:

$$x' = \sum_\iota \frac{\partial x}{\partial p_\iota}\,p_\iota'\,, \qquad y' = \sum_\iota \frac{\partial y}{\partial p_\iota}\,p_\iota'\,, \qquad z' = \sum_\iota \frac{\partial y}{\partial p_\iota}\,p_\iota'\,.$$

Therefore, T_2 is a quadratic form that is homogeneous in the p_ι' that is essentially positive, and consequently its discriminant will necessarily be positively non-zero. The linear equations [II], when written in the form:

$$\frac{\partial T_2}{\partial p_\iota'} = q_\iota - \frac{\partial T_1}{\partial p_\iota}\,,$$

will then express the p_ι' as linear functions of the q_ι that are generally inhomogeneous and have coefficients that depend upon the p_ι and t.

Set:

$$K = \sum_\iota p_\iota' q_\iota - T$$

and imagine that K is expressed in terms of the p_ι, q_ι, and t. One will then have:

$$\delta K = \delta \sum_{\iota} p'_{\iota} q_{\iota} - \delta T .$$

When the q_ι are introduced into [I], it will become:

$$\sum_{\iota} q_{\iota} \, \delta p_{\iota} = \delta T + \delta \mathcal{L} ,$$

and when that is added to the previous one, that will give:

$$\frac{d}{dt} \sum_{\iota} q_{\iota} \, \delta p_{\iota} - \delta \sum_{\iota} q_{\iota} \frac{dp_{\iota}}{dt} = \delta \mathcal{L} - \delta K .$$

Therefore, the equation of virtual work can be written:

$$\text{[III]} \qquad d \sum_{\iota} q_{\iota} \, \delta p_{\iota} - \delta \sum_{\iota} q_{\iota} \, dp_{\iota} = (\delta \mathcal{L} - \delta K) \, dt$$

or also

$$\text{[III}'\text{]} \qquad \sum_{\iota} (dq_{\iota} \, \delta p_{\iota} - \delta q_{\iota} \, dp_{\iota}) = (\delta \mathcal{L} - \delta K) \, dt .$$

That is the Hamiltonian form of the work equation: The δp in it can be considered to be coupled by the constraint equations: $\varphi_{\delta}^{(\nu)} = 0$, while the δq, like the $\delta p'$ in Lagrangian form, *remain arbitrary*.

I shall not repeat how one deduces the $N + n$ first-order differential equations of motion, like [III], but I shall only observe that the constraint equations: $\varphi_{\delta}^{(\nu)} = 0$, will now become finite equations between the p_ι, q_ι, and t, and that one will then have:

$$\text{[IV]} \qquad \varphi^{(\nu)} \equiv \varphi_0^{(\nu)} + \sum_{\iota=1}^{N} \varphi_{\iota}^{(\nu)} \frac{\partial K}{\partial q_{\iota}} = 0 \qquad (\nu = 1, 2, \ldots, N - n).$$

2. – In the case in which the forces admit a force function U, which might also depend upon time, if one sets:

$$H = K - U$$

then [III] will become:

$$\text{[III}'\text{]} \qquad \delta \left[\sum_{\iota} q_{\iota} \, dp_{\iota} - H \, dt \right] - d \left[\sum_{\iota} q_{\iota} \, \delta p_{\iota} - H \, \delta t \right] = 0 .$$

The differential equations of motion are obtained by setting $\delta t = 0$ and equating the coefficients of the δp and δq to zero in the equation that results by adding $\varphi_\delta^{(\nu)} = 0$ to [III′], when it is first multiplied by the indeterminate multipliers $\lambda_\nu\, dt$.

However, one can arrive at the same result by setting the coefficients of the δp_ι, δq_ι, δt in [III″] equal to zero, by first adding products of the:

$$\varphi_0^{(\nu)}\,\delta t + \sum_\iota \varphi_\iota^{(\nu)}\,\delta p_\iota = 0$$

with the multipliers $\lambda_\nu\, dt$. If one indeed proceeds in that way then one will obtain the equations:

$$\left.\begin{aligned} dp_\iota - \frac{\partial H}{\partial q_\iota}\,dt = 0\,, \\ -\frac{\partial H}{\partial q_\iota}\,dt - dq_\iota + \sum_\nu \lambda_\nu\,\varphi_\iota^{(\nu)} = 0\,, \end{aligned}\right\} \qquad (i = 1, 2, \ldots, N),$$

$$-\frac{\partial H}{\partial q_\iota}\,dt - dq_\iota + \sum_\nu \lambda_\nu\,\varphi_\iota^{(\nu)} = 0\,.$$

However, the last one can be written:

$$\sum_\iota \left(\frac{\partial H}{\partial q_\iota}\,dq_\iota + \frac{\partial H}{\partial p_\iota}\,dp_\iota\right) = \sum_\nu \lambda_\nu \sum_\iota \varphi_\iota^{(\nu)}\,\frac{\partial H}{\partial q_\iota}\,dt\,,$$

due to [IV], and in that form it appears to be a consequence of the previous 2*N*.

Therefore:

The differential equations of motion are the first Pfaff system of the differential expression $\sum_\iota p_\iota\,dq_\iota - H\,dt$, *in which the variables are coupled by the total differential equations:*

$$\varphi_0^{(\nu)}\,dt + \sum_{\iota=1}^{N} \varphi_\iota^{(\nu)}\,dp_\iota = 0 \qquad (\nu = 1, 2, \ldots, N - n).$$

That theorem is fundamental for an invariant theory of the transformation of the most general dynamical equations.

In particular, if a moving body is subject to holonomic constraints then one can assume that $p_1, \ldots, p_N$ are a system of independent parameters and then that $N = n$, and the differential equations of motion will have the canonical Hamiltonian form:

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_\iota}, \qquad \frac{dq_i}{dt} = -\frac{\partial H}{\partial p_\iota} \qquad (i = 1, 2, \ldots, n).$$

They are known to be the first Pfaff system of the differential expression:

$$\sum_\iota q_\iota \, dp_\iota - H \, dt .$$

3. – Let us return to the Hamiltonian equations of virtual work:

$$\text{[III]} \qquad \delta \sum_\iota q_\iota \, dp_\iota - d \sum_\iota q_\iota \, \delta p_\iota = (\delta K - \delta \mathcal{L}) \, dt .$$

One applies a special contact transformation that also depends upon time to the variables p_ι, q_ι, and therefore a transformation:

$$\left. \begin{aligned} p_\iota &= p_\iota(p_1^*, \ldots, p_N^*; q_1^*, \ldots, q_N^*; t), \\ q_\iota &= q_\iota(p_1^*, \ldots, p_N^*; q_1^*, \ldots, q_N^*; t), \end{aligned} \right\}$$

such that *if one regards t as a constant* then a relation of the form:

$$\sum_\iota q_\iota \, \delta p_\iota = \sum_\iota q_\iota^* \, \delta p_\iota^* + \delta \Omega$$

will be verified identically, in which Ω denotes of function of the variables: $p_1^*, \ldots, q_N^*$, and t.

If one also regards the t as variable then one will have:

$$\sum_\iota q_\iota \, dp_\iota = \sum_\iota q_\iota^* \, dp_\iota^* + r \, dt + d\Omega$$

identically, in which:

$$r = \sum_\iota q_\iota \frac{\partial p_\iota}{\partial t} - \frac{\partial \Omega}{\partial t},$$

and therefore:

$$\delta \sum_\iota q_\iota \, dp_\iota - d \sum_\iota q_\iota \, \delta p_\iota = \delta \sum_\iota q_\iota^* \, dp_\iota^* - d \sum_\iota q_\iota^* \, \delta p_\iota^* + \delta r \cdot dt .$$

Therefore, the work equation will become:

$$\text{[IV]} \qquad \delta \sum_\iota q_\iota^* \, dp_\iota^* - d \sum_\iota q_\iota^* \, \delta p_\iota^* = (\delta K^* - \delta \mathcal{L}^*) \, dt ,$$

in which:

$$K^* = K - r ,$$

and in which $\delta\mathcal{L}^*$ denotes the expression for the elementary work in the new variables, which will have the form:

$$\delta\mathcal{L}^* = \sum_\iota \mathcal{P}_\iota^* \, \delta p_\iota^* + \sum_\iota \mathcal{Q}_\iota^* \, \delta q_\iota^* .$$

In other words:

Special contact transformations will not alter the Hamiltonian form of the equation of virtual work.

In particular, if the moving body is a holonomic system whose position is defined by the independent parameters p_ι then one will have the following proposition as a corollary:

An arbitrary special contact transformation on the p and q will convert any canonical system of differential equations into another system that is also canonical.

4. – Consider a transformation that does not contain time, i.e., one of the form:

(S) $$p_\iota = p_\iota \, (P_1, P_2, \ldots, Q_n) : \qquad q_\iota = q_\iota \, (P_1, P_2, \ldots, Q_n) ,$$

and apply it to an arbitrary Hamiltonian system with a characteristic function h.

One will obviously have:

$(dp \, \delta q - dq \, \delta p)$

$$= \sum_k \sum_l \frac{\partial(p,q)}{\partial(P_k, P_l)} dP_k \, \delta P_l + \sum_k \sum_l \frac{\partial(p,q)}{\partial(Q_k, Q_l)} dQ_k \, \delta Q_l + \sum_k \sum_l \frac{\partial(p,q)}{\partial(P_k, Q_l)} (dP_k \, \delta Q_l - dQ_k \, \delta P_l) \; .$$

Therefore, if one sets:

$$\sum_i \frac{\partial(p_i, q_i)}{\partial(P_k, Q_l)} = [P_k, P_l] , \qquad \sum_i \frac{\partial(p_i, q_i)}{\partial(Q_k, Q_l)} = [Q_k, Q_l] , \qquad \sum_i \frac{\partial(p_i, q_i)}{\partial(P_k, P_l)} = [P_k, Q_l] ,$$

to abbreviate then it will result that:

$$\text{[V]}\qquad \begin{cases} \sum_{\iota} (dp_\iota\, \delta q_\iota - dq_\iota\, \delta p_\iota) = \sum_{(k,l)} [P_k, P_l](dP_k\, \delta P_l - dP_l\, \delta P_k) \\ \quad + \sum_k \sum_l [P_k, Q_l](dP_k\, \delta Q_l - dQ_l\, \delta P_k) \\ \quad + \sum_{(k,l)} [Q_k, Q_l](dQ_k\, \delta Q_l - dQ_l\, \delta Q_k) = \delta h \cdot dt. \end{cases}$$

In order for the transform of the Hamiltonian system with the characteristic function h to be another system that is also Hamiltonian with the characteristic H when one replaces the dP_ι , dQ_ι in the preceding equation with the expressions:

$$dP_\iota = \frac{\partial H}{\partial Q_\iota}\, dt , \qquad dQ_\iota = -\frac{\partial H}{\partial P_\iota}\, dt ,$$

respectively, it will be converted into an identity, and therefore it must be:

$$\text{[V}_1\text{]}\qquad \begin{cases} \sum_{(k,l)} [P_k, P_l]\left(\frac{\partial H}{\partial Q_k}\delta P_l - \frac{\partial H}{\partial Q_k}\delta P_k\right) + \sum_{(k,l)} [Q_k, Q_l]\left(\frac{\partial H}{\partial P_l}\delta Q_k - \frac{\partial H}{\partial P_k}\delta Q_k\right) \\ \qquad + \sum_k \sum_l [P_k, P_l]\left(\frac{\partial H}{\partial q_k}\delta Q_l + \frac{\partial H}{\partial p_k}\delta P_k\right) = \delta h. \end{cases}$$

In other words, the expression on the left-hand side must be an exact differential, and if H is arbitrary then there will be no other possibility than that one must have:

$$\text{[VI]}\qquad [P_k, P_l] = [Q_k, Q_l] = 0 , \qquad [P_k, Q_k] = c , \quad [P_k, Q_l] = 0$$

identically for them, in which c denotes a constant, and the indices k, l are distinct in the last relation.

Indeed, if one replaces H with H^2 in [V$_1$] then it will become:

$$2\, H\, \delta h = \text{exact differential},$$

so H will be a function of only h. In addition, if H' is the function that corresponds to another arbitrary function h' then when one replaces the H in [V$_1$] with, one will have:

$$H'\, \delta h + H\, \delta h' ,$$

and that expression must also be an exact differential, so:

$$\frac{dH}{dh} = \frac{dH'}{dh'} = \frac{1}{c} .$$

If one substitutes $c\,\dfrac{\partial H}{\partial \cdots} = \dfrac{\partial h}{\partial \cdots}$ in [V_1] and takes into account the arbitrariness in the derivatives of h and the δ then one will conclude [VI] immediately.

[V], along with [VI], will become:

$$c\left\{\delta \sum_i Q_i\, dP_i - d \sum_i Q_i\, \delta P_i\right\} = \delta \sum_i q_i\, dp_i - d \sum_i q_i\, \delta p_i \;,$$

so (*):

[VII] $$c \sum_i Q_i\, dP_i = \sum_i q_i\, dp_i + d\,\Omega .$$

We have then obtained the following theorem:

A transformation of the p and q that is independent of time and converts any canonical system into another system that is also canonical will necessarily be a contact transformation.

5. – The contact transformation [VII], which does not contain *t*, will convert the arbitrary Hamiltonian system:

$$\frac{dp_\iota}{dt} = \frac{\partial h}{\partial q_\iota}, \qquad \frac{dq_\iota}{dt} = -\frac{\partial h}{\partial p_\iota}$$

into the Hamiltonian system:

$$\frac{dP_\iota}{dt} = \frac{\partial H}{\partial Q_\iota}, \qquad \frac{dQ_\iota}{dt} = -\frac{\partial H}{\partial P_\iota} \qquad\qquad \left(H = \frac{h}{c}\right),$$

and as a consequence, the partial differential equation that is added to the first one will be converted into the one that is added to the second, and therefore:

$$\frac{\partial f}{\partial t} + (f, h) = 0$$

will be converted into:

$$\frac{\partial f}{\partial t} + \left(f, \frac{h}{c}\right)_{P,Q} = 0 .$$

However, since:

$$(f, h) = \sum_{(k,l)} \left\{\frac{\partial(f,h)}{\partial(P_k, P_l)}(P_k, P_l) + \frac{\partial(f,h)}{\partial(P_k, Q_l)}(Q_k, Q_l)\right\} + \sum_k \sum_l \frac{\partial(f,h)}{\partial(P_k, Q_l)}(P_k, Q_l) = \frac{1}{c}(f, h)_{P,Q} \;,$$

(*) Bear in mind that the bilinear covariant of a differential expression is annulled identically when subjects the variables to the relations that make that expression become an exact differential.

one will conclude:

$$\text{[VIII]} \qquad (P_k, P_l) = (Q_k, Q_l) = 0\,, \qquad (P_k, Q_k) = \frac{1}{c}\,, \quad (P_k, Q_l) = 0 \qquad (k \neq l)\,,$$

which are equations that must be a consequence of [VI]. In fact, with a very elegant procedure that is due to Professor A. MAYER (Math. Ann. vol. VIII, pp. 307), one can transform [VI] into the one that one deduces from [VIII] by exchanging the variables (P, Q) with (p, q), and c with $1/c$. In addition to [VI], one can deduce, by a known artifice, that:

$$\frac{\partial(p_1, \ldots, q_n)}{\partial(P_1, \ldots, Q_n)} = \pm c^n\,,$$

and as a consequence of verifying [VI] (or also [VIII]), the p, q will be mutually-independent functions of the P, Q.

From the considerations of the preceding §, it would seem that [VI], or its equivalent [VIII], are *sufficient* conditions for the differential expressions $\sum_i q_i\, dp_i$, $c\sum_i Q_i\, dP_i$ to have identical bilinear covariants, and with the aid of [V], one will see immediately that those conditions are also *necessary,* regardless of whether the transformation between the (p, q) and the (P, Q) is or is not independent of time. In other words, [VI], or its equivalent [VIII], is the necessary and sufficient condition for the transformation to be a contact transformation, or also for one to have:

$$(f, h)_{P,\,Q} = c\,(f, h)$$

identically, no matter what the functions f and h might be.

We would not like to take into account those transformations that have the effect of multiplying all of the q and the characteristic function h of any Hamiltonian system by the same constant c, so we shall assume that $c = 1$, or as we already did in § **3**, we shall define the contact transformations between the (p, q) and (P, Q) to be the ones that verify the relation:

$$\sum_i q_i\, dp_i = \sum_i Q_i\, dP_i + d\,\Omega$$

identically.

A characteristic property of those transformation is that the bilinear covariant of the differential expression $\sum_i q_i\, dp_i$ is invariant or that the Poisson functions are invariants.

I though it would be appropriate to present some things is this § that are well known today solely for the reader's convenience.

6. – Consider a holonomic system whose position is defined by the independent parameters p_1, p_2, …, p_n . We subdivide the applied forces on the moving body into two categories: The ones of the first category are what we call *principal forces*, which are derived from the force function U. The ones of the second category are the ones that we call *perturbing forces*, which are arbitrary and do virtual work $\delta\mathcal{L}$.

Set:

$$K - U = H ,$$

so the Hamiltonian equation of virtual work will become:

$$\delta \sum_i q_i \, dp_i - d \sum_i q_i \, \delta p_i = (\delta H - \delta\mathcal{L}) \, dt .$$

Apply the contact transformation that is defined by the equations:

$$q_i = \frac{\partial \Omega}{\partial p_i} , \qquad - q_i^* = \frac{\partial \Omega}{\partial p_i^*} ,$$

in which Ω denotes an arbitrary function of the p_i, p_i^*, and t that is subject to the single limitation that the determinant:

$$\frac{\partial \left(\dfrac{\partial \Omega}{\partial p_1}, \ldots, \dfrac{\partial \Omega}{\partial p_n} \right)}{\partial (p_1^*, \ldots, p_n^*)}$$

does not prove to be zero identically.

For such transformations, it will result that:

$$\sum_i q_i \, dp_i - \sum_i q_i^* \, dp_i^* + \frac{\partial \Omega}{\partial t} dt = d\,\Omega ,$$

and therefore, with the notation of § **3**:

$$- r = \frac{\partial \Omega}{\partial t} .$$

Therefore, the transform of the equation of virtual work will be:

$$\text{[IX]} \qquad \delta \sum_i q_i^* \, dp_i^* - d \sum_i q_i^* \, \delta p_i^* = \delta \left(\frac{\partial \Omega}{\partial t} + H \right) dt - \delta\mathcal{L}^* \, dt .$$

Assume that Ω is a complete solution of the partial differential equation:

$$\frac{\partial \Omega}{\partial t} + H \left(t; p_1, \ldots, p_n ; \frac{\partial \Omega}{\partial p_1}, \ldots, \frac{\partial \Omega}{\partial p_n} \right) = 0$$

with the non-additive arbitrary constants $p_1^*, \ldots, p_n^*$. [IX] will then become:

$$\sum_i (\delta q_i^* \, dp_i^* - \sum_i \delta p_i^* \, dq_i^*) = -\delta \mathcal{L}^* \cdot dt .$$

and since the δq^*, δp^* are arbitrary, that will imply the equations:

$$\frac{dp_i^*}{dt} = -\mathcal{Q}_i^* , \qquad \frac{dq_i^*}{dt} = \mathcal{P}_i^* \qquad (i = 1, 2, \ldots, n).$$

In particular, if the perturbing forces are zero then one will have $\mathcal{P}_i^* = \mathcal{Q}_i^* = 0$, and therefore, *the dynamical equations will assume the solved form*, i.e., the p^*, q^* will be integrals of the Hamiltonian system:

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i} , \quad \frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i} ,$$

and one therefore obtains Hamilton's theorem as a corollary.

In summary, we have the following *theorem:*

Divide the applied forces on the moving body into two categories: Principal forces, which are derived from a force function U, and perturbing forces, which can be arbitrary. In addition to the independent position parameters $p_1, \ldots, p_n$ *, introduce the conjugate variables:*

$$q_1 = \frac{\partial T}{\partial p_1} , \qquad \ldots, \qquad q_n = \frac{\partial T}{\partial p_n} ,$$

and form Hamilton's partial differential equation:

$$\frac{\partial \Omega}{\partial t} + H\left(t; p_1, \ldots, p_n ; \frac{\partial \Omega}{\partial p_1}, \ldots, \frac{\partial \Omega}{\partial p_n} \right) = 0$$

$$(H = \sum_i p_i' q_i - T - U) .$$

If one finds a complete solution of that equation with non-additive arbitrary constants $p_1^*, \ldots,$ p_n^* *then the integral equations of the unperturbed motion:*

$$q_i = \frac{\partial \Omega}{\partial p_i} , \qquad -q_i^* = \frac{\partial \Omega}{\partial p_i^*}$$

will give the p_i, q_i *as function of the* p_i^*, q_i^*, *and t, and if one introduces the* p_i^*, q_i^* *as new variables then the differential equations of perturbed motion will become:*

$$\frac{dp_i^*}{dt} = -\mathcal{Q}_i^*, \qquad \frac{dq_i^*}{dt} = \mathcal{P}_i^* \qquad (i = 1, 2, \ldots, n),$$

in which the $\mathcal{P}_i^*$, $\mathcal{Q}_i^*$ *are the coefficients of the* δq_i^*, δp_i^* *in the expression for the virtual work of the perturbing force, when transformed into the new variables.*

In particular, if the perturbing force is also derived from a force function Θ then when one expresses that as a function Θ^* of the new variables, the differential equations of perturbed motion will again assume the canonical form, as is known, and one will then have:

$$\frac{dp_i^*}{dt} = -\frac{\partial \Theta^*}{\partial q_i^*}, \qquad \frac{dq_i^*}{dt} = \frac{\partial \Theta^*}{\partial p_i^*}.$$

7. – As a complement to the results that were obtained in §§ **3** and **4**, we shall prove the following theorem:

Any transformation of the p and q that converts any canonical system of differential equations into another system that is also canonical will be a contact transformation.

First of all, observe that a system in the solved form:

$$dp_i = dq_i = 0$$

will be converted into a system that is also in solved form, and none of the transformations *S* will contain time.

Let *T* be a transformation that converts any canonical system into another canonical system. Consider a particular canonical system that can be reduced to the solved form by the contact transformation C_1, and with the help of *T*, one converts it into another canonical system that can be, in turn, reduced to solved form by the contact transformation C_2.

The transformation $C_1^{-1} T C_2$ will then convert a system in solved form into another one that is also in solved form, so one will have:

$$C_1^{-1} T C_2 = S.$$

However, when the transformation *S* is applied to an arbitrary canonical system, it will convert it into another one that is also canonical, as will its components C_1^{-1}, *T*, C_2. Therefore, *S* will be a contact transformation (§ **4**), and thus:

$$T = C_1 S C_2^{-1}$$

will also be a contact transformation.

As I have observed on another occasion (*), in general, for the systems of canonical equations in total differentials, there will also be an infinitude of *non-contact transformations* then that convert a given canonical system into another one that is also canonical. In truth, the transformation $C_1 S C_2^{-1}$, in which S is an arbitrary transformation that does not contain time, will convert a canonical system that is reducible to solved form by the transformation C_1 into the canonical system that is reducible to solved form by the transformation C_2 . However, in general, when the aforementioned transformation is applied to another Hamiltonian system, it will convert it into a system of non-canonical differential equations.

(*) Cf., my article: "I sistemi canonici di equaz. ai diff. totali, ecc.," Atti dell'Accademia, vol. XXXVIII, page 940.